THE COMPLETE CANONICAL PIECEWISE-LINEAR REPRESENTATION

by

Claus Kahlert and Leon O. Chua

Memorandum No. UCB/ERL M89/32

23 March 1989

THE COMPLETE CANONICAL PIECEWISE-LINEAR REPRESENTATION

by

Claus Kahlert and Leon O. Chua

Memorandum No. UCB/ERL M89/32

23 March 1989

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

TITLE PAGE

THE COMPLETE CANONICAL PIECEWISE-LINEAR REPRESENTATION

by

Claus Kahlert and Leon O. Chua

Memorandum No. UCB/ERL M89/32

23 March 1989

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

The Complete Canonical Piecewise-Linear Representation†

CLAUS KAHLERT AND LEON O. CHUA††

Abstract

An extension of the well-known *canonical representation* for *continuous piecewise-linear functions* is introduced. This form is no longer subject to any restrictions, moreover, it is shown that just one nesting of absolute-value functions is sufficient to describe the whole class.

† This research was supported in part by the Alexander von Humboldt-Foundation (CK), the Office of Naval Research under Contract N00014-89-J-1402, and by the National Science Foundation under Grant MIP 86 14000 (LOC), which is gratefully acknowledged.
†† C. Kahlert is with the Institute for Physical and Theoretical Chemistry, University of Tübingen, D–7400 Tübingen, West Germany; he is currently visiting the University of California at Berkeley. L.O. Chua is with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA.

1. Introduction

Piecewise-linear (PWL) models has proven very helpful in analyzing nonlinear circuits, not only from a computational point of view but also because they are much more amenable to analysis than general nonlinear equations. Specifically the *canonical representation* of continuous PWL functions

$$\mathbf{F}(\mathbf{x}) = \mathbf{b} + \mathbf{B}\mathbf{x} + \sum_{i=1}^{\sigma} \mathbf{c}^i \left| \langle \alpha^i, \mathbf{x} \rangle - \beta^i \right| \quad , \tag{1}$$

introduced in [1] gains an extra advantage compared to a conventional PWL description, both with respect to analytical and computational purposes simultaneously — a remarkable goal, which can be achieved only very rarely. These superior properties emerge primarily from the compact form, which takes advantage of the continuity of the described function, making its behavior in the different regions far from being completely independent. Thus a canonical representation may be interpreted as the *minimal* formulation, which makes full use of the "parametric degrees of freedom", i.e., no more parameters than necessary appear. This makes such a representation very satisfactory from a theoretical point of view. As far as concrete computations are concerned, the relatively small number of terms involved are moreover highly desirable for a computer (especially for vector-processors) since they involve mainly calculating inner products and absolute-values in $\mathbf{R}^n$.

The above canonical representation for continuous PWL functions describes a very large class of systems. In fact, it can be proved [2] that any "non-degenerate" (where no more than two boundaries have a common intersection) linear partition of the domain space gives rise to such a representation. Nevertheless, there exist numerous counter-examples, which show that this representation does not cover the whole class of continuous PWL functions. Unfortunately, the functions described by it are not even "dense" in the class, as might be inferred from the adjective "degenerate". At first glance, one might think that a situation with three or more boundaries possessing a common intersection is not generic and can thus be removed by a slight perturbation of the boundaries. However, any such perturbation produces new regions (reflecting new properties of the circuit or even new circuit elements) and neither circuit theory nor mathematics offers any information concerning the function's behavior inside the additional regions thus created. In the general case they do not fit into the present canonical representation, thus there is no *canonical unfolding* within the framework on hand.

In the subsequent sections we are going to show that the following *complete canonical representation* overcomes these shortcomings and is capable of representing *all kinds of piecewise-linear continuous functions with arbitrary region boundaries.*

$$\mathbf{F}(\mathbf{x}) = \mathbf{b} + \mathbf{B}\mathbf{x} + \sum_{i=1}^{\sigma} \mathbf{c}^i \left|\langle \alpha^i, \mathbf{x}\rangle - \beta^i\right| + \tag{2}$$
$$\sum_{j=1}^{\varrho} \sum_{k=3}^{\delta^j+2} \tilde{\mathbf{c}}^{jjk} \left\{ \left| \left| a^j_{j_k j_1}(\langle \alpha^{j_1}, \mathbf{x}\rangle - \beta^{j_1}) \right| + a^j_{j_k j_2}(\langle \alpha^{j_2}, \mathbf{x}\rangle - \beta^{j_2}) \right| - \right.$$
$$\left. \left| a^j_{j_k j_1}(\langle \alpha^{j_1}, \mathbf{x}\rangle - \beta^{j_1}) + \left| a^j_{j_k j_2}(\langle \alpha^{j_2}, \mathbf{x}\rangle - \beta^{j_2}) \right| \right| \right\}$$

Figure 1

In order to demonstrate the capabilities of this generalized representation, let us look at an example, illustrated in *Fig.* 1*a*: A PWL resistive three-port (**N**), being connected to a linear two-terminal resistor (**R**′). The constitutive relation of **N** is defined by the following canonical PWL representation:

$$\begin{aligned} v_1 &= a \cdot i_1 + b \cdot |i_3| \\ v_2 &= c \cdot i_2 \\ v_3 &= d \cdot i_2 - e \cdot i_3 + f \cdot |i_1| \quad , \end{aligned} \tag{3a}$$

while **R**′ is characterized by

$$v_4 = g \cdot i_4 \quad . \tag{3b}$$

As the third port of $\mathbf{N}$ is connected across $\mathbf{R}'$, we immediately obtain

$$\begin{aligned} v_1 &= a \cdot i_1 + b \cdot \left| \frac{d \cdot i_2 + f \cdot |i_1|}{g - e} \right| \\ v_2 &= c \cdot i_2 \end{aligned} \tag{4}$$

for the resulting two-port $\mathbf{N}'$. The behavior of the latter is visualized in *Fig.* 1*b*, using the parameter values $a, b, c, d, e, f = 1$; $g = 2$. *Equation* (4) with its nested absolute-value functions cannot be described in terms of (1), since it violates the *consistent variation* property (i.e., here the change in the Jacobian of the PWL function is not constant along the boundaries), which was shown in [2] to be a *necessary and sufficient* condition for a function to possess this sort of representation. The example also demonstrates that degenerate intersections of region boundaries can arise in *generic* setups; namely, perturbing the parameters in (3) will *not* yield a non-degenerate situation. Since the special parameter values chosen for *Fig.* 2 do not describe any exceptional situation, to avoid clutter, we shall use $a, b, c, d, e, f = 1$ further on.

Another property illustrated by the above (and the next) example is the occurrence of *terminating or piecewise-linear boundaries*. Such a situation cannot be described in terms of (1). In contrast, it is an easy task for our generalized representation to make the function's behavior the same along both sides of a part of a "boundary", thereby rendering it "invisible", while the function changes behavior along another portion of the same hyperplane.

With the fixed parameters mentioned, *Eq.* (4) may be processed by the algorithm described below, which yields for the first line

$$\begin{aligned} v_1 = i_1 + i_2 + \frac{1}{2} \cdot \Big\{ &|i_1 + i_2| + |i_1 - i_2| + \\ &\big| |i_1 + i_2| + i_1 - i_2 \big| - \big| i_1 + i_2 + |i_1 - i_2| \big| \Big\} \end{aligned} \tag{5}$$

(the second line of (4) remains unchanged). Although the equivalent representation (5) appears much longer and less elegant than (4), it has the form of the completely general canonical representation (2), which covers *all* piecewise-linear continuous functions and thus may be analyzed immediately by the tools to be presented in the subsequent sections.

Since the nested absolute-value of (4) could not be represented in terms of (1), one might still think that for every nesting appearing in the constitutive relation a new canonical representation has to be found. However, in this paper, we shall show how arbitrarily deep nestings or complicated boundaries can be formulated in terms of (2). To demonstrate the significance of our finding and of the algorithm (to be introduced in *Sections* 4 and 5), let us look at another example, where the third port of the resistive three-port **N** described above, is connected across a two terminal *piecewise-linear resistor* $\mathbf{R}''$ with the constitutive relation

$$v_4 = 3 \cdot i_4 + |i_4| \quad . \tag{6}$$

In the present case, for the resulting two-port $\mathbf{N}''$

$$\begin{aligned} v_1 &= i_1 + \frac{1}{3}\Big|2(i_2 + |i_1|) - \big|i_2 + |i_1|\big|\Big| \\ v_2 &= i_2 \end{aligned} \tag{7}$$

is obtained, containing already two nesting levels of the absolute-value function.

Figure 2

Utilizing again the algorithm to be presented below, the first line of (7) can be recast into the new canonical form containing just one level of nesting; namely,

$$\begin{aligned} v_1 = \frac{1}{3}\Big\{&3i_1 + i_2 - |i_1| + |i_1 + i_2| + |i_1 - i_2| + \\ &\big||i_1 + i_2| + i_1 - i_2\big| - \big|i_1 + i_2 + |i_1 - i_2|\big|\Big\} \quad . \end{aligned} \tag{8}$$

This second example again demonstrates how nested absolute-values are represented canonically. In fact, it does not matter at all how a function is represented in a concrete problem. The only information required for the canonical representation is the exact behavior in one arbitrary region together with the changes which the Jacobian of **F** undergoes along most (see below for details) boundaries between regions. The latter properties may be extracted from a nested absolute-value function as well as from a conventional representation, where for every single region the function is given separately, e.g., as a table of matrices.

It will also turn out that the complete canonical representation contains the minimal number of parameters. All redundancy resulting from the continuity of the piecewise-linear function to be described is already absorbed in the functional form of the new canonical representation.

Our approach for handling *degenerate* intersections will make use of the mutual dependence of boundaries with common intersections; thereby utilizing the fact that the changes in the Jacobian which occur when a boundary is crossed can either be cancelled or enhanced by the effects of another boundary. (This cannot happen in the non-degenerate case where there are only two intersecting boundaries. There the change in the function's behavior which appears as a boundary is crossed has to be compensated exactly, when the *same* boundary is crossed again in the other direction.)

2. A Quick Review of the Canonical Representation

Any continuous piecewise-linear function $\mathbf{F}$ with *non-degenerate* intersections of boundaries between regions may be written in the form given by *Eq.* (1), with $\beta^i \in \mathbf{R}$; $\mathbf{b}, \alpha^i, \mathbf{c}^i, \mathbf{x} \in \mathbf{R}^n$; and $\mathbf{B} \in \mathbf{R}^{n \times n}$. Here we adopted the notation and the definitions from [2] with the exception that coordinates in $\mathbf{R}^n$ are denoted by lower indices while upper indices will be used for "boundary-, intersection-, and region-labels". The boundaries themselves are linear manifolds (hyperplanes) defined by $\langle \alpha^i, \mathbf{x} \rangle = \beta^i$. The only assumption we make is that all regions in domain space have a non-zero volume in $\mathbf{R}^n$ (which may in fact be infinite).

A setup with affine boundaries might appear a rather special situation, however, it represents the most general case, as can be seen immediately from one key result to be used frequently further on. It comes from *Proposition* 1 of [3], which says that the difference in the Jacobians of regions adjacent to a boundary characterized by the vector α can always be written as a dyadic product $\mathbf{c}\alpha^T$, with a uniquely defined vector $\mathbf{c}$. This may be expressed briefly as "*continuity implies a dyad.*"

Since the latter result is crucial for the understanding of continuous PWL functions, we present the sketch of a proof, in order to gain some insight:

Let us assume that the function $\mathbf{F}$ is defined in two regions $R^{(1)}$ and $R^{(2)}$ with Jacobians $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$, respectively. For simplicity, let us further assume that the boundary between these regions is defined by $\langle \alpha, \mathbf{x} \rangle = 0$, i.e., $\beta = 0$, which can always be achieved by a simple translation (thus it is sufficent to look at *linear* boundaries instead of the more general *affine* ones). Since $\mathbf{F}$ $(= \mathbf{J}^{(i)}\mathbf{x} + \mathbf{w}^{(i)})$ is *continuous*, the function values have to match at the boundary (whose points will be denoted by $\mathbf{x}'$), hence

$$\Delta\mathbf{J}\mathbf{x}' := \mathbf{J}^{(2)}\mathbf{x}' - \mathbf{J}^{(1)}\mathbf{x}' = \mathbf{w}^{(1)} - \mathbf{w}^{(2)} \tag{9a}$$

has to be fulfilled. Specifically at the origin, which is a point of the boundary,

$$\Delta\mathbf{J}\,0 = 0 = \mathbf{w}^{(1)} - \mathbf{w}^{(2)} \tag{9b}$$

is obtained, which immediately gives us

$$\Delta\mathbf{J}\mathbf{x}' = 0 \tag{9c}$$

for all $\mathbf{x}'$ from the boundary. Let $\Delta\mathbf{J}_i$ be the i-th row of $\Delta\mathbf{J}$, then (9c) may be rewritten as

$$\begin{pmatrix} \langle \Delta\mathbf{J}_1^T, \mathbf{x}' \rangle \\ \vdots \\ \langle \Delta\mathbf{J}_n^T, \mathbf{x}' \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad . \tag{10}$$

Since this equation holds for *every* $\mathbf{x}'$ on the boundary, the $\Delta\mathbf{J}_i$ can differ from α^T only by a real constant c_i, yielding $\Delta\mathbf{J}_i = c_i\alpha^T$. Hence we have

$$\Delta\mathbf{J} = \mathbf{c}\alpha^T \quad , \tag{11}$$

which immediately (upon applying it to an *arbitrary* vector $\mathbf{x}$, which need not be from the boundary) gives us

$$\Delta\mathbf{J}\,\mathbf{x} = (\mathbf{c}\alpha^T)\,\mathbf{x} = \mathbf{c}\langle \alpha, \mathbf{x} \rangle \quad . \tag{11a}$$

•

Before we proceed, it is instructive to examine some consequences of the above analysis. Note first that *Eq.* (9*a*) is valid for arbitrary (not necessarily linear) boundaries. Nevertheless, (9*c*), which requires only the origin to be a point of the boundary (an assumption which was introduced above just for convenience), immediately implies that the boundary is the *kernel* of $\Delta\mathbf{J}$, i.e., a linear subspace. Thus, although frequently stated as an extra assumption, *linear boundaries* are an immediate *consequence of continuity*. As was mentioned above, the property also holds for general *affine* boundaries, which can be achieved by a translation.

