EXACT NONLINEAR ATTITUDE CONTROL LAWS FOR A FLEXIBLE SPACECRAFT

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Memorandum No. UCB/ERL M89/14

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EXACT NONLINEAR ATTITUDE CONTROL LAWS FOR A FLEXIBLE SPACECRAFT

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ABSTRACT

This paper presents exact nonlinear control laws for a flexible spacecraft consisting of a rigid body with an attached Euler-Bernoulli modelled flexible beam. Equations of motion and kinematics for the structure are derived. In the case of significant beam damping, it is shown that a rigid body control law (derived using the methods of exact linearization) is sufficient for performing the desired maneuvers. In the case of negligible beam damping, it is shown that a modified control law consisting of a rigid body control law and a beam boundary control law also performs the desired attitude maneuvers. Implementation issues are also discussed.

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1. INTRODUCTION

It is now well known that a number of nonlinear control systems of engineering interest can be transformed by a static state feedback and a nonlinear change of coordinates into an equivalent linear system [De L. 1], [De L. 2], [Mey. 1]. In particular, in the area of attitude control, the method has proved to be quite useful. Dwyer [Dwy. 1] used this method of linearizing transformations to obtain exact nonlinear continuous time control laws for large angle rotational maneuvers for a rigid body by use of external thrusters. Similar methods are employed in [Dwy. 3] to design control laws for a rigid body controlled by both external thrusters and momentum wheels.

This method has also been successfully been used for designing a nonlinear attitude control law for a satellite with flexible appendages. In [Mon. 1], the control law was derived for a satellite with its flexible appendages modelled by their finite dimensional modal approximation. However, implementation of the control scheme required information about the beam velocities and displacements at several points of the beam. In practice, these are difficult measurements to make.

The purpose of this paper is to outline the design and implementation of a nonlinear feedback control law for a satellite with flexible appendages without the restrictions of [Mon. 1]. The spacecraft to be considered will be a rigid body with a single flexible appendage attached to the rigid body. The appendage will be modelled as an Euler-Bernoulli type beam, rather than its finite dimensional approximation. The control law will be derived using linearizing transformations in the spirit of the above papers, but the implementation will be considerably different than [Mon. 1] in that it will not depend on the beam displacements and velocities, but rather on the forces and moments at the point of attachment. These quantities can easily be determined by the use of strain rosettes.

2. PROBLEM DESCRIPTION AND MATHEMATICAL MODEL

The problem considered in this paper is the so-called attitude control problem. This consists of finding a control law to change the orientation of the spacecraft to that of a specified orientation. This might be the case, for instance, if the satellite were to be pointed at an earth based ground station. We now consider the spacecraft model, and the resulting kinematics and dynamics in order to solve the problem.

The physical model is depicted in Figure 1. The structure consists of a rigid body in which a thin, flexible, cantilevered beam-like appendage of length L is attached.

Affix a "body" coordinate frame, denoted $\{O_B,\ \underline{b}_1, \underline{b}_2, \underline{b}_3\}$, to the rigid body center of mass O_B. $\underline{b}_1$, $\underline{b}_2$ and $\underline{b}_3$ are orthonormal vectors which coincide with the rigid body principal axes of inertia; in addition, assume the $\underline{b}_3$ axis coincides with the centroidal axis of the undeflected beam. Control inputs (not shown in Figure 1) consist of three thrusters and three momentum wheels, where for i=1, 2, 3, the torque jets J_i producing a torque τ_i about the $\underline{b}_i$ axis, and the ith momentum wheel spins about an axis parallel to $\underline{b}_i$, thus also producing a torque τ_i' about $\underline{b}_i$.

Finally, let $\{O_E,\ \underline{e}_1, \underline{e}_2, \underline{e}_3\}$ denote the inertial frame. The attitude problem thus becomes that of aligning $\underline{b}_1, \underline{b}_2, \underline{b}_3$ with $\underline{e}_1, \underline{e}_2, \underline{e}_3$ using the control jets..

Change of Basis

As shown in [Kane 1, p. 4] the orientation of the body frame with respect to the inertial frame may be determined at each instant t by the direction cosine matrix $Y \in \mathbb{R}^{3x3}$ defined as follows:

$$[\underline{b}_1\ \underline{b}_2\ \underline{b}_3] = [\underline{e}_1\ \underline{e}_2\ \underline{e}_3]Y. \tag{2.1}$$

Note that Y is a coordinate transformation that maps vectors from the body frame to the inertial frame. That is, (assuming O_B coincides with O_E) if $\underline{v}=v_1\underline{b}_1 + v_2\underline{b}_2 + v_3\underline{b}_3$, then the components of $Y(v_1\ v_2\ v_3)^T$ are the components of the vector $\underline{v}$ resolved along the inertial axes $\underline{e}_1, \underline{e}_2, \underline{e}_3$.

This matrix may be parametrized by the so called "Gibbs Vector", or "Rodrigues vector" [Kane 1, p. 16] $\underline{\xi} \in \mathbb{R}^3$ defined as

$$\underline{\xi}=\tan(\phi/2)\underline{e} \tag{2.2}$$

where ϕ is the angle of rotation (in radians) of the body frame about the instantaneous axis of rotation $\underline{e} \in \mathbb{R}^3$. Note that $\underline{\xi} \to \pm\infty$ as $\phi \to \pm\pi$. The parametrization of Y by $\underline{\xi}$ is [Kane 1, p. 17]

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}x\,] - I \tag{2.3}$$

where I is the 3x3 identity matrix, T denotes transpose and $\underline{\xi}x$ is the matrix representation of the cross-product with $\underline{\xi}$ i.e., if $\underline{\xi} = (\xi_1, \xi_2, \xi_3)^T$, then

$$\underline{\xi}x = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}.$$

There are, of course, other parametrizations of Y by attitude variables. In fact, most authors use either Euler quaternions or Euler angles for the parametrization [Dwy. 1, 3], [Mon. 1], [Vad. 1]. However, as discussed in [Dwy. 2], the Gibbs vector is probably the best choice of kinematic variable for control synthesis in that it avoids state constraints and/or feedback singularities that are usually present when other variables are used.

Kinematics

If $\underline{\omega} \in \mathbf{R}^3$ denotes the angular velocity of the body frame with respect to the inertial frame (in body coordinates), then the differential equation satisfied by $\underline{\xi}$ is [Kane 1, p. 62]

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x]\underline{\omega}. \tag{2.4}$$

Solution of this differential equation (starting from some initial attitude $\xi(t_0)$) allows computation of $\underline{\xi}(t)$ for all $t \geq t_0$, and hence by (2.3) $Y(\underline{\xi})$ for all $t \geq t_0$. In other words, solution of this differential equation allows determination of the orientation of the body frame with respect to the inertial frame for all $t \geq t_0$.

Spacecraft and Reaction Wheel Dynamics

First, some notation will be needed. Let I_0 be the rigid body inertia tensor (including torque jets and locked wheels) calculated with respect to the body frame $\{O_B, \underline{b}_1, \underline{b}_2, \underline{b}_3\}$. Let $I_A = \mathrm{diag}(I_{w1}, I_{w2}, I_{w3}) \in \mathbb{R}^{3x3}$ where I_{wi} is a component of the inertia tensor associated with wheel i, calculated about a frame located at the center of mass of each wheel. Let $\underline{\Omega}_w \in \mathbb{R}^3$ denote the angular velocity of wheel i about its axle (in body coordinates), let m_B be the mass of the rigid body, and let $\underline{\tau}=(\tau_1, \tau_2, \tau_3)^T$ denote the torque due to the thrusters. Finally, $\underline{F}_{bB}$ is the force the beam exerts on the body at $c\underline{b}_3$ (in body coordinates), while $\underline{M}_{bB}$ (also in body coordinates) is the moment the beam exerts on the body at $c\underline{b}_3$.

Now consider a free-body diagram drawn around the rigid body portion (excluding thrusters) of the spacecraft. Note that the angular momentum of the rigid body, calculated about O_B and, denoted by $\underline{h}$, is $I_0\underline{\omega} + I_A\underline{\Omega}_w$. Taking the rate of change of $\underline{h}$ with respect to the inertial frame yields

$$I_0\dot{\underline{\omega}} + I_A\dot{\underline{\Omega}}_w + \underline{\omega}xI_0\underline{\omega} + \underline{\omega}xI_A\underline{\Omega}_w = \underline{\tau} + c\underline{b}_3 \times \underline{F}_{bB} + \underline{M}_{bB}. \tag{2.5}$$

The right-hand side of (2.5) is the net external torque (calculated about O_B) applied to the rigid body. It is composed of the torque due to the torque jets, and the net moment that the beam applies to the rigid body. Next, apply Newton's third law of motion to the free-body with respect to O_E, an inertial frame. Since $\underline{y}$ gives the coordinates of O_B with respect to the inertial frame,

$$m_B\ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} \tag{2.6}$$

where $\underline{F}_{bB}$ is multiplied by Y since $\underline{F}_{bB}$ is in body coordinates. Note also that it is explicitly assumed that there is no net force due to the torque jets.

Now draw a free-body diagram about the momentum wheels alone. Compute the rate of change of the angular momentum associated with the momentum wheels with respect to the inertial frame, and write out the components associated with the wheel axles, to obtain, in matrix form,

$$I_A(\dot{\underline{\omega}} + \dot{\underline{\Omega}}_w) = \underline{\tau}' \tag{2.7}$$

where $\underline{\tau}' = (\tau_1', \tau_2', \tau_3')^T$, τ_i' is the torque exerted by the ith motor on the rotor of the ith wheel. Complete details for this calculation, and the others above, can be found in many sources, for example [Hug. 1, p. 67].

Finally, substituting (2.7) into (2.5) yields

$$(I_0 - I_A)\dot{\underline{\omega}} + \underline{\omega} x I_0 \underline{\omega} + \underline{\omega} x I_A \underline{\Omega}_w = \underline{\tau} - \underline{\tau}' + c\underline{b}_3 \times \underline{F}_{bB} + \underline{M}_{bB}. \tag{2.8}$$

BEAM DYNAMICS

Consider now a free-body diagram drawn around an infinitesimal segment of the beam located between $z\underline{b}_3$ and $(z+dz)\underline{b}_3$. Let $\underline{u} = (u_1, u_2, u_3+z)^T$ denote the position (in body cooridinates) of a point p whose undeformed position is $z\underline{b}_3$, and let $\dot{\underline{u}}$ denote the rate of change of $\underline{u}$ with respect to the inertial frame. Let $\underline{F}(z)$ denote the force acting on the section of the beam at $\underline{z}$, and $\underline{F}(z + dz)$ the force acting on the segment of the beam at $(z+dz)\underline{b}_3$. Then writing Newton's third law with respect to the inertial frame yields

$$\ddot{\underline{u}} + \dot{\underline{\omega}} \times \underline{u} + 2\underline{\omega} x \dot{\underline{u}} + \underline{\omega} x (\underline{\omega} x \underline{u}) + d\underline{F}(\underline{z}) + Y^{-1}\underline{\dot{y}} = 0. \tag{2.9}$$

where we have assumed a beam mass per unit length of unity, and

$$d\underline{F}(z) := \lim_{dz \to 0} \frac{\underline{F}(z + dz) - \underline{F}(z)}{dz}.$$

We will model the beam as an Euler-Bernoulli type beam, with Voight-Kelvin damping [Pop. 1, p.116] (often referred to as viscous damping), and for simplicity we will ignore torsion. Let μ_i denote the flexural rigidity of the rod in the ith direction, and let k_i be a positive constant reflecting the rate of energy dissipation of the beam in the ith direction, i=1, 2, 3. Assume for simplicity that the beam has its principal axes of inertia parallel to the principal axes of the rigid body, so that the expression for $\underline{F}(\underline{z})$ becomes [Pop. 1, pp. 116, 381-383]

$$\underline{F}(\underline{z}) = \mu\partial'(\underline{u}) + k\partial'(\dot{\underline{u}}) \tag{2.10}$$

where $\mu = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$, $k = \mathrm{diag}(k_1, k_2, k_3)$ and $\partial'(\cdot) := \left(\frac{\partial^3(\cdot)}{\partial z^3}, \frac{\partial^3(\cdot)}{\partial z^3}, -\frac{\partial(\cdot)}{\partial z}\right)^T$. Hence, for such a beam $d\underline{F}(\underline{z})$ becomes

$$d\underline{F}(\underline{z}) = \mu\partial(\underline{u}) + k\partial(\dot{\underline{u}}) \tag{2.11}$$

where $\partial(\cdot)=\left(\frac{\partial^4(\cdot)}{\partial z^4}, \frac{\partial^4(\cdot)}{\partial z^4}, -\frac{\partial^2(\cdot)}{\partial z^2}\right)^T$. Insert (2.11) into (2.9) to obtain

$$\ddot{\underline{u}} + \dot{\underline{\omega}} \times \underline{u} + 2\underline{\omega} \times \dot{\underline{u}} + \underline{\omega} \times (\underline{\omega} \times \underline{u}) + \mu\partial(\underline{u}) + k\partial(\dot{\underline{u}}) + Y^{-1}\ddot{\underline{y}} = 0 \tag{2.12}$$

The boundary conditions for this fixed end - free end beam are (see [Pop. 1, pp. 385-386, 124, 128])

$$\begin{aligned} &\underline{u}(c)=0,\ u_1'(c)=u_2'(c)=0 \qquad u_1''(c+L)=0\ \ u_2''(c+L)=0 \\ &u_1'''(c+L)=0,\ u_2'''(c+L)=0,\ u_3'(c+L)=0 \end{aligned} \tag{2.13}$$

A derivation of these equations using Lagrangian techniques, rather than free-body diagrams can be found in [Bai. 1].

3. CONTROL LAW DETERMINATION

To achieve the objective of aligning the body frame with the inertial frame, which means that Y=I, examination of (2.3) shows that $\xi(t)$ should become 0. This indicates the control law strategy: find a control law in which $\xi(t)\rightarrow 0$. Note from (2.4) that necessarily $\underline{\omega}\rightarrow 0$, and from physical intuition, the beam stops vibrating and returns to its undeformed state. In other words, the control law should essentially stabilize the rigid body/beam system.

For simplicity, we will control the satellite by using the torque jets only. (See the second remark at the end of Theorem 3.1 for further comments). This corresponds to setting $\underline{\tau}'=0$, $I_A=0$ and $\underline{\Omega}_w=0$ in (2.8) above.

To design the control law, the method of linearizing transformations will be used. (For a thorough explanation of this procedure, see [De L. 1], [Sas. 1], and, in particular, [Isi. 1].) More precisely, we desire to find a static state feedback and a nonlinear change of coordinates to transform the coupled nonlinear-partial differential equations (2.4), (2.6), and (2.8) into a "normal form" [Isi. 1, p. 8], i.e. a system with linear input-output dynamics, and a corresponding unobservable, possibly nonlinear subsytem. Strictly speaking, since (2.4), (2.6), and (2.8) contain partial differential equations, the methods mentioned above do not necessarily apply. However, we will proceed blindly along these lines and investigate what happens.

Theorem 3.1. - Consider the system described above by equations (2.4), (2.5), (2.6). (2.3), (2.12) and (2.13). Assume that $\underline{\tau}'=0$, so that these equations become

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}\times]\underline{\omega}. \tag{2.4}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega}\times I_0\underline{\omega} = \underline{\tau} + c\underline{b}_3 \times \underline{F}_{bB} + \underline{M}_{bB}. \tag{2.5}$$

$$m_B\ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} \tag{2.6}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}x\,] - I \tag{2.3}$$

$$\underline{\ddot{u}} + \underline{\dot{\omega}} \times \underline{u} + 2\underline{\omega} x\, \underline{\dot{u}} + \underline{\omega} x(\underline{\omega} x \underline{u}) + \mu\partial(\underline{u}) + k\partial(\,\underline{\dot{u}}\,) + Y^{-1}\underline{\ddot{y}} = 0 \tag{2.12}$$

$$\underline{u}(c)=0,\ u_1'(c)=u_2'(c)=0 \qquad u_1''(c + L) = 0\ \ u_2''(c + L) = 0 \tag{2.13}$$
$$u_1'''(c + L) = 0,\ u_2'''(c + L) = 0,\ \ u_3'(c + L) = 0$$

Suppose now that we can determine $\underline{F}_{bB}(t)$ and $\underline{M}_{bB}(t)$ by on-board measurments. (See Appendix A for an example of how this might be done). Suppose also that the mass of the rigid body is much larger than the mass of the beam. If we apply the control law

$$\underline{\tau} = \underline{\omega} x I_0 \underline{\omega} + 2(1 + \underline{\xi}^T\underline{\xi}) I_0 (I - \underline{\xi}x)(-\beta\underline{\xi} - \gamma\underline{\dot{\xi}}) - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} - c\underline{b}_3 x \underline{F}_{bB} - \underline{M}_{bB} \tag{3.1}$$

where $\beta > 0$ and $\gamma > 0$ then

(i) The attitude $\underline{\xi}(t) \to 0$ exponentially, and $\underline{\dot{\xi}}(t) \to 0$ exponentially;

(ii) The angular velocity $\underline{\omega} \to 0$ exponentially, and $\underline{\dot{\omega}} \to 0$ exponentially;

(iii) The beam deflections $\underline{u}$ and beam velocities $\underline{\dot{u}}$ both go to zero exponentially.

Proof of (i) -We choose $\underline{\xi} = (\xi_1, \xi_2, \xi_3)^T$ to be the "dummy" output function. Then, following the linearization procedure given in [Isi. 1, sec. 2.3, 3.3], we intend to "differentiate the output until an input appears". Differentiating $\underline{\xi}$ yields (2.4), which will be repeated here for convenience.

$$\underline{\dot{\xi}} = \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)\underline{\omega} \tag{2.14}$$

Since no input appears in this expression differentiate again

$$\underline{\ddot{\xi}} = \frac{d}{dt}\Big(\frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)\Big)\underline{\omega} + \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)\frac{d\underline{\omega}}{dt}\ .$$

The calculation of the derivative in the first term is rather tedious; after computation we plug its value in and obtain

$$\begin{aligned}\underline{\ddot{\xi}} &= \frac{1}{2}(\underline{\xi}^T\underline{\omega})(\underline{\omega} + \underline{\xi}\underline{\xi}^T\underline{\omega} + \underline{\xi}x\underline{\omega}) + \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)\underline{\dot{\omega}} \\ &= \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + \underline{\dot{\omega}}].\end{aligned} \tag{3.2}$$

Insert (2.8) and use the fact that I_0 is invertible to obtain

$$\underline{\ddot{\xi}} = \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + I_0^{-1}\{-\underline{\omega}xI_0\underline{\omega} + c\underline{b}_3x\underline{F}_{bB} + \underline{M}_{bB} + \underline{\tau}\}]. \tag{3.3}$$

Now set,

$$\underline{\tau} = \underline{\tilde{\tau}} - c\underline{b}_3 x \underline{F}_{bB} - \underline{M}_{bB} \tag{3.4}$$

where, again, $\underline{F}_{bB}$ and $\underline{M}_{bB}$ have been measured, and $\underline{\tilde{\tau}}$ is a vector of real valued functions. ($\underline{\tilde{\tau}}$ can be thought of as the new exogenous input.) Insert (3.4) into (3.3) to obtain

$$\ddot{\underline{\xi}} = \frac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)[(\underline{\xi}^T\underline{\omega})\underline{\omega} + I_0^{-1}\{-\underline{\omega}xI_0\underline{\omega} + \underline{\tilde{\tau}}\}]. \tag{3.5}$$

But this is exactly the form of the equation one gets for a rigid body without flexible appendages [Dwy. 2]. Since the term outside the square brackets is nonsingular, we can apply the following control law

$$\begin{aligned} \underline{\tilde{\tau}} &= \underline{\omega}xI_0\underline{\omega} + I_0[\tfrac{1}{2}(I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x)]^{-1}\underline{w} - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} \\ &= \underline{\omega}xI_0\underline{\omega} + 2(1 + \underline{\xi}^T\underline{\xi})I_0(I - \underline{\xi}x)\underline{w} - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} \end{aligned} \tag{3.6}$$

where $\underline{w}$ is a vector of real valued functions, and again can be thought of as a new exogenous input. Applying (3.6) to (3.5) then yields the linear system

$$\ddot{\underline{\xi}} = \underline{w}. \tag{3.7}$$

On this controllable linear system, the poles can be placed as desired. For example, let

$$\underline{w} = -\beta\underline{\xi} - \gamma\dot{\underline{\xi}}. \tag{3.8}$$

where $\beta > 0$ and $\gamma > 0$. Inserting (3.8) into (3.7) and writing (3.7) in state space form yields

$$\begin{bmatrix} \dot{\underline{\xi}} \\ \ddot{\underline{\xi}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\beta I & -\gamma I \end{bmatrix} \begin{bmatrix} \underline{\xi} \\ \dot{\underline{\xi}} \end{bmatrix} \tag{3.9}$$

which implies (since $\beta > 0$ and $\gamma > 0$) that $\underline{\xi}(t) \to 0$ exponentially, and $\dot{\underline{\xi}}(t) \to 0$ exponentially. From (2.4), $\underline{\omega} = 2(1 + \underline{\xi}^T\underline{\xi})(I - \underline{\xi}x)\dot{\underline{\xi}}$ whence it follows that $\underline{\omega} \to 0$ exponentially. From (3.2) and (3.9) we also have that $\dot{\underline{\omega}} \to 0$ exponentially. Since a combination of (3.4), (3.6), and (3.8) yield (3.1), (i) and (ii) are proved.

Finally, since the beam is damped and $\underline{\omega}$, $\dot{\underline{\omega}} \to 0$ exponentially, it can be shown that if the rigid body has a much larger mass than the beam then the beam deflections and velocities go to zero exponentially. (See Appendix B for the proof). Thus (iii) is proved. ■

Remark - The interpretation of the control law is simple. First, the effect of the flexible body on the rigid body is removed by (3.3), and then the decoupled rigid body is controlled by (3.5) (which of course must be the same as the control for the rigid body alone as given in [Dwy. 2]).

Remark - In the case where momentum wheels alone are used to control the structure, the control law is very similar to the one in Theorem 3.1. The equations of motion and kinematics for this structure are nearly identical to those in the statement of Theorem 3.1, with the exception that (2.8) should be used instead of (2.12). In this case, it is easy to verify using exactly the methods in the proof of Theorem 3.1 that the control law

$$\underline{\tau} = -\underline{\omega} x I_0 \underline{\omega} - \underline{\omega} x I_A \underline{\Omega}_w - 2(1+\underline{\xi}^T\underline{\xi})(I_0 - I_A)(I - \underline{\xi}x)(-\beta\underline{\xi} - \gamma\dot{\underline{\xi}}) + (I_0 - I_A)(\underline{\xi}^T\underline{\omega})\underline{\omega} + c\underline{b}_3 x \underline{F}_{bB} + \underline{M}_{bB} \tag{3.10}$$

with $\beta>0$ and $\gamma>0$ satisfies the conditions (i), (ii), and (iii) of Theorem 3.1.

4. ATTITUDE CONTROL WITH BEAM BOUNDARY CONTROL

In the previous section, attitude control was obtained by decoupling the rigid body from the beam, and applying a rigid body control law. By decoupling the two components we are then left with an uncontrolled, damped beam. However, if the damping is small, or essentially negligible, then oscillations in the beam can continue for an undesirably long time. In this section we will consider the problem when the beam damping is assumed to be zero, i.e., k=0 in (2.12). Since there is no damping in the beam, it is easy to see that the control law of Theorem 3.1 will not work because the beam oscillations will not die off. Thus, if we are to employ a decoupling linearization law in the spirit of Theorem 3.1, beam control will be needed to stabilize the beam.

With these ideas in mind, now consider the structure shown in Figure 1, with the following modifications: At the beam tip (undeflected position is $(c+L)\underline{b}_3$) append 3 point force actuators and 3 point velocity sensors. The three actuators are situated with directions parallel to the three coordinate axes, while the three velocity sensors measure the velocity of the beam tip with respect to the inertial frame. Finally, assume that no damping is present in the beam. With these modifications the equations in Theorem 3.1 become

$$\dot{\underline{\xi}} = \frac{1}{2}[I + \underline{\xi}\underline{\xi}^T + \underline{\xi}x]\underline{\omega}. \tag{2.4}$$

$$I_0\dot{\underline{\omega}} + \underline{\omega} x I_0 \underline{\omega} = \underline{\tau} + c\underline{b}_3 \, x \, \underline{F}_{bB} + \underline{M}_{bB}. \tag{2.5}$$

$$m_B \ddot{\underline{y}} = Y(\underline{\xi})\underline{F}_{bB} \tag{2.6}$$

$$Y(\underline{\xi}) = 2(1 + \underline{\xi}^T\underline{\xi})^{-1}[\, I + \underline{\xi}\,\underline{\xi}^T + \underline{\xi}x\,] - I \tag{2.3}$$

$$\ddot{\underline{u}} + \dot{\underline{\omega}} \, x \, \underline{u} + 2\underline{\omega} x \, \dot{\underline{u}} + \underline{\omega} x(\underline{\omega} x \underline{u}) + \mu\partial(\underline{u}) + Y^{-1}\ddot{\underline{y}} = 0 \tag{2.12}$$

$$\underline{u}(c)=0,\ u_1'(c)=u_2'(c)=0 \qquad u_1''(c+L)=0 \ \ u_2''(c+L)=0 \tag{2.13}$$
$$u_1'''(c+L) = -F_1(t),\ u_2'''(c+L) = -F_2(t),\ \ u_3'(c+L) = -F_3(t)$$

where $F_i(t)$, i=1, 2, 3, is the the point force actuator associated with the ith axis.