Next take an $\mathbf{x} \in R^{(2)}$ and look at the effect of a switching Jacobian, which is due to $\Delta\mathbf{J}$. For this purpose, $\mathbf{x}$ is decomposed (which can be done uniquely) into $\mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$, where $\mathbf{x}^{\parallel}$ is along the boundary and $\mathbf{x}^{\perp}$ is normal to it. Now, from (11*a*) it becomes obvious that

$$\Delta\mathbf{J}\mathbf{x} = \Delta\mathbf{J}\mathbf{x}^{\perp} \tag{12}$$

and thus, upon "walking around" in the domain space, a change in the Jacobian can only be recognized in the direction normal to the boundary. In other words, upon crossing the boundary, only the derivative of $\mathbf{F}$ in the direction of α changes by $\mathbf{c}$, while the derivatives in all the other $n-1$ linearly independent directions remain unchanged. This demonstrates that, in order to preserve continuity, for a given boundary only n degrees of freedom (the components of $\mathbf{c}$) are present among of the n^2 entries of the Jacobian.

Since only one direction in space, namely the one given by α, is involved in the change of the function's behavior as a boundary is crossed, the latter "process" is essentially a one-dimensional problem. This is a trademark of the "canonical representation" with a constant jump in the Jacobian throughout the whole boundary, yielding the *consistent variation* property. At a first glance, such a property seems to be a rather severe restriction; however, as was shown in [2], all continuous PWL functions without degenerate intersections of boundaries can be described by (1).

In order to visualize the situation, the intersection of two boundaries will be treated next. However, first we are going to introduce a notion, which will prove very helpful and gain much insight. Let Γ be an arbitrary closed path in the domain space, possessing the parametric representation

$$\Gamma : [0,1] \rightarrow \mathbf{R}^n \ ; \ s \mapsto \Gamma(s) \quad . \tag{13}$$

Along any such path, the Jacobian of $\mathbf{F}$ is piecewise-constant. This yields the motivation to formulate the Jacobian as a function over the unit interval

$$\mathcal{J}_\Gamma : [0,1] \rightarrow \mathbf{R}^{n\times n} \; ; \; s \mapsto \mathcal{J}_\Gamma(s) = \left(\frac{\partial F_i}{\partial x_j}\right)\bigg|_{\mathbf{x}=\Gamma(s)} \tag{14}$$

with values in the class of all real $n \times n$ matrices. Obviously $\mathcal{J}_\Gamma$ is piecewise-constant with jumps of "height" $\mathbf{c}\alpha^T$ on the boundaries. Moreover, since $\mathbf{F}$ is single-valued, we have

$$\mathcal{J}_\Gamma(1) = \mathcal{J}_\Gamma(0) \quad , \tag{15}$$

which can be considered as a *global continuity condition*, in contrast to the local condition (11), found along the boundaries. *Equation* (15) further reduces the number of free parameters available in the continuous PWL functions. This constraint is a global consequence of the continuity of $\mathbf{F}$ and should not be confused with the limitations in the canonical PWL representation.

Figure 3

Let us now turn to the case of two intersecting boundaries. Such a situation can be characterized by two linearly independent α's (say α^1 and α^2), which form a basis for the jumps that can occur. In consequence, no change appearing along the boundary defined by α^1 can be influenced by the change in the Jacobian along the other boundary. In other words, for any two non-zero vectors $\alpha^1 \neq \alpha^2$ there exist no vectors $\mathbf{c}^1, \mathbf{c}^2 \neq 0$ such that $\mathbf{c}^1{\alpha^1}^T = \mathbf{c}^2{\alpha^2}^T$. This gives a geometrical interpretation of the consistent variation property for non-degenerate intersections of boundaries. The situation changes immediately as degenerate intersections (with three or more boundaries sharing a common $(n-2)$-dimensional manifold) appear, since then the α's are *no longer linearly independent.*

3. The Complete Canonical Representation

In this section we are going to state our major result and demonstrate some properties of the *complete canonical representation*. First, however, we need some definitions, which extend those of [2].

Def. 1 *A connected open set of non-zero n-dimensional volume in domain space, where the Jacobian of the PWL function* $\mathbf{F}$ *is constant throughout is called a region* $R^{(i)}$ *of* $\mathbf{F}$.

As long as just one function will be considered, these sets will just be called regions.

Def. 2 *A finite family of regions is called a (complete) covering of the domain space if the union of the closures of the regions is the entire space.*

Def. 3 *Let* $R^{(i)}$ *and* $R^{(j)}$ *be regions with Jacobians* $\mathbf{J}^{(i)}$ *and* $\mathbf{J}^{(j)}$*, respectively, of* $\mathbf{F}$. *Then the matrix*

$$\Delta\mathbf{J}^{(ij)} := \mathbf{J}^{(i)} - \mathbf{J}^{(j)} \tag{16}$$

is called the difference of or the jump in the Jacobian of $\mathbf{F}$ *between regions* $R^{(i)}$ *and* $R^{(j)}$.

This difference can of course be calculated for arbitrary pairs of regions, for *adjacent regions*, however, it has the special form given by *Eq.* (11).

Def. 4 *Two regions* $R^{(i)}$ *and* $R^{(j)}$ *are called adjacent, if there exists a point* $\mathbf{x}$ *and an* $r_0 \in \mathbf{R}^+$ *such that for all* $0 < r \leq r_0$ *the open balls* $\mathcal{B}_r(\mathbf{x})$ *centered at* $\mathbf{x}$ *with radius* r *contain both points of* $R^{(i)}$ *and* $R^{(j)}$ *and are proper subsets of the union of the closures of the two regions.*

This definition rules out situations, when boundaries of regions have just isolated points in common, as is depicted in *Fig.* 4*a*.

Def. 5 *A finite family of (disjoint) regions is called a minimal (complete) covering if it fulfills Definition* 2 *and for any two adjacent regions* $R^{(i)}$ *and* $R^{(j)}$ *the matrix* $\Delta\mathbf{J}^{(ij)}$ *does not vanish.*

Although all possible coverings of the domain space can be employed to describe a PWL function, the *minimal covering* plays a prominent role, since it yields the largest connected regions where the function's Jacobian is constant. To avoid clutter, we shall treat only *minimal complete coverings.* This does not restrict us in any way, since all other coverings can be derived directly from it.

Def. 6 *An $(n-1)$-dimensional linear manifold (hyperplane)*

$$S^i := \{\mathbf{x} \,|\, \langle \alpha^i, \mathbf{x} \rangle = \beta^i\} \tag{17}$$

is called a pre-boundary for the continuous PWL function $\mathbf{F}$ *if there exists an* $\mathbf{x} \in S^i$ *and an open ball* $\mathcal{B}_{r_0}(\mathbf{x})$ $(r_0 \in \mathbf{R}^+)$, *such that for every* $\mathbf{x}' \in \mathcal{B}_{r_0}(\mathbf{x}) \cap S^i$ *all open balls* $\mathcal{B}'_r(\mathbf{x}')$ $(0 < r \leq r_0)$ *in domain space have non-empty intersection with at least two different regions of the minimal covering.*

Figure 4

Note that this definition allows the possibility that a portion of a pre-boundary may be "invisible", as was demonstrated in the examples of *Section* 1, while other parts are actual region boundaries. Conceptually an invisible boundary, i.e., a merging of regions in a minimal covering, means that regions need no longer be *convex sets.* In what follows, we assume that σ pre-boundaries of $\mathbf{F}$ are present in the domain space. When no confusion can arise, pre-boundaries will frequently be denoted just as boundaries. For convenience, sometimes the $\Delta\mathbf{J}$ will also be labeled by using the boundary-index carrying a further index. For example $\Delta\mathbf{J}^{i_k}$ means the k-th jump in the Jacobian appearing at the boundary S^i. In these cases we implicitly assume that the boundary is crossed from a region where $\langle \alpha^i, \mathbf{x} \rangle - \beta^i$ is less than zero towards a region where this expression is positive (see for example *Fig.* 6*a* below).

Next intersecting boundaries will be considered:

Def. 7 *An $(n-2)$-dimensional linear manifold I^j is called a direct boundary intersection if $I^j = S^k \cap S^l$ with $k \neq l$.*

Now that the notion of *direct boundary intersections* has been introduced, of which we assume to have ϱ present, and which will frequently be denoted just as intersections or nodes, we have to characterize them. For this purpose one more definition is required:

Def. 8 *The degeneracy δ^j of a direct intersection I^j is the number of boundaries S^i containing I^j as subsets, minus two. Thus*

$$\delta : [1 \ldots \varrho] \rightarrow [0 \ldots \sigma - 2] \; ; \; j \mapsto \delta^j \quad . \tag{18}$$

This reflects the fact that an intersection is called degenerate as soon as it is shared by three or more boundaries. (non-degenerate intersections have degeneracy zero.) The next step will be to introduce the geometry of the boundaries in domain space into the formalism. To this end an *index function* (parametrized by the "node index" j) has to be defined

$$\mathcal{I} : [1 \ldots \delta^j + 2] \rightarrow [1 \ldots \sigma] \; ; \; (i;j) \mapsto \mathcal{I}(i;j) \quad , \tag{19}$$

which connects the intersection I^j and the boundaries involved there, being labeled by $1 \ldots \delta^j + 2$, to the original labeling of the boundaries S^i. Note that $\mathcal{I}$ is not surjective if one or more boundaries have no intersection with others and the map is not injective if at least one boundary is involved in more than one node. Further on we shall frequently refer to $\mathcal{I}(j,i)$ loosely as j_i (meaning the original label of the i-th boundary involved in the intersection I^j).

With all these tools on hand, the complete canonical representation may now be presented in a compact formulation:

Theorem 1 *All continuous PWL functions can be represented in the form*

$$\mathbf{F}(\mathbf{x}) \; = \; \mathbf{b} \, + \, \mathbf{B}\mathbf{x} \, + \, \sum_{i=1}^{\sigma} \mathbf{f}^i(\mathbf{x}) \, + \, \sum_{j=1}^{\varrho} \sum_{k=3}^{\delta^j+2} \mathbf{g}^{jj_k}(\mathbf{x}) \tag{20}$$

with

$$\mathbf{f}^i(\mathbf{x}) := \mathbf{c}^i \left| \langle \alpha^i, \mathbf{x} \rangle - \beta^i \right| \tag{21}$$

and

$$\begin{aligned} \mathbf{g}^{jj_k}(\mathbf{x}) \; := \; \tilde{\mathbf{c}}^{jj_k} \Big\{ & \Big| \big| a^j_{j_k j_1} (\langle \alpha^{j_1}, \mathbf{x} \rangle - \beta^{j_1}) \big| + a^j_{j_k j_2} (\langle \alpha^{j_2}, \mathbf{x} \rangle - \beta^{j_2}) \Big| - \\ & \Big| a^j_{j_k j_1} (\langle \alpha^{j_1}, \mathbf{x} \rangle - \beta^{j_1}) + \big| a^j_{j_k j_2} (\langle \alpha^{j_2}, \mathbf{x} \rangle - \beta^{j_2}) \big| \Big| \Big\} \end{aligned} \tag{22}$$

where

$$\alpha^{j_k} := a^j_{j_k j_1}\alpha^{j_1} + a^j_{j_k j_2}\alpha^{j_2} \quad . \tag{23}$$

(Note that the latter function carries two indices, one from the node and the other describing the boundary.)

Moreover, (20) contains no redundant parameters.