For the applications in this paper, it turns out that a very desirable form for the $F_i(t)$, i=1, 2, 3, is choosing simple velocity feedback, i.e.

$$F_i(t) = -\alpha_i \dot{u}_i\,(c+L)\,, \quad \alpha_i>0 \qquad i=1, 2, 3 \tag{4.1}$$

Theorem 4.1 - Consider the modified system described above, together with the control law

$$\underline{\tau} = \underline{\omega} x I_0 \underline{\omega} + 2(1 + \underline{\xi}^T\underline{\xi})I_0(I - \underline{\xi}x)(-A\underline{\xi} - B\dot{\underline{\xi}}) - I_0(\underline{\xi}^T\underline{\omega})\underline{\omega} - c\underline{b}_3 x \underline{F}_{bB} - \underline{M}_{bB} \tag{4.2}$$

$$u_1'''(c + L) = \alpha\dot{u}_1(c + L) \quad u_2'''(c + L) = \beta\dot{u}_2(c + L) \quad u_3'(c + 1) = \gamma\dot{u}_3(c + L) \tag{4.3}$$

where A>0, B>0, α>0, β>0, γ>0. Also assume that $\ddot{\underline{y}}=0$. Then the attitude is corrected (i.e., $\underline{\xi}(t)\rightarrow 0$ exponentially), and the beam velocities and deflections go to zero exponentially.

Proof of Theorem 4.1 - See Appendix C. ■

Remark - The interpretation of the control law is again simple. The rigid body torque law (4.2) decouples the rigid body from the beam, and then stabilizes the rigid body. The beam boundary control law (4.3) exponentially stabilizes the beam. Thus we are again left with two decoupled exponentially stable systems, as in Theorem 3.1.

5. CONCLUSIONS

This paper has considered the attitude control problem for a flexible satellite consisting of a rigid hub and a elastic beam, modelled as an infinite dimensional Euler-Bernoulli beam. A control law was demonstrated for the case where the beam contained viscous damping, and a modified law in the case where beam damping was absent. The novelty in the proposed law was twofold: it was seen to be easily implementable using strain rosettes, and it stabilized all the beam modes, rather than a finite number of them. Several problems still are worth investigating, among them robustness, shaping of beam response, sensor and actuator placement, and implementation issues dealing with limitations on achievable torque in the control jets.

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APPENDIX A - DETERMINATION OF SHEAR FORCES AND MOMENTS

Determination of beam forces and moments is highly problem specific. In this appendix, we will consider the determination of forces and moments due to a rectangular beam attached to a rigid body. See Figure 2.

The problem with determining these quantities is they cannot be directly measured, but rather must be determined through some other quantity which can be measured. The simplest way of doing this is by use of strain gauges and rosettes. The reader unfamiliar with these devices can find a simple discussion in [Pop. 1, p. 311] or a more complete discussion in [Het. 1, chapt. 5-9].

1. STRESS AND STRAIN TENSORS

Only a very brief discussion of material properties will be given here, mainly to fix notation. Readers interested in a more detailed exposition are referred to [Pop. 1, Chapters 3, 4] or [Lan. 1, Chapter 1].

Let the position of a particle P in the beam be $r_1\underline{i} + r_2\underline{j} + r_3\underline{k}$ (where $\underline{i}$, $\underline{j}$, $\underline{k}$ refer to the unit vectors along some x, y, z coordinate axes). Upon application of forces to the beam, deformation occurs and the point P moves to $(r_1 + u_1)\underline{i} + (r_2 + u_2)\underline{j} + (r_3 + u_3)\underline{k}$.

Let $\varepsilon_{x_i x_j}$ denote the ij-th component of the strain tensor defined as

$$\varepsilon_{x_i x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k} \right) \qquad \text{(A1.1)}$$

with summation over k, where $x_1{:=}x$, $x_2{:=}y$, $x_3{:=}z$.

Now consider an infinitesimal cubic volume element centered about a point P of the beam, with faces of area ΔA. Let $\sigma_{x_i x_j}$ denote the ij-th member of the stress tensor defined as

$$\sigma_{x_i x_j} = \lim_{\Delta A \to 0} \frac{\Delta F_{x_i j}}{\Delta A} \quad i, j = 1,2,3 \qquad \text{(A1.2)}$$

where $\Delta F_{x_i j}$ is the x_ith component of the force acting on face j of the cube. (Faces 1 and 4 have outward normals parallel to the x and -x axes, respectively, faces 2 and 5 refer simarily to y and -y, and 3 and 6 refer to z and -z.)

By assuming homogeneous, isotropic material, and also assuming small strains, we get Hooke's Law relations between stress and strain

$$\varepsilon_{xx} = \sigma_{xx}/E - \nu\sigma_{yy}/E - \nu\sigma_{zz}/E \qquad \text{(A1.3)}$$

$$\varepsilon_{yy} = \sigma_{yy}/E - \nu\sigma_{xx}/E - \nu\sigma_{zz}/E \qquad \text{(A1.4)}$$

$$\varepsilon_{zz} = \sigma_{zz}/E - \nu\sigma_{xx}/E - \nu\sigma_{yy}/E \qquad \text{(A1.5)}$$

$$\varepsilon_{xy} = \sigma_{xy}/G \qquad \text{(A1.6)}$$

$$\varepsilon_{yz} = \sigma_{yz}/G \tag{A1.7}$$
$$\varepsilon_{xz} = \sigma_{xz}/G \tag{A1.8}$$

where E is the Young's modulus for the material, ν is Poisson's ratio, and G is the shear modulus.

In general, the contributions to Poisson's ratio is small and hence for simplicity it will be ignored. Then equations (A1.3 - A1.5) simplify to

$$\varepsilon_{xx} = \sigma_{xx}/E \tag{A1.3a}$$
$$\varepsilon_{yy} = \sigma_{yy}/E \tag{A1.4a}$$
$$\varepsilon_{zz} = \sigma_{zz}/E \tag{A1.5a}$$

2. FORCES AND MOMENTS AFFECTING BEAM

Consider a rectangular beam as shown in figure 2. Recall that the neutral surface (or elastic line) is the portion of the beam which does not change length during deformation. In the case shown here, it is simply the z-axis. In determining stresses due to bending moments, the fundamental assumption is that the strains vary linearly as their respective distances from the neutral surface. With such an assumption, and using equilibrium conditions for an arbitrary beam segment, it is easy to show that the bending moment about the x-axis =: M_x is [Pop. 1, p. 182]

$$M_x = I_x \sigma_{zz}/x \tag{A2.1}$$

where $I_x = \int x^2 dA = ab^3/12$. Similarily,

$$M_y = I_y \sigma_{zz}/y \tag{A2.2}$$

where M_y is the bending moment about the y-axis and $I_y = \int y^2 dA = ba^3/12$.

If the bar undergoes a moment M_z about the z-axis, the torsional shear distribution is somewhat difficult to compute. However, it turns out that the distribution (see Figure 3) has a maximum occuring at the midpoint of the longest side (in this case, the side parallel to the y-axis). The maximum shear stress turns out to be [Pop.1, p. 167]

$$(\sigma_{zx})_{max} = M_z/ba^2\alpha$$

where it is assumed that $a \geq b$, and α is a parameter depending on the b/a ratio, and M_z is the moment about the z-axis. Hence,

$$M_z = (\sigma_{zx})_{max} ba^2\alpha \tag{A2.3}$$

By symmetry, it is also clear that $(\sigma_{zy})_{max} = M_z\alpha/b^2a$, which occurs at the midpoint of the shorter side (the side of length b). Hence we must also have

$$M_z = (\sigma_{zy})_{max}\alpha/b^2a \tag{A2.2b}$$

To determine the shear stresses in the beam, recall that the shear distribution for a rectangular bar subjected to a shear force in the x-direction V_x is parabolic in nature and given by [Pop. 1, p. 232]

$$\sigma_{zy}(x_1) = \frac{V_x}{2I_x}\left(\left(\frac{b}{2}\right)^2 - x_1^2\right) \quad x_1 \in [0, b/2]$$

(See figure 4) This shows that the shear stress is zero at the boundary (x_1=b/2) and has a maximum at x_1=0 of value $(\sigma_{zy})_{max} = V_x b^2/8\, I_x$. Solving for V_x,

$$V_x = -\frac{8}{a^2} I_x(\sigma_{zy})_{max} = -\frac{3}{2ab}(\sigma_{zy})_{max} \tag{A2.4}$$

Similarily,

$$V_y = -\frac{8}{b^2} I_y(\sigma_{zx})_{max} = -\frac{3}{2ab}(\sigma_{zx})_{max} \tag{A2.5}$$

where V_y is the shear force in the y-direction.

Finally, to determine the axial stress induced by a tensile or compressive force, note that the average stress over a cross-section is simply $F_z/A = -\sigma_{zz}$ since the cross-section is constant over the length of the beam (when considering axial forces alone). Hence, F_z, the axial force in the z-direction, is

$$F_z = -A\sigma_{zz} \tag{A2.6}$$

In the following, only small deflections will be considered, so that the principle of superposition holds. That is, the resultant strain in the system is the algebraic sum of the individual strains when applied separately. Superposition of stresses as well as strains also follows from the previous assumption of Hooke's Law.

3. FORCE AND MOMENT DETERMINATION FROM STRAIN ROSETTES

In order to determine the forces and moments affecting the beam, strain rosettes are mounted on the beam as shown in Figure 2. With the rosettes placed as shown, the following information is obtained:

Rosette 1: $$\begin{bmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \Bigg|_{x=\ b/2,\ y=0,\ z=c} \tag{A3.1}$$

Rosette 2: $$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zz} \end{bmatrix} \Bigg|_{x=\ 0,\ y=a/2,\ z=c} \tag{A3.2}$$

Rosette 3: $$\begin{bmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \Bigg|_{x=\ -b/2,\ y=0,\ z=c} \tag{A3.3}$$

Rosette 4: $$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zz} \end{bmatrix} \Bigg|_{x=\ 0,\ y=-a/2,\ z=c} \tag{A3.4}$$

From these measurements, the forces and moments affecting the beam can be determined by use of equations (A2.1)-(A2.6), superposition, and Hooke's Law relationships. Specifically, at rosettes 1 and 3 equations (A2.1), (A2.2), and (A2.6) shows that (y=0 at both 1 and 3 so that there is no contribution due to M_x)

$$\sigma_{zz} = -F_z/A + M_y x/I_x$$

$$\Rightarrow \begin{cases} \sigma_{zz}|_1 + \sigma_{zz}|_3 = -2F_z/A \\ \sigma_{zz}|_1 - \sigma_{zz}|_3 = 2M_y b/2I_x = M_y b/I_x \end{cases}$$

$$\Rightarrow \begin{cases} F_z = -\dfrac{A}{2}(\sigma_{zz}|_1 + \sigma_{zz}|_3) \\ M_y = I_x/b(\sigma_{zz}|_1 - \sigma_{zz}|_3) \end{cases}$$

Similar arguments show that

$$M_x = I_y/a(\sigma_{zz}|_2 - \sigma_{zz}|_4)$$

By the Hooke's Law relationships, $\sigma_{zz} = E\varepsilon_{zz}$. Hence,

$$F_z = -\frac{A}{2E}(\varepsilon_{zz}|_1 + \varepsilon_{zz}|_3) \tag{A3.5}$$

$$M_y = I_x/bE(\varepsilon_{zz}|_1 - \varepsilon_{zz}|_3) \tag{A3.6}$$

$$M_x = I_y/Ea(\varepsilon_{zz}|_2 - \varepsilon_{zz}|_4) \tag{A3.7}$$

Since the rosettes at each of these positions determine the strains in the parentheses, F_z,

M_y, M_x, are determinable from the experimental data.

Finally, to determine M_z, V_x, and V_y consider figure 3. Since the stresses are additive at one side of the cross-section, but subtract from one another on the other side, it is easy to solve for the quantities M_z, V_x, and V_y. Proceeding along these lines, use figure 3 and equations (A2.3), (A2.4) and (A2.5) to obtain

$$\varepsilon_{zy}|_1 G = \sigma_{zy}|_1 = +M_z\alpha/b^2a + 3V_y/2ab \tag{A3.8}$$
$$\varepsilon_{zy}|_3 G = \sigma_{zy}|_3 = -M_z\alpha/b^2a + 3V_y/2ab \tag{A3.9}$$
$$\varepsilon_{zx}|_2 G = \sigma_{zx}|_2 = +M_z/ba^2\alpha + 3V_x/2ab \tag{A3.10}$$
$$\varepsilon_{zx}|_4 G = \sigma_{zx}|_4 = -M_z/ba^2\alpha + 3V_x/2ab \tag{A3.11}$$

Since $\varepsilon_{zx}|_1$, $\varepsilon_{zx}|_3$, $\varepsilon_{zx}|_2$, $\varepsilon_{zx}|_4$ are known by measurement, and since b, a, and α are known, equations (A3.8)-(A3-11) is a system of 4 equations in 3 unknowns. From this system, a least-squares solution for M_z, V_x, and V_y can be found.

Note that if c is small, the moments and forces acting on the body by the beam are close to the corresponding values at the point of attachment. Thus,

$$\underline{M}_{bB}' \cong \underline{M}_x i + \underline{M}_y j + \underline{M}_z k \tag{A3.12}$$
$$\underline{F}_{bB}' \cong \underline{F}_x i + \underline{F}_y j + \underline{F}_z k \tag{A3.13}$$

If $\underline{i}$, $\underline{j}$, $\underline{k}$ are parallel to the $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ axes, respectively, then $\underline{M}_{bB}' = \underline{M}_{bB}$ and $\underline{F}_{bB}' = \underline{F}_{bB}$. Otherwise there is a (fixed) rotation matrix Q, as in the discussion of kinematics in section 2, such that $[\underline{b}_1\ \ \underline{b}_2\ \ \underline{b}_3]Q = [\underline{i}\ \ \underline{j}\ \ \underline{k}]$. Then the components of $\underline{M}_{bB}$ and $\underline{F}_{bB}$ are Q[components of $\underline{M}_{bB}'$] and Q[components of $\underline{F}_{bB}'$], respectively.

Remark - If we add Kelvin-Voight damping to our model of the form, i.e. we add damping of the form

$$\sigma_{jj} = E\varepsilon_{jj} + \eta_{jj}\frac{d\varepsilon_{jj}}{dt}$$

$$\sigma_{jk} = G\varepsilon_{jk} + \eta_{jk}\frac{d\varepsilon_{jk}}{dt} \qquad j \neq k,$$

then the formulas change very simply:

$$F_z = -\frac{A}{2E}(\varepsilon_{zz}|_1 + \varepsilon_{zz}|_3) - \frac{A}{2\eta_{zz}}(\dot{\varepsilon}_{zz}|_1 + \dot{\varepsilon}_{zz}|_3).$$
$$M_y = I_x(\varepsilon_{zz}|_1 - \varepsilon_{zz}|_3)/Eb + I_x(\dot{\varepsilon}_{zz}|_1 - \dot{\varepsilon}_{zz}|_3)/\eta_{zz}b$$
$$M_x = I_y(\varepsilon_{zz}|_2 - \varepsilon_{zz}|_4)/Ea + I_y(\dot{\varepsilon}_{zz}|_2 - \dot{\varepsilon}_{zz}|_4)/\eta_{zz}a$$

For the torsion and shear calculations, just rewrite the LHS of (A3.8)-(A3.11) to obtain

$$
\begin{aligned}
\varepsilon_{zy}|_1 G + \eta_{zy}\dot{\varepsilon}_{zy}|_1 &= \sigma_{zy}|_1 = +M_z\alpha/b^2a + 3V_y/2ab \\
\varepsilon_{zy}|_3 G + \eta_{zy}\dot{\varepsilon}_{zy}|_3 &= \sigma_{zy}|_3 = -M_z\alpha/b^2a + 3V_y/2ab \\
\varepsilon_{zx}|_2 G + \eta_{zx}\dot{\varepsilon}_{zx}|_2 &= \sigma_{zx}|_2 = +M_z/ba^2\alpha + 3V_x/2ab \\
\varepsilon_{zx}|_4 G + \eta_{zx}\dot{\varepsilon}_{zx}|_4 &= \sigma_{zx}|_4 = -M_z/ba^2\alpha + 3V_x/2ab
\end{aligned}
$$

and again compute a least-squares solution for M_z, V_y and V_x.

All of these computations presuppose that the strain derivatives can be determined. Of course, one could get an approximation of these quantities by on-line finite differences, i.e.

$$\dot{\varepsilon}_{zz}(t) \approx \frac{\varepsilon_{zz}(t) - \varepsilon_{zz}(t - T)}{T}$$

where T is the time between strain samples.

APPENDIX B - Proof of Theorem 4.1, part (iii).

To show that the beam deflections go to zero exponentially, we will use methods of functional analysis. More exactly, we will show that the eigenvalues of a particular linear operator are all strictly negative (Lemma B.3), and then we use a theorem from analytic semigroup theory to conclude that the operator generates an exponentially stable semigroup (Theorem B.4). This result, combined with a Bellman-Gronwall type proof will yield the result.

It is convenient to write (2.12) in state space form as

$$\begin{bmatrix} \dot{\underline{u}} \\ \ddot{\underline{u}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) + \frac{1}{m_B}\mu\partial'(\cdot)\big|_c & -k\partial(\cdot) + \frac{1}{m_B}k\partial'(\cdot)\big|_c \end{bmatrix} \begin{bmatrix} \underline{u} \\ \dot{\underline{u}} \end{bmatrix} - \begin{bmatrix} 0 \\ \dot{\underline{\omega}} \times \underline{u} + 2\underline{\omega} \times \dot{\underline{u}} + \underline{\omega} \times (\underline{\omega} \times \underline{u}) \end{bmatrix}$$

$$\underline{u}(0)=\underline{u}_0, \dot{\underline{u}}(0)=\dot{\underline{u}}_0 \qquad \text{(B.1)}$$

where use has also been made of (2.6).

Notation: Let $H^k[c, c+L]$, k=0, 1, ... be defined as

$$H^0[c, c+L] = \{f \in L^2[c, c+L]\}$$

.

.

.

$$H^k[c, c+L] = \{f \in L^2[c, c+L] \mid f, f' \ldots f^k \in L^2[c, c+L]\}$$

For simplicity of notation H^k will denote $H^k[c, c+L]$, and similarily L^2 will denote $L^2[c, c+L]$.

Now let A denote the linear operator in the first term on the right side of (B.1), i.e.

$$A = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) + \frac{1}{m_B}\mu\partial'(\cdot)\big|_c & -k\partial(\cdot) + \frac{1}{m_B}k\partial'(\cdot)\big|_c \end{bmatrix}. \qquad \text{(B.2)}$$

Before considering the specifics of the operator A, it is convenient to first examine the operator A′ defined as

$$A' = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & -k\partial(\cdot) \end{bmatrix}, \qquad A' : (\underline{u}, \dot{\underline{u}}) \to (\dot{\underline{u}}, -\mu\partial(\underline{u}) - k\partial(\dot{\underline{u}})) \; . \qquad \text{(B.3)}$$

Note that since A′ is a differential operator, it is, in general, an unbounded operator. Let the space A′ operates on be $(L^2 x L^2 x L^2 x L^2 x L^2 x L^2)$, together with the corresponding inner product

$$[f, g] = [(f_1, f_2, f_3, f_4, f_5, f_6)^T, (g_1, g_2, g_3, g_4, g_5, g_6)^T] = [f_1, g_1] + [f_2, g_2] + [f_3, g_3] + [f_4, g_4] + [f_5, g_5] + [f_6, g_6]. \tag{B.4}$$

where $[\bullet, \bullet]$ denotes the ordinary inner product in $L^2[c, c+L]$. Let the domain of A', denoted $D(A')$, be

$$D(A') = \{f = (f_1, f_2, f_3, f_4, f_5, f_6)^T \mid f_1 \in H^4,\ f_2 \in H^4,\ f_3 \in H^2,\ f_4 \in H^4,\ f_5 \in H^4,\ f_6 \in H^2, \text{ together with the initial conditions } f_1(c)=f_2(c)=f_3(c)=f_4(c)=f_5(c)=f_6(c)=0,\ f_1'(c)=f_2'(c)=f_3'(c)=f_4'(c)=f_5'(c)=f_6'(c)=0,\ \mu_1 f_1''(c+L)+k_4 f_4''(c+L)=\mu_2 f_2''(c+L)+k_5 f_5''(c+L)=\mu_3 f_3''(c+L)+k_6 f_6''(c+L)=0,\ \mu_1 f_1'''(c+L)+k_4 f_4'''(c+L)=\mu_2 f_2'''(c+L)+k_5 f_5'''(c+L)=\mu_3 f_3'''(c+L)+k_6 f_6'''(c+L)=0\}. \tag{B.5}$$

Note that $D(A')$ is dense in $(L^2 x L^2 x L^2 x L^2 x L^2 x L^2)$ (See Appendix C , Proposition C.2 for a proof of this fact.) Now consider the "abstract" differential equation

$$\dot{x} = A'x \quad , x(0) = x_0 \in D(A') \tag{B.6}$$

Before proceeding, we need to introduce some definitions and notation from the semigroup literature. The interested reader can find excellent expositions on this subject in many texts, e.g. [Paz. 1], [Kat. 1], [Bal. 1].

Definition B.1 Let $T(t)$ for all $t \in [0, \infty)$ be a bounded linear operator in a Banach space X. $\{T(t)\}$ is said to be a <u>strongly continuous semigroup</u> (or, simply, <u>s. c. semigroup</u>) if

(i) $T(t+s) = T(t)T(s) = T(s)T(t)$, for any $t \geq 0$, any $s \geq 0$.
(ii) $T(0) = I$ (B.7)
(iii) $\|T(t)x - x\| \to 0$, as $t \to 0+$, for any $x \in X$.

Definition B.2 A strongly continuous semigroup $\{T(t)\}$ satisfying $\|T(t)\| \leq 1$ for all $t \in [0, \infty)$ is called a <u>contraction semigroup</u>. If there exists $K>0$, $\delta>0$ such that the semigroup satisfies $\|T(t)\| \leq Ke^{-\delta t}$, then $T(t)$ is termed an <u>exponentially stable semigroup</u>.

Definition B.3 A semigroup $\{T(t)\}$ is said to be <u>analytic</u> if there exists a sector Δ of the form

$\Delta = \{z \in C: \phi_1 < \arg(z) < \phi_2, \phi_1 < 0 < \phi_2\}$
containing the real axis with

(i) $z \to T(z)$ is analytic in Δ.
(ii) $T(0) = I$, $\lim_{\substack{z \to 0 \\ z \in \Delta}} T(z)x = x$, for any $x \in X$. (B.8)

(iii) $T(z_1+z_2) = T(z_1)T(z_2) = T(z_2)T(z_1)$, for any $z_1 \in \Delta$, and any $z_2 \in \Delta$.

Proposition B.1 Consider the linear operator A' defined in (B.3), together with the space $(L^2 x L^2 x L^2 x L^2 x L^2 x L^2)$ together with the corresponding inner product. Then A generates a strongly continuous semigroup T(t), in fact a contraction semigroup.

Proof of Proposition B.1 Since this proposition is a simple consequence of Theorem B.1, the proof of this proposition will be deferred until that time. ■

In order to show that beam deflections go to zero exponentially, it is necessary to show that $\|T(t)\| \leq M\exp(\omega_0(t))$ for some $\omega_0 < 0$. (Since A' generates a contraction semigroup, we are guaranteed that $\omega_0 \leq 0$). This has been shown in [Gib. 1, Thm 6.1] for a one dimensional beam, but the ω_0 obtained there has no clear relationship to the operator A. Analogous to the finite dimensional case, it will be shown that ω_0 is actually the maximum real part of the eigenvalues of the linear operator A'. Unfortunately, strong continuity of the semigroup is not sufficient to conclude that $\omega_0 = \sup\{Re(\lambda) \mid \lambda \in \sigma(A')\}$. However, if A generates an <u>analytic</u> semigroup, then we have the following result [Tri. 1, p. 387] .

Proposition B.2 - Suppose a linear operator A generates an analytic semigroup T(t) on the space X. Then T(t) satisfies $\|T(t)\| \leq M\exp(\omega_0(t))$, where $\omega_0 = \sup\{Re(\lambda) \mid \lambda \in \sigma(A)\}$.

For simplicity we will restrict our attention to the transverse beam vibrations in the $\underline{b}_1$ direction, which corresponds to the variables f_1, $f_4 \in D(A')$. Recall that the beam displacemnts are decoupled due to the assumption that the beam principal axes of inertia are parallel to the $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ directions. (See the "Beam Dynamics" subsection in section 2). So it is easy to see that no generality is lost by restricting attention to the f_1, f_4 terms. Strictly speaking, a separate proof should be given for the f_3, f_6 terms corresponding to axial displacements since the displacements are governed by a different differential equation. However, from the proof for the tranverse displacements, it will be easy to see the extension to the axial case. So now consider the operator

$$\tilde{A} = \begin{bmatrix} 0 & I \\ -\mu \dfrac{\partial^4(\cdot)}{\partial z^4} & -k\dfrac{\partial^4(\cdot)}{\partial z^4} \end{bmatrix}. \tag{B.9}$$

where the subscripts on k and μ have been suppressed for simplicity.

Theorem B.1 $\tilde{A}$ generates an analytic semigroup $\tilde{N}(t)$.