The complete proof of this theorem will be presented in *Section* 5. Here we are going to demonstrate the main properties of the functions $\mathbf{g}^{jj_k}$, in the next section a constructive proof in the form of an algorithm will be presented for calculating the **c**'s and $\tilde{\mathbf{c}}$'s for one degenerate node. And finally the case of several intersections will be treated.

To obtain some insight into the structure of the new canonical representation, let us first look at the structure of *Equation* (20). While the first three terms on the right-hand side are identical to (1), the final expression contains a summation over all nodes and over all boundaries making these intersections degenerate. Thus, non-degenerate I^j yield no contribution and a setup where all $\delta^j = 0$ gives us back the old canonical representation.

This yields the motivation for a step by step generalization of (1). First, the situation of one degenerate intersection containing the origin (i.e., all β's vanish, which can be achieved by a translation) will be investigated. This is already a fairly general case, since the "local" contributions from the different nodes independently sum up in **F**. In fact, since the different **g**'s also appear linearly in (20), it is sufficient to investigate the case of a *degeneracy one* intersection. (Since only one node will be treated until *Section* 5, the node-index on the **g**'s, **c**'s, and a's will be suppressed.)

As we saw in the preceding section, upon crossing a boundary, the only direction of relevance is the one normal to the boundary, while those directed along the boundary yield no contribution to $\Delta\mathbf{J}$. Hence, without loss of generality, the theory of one intersection can be developed in the *two-dimensional quotient space* of the domain space $\mathbf{R}^n$ and the intersection (that is the space spanned by the α's of the boundaries contributing to the node). Since only this space will be considered further on, we shall also call it "domain space" and use the same symbols as in the original description. Here the crucial property of a degenerate intersection becomes clear: All the *linearly dependent* α's, together with the jumps in the Jacobian caused by them, can be expressed as linear combinations of *two* basis vectors.

For the prototype situation described above, we are left with the two "bias terms" $\mathbf{b}$, the constant, and $\mathbf{Bx}$, the linear contribution, next come the $\mathbf{f}$'s, describing the constant jumps of the Jacobian along the boundaries — this may again be interpreted as a bias, now for the jump of the Jacobian. The final contribution is due to $\mathbf{g}^j$ (in order to keep the formulas general, we use here a generic index "j" instead of the special "3", which would appear in the present case), this function describes the switching of the jumps in the Jacobians, as the boundary "passes" the intersection. In order to maintain continuity of $\mathbf{F}$, $\mathbf{g}^j$ has to represents the *interaction of boundaries* rather than single boundary contributions as it is known from the $\mathbf{f}^i$.

To see how this works, let us look at the behavior of $\mathbf{g}^j$. The basic structure found is the *nested absolute-value functions.* Without the "inner" $|\cdot|$'s, every single $\mathbf{g}^j(\mathbf{x})$ would vanish. The first property, which can be seen immediately, is that $\mathbf{g}^j(\mathbf{x})$ vanishes if both $a_{j1}\langle\alpha^1,\mathbf{x}\rangle$ and $a_{j2}\langle\alpha^2,\mathbf{x}\rangle$ have the same sign. Thus $\mathbf{g}^j(\mathbf{x})$ is equal to zero in two regions. This is a nice feature, since it shows that $\mathbf{g}^j$ adds no bias term to $\mathbf{F}$ but rather describes the *pure deviation from the consistent variation.* Note that $\mathbf{g}^j$ vanishes if the deviation is zero and (20) reverts back to the old canonical representation.

Next, the lines (hyperplanes in $\mathbf{R}^2$) where $\mathbf{g}^j(\mathbf{x})$ changes behavior have to be determined. A superficial inspection of (21)–(23) suggests that the boundaries are defined by $\langle\alpha^1,\mathbf{x}\rangle = 0$, $\langle\alpha^2,\mathbf{x}\rangle = 0$, $\langle\alpha^j,\mathbf{x}\rangle = 0$ and $\langle a_{j1}\alpha^1 - a_{j2}\alpha^2,\mathbf{x}\rangle = 0$. The latter line is, however, not a region boundary of the function $\mathbf{F}$, hence there should be no change in $\mathbf{g}^j$ along it. This is guaranteed by the following Lemma:

Lemma 1 *The function $\mathbf{g}^j$ does not change its Jacobian along the hyperplane defined by $\langle(a_{j1}\alpha^1 - a_{j2}\alpha^2),\mathbf{x}\rangle = 0$.*

Proof From the definition of the line we find immediately (by taking squares):

$$\langle a_{j1}\alpha^1,\mathbf{x}\rangle \cdot \langle a_{j2}\alpha^2,\mathbf{x}\rangle = \frac{1}{2}\Big(\langle a_{j1}\alpha^1,\mathbf{x}\rangle^2 + \langle a_{j2}\alpha^2,\mathbf{x}\rangle^2\Big) > 0 \qquad (24)$$

thus both $\langle a_{j1}\alpha^1,\mathbf{x}\rangle$ and $\langle a_{j2}\alpha^2,\mathbf{x}\rangle$ must have the same sign, i.e., $\mathbf{g}^j(\mathbf{x})$ vanishes along this line. Moreover, since both a_{j1} and a_{j2} are non-zero, the line in question is from the regions, where $\mathbf{g}^j(\mathbf{x})$ vanishes throughout, hence there can be no change by crossing this hyperplane, i.e., it yields no additional boundary, which would have voided the whole description. •

Similar arguments can be used to show that $\mathbf{g}^j$ changes along the three other lines. To clarify this, we actually calculate $\mathbf{g}^j$ and the jumps along the boundaries in the six sectors separated by the three boundaries in question (see *Fig.* 5*a*). For this purpose let us assume $a_{j1}, a_{j2} > 0$. This can be done without loss of generality, since $-\alpha^1$ and/or $-\alpha^2$ may equally be used as basis vectors. In the case of a higher degeneracy the signs of a_{j1} and a_{j2} are no longer free for all but one of the α^j's. Then the same entries appear in the table-representation of $\mathbf{g}^j$, however permuted. Interchanging the labels of α^1 and α^2 and the signs of the a's allows us to build eight tables similar to the one below.

Figure 5

Table 1 The explicit behavior of the three $\mathbf{f}^i$ and of $\mathbf{g}^j$ and their contribution to the jumps of the Jacobian along the region boundaries, calculated along a closed path Γ around the degenerate intersection, as indicated in *Fig.* 5*a*.

Region	$\mathbf{f}^1$	$\mathbf{f}^2$	$\mathbf{f}^j$	$\mathbf{g}^j$	$\Delta\mathbf{J}^{(kl)}$
$R^{(1)}$	$\mathbf{c}^1{\alpha^1}^T$	$\mathbf{c}^2{\alpha^2}^T$	$\mathbf{c}^j{\alpha^j}^T$	0	
					$-2(\mathbf{c}^1 + a_{j1}\tilde{\mathbf{c}}^j){\alpha^1}^T$
$R^{(2)}$	$-\mathbf{c}^1{\alpha^1}^T$	$\mathbf{c}^2{\alpha^2}^T$	$\mathbf{c}^j{\alpha^j}^T$	$-2a_{j1}\,\tilde{\mathbf{c}}^j{\alpha^1}^T$	
					$2(-\mathbf{c}^j + \tilde{\mathbf{c}}^j){\alpha^j}^T$
$R^{(3)}$	$-\mathbf{c}^1{\alpha^1}^T$	$\mathbf{c}^2{\alpha^2}^T$	$-\mathbf{c}^j{\alpha^j}^T$	$2a_{j2}\,\tilde{\mathbf{c}}^j{\alpha^2}^T$	
					$-2(\mathbf{c}^2 + a_{j2}\tilde{\mathbf{c}}^j){\alpha^2}^T$
$R^{(4)}$	$-\mathbf{c}^1{\alpha^1}^T$	$-\mathbf{c}^2{\alpha^2}^T$	$-\mathbf{c}^j{\alpha^j}^T$	0	
					$2(\mathbf{c}^1 - a_{j1}\tilde{\mathbf{c}}^j){\alpha^1}^T$
$R^{(5)}$	$\mathbf{c}^1{\alpha^1}^T$	$-\mathbf{c}^2{\alpha^2}^T$	$-\mathbf{c}^j{\alpha^j}^T$	$-2a_{j1}\,\tilde{\mathbf{c}}^j{\alpha^1}^T$	
					$2(\mathbf{c}^j + \tilde{\mathbf{c}}^j){\alpha^j}^T$
$R^{(6)}$	$\mathbf{c}^1{\alpha^1}^T$	$-\mathbf{c}^2{\alpha^2}^T$	$\mathbf{c}^j{\alpha^j}^T$	$2a_{j2}\,\tilde{\mathbf{c}}^j{\alpha^2}^T$	
					$2(\mathbf{c}^2 - a_{j2}\tilde{\mathbf{c}}^j){\alpha^2}^T$

This table clearly shows the properties of the functions $\mathbf{f}^i$ and $\mathbf{g}^j$. Unlike the $\mathbf{f}^i$, which represent one boundary each, the $\mathbf{g}^j$ represents a *pair-*

interaction of α^j with both of the basis vectors. The other major difference is found in the symmetry properties, while all $\mathbf{f}^i$ are *symmetric* with respect to a change of sign in x, i.e., $\mathbf{f}^i(-\mathbf{x}) = \mathbf{f}^i(\mathbf{x})$; $\mathbf{g}^j$ behaves in an *antisymmetric* manner, i.e., $\mathbf{g}^j(-\mathbf{x}) = -\mathbf{g}^j(\mathbf{x})$. (Note: These formulas hold only for an intersection at the origin. For an arbitrary node located at $\mathbf{y}$ the two functions are symmetric or antisymmetric with respect to that point, yielding $\mathbf{f}^i(\mathbf{y}-\mathbf{x}) = \mathbf{f}^i(\mathbf{y}+\mathbf{x})$ and $\mathbf{g}^j(\mathbf{y}-\mathbf{x}) = -\mathbf{g}^j(\mathbf{y}+\mathbf{x})$, respectively.) As a consequence the contribution to the jump of the Jacobian in one direction (say from $\langle\alpha^i,\mathbf{x}\rangle < 0$ to $\langle\alpha^i,\mathbf{x}\rangle > 0$) is constant for all the $\mathbf{f}^i$ while the contribution coming from $\mathbf{g}^j$ changes its sign at the intersection. This, of course, immediately requires that, after half a turn of Γ around the node, when the first, second and j-th boundary are crossed, the jumps in the Jacobian caused by $\mathbf{g}^j$ have to sum up to zero, both in the directions of α^1 and α^2 separately. By virtue of the symmetry of $\mathbf{g}^j$, the same argument applies to the second half of the path.

Now that we know the symmetry properties of $\mathbf{g}^j$, let us look at all eight setups mentioned above. Four of them can be obtained immediately by exchanging the indices of α^1 and α^2. The four others are characterized by (a) $(a_{j1} > 0, a_{j2} > 0)$ (the case treated in *Table* 1), (b) $(a_{j1} > 0, a_{j2} < 0)$, (c) $(a_{j1} < 0, a_{j2} > 0)$, and (d) $(a_{j1} < 0, a_{j2} < 0)$. Now it is an easy task to show that the cases (a) and (d) as well as (b) and (c) are mutually equivalent. Thus only two principal types are left, where a different set of basis vectors is used (see *Fig.* 5 above). The behavior of $\mathbf{g}^j$ and its Jacobian for the two principal setups is demonstrated in the following table:

Table 2 The two principal behaviors of the functions $\mathbf{g}^j$ that appear with different signs of the expansion coefficients a_{jk} $(k = 1, 2)$.

Region	$a_{j1} > 0,\ a_{j2} > 0$: $\mathbf{g}^j$	$a_{j1} > 0,\ a_{j2} > 0$: $\Delta\left(\frac{\partial g^j_k}{\partial x_l}\right)$	$a_{j1} > 0,\ a_{j2} < 0$: $\mathbf{g}^j$	$a_{j1} > 0,\ a_{j2} < 0$: $\Delta\left(\frac{\partial g^j_k}{\partial x_l}\right)$
$R^{(1)}$	0		$-2a_{j1}\,\tilde{\mathbf{c}}^j {\alpha^1}^T$	
		$-2a_{j1}\,\tilde{\mathbf{c}}^j {\alpha^1}^T$		$2a_{j1}\,\tilde{\mathbf{c}}^j {\alpha^1}^T$
$R^{(2)}$	$-2a_{j1}\,\tilde{\mathbf{c}}^j {\alpha^1}^T$		0	
		$2\tilde{\mathbf{c}}^j {\alpha^j}^T$		$2a_{j2}\,\tilde{\mathbf{c}}^j {\alpha^2}^T$
$R^{(3)}$	$2a_{j2}\,\tilde{\mathbf{c}}^j {\alpha^2}^T$		$2a_{j2}\,\tilde{\mathbf{c}}^j {\alpha^2}^T$	
		$-2a_{j2}\,\tilde{\mathbf{c}}^j {\alpha^2}^T$		$-2\tilde{\mathbf{c}}^j {\alpha^j}^T$

This table nicely depicts the symmetry properties of $\mathbf{g}^j$. Moreover, it yields a *complete* list of the functional behaviors and the jumps of the Jacobian caused by this function.