Remark - The proof of this result can be found in [Mas. 1, Theorem 1.1], and [Hua. 1, Theorem 4.1]. A conceptually simpler proof, following ideas of [Che. 1], will be given below.

The following theorem (see [Kato 1, pp. 489-490], also see proof of [Paz. 1, Thm 5.2, p. 61]) will be crucial in proving the analyticity of $\tilde{N}(t)$.

Theorem B.2 Let A be a closed linear operator on a Hilbert space X. Then A generates an analytic semigroup T(t) if $\exists$ M>0, $\theta \in (0, \pi/2)$ such that

$$\|(\lambda I - A)^{-1}\| \leq M/|\lambda| \tag{B.10}$$

for all $\lambda \in C/\{0\}$ sufficiently large contained in the sector $|\arg(\lambda)| \leq \pi/2 + \theta$.

Before proving Theorem B.1, two more lemmas will be needed.

Lemma B.1 Let $\mathfrak{I}$ denote the operator $\frac{\partial^4(\cdot)}{\partial z^4}$. For $\tilde{A}$ given in (B.9),

$$(\lambda I - \tilde{A})^{-1} = (\lambda^2 I + (\mu + k\lambda)\mathfrak{I})^{-1} \begin{bmatrix} (\lambda I + k\mathfrak{I}) & I \\ -\mu\mathfrak{I} & \lambda I \end{bmatrix} \tag{B.11}$$

Proof of Lemma B.1: Brute force calculation. ■

Lemma B.2 With $\mathfrak{I}$ as above, $\exists$ C>0, C'>0, and $\theta \in (0, \pi/2)$ such that

$$\|(\lambda^2 I + (\mu + k\lambda)\mathfrak{I})^{-1}\| \leq C/|\lambda|^2 \tag{B.12}$$
$$\|(\lambda^2 I + (\mu + k\lambda)\mathfrak{I})^{-1}\mathfrak{I}\| \leq C'/|\lambda| \tag{B.13}$$

for all $\lambda \in C/\{0\}$ sufficiently large contained in the sector $|\arg(\lambda)| \leq \pi/2 + \theta$.

Proof of Lemma B.2: An easy integration by parts shows that $\mathfrak{I}$ is a self adjoint operator in L^2. Let E_α denote the resolution of the identity associated to $\mathfrak{I}$. (See [Kat. 1, p.356] for the definition of E_α and the associated spectral theory.) Then, using the spectral decomposition theorem for normal operators [Kat. 1, p. 356], one obtains

$$(\lambda^2 + (\mu + k\lambda)\mathfrak{I})^{-1} = \int(\lambda^2 + (\mu + k\lambda)\alpha)^{-1} dE_\alpha \quad \text{and} \tag{B.14}$$

$$(\lambda^2 + (\mu + k\lambda)\mathfrak{I})^{-1}\mathfrak{I} = \int\alpha(\lambda^2 + (\mu + k\lambda)\alpha)^{-1} dE_\alpha \tag{B.15}$$

(This representation is the infinite dimensional analog of the the dyadic expansion for self-adjoint matrices.) To prove the lemma, it is enough to show that

$$|(\lambda^2 + (\mu + k\lambda)\alpha)^{-1}| \leq C/|\lambda|^2 \quad \text{and} \tag{B.16}$$

$$|(\lambda^2 + (\mu + k\lambda)\alpha)^{-1}\alpha| \leq C/|\lambda| \tag{B.17}$$

for $\alpha \in \sigma(\partial)$ ($\sigma(\partial)$ is the spectrum of ∂) and λ in an appropriate sector. Note also that

$$(\lambda^2 + (\mu + k\lambda)\alpha) = \left(\lambda + \frac{k\alpha + \sqrt{k^2\alpha^2 - 4\mu\alpha}}{2}\right)\left(\lambda + \frac{k\alpha - \sqrt{k^2\alpha^2 - 4\mu\alpha}}{2}\right)$$
$$=: (\lambda + r^+(\alpha))(\lambda + r^-(\alpha)) \tag{B.18}$$

Recall that $\sigma(\partial) = \{\gamma_n^4 |\ \gamma_n$ satisfies $\cos\gamma_n 1 \cosh\gamma_n 1 = -1\}$, (use the method of [Cou. 1, p. 296], but change the boundary conditions) and note that $|\gamma_n| \to \infty$. Suppose now that the structure is lightly damped, i.e. k is small. Assume that k is sufficiently small so that $k^2\alpha^2 - 4\mu\alpha < 0$ for some $\alpha \in \sigma(\partial)$. Figure 5 shows a possible distribution of the zeroes of $(\lambda^2 + (\mu + k\lambda)\alpha)$ for the case of light damping. For $\alpha < 4\mu/k^2$, the zeroes are distributed on a circle of radius μ/k. For $\alpha \geq 4\mu/k^2$, the zeroes are real, with $|r^+(\alpha)| \geq 2\mu/k$ and $\mu/k < |r^-(\alpha)| < 2\mu/k$. For example, in Figure 5, $\{-r^+(\alpha'), -r^-(\alpha')\}$ and $\{-r^+(\alpha^*), -r^-(\alpha^*)\}$ are possible zero locations when $\alpha_1 < 4\mu/k^2$ and $\alpha_2 \geq 4\mu/k^2$, respectively. Let α^* correspond to the zeroes with minimum argument ($r^+(\alpha^*)$ and $r^-(\alpha^*)$ in Figure 5). Now choose ε sufficiently small so that

$$\arg(k\alpha + j(\sqrt{4\mu\alpha - k^2\alpha^2})) - \pi/2 - \varepsilon > 0$$

This can always be done since $\arg(k\alpha + j(\sqrt{4\mu\alpha - k^2\alpha^2})) - \pi/2$ is bounded away from zero. So now define $\theta = \arg(k\alpha + j(\sqrt{4\mu\alpha - k^2\alpha^2})) - \pi/2 - \varepsilon$ and note that $\theta \in (0, \pi/2)$. Suppose now λ is in the sector

$$|\arg(\lambda)| \leq \pi/2 + \theta$$

For the purposes of proof, we can assume by symmetry that λ lies in the half plane $\mathrm{Im}(\lambda) \geq 0$.

Case 1: First consider any $\alpha' \in \sigma(\partial)$ such that $\alpha' \geq 4\mu/k^2$ (see Figure 6). Then $r^+(\alpha')$ and $r^-(\alpha')$ are both real. From Figure 6 we see that the distance from $r^{\pm}(\alpha')$ to λ is less than or equal to the distance from $r^{\pm}(\alpha')$ to the ray determined by θ_1. Hence,

$$|r^{\pm}(\alpha')||(\lambda + r^{\pm}(\alpha'))^{-1}| \leq \csc(\pi/2 - \theta_1) = \sec(\theta_1) \tag{B.19}$$

Since $|r^{\pm}(\alpha')| = \dfrac{k\alpha' + \sqrt{k^2\alpha'^2 - 4\mu\alpha'}}{2}$, and $\sqrt{k^2\alpha'^2 - 4\mu\alpha'} \geq 0$, we must have

$$k\alpha'|(\lambda + r^{\pm}(\alpha'))^{-1}|/2 \leq \sec(\theta_1) \tag{B.20}$$

which implies

$$\alpha'|(\lambda + r^{+}(\alpha'))^{-1}| \leq 2\sec(\theta_1)/k \tag{B.21}$$

Next, by the law of cosines,

$$|\lambda + r^{\pm}(\alpha')| = |\lambda|^2 + |r^{\pm}(\alpha')|^2 - 2\cos(\psi)|\lambda||r^{\pm}(\alpha')|$$

so that

$$\left(\frac{|\lambda|}{|\lambda + r^{\pm}(\alpha')|}\right)^2 + \left(\frac{|r^{\pm}(\alpha')|}{|\lambda + r^{\pm}(\alpha')|}\right)^2 - 2\cos(\psi)\left(\frac{|\lambda|}{|\lambda + r^{\pm}(\alpha')|}\right)\left(\frac{|r^{\pm}(\alpha')|}{|\lambda + r^{\pm}(\alpha')|}\right) = 1$$

Now, if $\psi > \pi/2$, $|\lambda||(\lambda + r^{\pm}(\alpha'))^{-1}| \leq 1$. If $\psi < \pi/2$, use (B.19) in the above to obtain

$$\left(\frac{|\lambda|}{|\lambda + r^{+}(\alpha')|}\right)^2 - 2\cos(\psi)\sec(\theta_1)\left(\frac{|\lambda|}{|\lambda + r^{+}(\alpha')|}\right) \leq 1$$

which implies

$$|\lambda||(\lambda + r^{\pm}(\alpha'))^{-1}| \leq \cos(\psi)\sec(\theta_1) + \sqrt{\cos^2(\psi)\sec^2(\theta_1) + 1} \tag{B.22}$$

Hence,

$$|(\lambda^2 + (\mu + k\lambda)\alpha')^{-1}| \leq K/|\lambda|^2 \tag{B.23}$$

where $K=1$ if $\psi > \pi/2$, and $K = \sec(\theta_1) + \sqrt{\sec^2(\theta_1) + 1}$ if $\psi < \pi/2$. Combine (B.22) with (B.21) to obtain

$$|(\lambda^2 + (\mu + k\lambda)\alpha')^{-1}\alpha'| \leq C/|\lambda|. \tag{B.24}$$

where C is an appropriate constant. Since both (B.23) and (B.24) are bounded by $C/|\lambda|$ for $|\lambda|$ sufficiently large, the lemma is proved for case 1.

Case 2: Now consider any $\alpha'' \in \sigma(\partial)$ such that $\alpha'' < 4\mu/k^2$. Then $r^{+}(\alpha'')$ and $r^{-}(\alpha'')$ lie on the circle in Figure 5. Performing calculations precisely the same way as before, one obtains (see Figure 7)

$$|r^{\pm}(\alpha'')||\,(\lambda + r^{\pm}(\alpha''))^{-1}| \le \csc(\theta_1 - \theta) \le \csc(\varepsilon) \qquad \text{and} \tag{B.25}$$

$$\alpha''|\,(\lambda + r^{\pm}(\alpha''))^{-1}|/2 \le 2\sec(\theta)/k \tag{B.26}$$

Again, if $\psi > \pi/2$, $|\lambda||\,(\lambda + r^{\pm}(\alpha''))^{-1}| \le 1$, whereas if $\psi < \pi/2$,

$$|\lambda||\,(\lambda + r^{\pm}(\alpha''))^{-1}| \le \cos(\psi)\csc(\theta_1 - \theta) + \sqrt{\cos^2(\psi)\sec^2(\theta_1 - \theta)\,\cdot}$$

$$\le \cos(\psi)\csc(\varepsilon) + \sqrt{\cos^2(\psi)\sec^2(\varepsilon) + 1} \le \csc(\varepsilon) + \sqrt{\sec^2(\varepsilon) + 1} \tag{B.27}$$

So that as before

$$|(\lambda^2 + (\mu + k\lambda)\alpha'')^{-1}| \le K'/|\lambda|^2 \tag{B.28}$$

$$|(\lambda^2 + (\mu + k\lambda)\alpha'')^{-1}\alpha''| \le C'/|\lambda|. \tag{B.29}$$

which concludes the proof of the lemma for the case where the damping is small. If the damping is large, (there are no imaginary zeroes of $\lambda^2 + (\mu + k\lambda)\alpha$ for any $\alpha \in \sigma(\tilde{\partial})$), then the proof can be duplicated exactly as in Case 2 by using any $\theta < \pi/2$. This concludes the proof of Lemma B.2. ■

Proof of Theorem B.1: Again consider the operator $\tilde{A}$ given by (B.9). Now, for λ in the appropriate sector, Lemma B.2 shows that

$$\begin{aligned}\|(\lambda I - \tilde{A})^{-1}\| = &\sup_{f \neq 0} \frac{\|\,(\lambda I - \tilde{A})^{-1} f\,\|}{\|\,f\,\|} \\ \le\ &\max\{\|(\lambda^2 I + (\mu + k\lambda)\tilde{\partial}\,)^{-1}(\lambda I + k\partial)\|,\ \|(\lambda^2 I + (\mu + k\lambda)\tilde{\partial}\,)^{-1}\|, \\ &\|(\lambda^2 I + (\mu + k\lambda)\tilde{\partial}\,)^{-1}\mu\tilde{\partial}\|,\ \|\lambda(\lambda^2 I + (\mu + k\lambda)\tilde{\partial}\,)^{-1}\|\} \\ \le\ &K'/|\lambda| \end{aligned} \tag{B.30}$$

where K' is the appropriate constant, and $|\lambda|$ is sufficiently large. This inequality, together with Theorem B.2 shows that $\tilde{A}$ generates an analytic semigroup. ■

The operator we are really concerned with is A, not $\tilde{A}$. Let

$$A = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & -k\partial(\cdot) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_B}\mu\partial'(\cdot)|_c & +\frac{1}{m_B}k\partial'(\cdot)|_c \end{bmatrix}. \tag{B.31}$$

$$=: A' + B$$

By using Theorem B.1, and the analogous result for the axial displacements, A' generates an analytic semigroup. The question arises: Does A generate an analytic semigroup? Thinking of B as a perturbation of A', the question becomes that of whether analyticity is preserved under perturbations. As one might expect, only certain classes of perturbations are allowed in order for this to be true. To show that indeed the B of (B.31) is contained in this class, the following theorem is needed: [Paz. 1, p.80]

Theorem B.3: Let A be the infinitesimal generator of a uniformly bounded, analytic semigroup. Let B be a closed linear operator satisfying $D(B) \supset D(A)$ and

$$\|Bx\| \le a\|Ax\| \qquad \text{for } x \in D(A). \tag{B.32}$$

Then there exists a positive constant δ such that for $0 \le a \le \delta$, $A + B$ is the infinitesimal generator of an uniformly bounded, analytic semigroup.

A careful examination of the proof of this lemma shows that an estimate of the constant δ is given by

$$\delta = \frac{1}{2}(1 + M)^{-1} \qquad \text{where } M \text{ satisfies } \|(\lambda I + A)^{-1}\| \le M/|\lambda| \tag{B.33}$$

Theorem B.4 The linear operator A given in (B.2) generates an analytic semigroup T(t) satisfying $\|T(t)\| \le M'\exp(\omega_0(t))$ with $\omega_0 = \sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\}$.

Proof of Theorem B.4: By Theorem B.1, A' generates an analytic semigroup. Since all the eigenvalues of A' are negative and bounded away from zero (see Figure 5), Theorem B.1 and Proposition B.2 shows that A' generates an exponentially stable semigroup. Thus A' generates a uniformly bounded semigroup. A simple calculation shows that B is closed while clearly $D(B) \supset D(A')$. So we are left to show that the inequality (B.32) is satisfied for the given A' and B. For the B defined in (B.31) note that

$$\begin{aligned} \|Bx\| &= \frac{1}{m_B} \|\mu\partial'(x_1)|_c + k\partial'(x_2)|_c\| = \frac{1}{m_B} \left\| \int_c^{c+L} \mu\partial(x_1) + k\partial(x_2)\, dz \right\| \\ &= \frac{1}{m_B} \|C(\mu\partial(x_1) + k\partial(x_2))\| \end{aligned} \tag{B.34}$$

where $C: L^2[c, c+L] \to L^2[c, c+L]$ is the linear integral operator

$$C(f) = \int_c^{c+L} f(z)\, dz$$

It is easy to see that $\|C\| = \sup_{f \neq 0} \frac{\| Cf \|}{\| f \|} = L$. Hence (B.34) becomes

$$\|Bx\| \leq \frac{1}{m_B} \|C\| \, \|(\mu\partial(x_1) + k\partial(x_2))\| = \frac{L}{m_B} \|(\mu\partial(x_1) + k\partial(x_2))\|$$

$$\leq \frac{L}{m_B} \|A'x\| \qquad \text{(B.35)}$$

So, if L/m_B is sufficiently small (i.e., the rigid body has a much larger mass than the beam) then the hypotheses of Theorem B.3 are satisfied so that A generates an analytic semigroup T(t). To get an estimate of the required L/m_B ratio, one would set $L/m_B = \delta$ of (B.33) with M=K′ of (B.30). To get an accurate estimate of K′ requires bookkeeping of the various constants in the proof of Lemma B.2. The last claim follows from the the analyticity of T(t) and Proposition B.2.

■

Finally, to get an estimate of the exponential decay rate of the generated semigroup T(t), the eigenvalues must be computed.

Lemma B.3 Consider the linear operator A given by equation (B.2). Consider the eigenvalue problem

$$Ax = \lambda x \qquad \text{(B.36)}$$

Then A has eigenvalues

$$\lambda_i^{\pm} = \frac{k_i v_i^4 \pm \sqrt{k_i^2 v_i^8 - 4\mu_i v_i^4}}{2} \qquad i = 1, 2 \qquad \text{(B.37)}$$

$$\lambda_3^{\pm} = \frac{k_3 v_3^2 \pm \sqrt{k_3^2 v_3^4 - 4\mu_3 v_3^2}}{2} \qquad \text{(B.38)}$$

where the v_i satisfy

$$\cosh v_i L \cos v_i L + \frac{1}{v_i m_B}(\cos v_i L \sinh v_i L + \cosh v_i L \sin v_i L) = 0 \qquad i = 1, 2 \qquad \text{(B.39)}$$

$$\cos v_3 L = (\sin v_3 L) / v_i m_B. \qquad \text{(B.40)}$$

The eigenvectors corresponding to these eigenvalues are

$$x_1^{\pm} = ([\ ]_1, 0, 0, \lambda_1^{\pm}[\ ]_1, 0, 0)^T \tag{B.41}$$

$$x_2^{\pm} = (0, [\ ]_2, 0, 0, \lambda_2^{\pm}[\ ]_2, 0)^T \tag{B.42}$$

$$x_3^{\pm} = (0, 0, [\ ]_3, 0, 0, \lambda_3^{\pm}[\ ]_3)^T \tag{B.43}$$

where $[\ ]_i = (2c_{i2}/v_1 - c_{i1})\cos v_i(z-c) - c_{i2}(\sin v_i(z-c) + 1/(v_i m_B)) + c_{i1}\cosh v_i(z-c) + c_{i2}(\sin v_i(z-c) - 1/(v_i m_B))$, $i=1, 2$, and

$[\ ]_3 = (\sin v_3(z-c) - 1/(v_3 m_B)) + \cosh v_3(z-c) 1/(v_3 m_B)$. Also, the c_{ij} satisfy

$$\begin{bmatrix} \cos v_i L + \cosh v_i L & \sin v_i L + \sinh v_i L - 2\cosh v_i L/(v_i m_B) \\ \sinh v_i L - \sin v_i L & \cos v_i L + \cosh v_i L - 2\cosh v_i L/(v_i m_B) \end{bmatrix} \begin{bmatrix} c_{i1} \\ c_{i2} \end{bmatrix} = 0, \quad i=1,2 \tag{B.44}$$

Proof of Lemma B.3: Direct computation shows that $Ax_i^{\pm} = \lambda_i^{\pm} x_i^{\pm}$. The conditions that the v_i and c_{ij} satisfy come from the beam boundary conditions

$$u_i(c) = 0, \ u'_i(c) = 0 \tag{B.45}$$

$$u''_i(c+L) = 0, \ u'''_i(c+L) = 0 \tag{B.46}$$

which correspond to zero deflection and velocity at the fixed end, and zero moment and force at the free end.

To obtain the c_{ij}, write an arbitrary eigenvector as a linear combination of $\cosh v_i(z-c)$, $\cos v_i(z-c)$, $\sinh v_i(z-c)$ and $\sin v_i(z-c) = 0$, $i = 1, 2$ and similarily for the v_3 term. These combinations must satisfy the boundary conditions if they are to be eigenvectors. Since there are 4 boundary conditions and four unknown coefficients, we get a homogenous system of 4 equations and 4 unknowns. If the coefficients are to be nonzero, the determinant of the corresponding matrix must be zero. The determinant of the system is precisely the conditions (B.39) and (B.40). Partially solving the resulting system, and inserting the partial solution results in the system (B.44). ■

We are finally prepared to prove Theorem 3.1, part(iii).

Proof of Theorem 3.1, part(iii) - Let $T(t)$ denote the semigroup generated by A. Since the perturbation terms in (B.1) are Lipshitz in $\underline{u}$ and $\dot{\underline{u}}$, the use of the "variation of constants" formula is admissable (see [Paz. 1, Thm 1.2, p.184]) so that the solution of (B.1) can be written

$$\begin{bmatrix} \underline{u}(t) \\ \dot{\underline{u}}(t) \end{bmatrix} = T(t-t_0) \begin{bmatrix} \underline{u}(t) \\ \dot{\underline{u}}(t) \end{bmatrix} - \int_{t_0}^{t} T(t-\tau) \begin{bmatrix} 0 \\ \dot{\underline{\omega}}(\tau) x \underline{u}(\tau) + 2\underline{\omega}(\tau) x \ \dot{\underline{u}}(\tau) + \underline{\omega}(\tau) x (\underline{\omega}(\tau) x \underline{u}(\tau)) \end{bmatrix} d\tau \tag{B.47}$$

Take norms on both sides and use the fact that $\|\underline{a}\times\underline{b}\| = \|\underline{a}\|\ \|\underline{b}\|\ |\sin(\theta)| \le \|\underline{a}\|\ \|\underline{b}\|$ where $\theta\in[0,\pi]$ is the angle in radians between $\underline{a}$ and $\underline{b}$ to obtain

$$\left\|\begin{matrix}\underline{u}(t)\\ \dot{\underline{u}}(t)\end{matrix}\right\| \le \|T(t-t_0)\| \left\|\begin{matrix}\underline{u}(t_0)\\ \dot{\underline{u}}(t_0)\end{matrix}\right\| + \int_{t_0}^{t} \|T(t-\tau)\| \left\{\begin{matrix}\|\dot{\underline{\omega}}(\tau)\|\|\underline{u}(\tau)\| + 2\|\underline{\omega}(\tau)\|\|\dot{\underline{u}}(\tau)\| + \\ \|\underline{\omega}(\tau)\|^2\|\underline{u}(\tau)\|\end{matrix}\right\} d\tau \tag{B.48}$$

Using Theorem B.4, together with Lemma B.3 then yields

$$\le M\exp(\omega_0(t-t_0)) \left\|\begin{matrix}\underline{u}(t_0)\\ \dot{\underline{u}}(t_0)\end{matrix}\right\| + \int_{t_0}^{t} M\exp(\omega_0(t-\tau)) \left\{\begin{matrix}\|\dot{\underline{\omega}}(\tau)\|\ \|\underline{u}(\tau)\| + 2\|\underline{\omega}(\tau)\|\ \|\dot{\underline{u}}(\tau)\| + \\ \|\underline{\omega}(\tau)\|^2\|\underline{u}(\tau)\|\end{matrix}\right\} d\tau \tag{B.49}$$

where $\omega_0 = \sup\{\ \mathrm{Re}(\lambda) \mid \lambda\in\sigma(A)\}$. Now define

$$x(t) := \exp(-\omega_0(t-t_0)) \left\|\begin{matrix}\underline{u}(t)\\ \dot{\underline{u}}(t)\end{matrix}\right\| = \exp(-\omega_0(t-t_0))[\ \|\underline{u}(t)\| + \|\dot{\underline{u}}(t)\|\ ] \tag{B.50}$$

Then insert (B.50) into (B.49) and simplify to get

$$x(t) \le Mx(t_0) + \int_{t_0}^{t} Mx(\tau)\{\|\ \dot{\underline{\omega}}(\tau)\| + 2\|\underline{\omega}(\tau)\| + \|\underline{\omega}(\tau)\|^2\} d\tau \tag{B.51}$$

Now apply the Bellman-Gronwall lemma to obtain

$$x(t) \le Mx(t_0)\exp\Big(\int_{t_0}^{t} M\{\|\ \dot{\underline{\omega}}(\tau)\| + 2\|\underline{\omega}(\tau)\| + \|\underline{\omega}(\tau)\|^2\} d\tau\Big) \tag{B.52}$$

Since both $\underline{\omega}(t)$ and $\dot{\underline{\omega}}(t)$ go to zero exponentially by design we can choose t_0 sufficiently large so that $\max\{\|\underline{\omega}(t)\|, \|\dot{\underline{\omega}}(t)\|\} \le \min\{1, -\omega_0/[8M]\}$, $\forall t\ge t_0$. Thus

$$x(t) \le Mx(t_0)\exp[-\omega_0(t-t_0)/2] \qquad \forall t\ge t_0.$$

Substituting (B.50) for x(t), we obtain

$$\left\|\begin{matrix}\underline{u}(t)\\ \dot{\underline{u}}(t)\end{matrix}\right\| \le M \left\|\begin{matrix}\underline{u}(t_0)\\ \dot{\underline{u}}(t_0)\end{matrix}\right\| \exp[\omega_0(t-t_0)/2] \qquad \forall t\ge t_0.$$

which, since $\omega_0<0$, proves the theorem. ■

Remark - One minor, but annoying, point should be briefly discussed. In order to prove that the beam deflections go to zero exponentially, it was necessary to assume that the rigid body mass was much larger than the beam mass. This plausible assumption was necessary in order to prove that A in (B.2) generated an analytic semigroup. Recall that $A = A' + B$ where B is the "perturbation". The perturbation is due to the coupling of the rigid body center of mass acceleration and the beam dynamics (see (B.1)). If the body mass is much larger than the beam mass, then (B.1) shows that the perturbation is small. It is intuitively clear that if the perturbation is small, then the deflections should go to zero exponentially since the beam is essentially decoupled from the rigid body. Heuristically, this is the content of Theorem B.5. Theorem B.5 does not rule out other dynamical behavior in the case where the perturbation is large. Further analysis will be required to determine whether other phenomenon can actually occur.