4. The Algorithm: A Constructive Proof

Here we are going to use the properties of the new canonical representation (20) to prove that it describes any intersection of degeneracy one. This will be presented in the form of an *algorithm*, which requires only continuity for the function to possess such a representation.

The idea of the proof is very simple. The jumps in the Jacobian along each boundary, which, in the general case, differ on both sides of a degenerate intersection, will be written as the sum of a constant term (emerging from $\mathbf{f}^i$) and one that switches its sign at the intersection (coming from $\mathbf{g}^j$). While every boundary contributes an f-function, the number of g's is given by the degeneracy δ of the node (being one in the present case); which is an immediate consequence of the *global continuity condition* (15). As in the preceding section, the generic index "j" will be used for the third boundary.

Lemma 2 *Any continuous PWL function with an intersection of degeneracy one at the origin can be written in the form*

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \mathbf{b} + \mathbf{B} + \sum_{i\in\{1,2,j\}} \mathbf{c}^i \left|\langle\alpha^i,\mathbf{x}\rangle\right| + \\ &\tilde{\mathbf{c}}^j\Big\{\Big|\left|a_{j1}\langle\alpha^1,\mathbf{x}\rangle\right| + a_{j2}\langle\alpha^2,\mathbf{x}\rangle\Big| - \Big|a_{j1}\langle\alpha^1,\mathbf{x}\rangle + \left|a_{j2}\langle\alpha^2,\mathbf{x}\rangle\right|\Big|\Big\}\end{aligned} \tag{25}$$

Proof We are going to present an algorithm which yields the vectors $\mathbf{c}^i$ and $\tilde{\mathbf{c}}^j$. To avoid clutter, let us assume (without loss of generality) that the regions and boundaries are set as in *Table* 1 or *Fig.* 5*a*, respectively, with the Jacobian $\mathbf{J}^{(i)}$ $(i = 1 \ldots 6)$ given explicitly for every region $R^{(i)}$. The following steps will yield all parameters that are needed in (25):

Step 1 Check that $\mathbf{J}^{(k)}\mathbf{x} = \mathbf{J}^{(l)}\mathbf{x}$ along the boundaries of adjacent regions and write $\Delta\mathbf{J}^{(kl)}$ as $\tilde{\mathbf{c}}^{(kl)}{\alpha^i}^T$, with $i \in \{1,2,j\}$.

Step 2 Use *Table* 1 to calculate $\mathbf{c}^j$ and $\tilde{\mathbf{c}}^j$ from the jumps occurring along S^j, that is

$$-\mathbf{c}^j + \tilde{\mathbf{c}}^j = \frac{\bar{\mathbf{c}}^{(3,2)}}{2} \tag{26a}$$

$$\mathbf{c}^j + \tilde{\mathbf{c}}^j = \frac{\bar{\mathbf{c}}^{(6,5)}}{2} \quad . \tag{26b}$$

These 2×2 linear equations can be solved uniquely for each component of $\mathbf{c}^j$ and $\tilde{\mathbf{c}}^j$ separately.

Step 3 Again use *Table* 1 to find the jumps of the Jacobian along one portion of each S^1 and S^2 (here the jumps between the regions $R^{(2)}$ and $R^{(1)}$, respectively $R^{(4)}$ and $R^{(3)}$, were chosen). The two resulting one-dimensional linear equations

$$\mathbf{c}^1 + a_{j1}\,\tilde{\mathbf{c}}^j = -\frac{\bar{\mathbf{c}}^{(2,1)}}{2} \tag{27a}$$

and

$$\mathbf{c}^2 + a_{j2}\,\tilde{\mathbf{c}}^j = -\frac{\bar{\mathbf{c}}^{(4,3)}}{2} \tag{27b}$$

yield the components of $\mathbf{c}^1$ and $\mathbf{c}^2$ uniquely. Eventually

$$\mathbf{c}^1 - a_{j1}\,\tilde{\mathbf{c}}^j = \frac{\bar{\mathbf{c}}^{(5,4)}}{2} \tag{28a}$$

and

$$\mathbf{c}^2 - a_{j2}\,\tilde{\mathbf{c}}^j = \frac{\bar{\mathbf{c}}^{(1,6)}}{2} \tag{28b}$$

may be used to check that all jumps in the Jacobian along a closed path Γ sum up correctly to zero, in accordance with *Eq.* (15); namely in *both* directions of α^1 and α^2 for every single component of the $\mathbf{c}$-vectors. Hence, from (28) we have $2n$ *global continuity conditions.*

Step 4 Finally $\mathbf{b}$ and $\mathbf{B}$, the constant terms in $\mathbf{F}$, have to be determined. The first one is obtained from

$$\mathbf{b} = \mathbf{F}(0) \tag{29a}$$

In order to find $\mathbf{B}$, we can pick an arbitrary region, say $R^{(1)}$, and solve (25) for this matrix, yielding

$$\mathbf{B} = \mathbf{J}^{(1)} - \sum_{i \in \{1,2,j\}} \mathbf{f}^i - \mathbf{g}^j \quad , \tag{29b}$$

where the functions $\mathbf{f}^i$ and $\mathbf{g}^j$ have to be taken from the first region. This completes the proof that any degeneracy one intersection can be represented by (25). Since all associated equations have unique solutions, we have not only shown the completeness of (25) but also that the vectors $\bar{\mathbf{c}}$ (coming from the $\Delta\mathbf{J}$) correspond *one-to-one* with the $\mathbf{c}^i$ and $\tilde{\mathbf{c}}^j$ of the canonical representation.
$\bullet$

As an illustration, let us apply the above algorithm to determine the coefficients of *Example* 2 of [2]. With $\alpha^1 = (1,1)^T$, $\alpha^2 = (0,2)^T$, and $\alpha^3 = (-1,1)^T$ we obtain $\mathbf{c}^1 = \mathbf{c}^2 = \mathbf{c}^3 = \tilde{\mathbf{c}}^3 = -(\frac{1}{8}, \frac{1}{8})^T$, $\mathbf{b} = 0$, and $\mathbf{B} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$.

It is now an easy task to generalize our preceding result to an intersection having an *arbitrary degeneracy.*

Lemma 3 *Any continuous PWL function with an intersection of any order of degeneracy at the origin can be obtained from (25) by introducing a summation over the j-indices.*

Since the proof is completely similar to *Lemma* 3, we sketch only the differences: (1) Steps one and two have to be performed for every j. (2) In step three the terms $a_{jk}\tilde{\mathbf{c}}^j$ $(k = 1,2)$ have to be substituted by a sum over all j. (3) The same applies in the final step to $\mathbf{f}^j$ and $\mathbf{g}^j$.

To sum up this section, the $\mathbf{g}^j$-function represents the switching jump in the Jacobian along a boundary, as the latter reaches a degenerate intersection. With it we have obtained the whole hierarchy of structures appearing in continuous PWL functions. The vector $\mathbf{b}$ is the constant contribution, $\mathbf{B}$ yields the "average" Jacobian, the $\mathbf{f}^i$'s give the deviations from $\mathbf{B}$, which arise across the region boundaries, and finally $\mathbf{g}^j$ contributes a modulation of the jumps in the Jacobian. One might now surmise that even more subtle structures like intersections of direct intersections have to be employed in order to obtain a complete description of the class of continuous PWL functions.

This corresponding to possible complication, however, can be ruled out easily. Since the $\Delta \mathbf{J}^{ik}$'s at one boundary are already matrices of *rank one*, and moreover, are all constructed from the same α^i, any difference between them can arise only from the c-vectors which are described in terms of the functions $\mathbf{g}^j$, as will be demonstrated in the next section. Thus there is no need for any further elements in the complete canonical representation.

5. Several Degenerate Intersections

Consider now the most general situation when several different direct intersections of boundaries are present in the domain space. The first step towards this goal is to consider an intersection of boundaries located arbitrarily in the domain space. This can be achieved by a simple translation in (25), which will bring back the β's in the description of the boundaries (to avoid clutter, we shall use the same symbols for the transformed vectors).

Next, new boundaries have to be introduced. As long as they intersect in a non-degenerate manner, they are added in the same fashion as all boundaries in (1), i.e., by adding a new term $\mathbf{f}^i(\mathbf{x})$. Another set of boundaries, which form an additional degenerate node can also be introduced immediately by adding another sum over the g-functions involved in that intersection. This can be done because all the g's from other degenerate intersections are constant at the "new" intersection.

The preceding argument already pointed towards the final problem which has to be solved in order to prove *Theorem* 1; namely the situation when a boundary "connects" two degenerate intersections. We are going to show that this case is also covered by (20) and thus needs no new elements for its description.

First the linearity of the boundaries rules out that more than one boundary can participate in the same two nodes, i.e., if $I^k \subset S^i$ and $I^l \subset S^i$ $(k \neq l)$ then for all $j \neq i$ either $I^k \not\subset S^j$ or $I^l \not\subset S^j$ or both are true. This is an immediate consequence of the fact that the solution of a linear equation is a *unique* linear manifold — in the present case it is S^i. Thus dealing with connected intersections is essentially a *one-dimensional* problem, where only one vector α is involved (this, of course, is the dimension in the domain space and not the dimension of the resulting algebraic problem).

Lemma 4 *The function* $\mathbf{F}$ *restricted to the boundary* S^i*, which may connect* m *degenerate intersections* I^{j_l} $(l = 1 \dots m)$*, can be described by*

$$\mathbf{F}\big|_{S^i}(\mathbf{x}) = \mathbf{b} + \mathbf{B}\,\mathbf{x} + \mathbf{f}^i(\mathbf{x}) + \sum_{l=1}^{m} \mathbf{g}^{j_l i}(\mathbf{x}) \quad . \tag{30}$$

(Note that the the first index in the $\mathbf{g}$'s reappeared.)

This lemma supplements *Lemma* 3 nicely, since here we see a summation over all the node-indices of one fixed boundary.

Figure 6

At this point one might be concerned whether a situation, like the one shown in *Fig.* 6*b*, can be described in terms of (20). As can be seen from *Fig.* 6*c*, the boundary S^i decomposes into four sectors separated by the two degenerate intersections I^j and I^k. At first glance it appears as if the jumps $\Delta\mathbf{J}^{i_l}$ $(l = 1 \dots 4)$ in the Jacobian for the four sectors could be chosen independently while only three free parameter vectors ($\mathbf{c}^i$, $\tilde{\mathbf{c}}^j$ and $\tilde{\mathbf{c}}^k$) are available.

In this case again, the technique of closed paths proves very helpful. Let us look at Γ and Γ' in *Fig.* 6*b*, which both go around I^j, on different sides of I^k. In the present case, the boundaries S^{i_1} and S^{i_2} have non-degenerate intersections with S^{i_3}, S^{i_4}, and S^{i_5} hence the $\Delta\mathbf{J}$'s do not change as these boundaries intersect, and for both Γ and Γ' the same changes in the Jacobian are found as they cross S^{i_1} and S^{i_2}. This, however, immediately implies that the jumps of the Jacobian on S^i, as they are seen by the two closed paths Γ and Γ', respectively, can differ by just *one* constant $\mathbf{c}'{\alpha^i}^T$. The same argument applies to a "symmetric" situation with closed paths around I^k. Thus, due to continuity, the four $\Delta\mathbf{J}^{i_l}$ in S^i can be described by three parametric degrees of freedom, thus

$$\Delta \mathbf{J}^{i_1} = \mathbf{c}\alpha^{i^T} \tag{31a}$$

$$\Delta \mathbf{J}^{i_2} = (\mathbf{c}+\mathbf{c}')\alpha^{i^T} \tag{31b}$$

$$\Delta \mathbf{J}^{i_3} = (\mathbf{c}+\mathbf{c}'+\mathbf{c}'')\alpha^{i^T} \tag{31c}$$

$$\Delta \mathbf{J}^{i_2} = (\mathbf{c}+\mathbf{c}'')\alpha^{i^T} \tag{31d}$$

is obtained, with $\mathbf{c}$, $\mathbf{c}'$ and $\mathbf{c}''$ as free parameters.

Next, we can look at the situation where an intersection of S^{i_1} or of S^{i_2} with S^{i_3}, S^{i_4}, or S^{i_5}, respectively, is degenerate. This setup, however, we already know from the previous section. The constant added to the jump in the Jacobian as this degenerate node is passed applies both "below" and "above" S^i, hence the extra terms cancel in the sum taken along a closed path. The rest of the argument remains the same.

In order to calculate the $\mathbf{g}^{j_l i}$, *Equation* (31) allows us, to choose an arbitrary one-dimensional path in the boundary S^i, which crosses all degenerate nodes of this boundary.