APPENDIX C - Proof of Theorem 4.1.

In order to prove the result, we need to resort to the methods of semigroup theory similar to the proof in Appendix B. We first need to show that the solutions to the set of differential equations exist (Proposition C.3), and that these solutions are exponentially stable (Theorem C.1). Actually, Theorem C.1 is a known result due to Chen [Che. 1], and the reader uninterested in the details may proceed to that point. The proof is included for two reasons. First, the proof of this result in [Che. 1] is very terse. Secondly, the method of proof given here explains how and why various quantities are chosen, which might be helpful for the reader choosing to apply these methods to another problem.

Consider now the linear operator

$$A = \begin{bmatrix} 0 & I \\ -\mu\partial(\cdot) & 0 \end{bmatrix}, \tag{C.1}$$

(where $\partial(\cdot)=(\frac{\partial^4(\cdot)}{\partial z^4}, \frac{\partial^4(\cdot)}{\partial z^4}, -\frac{\partial^2(\cdot)}{\partial z^2})^T$, and $\mu=\mathrm{diag}(\mu_1, \mu_2, \mu_3)$) which corresponds to (B.2) with zero viscous damping (k), and an ignoring of the $\frac{1}{m_B}\mu\partial'(\cdot)|_c$ term.

Since we have a (unbounded) linear operator, we must define the space it operates on, its domain, and since Hilbert spaces are very convenient, an inner product. Let the space A operates on, X, be defined as

$$X = \{(x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T |\ x_1 \in H^2, x_2 \in H^2, x_3 \in H^1,\ x_4 \in L^2, x_5 \in L^2, x_6 \in L^2, \\ x_1(c)=x_2(c)=x_3(c)=0, x_1'(c)=x_2'(c)=0\ \} \tag{C.2}$$

where the H^k are defined in Appendix B, and let the corresponding "energy" inner product be

$$[f, g]_E = [(f_1, f_2, f_3, f_4, f_5, f_6)^T, (g_1, g_2, g_3, g_4, g_5, g_6)^T]_E = \mu_1[f_1'', g_1''] + \mu_2[f_2'', g_2''] + \\ \mu_3[f_3'', g_3''] + [f_4, g_4] + [f_5, g_5] + [f_6, g_6]. \tag{C.3}$$

Let the domain of A, D(A), be defined as

$$D(A) = \{(x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T |\ x_1 \in H^4, x_2 \in H^4, x_3 \in H^2,\ x_4 \in H^2, x_5 \in H^2, x_6 \in H^1, \\ x_1(c)=x_2(c)=x_3(c)=x_4(c)=x_5(c)=x_6(c)=0,\quad x_1'(c)=x_2'(c)=x_4'(c)=x_5'(c)=0,\quad x_1''(c+L) \\ = x_2''(c+L) = 0, x_1'''(c+L)=\alpha x_4(c+L), x_2'''(c+L)=\beta x_5(c+L), x_3'(c+L)=\gamma x_6(c+L)\} \tag{C.4}$$

where $\alpha>0$, $\beta>0$, and $\gamma>0$.

Remark - A reasonable question at this point is whether the space X with the corresponding inner product (C.3) is complete or not. If it wasn't, we would have to complete it, which woud make the succeeding proofs more complicated. This leads us to the following proposition.

Proposition C.1 - The space X defined in (C.2) together with the corresponding inner product (C.3) is a complete Hilbert space.

Proof of Proposition (C.1) - Recall $\{x \mid x \in H^k,\ x(c)=x'(c)= \ldots =x^{k-1}(c)=0\}$ with corresponding inner product

$$[x, y]_S = [x, y] + [x', y'] + \ldots [x^k, y^k] \tag{C.5}$$

is a complete inner product space for k=0, 1, 2, ...(it is a Sobelev space). If we can show that the norm (C.3) is equivalent to the norm (C.5), then the proposition follows. Clearly,

$$\|x\|_E \leq K\|x\|_S \tag{C.6}$$

Conversely,

$$\|x\|^2_S = \int_c^{c+L} [(x_1'')^2 + (x_1')^2 + x_1^2 + (x_2'')^2 + (x_2')^2 + x_2^2 + (x_3')^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2]\, dx$$

$$\leq (2-K)\|x\|^2_E + \int_c^{c+L} [(x_1')^2 + x_1^2 + (x_2')^2 + x_2^2 + (x_3)^2]\, dx \tag{C.7}$$

where $K=\min\{\mu_1, \mu_2, \mu_3, 1\}$. Thus

$$\|x\|^2_S \leq 2\|x\|^2_E + \int_c^{c+L} \left\{ \left(\int_c^x x_1''dx\right)^2 + \left(\int_c^x x_1'dx\right)^2 + \left(\int_c^x x_2''dx\right)^2 + \left(\int_c^x x_2''dx\right)^2 + \left(\int_c^x x_3'dx\right)^2 \right\} dx \tag{C.8}$$

$$\leq 2\|x\|^2_E + \int_c^{c+L} \left\{ \left(\int_c^x |x_1''|dx\right)^2 + \left(\int_c^x |x_1'|dx\right)^2 + \left(\int_c^x |x_2''|dx\right)^2 + \left(\int_c^x |x_2''|dx\right)^2 + \left(\int_c^x |x_3'|dx\right)^2 \right\} dx \tag{C.9}$$

Next, using the Schwarz inequality, we obtain

$$\leq 2\|x\|^2_E + L\int_c^{c+L} \left\{ \int_c^{c+L} |x_1''|^2 dx + \int_c^{c+L} |x_1'|^2 dx + \int_c^{c+L} |x_2''|^2 dx + \int_c^{c+L} |x_2'|^2 dx + \int_c^{c+L} |x_3'|^2 dx \right\} dx \tag{C.10}$$

By using the same process as above, transform the integral terms involving x_1' and x_2' into terms involving x_1'' and x_2''. It is easy to see that we then obtain

$$\|x\|^2_S \leq K'\|x\|^2_E$$

for an appropriate K'. Hence, the norms are equivalent, and the space X and inner product given by (C.3) constitute a complete inner product space. ■

Remark - Another reasonable question at this point is why X and the inner product were chosen as they were. This point is important for applications and is never directly addressed in the literature. X was chosen as $H^4xH^4xH^2xL^2xL^2xL^2$ so that the operator A is closed (see below): the initial conditions were included so that the corresponding inner product (C.3) would be equivalent to the "natural" inner product (C.5). The inner product (C.3) was chosen in order to make A a dissipative operator (see below). This makes it easy to show that A generates a strongly continuous semigroup (Proposition C.3 below). Finally, the domain of A, D(A), was chosen not only to ensure that A is defined on it, but also to ensure that D(A) is dense in X . Generally, one wants to choose the domain as large as possible, because the domain specifies the allowable initial conditions in the differential equations.

We now consider properties of the domain of A, D(A).

Proposition C.2 - Consider D(A) defined as in (C.4) Then D(A) is dense in the space X.

Proof of Proposition C.2 - First, it should be clear that

$$Y = \{(y_1\ y_2\ y_3\ y_4\ y_5\ y_6)^T \mid y_1 \in C^\infty,\ y_2 \in C^\infty, y_3 \in C^\infty, y_4 \in C^\infty, y_5 \in C^\infty, y_6 \in C^\infty,$$
$$y_1(c)=y_2(c)=y_3(c)=0,\ y_1'(c)=y_2'(c)=0\}$$

is dense in X. It thus suffices to show that an arbitrary C^∞ function can be approximated arbitrarily closely by a C^∞ function satisfying homogeneous boundary conditions. Consider the following function

$$g(x)=\begin{cases} \displaystyle\int_0^x \exp\left(-\frac{1}{x^2}\right)\exp-\left(\frac{1}{x-\delta}\right)^2 dx \Big/ \int_0^\delta \exp\left(-\frac{1}{x^2}\right)\exp-\left(\frac{1}{x-\delta}\right)^2 dx & 0 \leq x < \delta \\ 1 & \delta \leq x < L-\delta \\ \displaystyle\int_{L-\delta}^x \exp-\left(\frac{1}{(x-L)^2}\right)\exp-\left(\frac{1}{x-L+\delta}\right)^2 dx \Big/ \int_{L-\delta}^x \exp-\left(\frac{1}{(x-L)^2}\right)\exp-\left(\frac{1}{x-L+\delta}\right)^2 dx & L-\delta \leq x < L \end{cases}$$

Note that g(x) is (i) C^∞ for any $\delta>0$, (ii) 1 for $\delta<x<L-\delta$ and (iii) $g^k(0)=g^j(L)=0$ for any k=0, 1, 2, ... and j=0, 1, 2, ... Then if f(x) is a C^∞ function, then f(x)g(x) is the approximating C^∞ function. ■

Proposition C.3 - Consider the operator A of (C.1) together with the corresponding space X and inner product given by (C.3). Then A generates a strongly continuous semigroup.

Proof of Proposition C.3 - By the Lumer-Phillips theorem [Paz. 1, p. 14, Theorem 4.3], a linear operator with dense domain generates a strongly semigroup if A is dissipative, and $\exists$ $\lambda > 0$ such that the range of $\lambda I - A$ is all of X. By Proposition C.2 D(A) is dense in X. Note that if $v= (v_1\ v_2\ v_3\ v_4\ v_5\ v_6)^T$, then using (C.3) and an integration by parts yields

$$(Av,v) = -\alpha(v_4(c + L))^2 - \beta(v_5(c + L))^2 - \gamma(v_6(c + L))^2 \le 0.$$

Hence A is dissipative. To complete the proof, we need only show that for some $\lambda > 0$ the range of $\lambda I - A$ is all of X. This is done in two steps: (i) $\forall\ \lambda > 0$, the range of $\lambda I - A$ is dense in X, and (ii) the range of $\lambda I - A$ is closed.

Proof of (i) - Take $\lambda > 0$ arbitrary, and suppose $\exists\ y \in X$ such that $((\lambda I - A)x, y) = 0$ for all $x \in D(A)$. If $x = (x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T$ and $y= (y_1\ y_2\ y_3\ y_4\ y_5\ y_6)^T$, then $((\lambda I - A)x, y) = 0$ implies

$$\begin{aligned}
&\mu_1(\lambda x_1'' - x_4'', y_1'') = 0 \\
&\mu_2(\lambda x_2'' - x_5'', y_2'') = 0 \\
&\mu_3(\lambda x_3' - x_6', y_3') = 0 \\
&(\mu_1 x_1'''' + \lambda x_4, y_4) = 0 \\
&(\mu_2 x_2'''' + \lambda x_5, y_5) = 0 \\
&(-\mu_3 x_3'' + \lambda x_6, y_6) = 0
\end{aligned} \qquad \text{(C.11)}$$

Set $x_4=x_5=x_6=0$. Now let x_1, x_2, and x_3 be arbitrary C^∞ functions satisfying the boundary conditions $x_1(c)=x_2(c)=x_3(c)=0$, $x_1'(c)=x_2'(c)=0$, $x_1'''(c+L)=x_2'''(c+L)= x_3'(c+L)=0$. Then clearly $(x_1\ x_2\ x_3\ 0\ 0\ 0)^T \in D(A)$, and the class of such elements is dense in $H^4 x H^4 x H^2$. Then the equations (C.11) imply $y_1''=y_2''=y_3'=y_4=y_5=y_6=0$. Since $y_1''(x)=y_2''(x)=y_3'(x)=0$, $\forall\ x \in [c, c + L]$, the boundary conditions satisfied by these y's imply $y_1=y_2=y_3=0$. Hence y=0, and thus the range of $\lambda I - A$ is dense in X.

Proof of (ii) - Let $y_n =(\lambda I - A)x_n$ converge to $y \in X$. We must show that $\exists\ x \in D(A)$ such that $y = (\lambda I - A)x$. Since A is dissipative, we have

$$\| y_n\|^2 = \|(\lambda I - A)x_n\|^2 = \lambda\| x_n\|^2 - 2\lambda(x_n, Ax_n) + \| Ax_n\|^2 \quad (C.12)$$

$$\geq \lambda\| x_n\|^2 + \| Ax_n\|^2 \quad (C.13)$$

$$\geq \lambda\| x_n\|^2 \quad (C.14)$$

Since y_n converges, this implies x_n converges to some value $x \in X$. (Consider a Cauchy sequence $y_n - y_m$). Hence, by (C.13), Ax_n converges. If $x_n = (x_{n1}\ x_{n2}\ x_{n3}\ x_{n4}\ x_{n5}\ x_{n6})^T$ and $x = (x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T$, then

$$Ax_n = (x_{n4}\ x_{n5}\ x_{n6}\ -\mu_1 x_{n1}''''\ -\mu_2 x_{n2}''''\ \mu_3 x_{n3}'')^T$$

which shows that $x_1 \in H^4$, $x_2 \in H^4$, $x_3 \in H^2$, $x_4 \in H^2$, $x_5 \in H^2$, $x_6 \in H^1$. This implies that $x \in D(A)$, from which it follows that $y = (\lambda I - A)x$. ■

Proposition C.3 shows that the operator A of (C.1) generates a strongly continuous semigroup, but says nothing about its dynamic behavior. The following important result makes the situation clear.