Proof of Lemma 4 Let us pick a straight line Λ in S^i, which meets all m degenerate intersections of this boundary and assume the intersections are ordered in such a way that jumps in the Jacobian found on the different segments of Λ can be written as $2\bar{\mathbf{c}}^k\alpha^{i^T}$ $(k = 1\ldots m+1)$. Moreover, let us assume that all $a^k_{j_1 j_1}$ and $a^k_{j_1 j_2}$ are positive, which can be done without loss of generality. Then to the left of the first node we find

$$\sum_{l=1}^{m} \tilde{\mathbf{c}}^{j_l i} + \mathbf{c}^i = \bar{\mathbf{c}}^1 \quad , \tag{32a}$$

after crossing it,

$$-\tilde{\mathbf{c}}^{j_1 i} + \sum_{l=2}^{m} \tilde{\mathbf{c}}^{j_l i} + \mathbf{c}^i = \bar{\mathbf{c}}^2 \quad , \tag{32b}$$

appears, etc., until after having crossed the final (m-th) intersection

$$-\sum_{l=1}^{m} \tilde{\mathbf{c}}^{j_l i} + \mathbf{c}^i = \bar{\mathbf{c}}^{m+1} \quad , \tag{32c}$$

is obtained. Thus for every component of the vectors $\mathbf{c}^i$ and $\tilde{\mathbf{c}}^k i$ we have to solve a linear equation of dimension $m+1$, which has the structure

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 1 & 1 & \dots & 1 \\ -1 & -1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \tilde{c}^{j_1 i} \\ \tilde{c}^{j_2 i} \\ \tilde{c}^{j_3 i} \\ \vdots \\ c^i \end{pmatrix} = \begin{pmatrix} \bar{c}^1 \\ \bar{c}^2 \\ \bar{c}^3 \\ \vdots \\ \bar{c}^{m+1} \end{pmatrix} \tag{33}$$

It is an easy exercise to show that the associated matrix is non-singular (possessing determinant 2^m) and thus yields unique solutions for all n equations of type (33), thereby yielding all the necessary parameters of the representation.
●

Now we are in the position to prove *Theorem* 1:

Proof of Theorem 1 The final statement of the theorem (concerning the non-redundant parameters) is obvious. For any arbitrary setting of the $\mathbf{c}$- and $\tilde{\mathbf{c}}$-vectors, the function $\mathbf{F}$ is always a continuous function.

To demonstrate the completeness of the description, we only have to combine *Lemma* 3 and *Lemma* 4. Since all summations appearing have to be taken over a finite range of indices, the terms may be rearranged to fit *Eq.* (20).
●

6. Concluding Remarks

Although the classical canonical representation (1) proved very helpful in the past, it does not describe the complete class of continuous PWL functions. This is due to its restriction by the consistent variation property. The representation introduced in this paper requires just one new element, the nested absolute-value functions, to cover the whole class. However, the appearance of nested absolute-values does not automatically imply that a representation in the form of (1) is not possible, as can be seen from the following example:

$$\big||x|-1\big| = -1 + |x+1| - |x| + |x-1| \quad , \tag{34}$$

which can be checked easily. In fact, in one dimension, any arbitrarily deep nesting of absolute-value functions can be rewritten in terms of unnested absolute-values. This may not be well known, but is a simple corollary of the consistent variation property, which *automatically* applies in one dimension. Hence *any one-dimensional* continuous PWL function can be represented by (1). In fact, explicit formulas for calculating the coefficients of *any* one-dimensional PWL function are given in [4].

Here we showed that in higher dimensions, where consistent variation may be violated, a rather simple form containing only one nesting of absolute-value functions is sufficient to describe any continuous PWL function. As an extra bonus, our generalization (20) also gives an immediate interpretation of the various terms in this equation in terms of the Jacobian of the function $\mathbf{F}$, which has to be represented, and the boundaries, where the Jacobian changes.

To sum up, any continuous PWL function in $\mathbf{R}^n$ is uniquely characterized by $\mathcal{N}$ ($\mathbf{c}$- and $\tilde{\mathbf{c}}$-) vectors in the n-dimensional domain space, where

$$\mathcal{N} = 1 + n + \sigma + \sum_{j=1}^{\varrho} \delta^j \quad . \tag{35}$$

These vectors represent the parametric degrees of freedom. Hence the function can be written in the form:

$$\begin{aligned} &Constant \;+\; Linear\ Map \;+ \sum_{(boundaries)} one\ term\ jumps \;+ \\ &\sum_{(intersections)} \;\; \sum_{(dependent\ directions)} pair\ interactions \end{aligned} \tag{36}$$

References

[1] S.M. Kang and L.O. Chua, "A global representation of multi-dimensional piecewise-linear functions", *IEEE Trans. Circuits Syst.* CAS–**25** (1978), pp. 938–940.

[2] L.O. Chua and A.-C. Deng, "Canonical piecewise-linear representation", *IEEE Trans. Circuits Syst.* **CAS–35**, 1 (1988), pp. 101–111.

[3] T. Ohutsuki, T. Fujisawa, and S. Kumagai, "Existence theorems and a solution algorithm for piecewise-linear resistor networks", *SIAM J. Math. Anal.* **8** (1977), pp. 69–99.

[4] L.O. Chua and S.M. Kang, "Section-Wise Piecewise-Linear Functions: Canonical Representation, Properties, and Applications", *Proceedings of the IEEE* Vol. **65**, 6 (1977), pp. 915–929.

Captions

Figure 1 As the third port of the PWL three-port resistor $\mathbf{N}$, which is defined by *Eq.*(3*a*), is terminated with a linear resistor $\mathbf{R}'$ (3*b*), the resulting two-port $\mathbf{N}'$ can no longer be described by a canonical representation of type (1), since it violates the *consistent variation property.* (a) Schematic representation of the circuit. (b) The behavior of the voltage v_1 as a function of the currents i_1 and i_2. The solid lines represent region boundaries while the dashed lines are "invisible" since the Jacobian of $(v_1, v_2)^T$ does *not* change there.

Figure 2 If the resistor $\mathbf{R}'$ in *Fig.* 1 is substituted by a PWL resistor $\mathbf{R}''$ defined by (6), doubly-nested absolute-value functions appear in $v_1(i_1, i_2)$ of the resulting two-port $\mathbf{N}''$. Nevertheless, this can be recast into the form of *Eq.*(2). (a) Schematic representation of the circuit $\mathbf{N}''$. (b) The behavior of the voltage v_1 as a function of the currents i_1 and i_2 in $\mathbf{N}''$.

Figure 3 A typical non-degenerate intersection of boundaries. Along the closed path Γ, the Jacobian is a piecewise-constant function, where the jumps $\Delta\mathbf{J}^{(ij)}$ have to sum up to zero, in order to obtain a single-valued Jacobian.

Figure 4 (a) Two regions are considered to be adjacent if they share a whole segment of their boundaries, not just isolated points. (b) A hyperplane S^i is a pre-boundary if at least one connected portion of it lies between different regions. This is fulfilled by S^5, where the segment from I^1 to I^2 is the boundary between $R^{(3)}$ and $R^{(7)}$. In contrast, S'^5 has just the "points" I'^3, I'^4, I'^5, and I'^6 where all open neighborhoods intersect different regions. Thus it is obviously *not* a boundary.

Figure 5 The two principal choices of the α-vectors for a direct boundary intersection of degeneracy one. In both cases $\alpha^3 := a_{3,1}\alpha^1 + a_{3,2}\alpha^2$. However, while both coefficients a_{ij} are positive in (a), in the second case (b) $a_{3,1} > 0$ and $a_{3,2} < 0$. The g-functions for these two setups are demonstrated explicitly in *Table* 2.

Figure 6 (a) The two degenerate intersections I^j and I^k are *connected* by the boundary S^i. The three different jumps in the Jacobian $(\Delta\mathbf{J}^{i_1}$, $\Delta\mathbf{J}^{i_2}$

and $\Delta \mathbf{J}^{i_3}$) can be represented by the parameters $\mathbf{c}^i$, $\tilde{\mathbf{c}}^{ji}$ and $\tilde{\mathbf{c}}^{ki}$ of the canonical represetation (2). (b) The general case of a boundary, connecting two degenerate intersections, can be represented in $\mathbf{R}^3$. The two closed paths Γ and Γ' indicate what constraint applies to the four $\Delta\mathbf{J}$'s of S^i shown in (c).

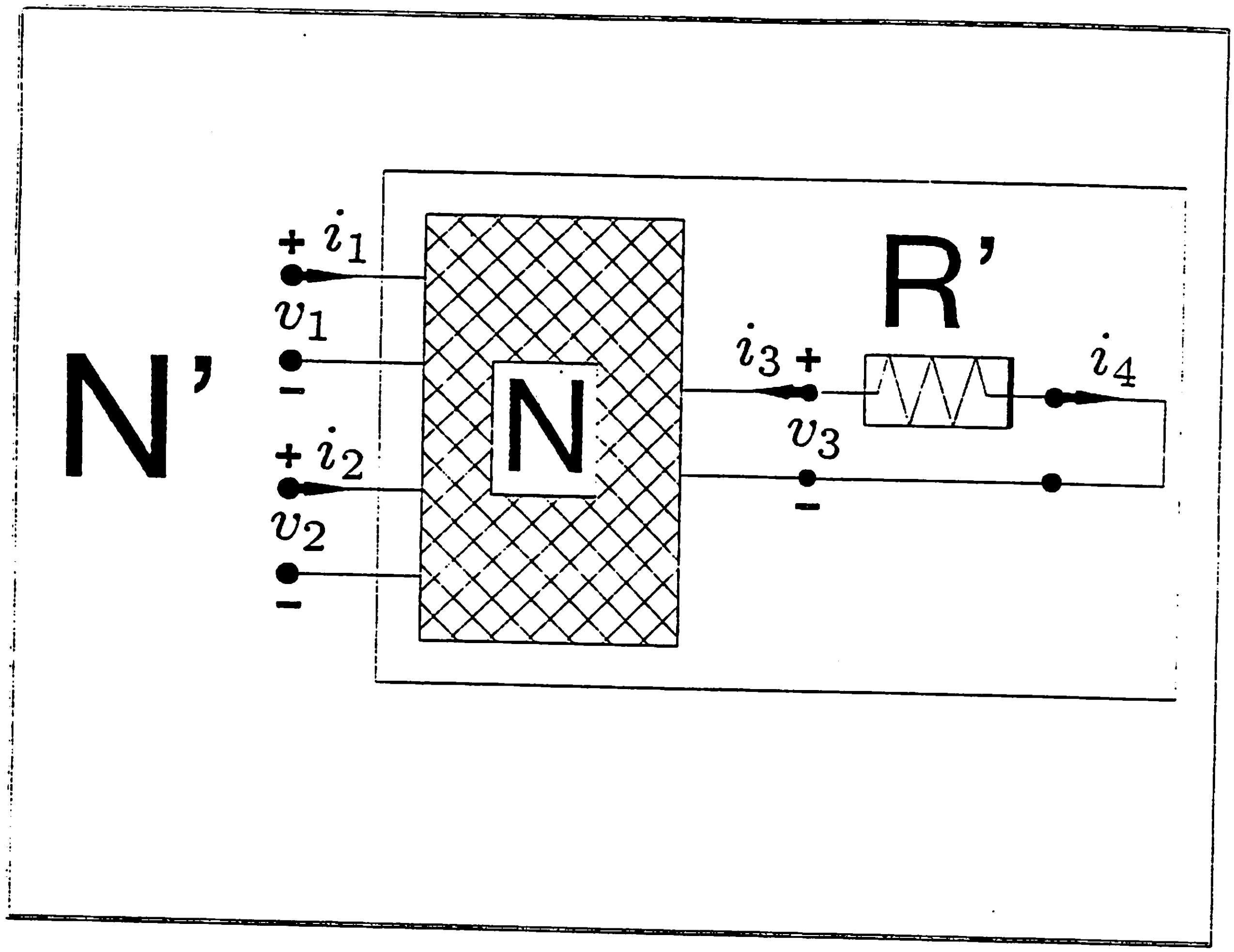


Figure 1 (*a*)

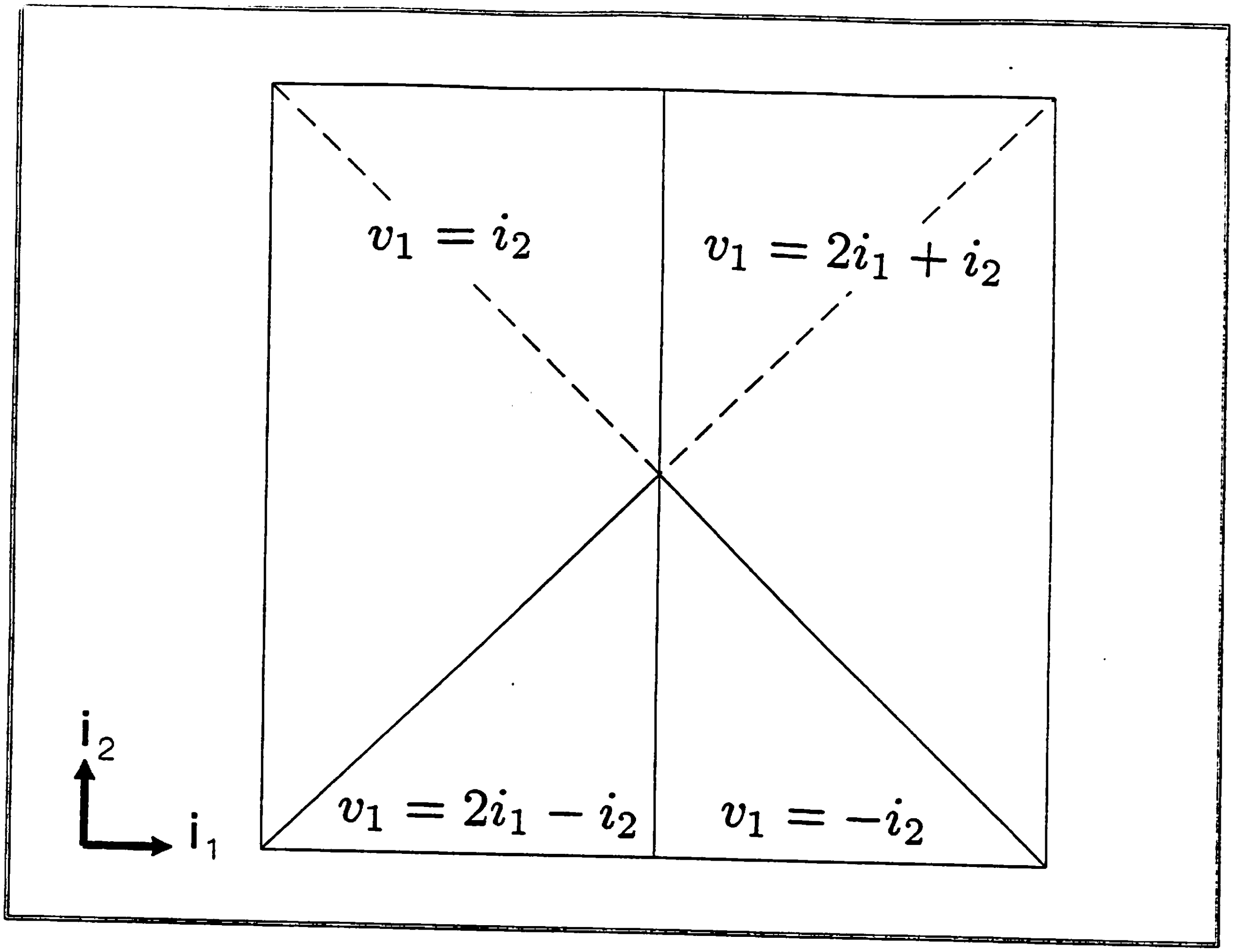


Figure 1 (*b*)

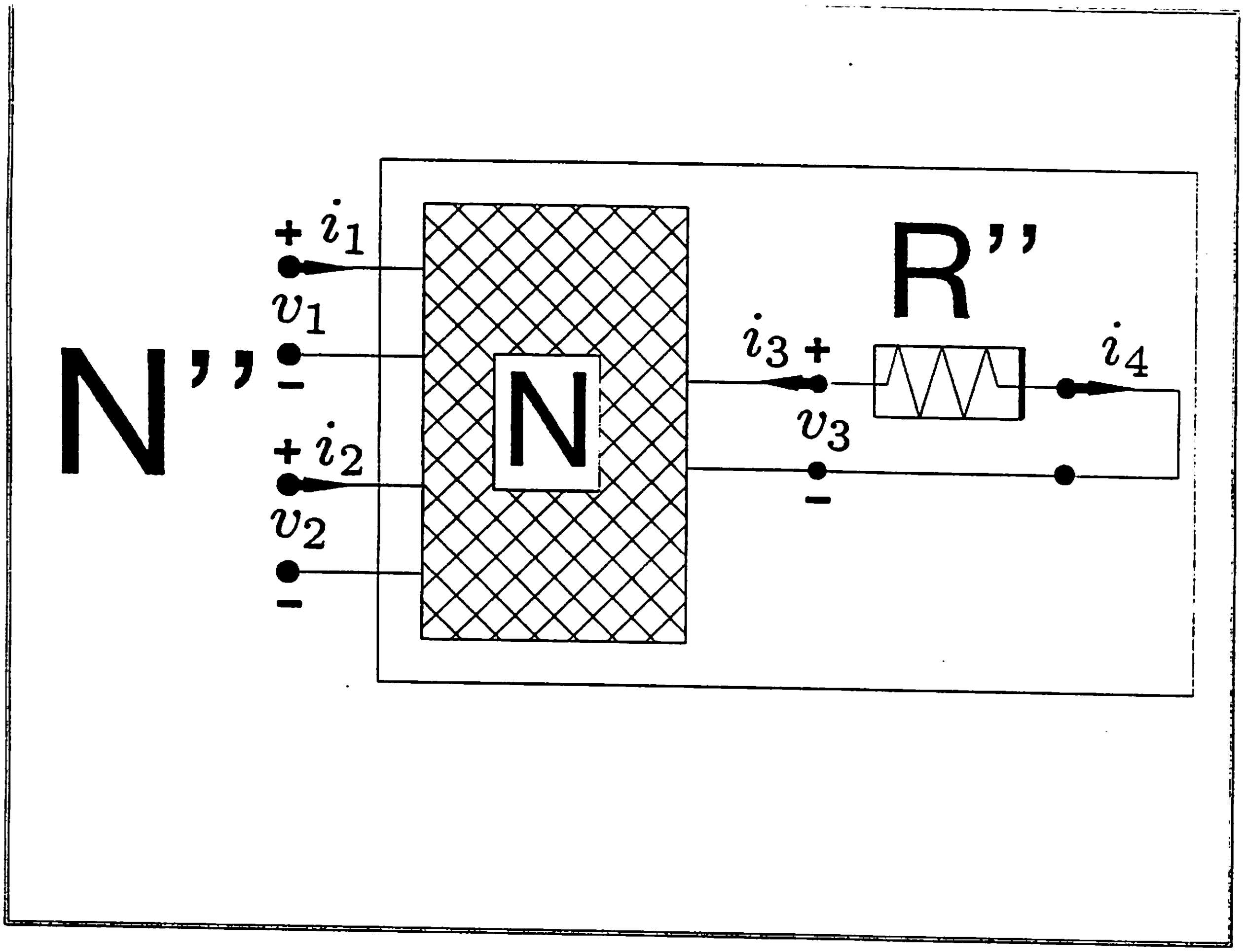


Figure 2 (a)

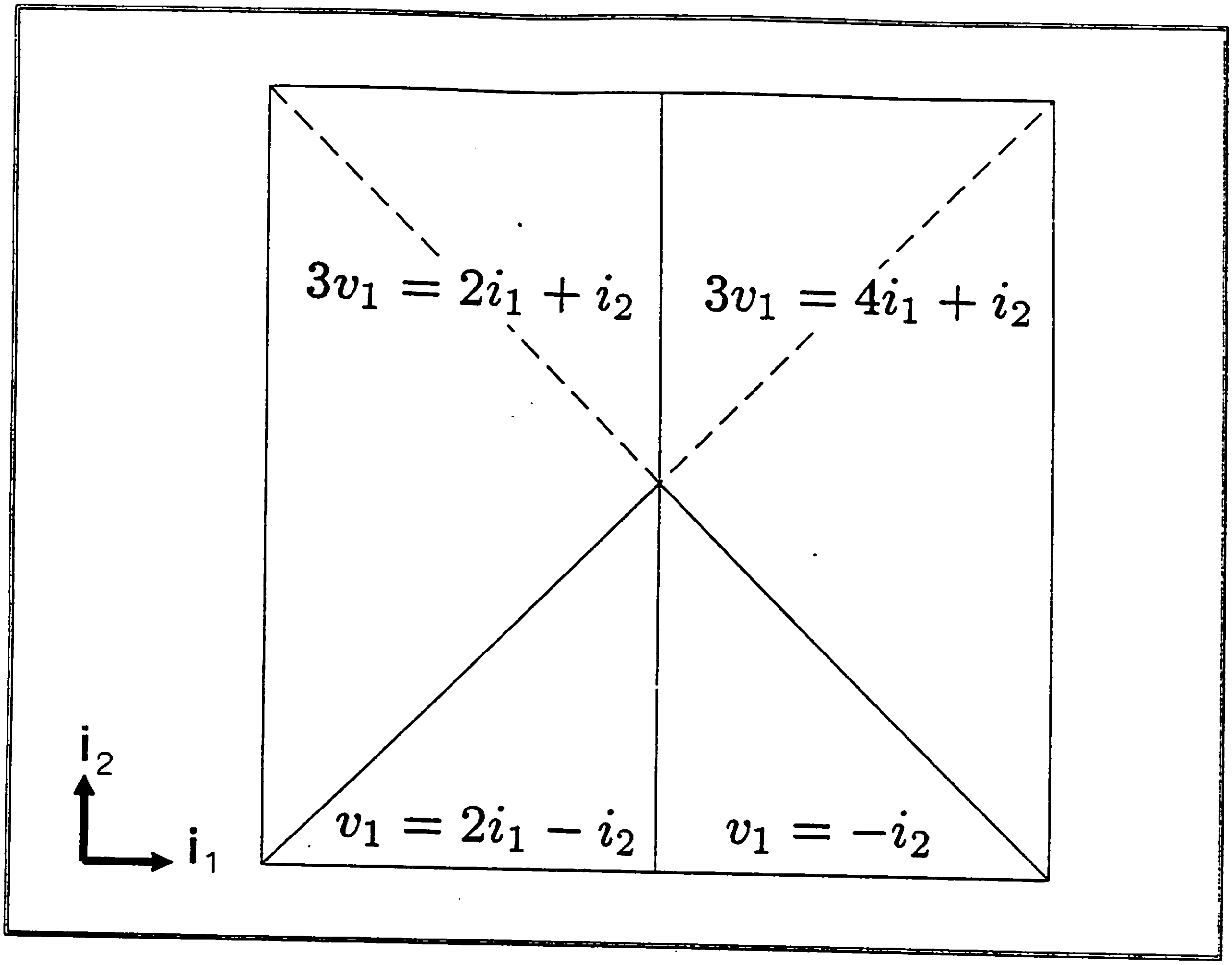
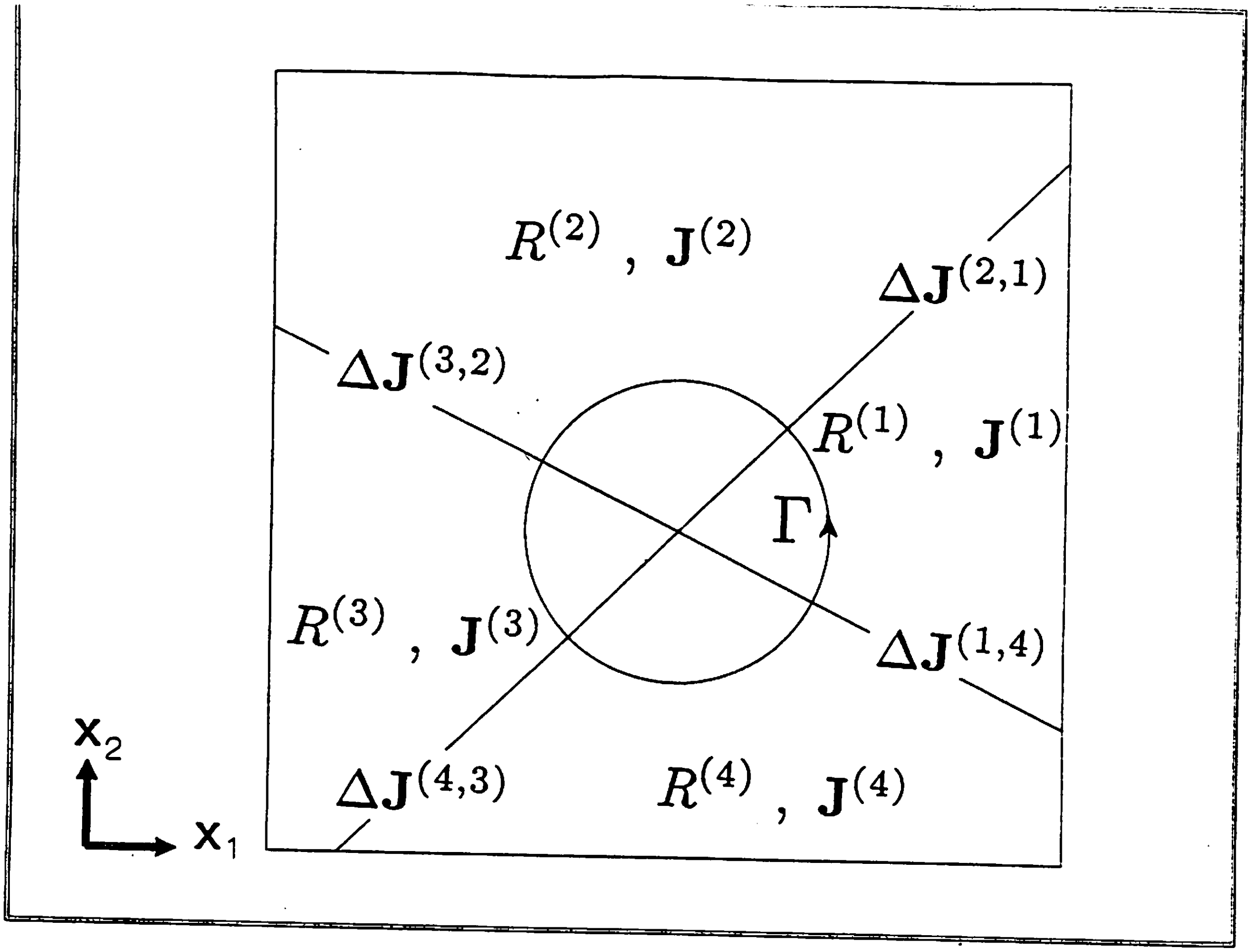
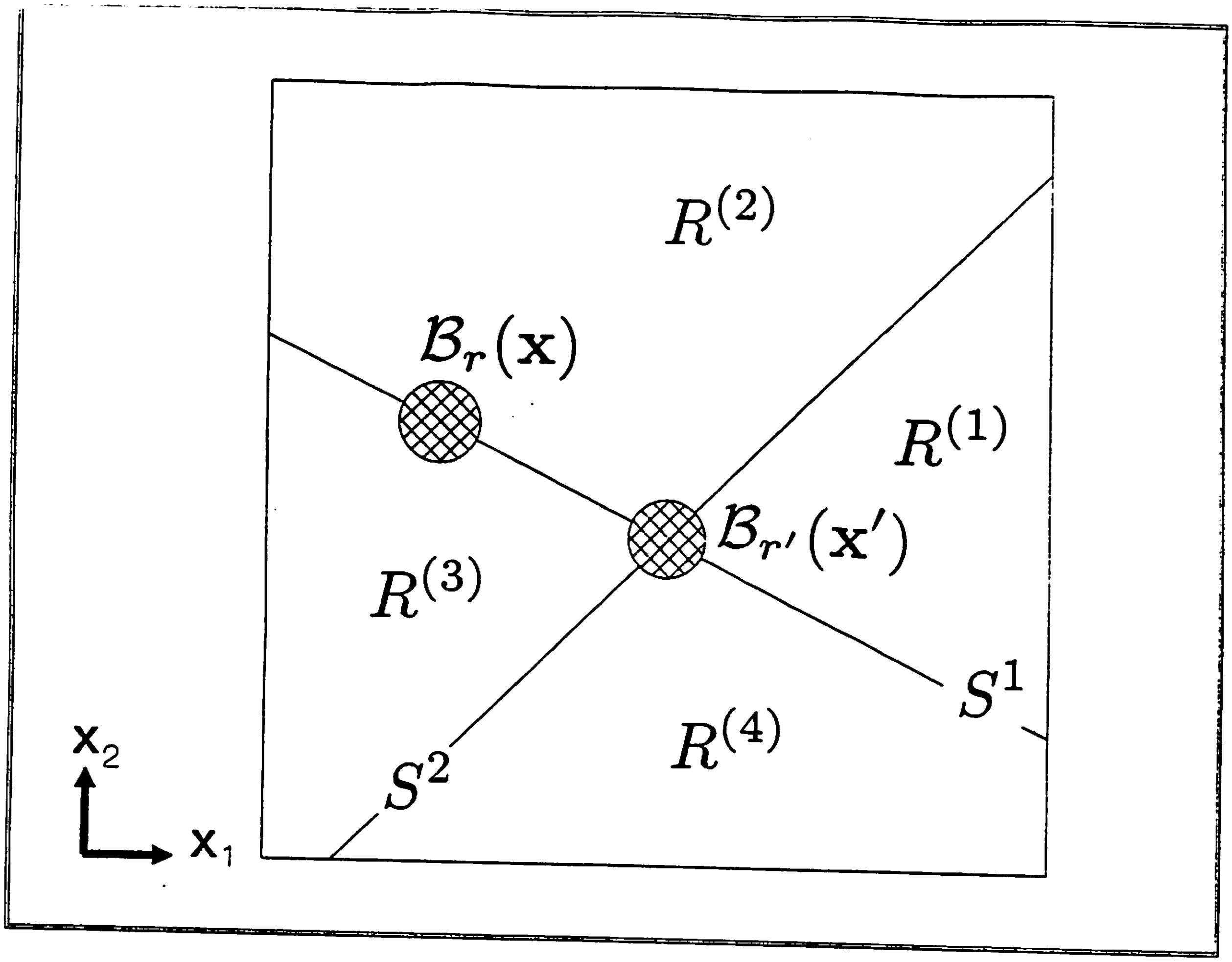