Theorem C.1 Consider the operator A of (C.1) together with the corresponding space (C.2) and inner product (C.3). Then A generates an exponentially stable semigroup.

Proof of Theorem C.1 - See [Che. 2, Thm. 3.1, page 533]. The idea is a Lyapunov type argument. ■

With this result in hand, we are finally able to prove Theorem 4.1.

Proof of Theorem 4.1 - The rigid body control law (4.1) forces $\xi(t)\rightarrow 0$ exponentially exactly as in Theorem 3.1. To show that the beam velocities and deflections go to zero, use exactly the methods of the proof of Theorem B.5. More explicitly, let T(t) denoted the exponentially stable semigroup generated by the A of (C.1). Then use the proof of Theorem B.5 line for line. ■

Remark - From the proof of Theorem 4.1, it is easy to see that any exponentially stable beam configuration (damped beam, undamped beam with boundary control, etc.) will be sufficient for Theorem 4.1 to be true. In particular, Theorem 4.1 holds if the Euler-Bernoulli beam model (with or without damping) is replaced by a Timoshenko beam model (with or without damping). This follows clearly if the Timoshenko beam contains Kelvin-Voight type damping, and if it is undamped, boundary velocity feedback control exponentially stabilizes it in exactly the same way as the Euler-Bernoulli beam (see [Kim 1]). From an engineering viewpoint, the only difference is in the calculation of the forces and moments at the point of attachment, and the calculation of beam response during manuevers, quantities which clearly depend on the beam model employed.

Remark - Theorem 4.1 explicitly assumes $\ddot{\underline{y}} = 0$. Unfortunately, in contrast to Theorem 3.1, we cannot guarantee that A generates an exponentially stable semigroup if $\ddot{\underline{y}} \neq 0$. However, if we assume some infinitesimally small viscous damping term, then clearly Theorem 4.1 holds if the rigid body mass is much larger than the beam mass. (The reasoning of Theorem 3.1). The only drawback is that the exponential time constant may be undesirably large. Force thrusters on the rigid body (which are usually present to control the spacecraft center of mass) could certainly alleviate this problem, but this is extra complexity and fuel consumption.

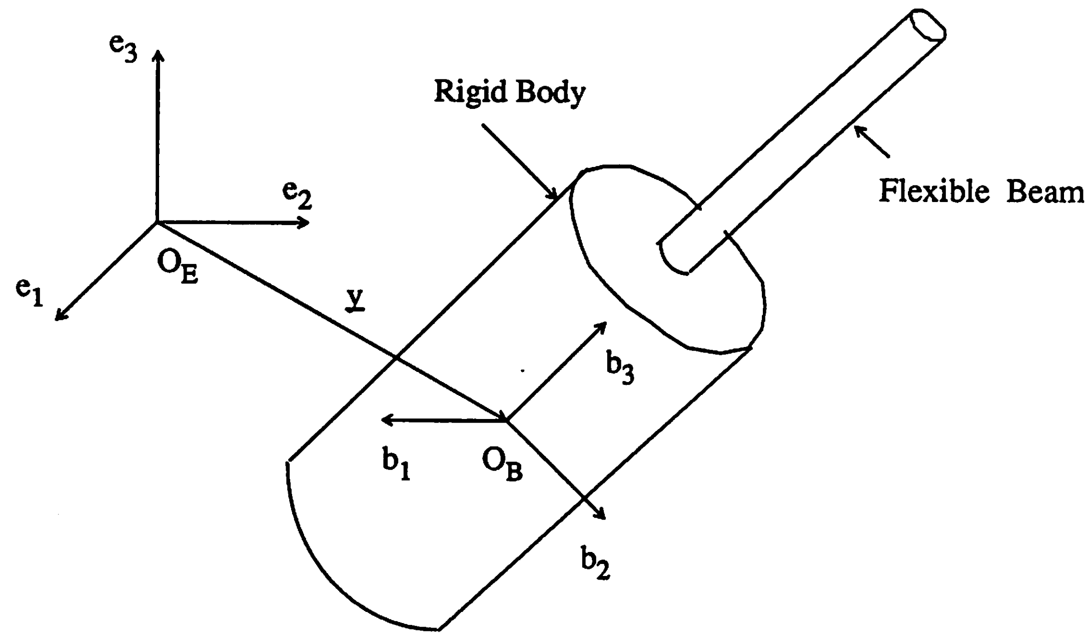

Figure 1 - Spacecraft Configuration

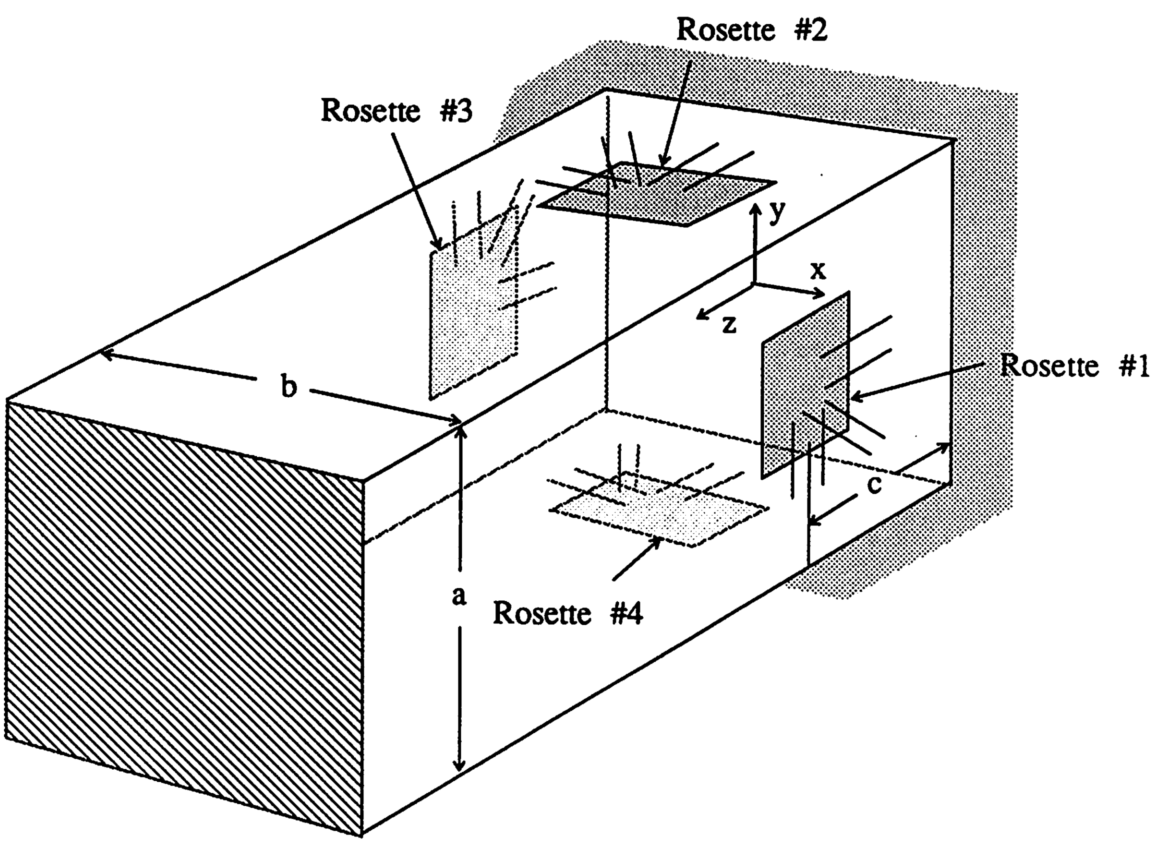

Figure 2 - Rectangular Beam under consideration

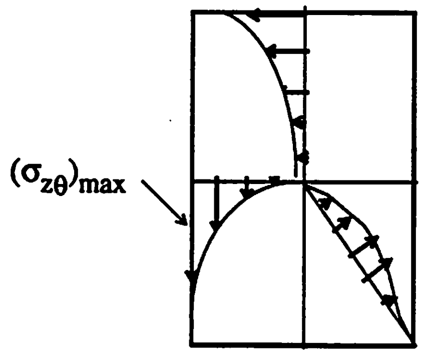

Figure 3 - Shear Stress Distribution
due to torsion

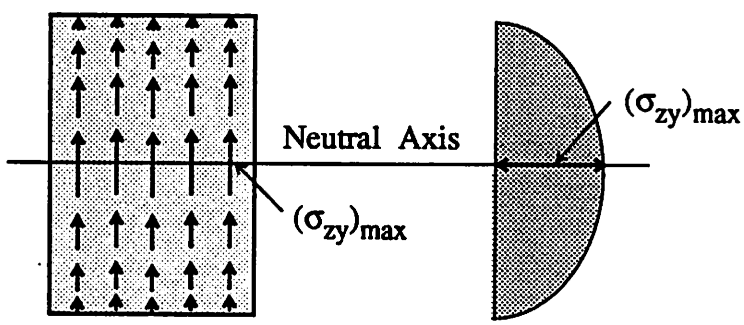

Figure 4 - Shear Stress Distributrion
due to vertical shear

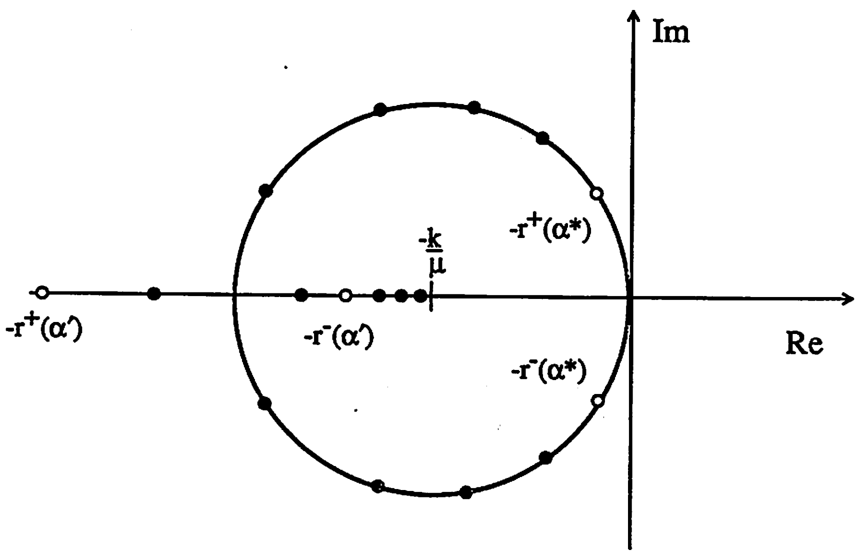

Figure 5 - Distribution of Zeroes of $\lambda^2 + (k + \mu\lambda)\alpha$

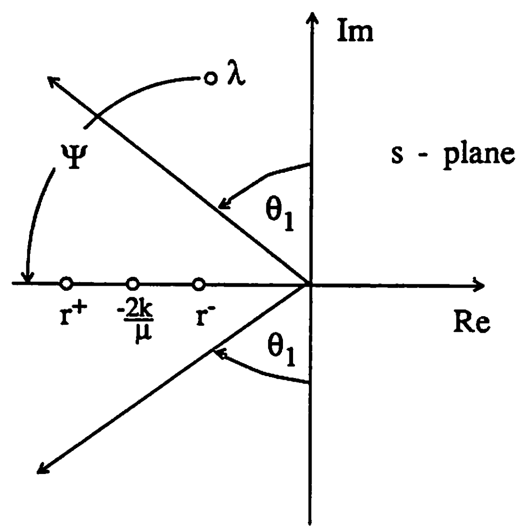

Figure 6 - Proof of Lemma B.2

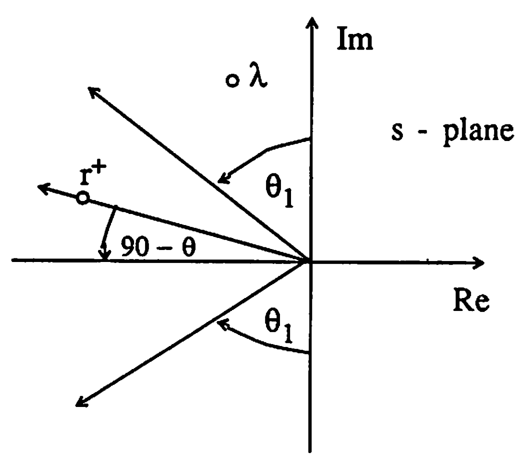

Figure 7 - Proof of Lemma B.2