Figure 2 (b)



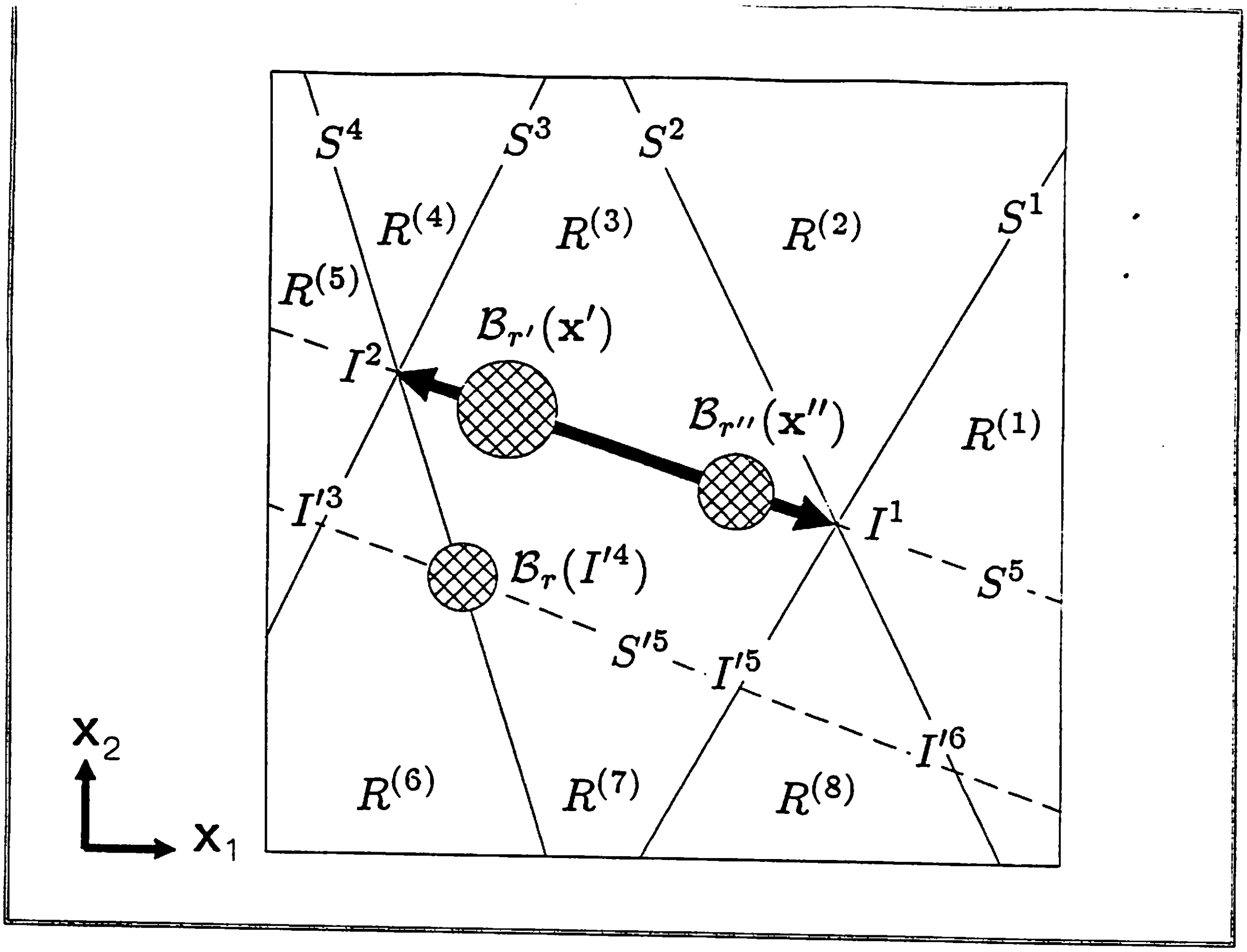
$R^{(2)}$, $\mathbf{J}^{(2)}$
$\Delta\mathbf{J}^{(2,1)}$
$\Delta\mathbf{J}^{(3,2)}$
$R^{(1)}$, $\mathbf{J}^{(1)}$
Γ
$R^{(3)}$, $\mathbf{J}^{(3)}$
$\Delta\mathbf{J}^{(1,4)}$
x_2
$\Delta\mathbf{J}^{(4,3)}$
$R^{(4)}$, $\mathbf{J}^{(4)}$
x_1

Figure 3



$R^{(2)}$
$\mathcal{B}_r(\mathbf{x})$
$R^{(1)}$
$\mathcal{B}_{r'}(\mathbf{x}')$
$R^{(3)}$
S^1
$R^{(4)}$
$\mathbf{x}_2$
S^2
$\mathbf{x}_1$

Figure 4 (a)



S^4
S^3
S^2
S^1
$R^{(4)}$
$R^{(3)}$
$R^{(2)}$
$R^{(5)}$
$\mathcal{B}_{r'}(\mathbf{x}')$
I^2
$\mathcal{B}_{r''}(\mathbf{x}'')$
$R^{(1)}$
I'^3
I^1
$\mathcal{B}_r(I'^4)$
S^5
S'^5
I'^5
I'^6
$\mathbf{x}_2$
$\mathbf{x}_1$
$R^{(6)}$
$R^{(7)}$
$R^{(8)}$

Figure 4 (*b*)

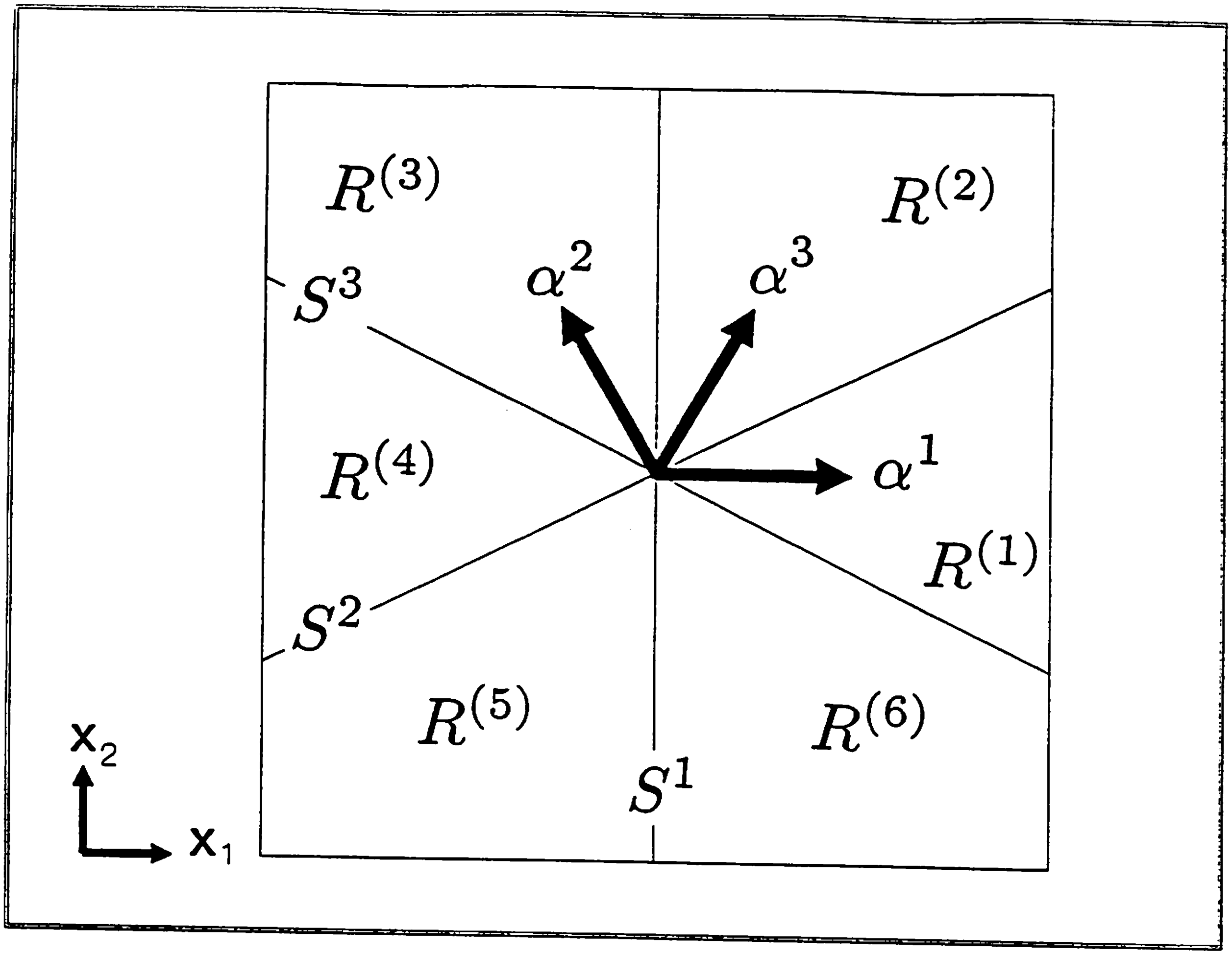


Figure 5 (*a*)

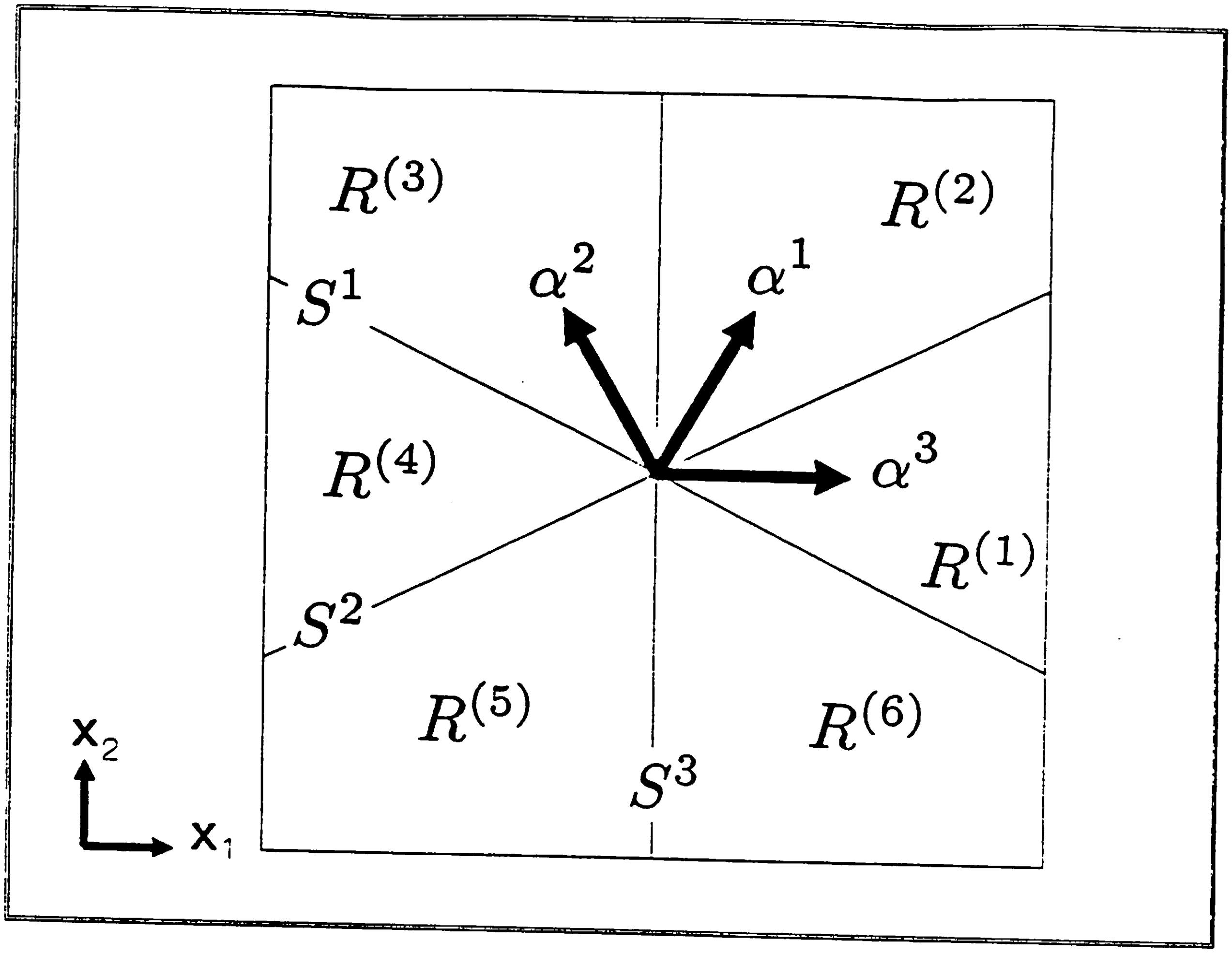
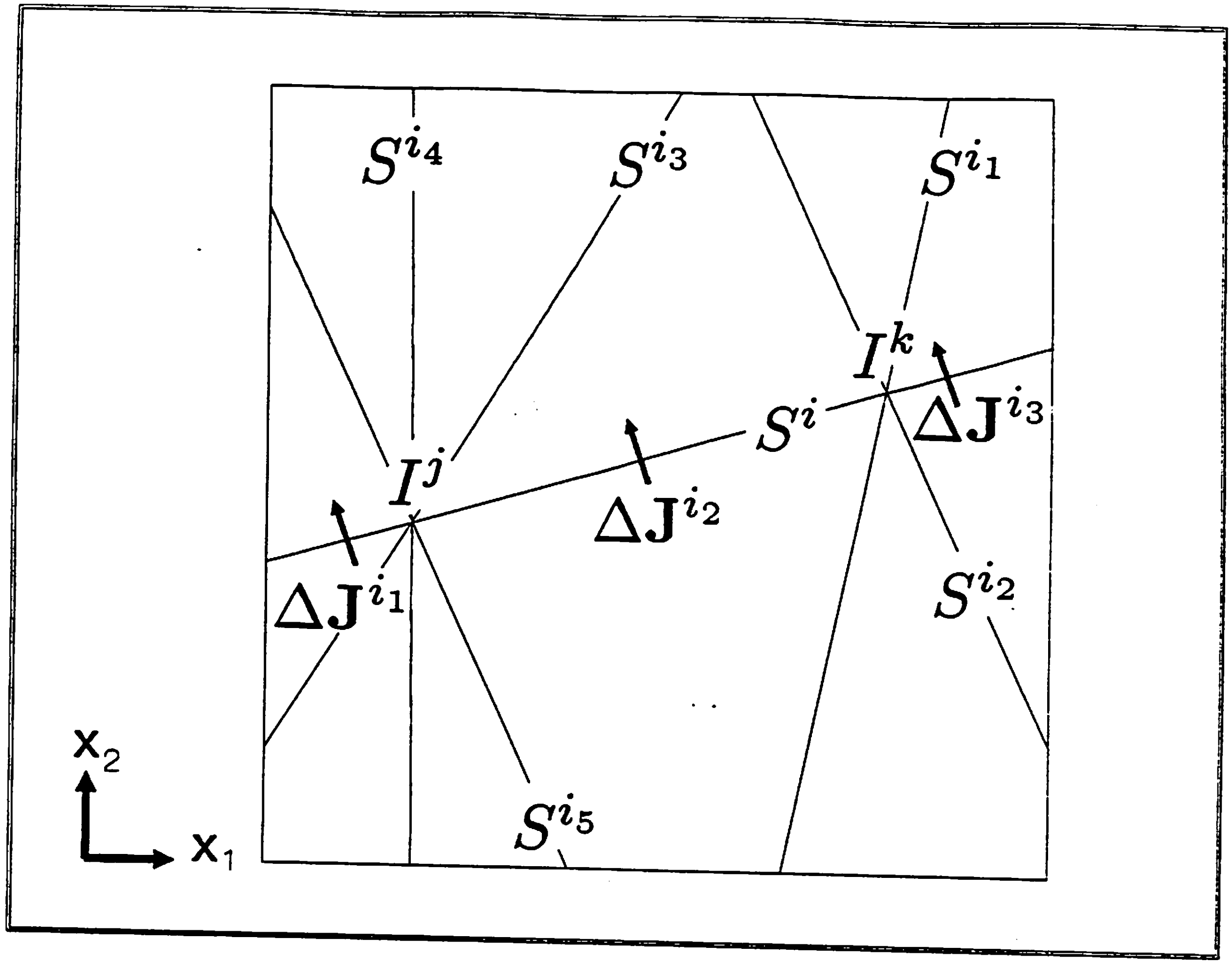


Figure 5 (*b*)



S^{i_4}
S^{i_3}
S^{i_1}
I^k
S^i
$\Delta\mathbf{J}^{i_3}$
I^j
$\Delta\mathbf{J}^{i_2}$
$\Delta\mathbf{J}^{i_1}$
S^{i_2}
x_2
S^{i_5}
x_1

Figure 6 (*a*)

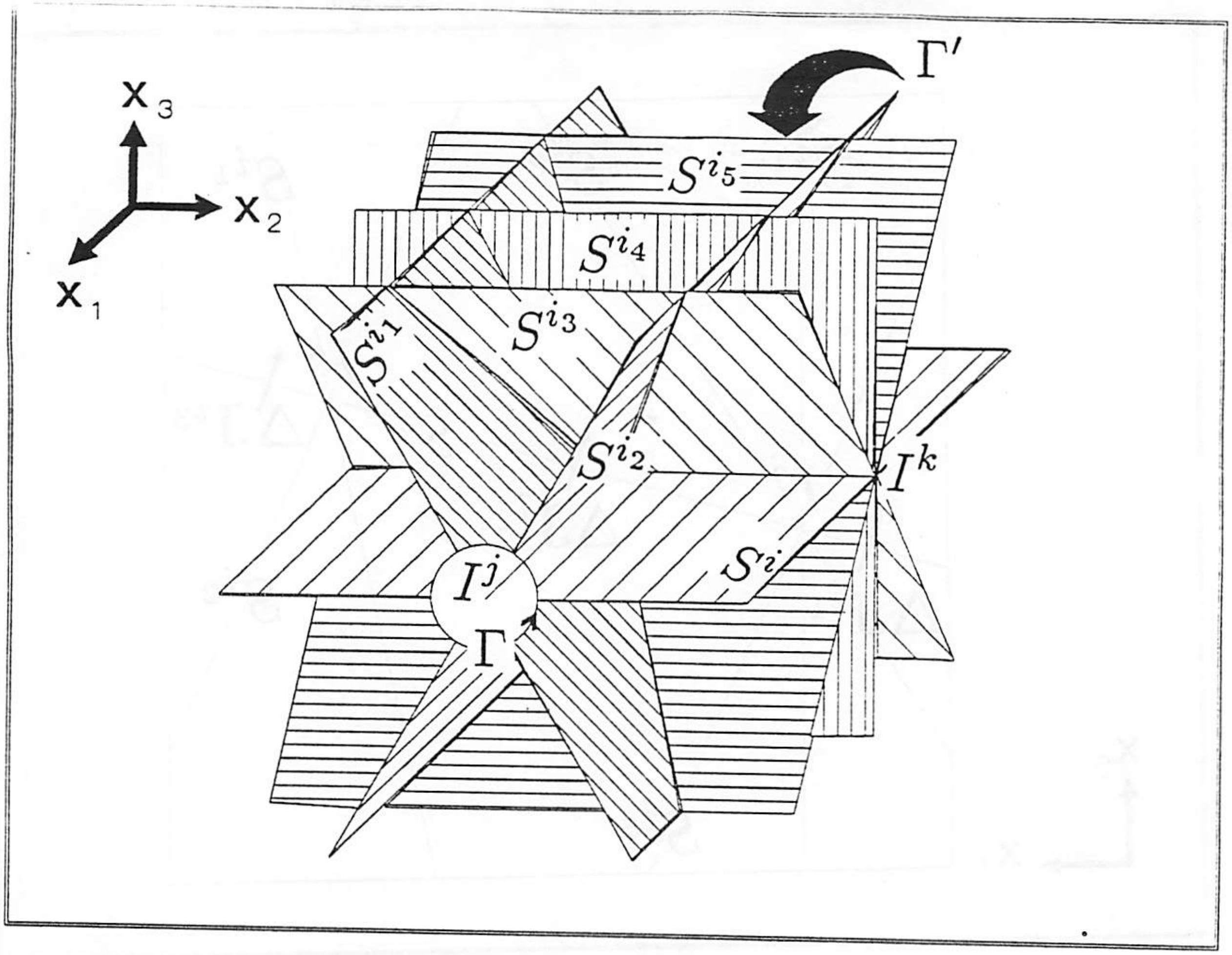
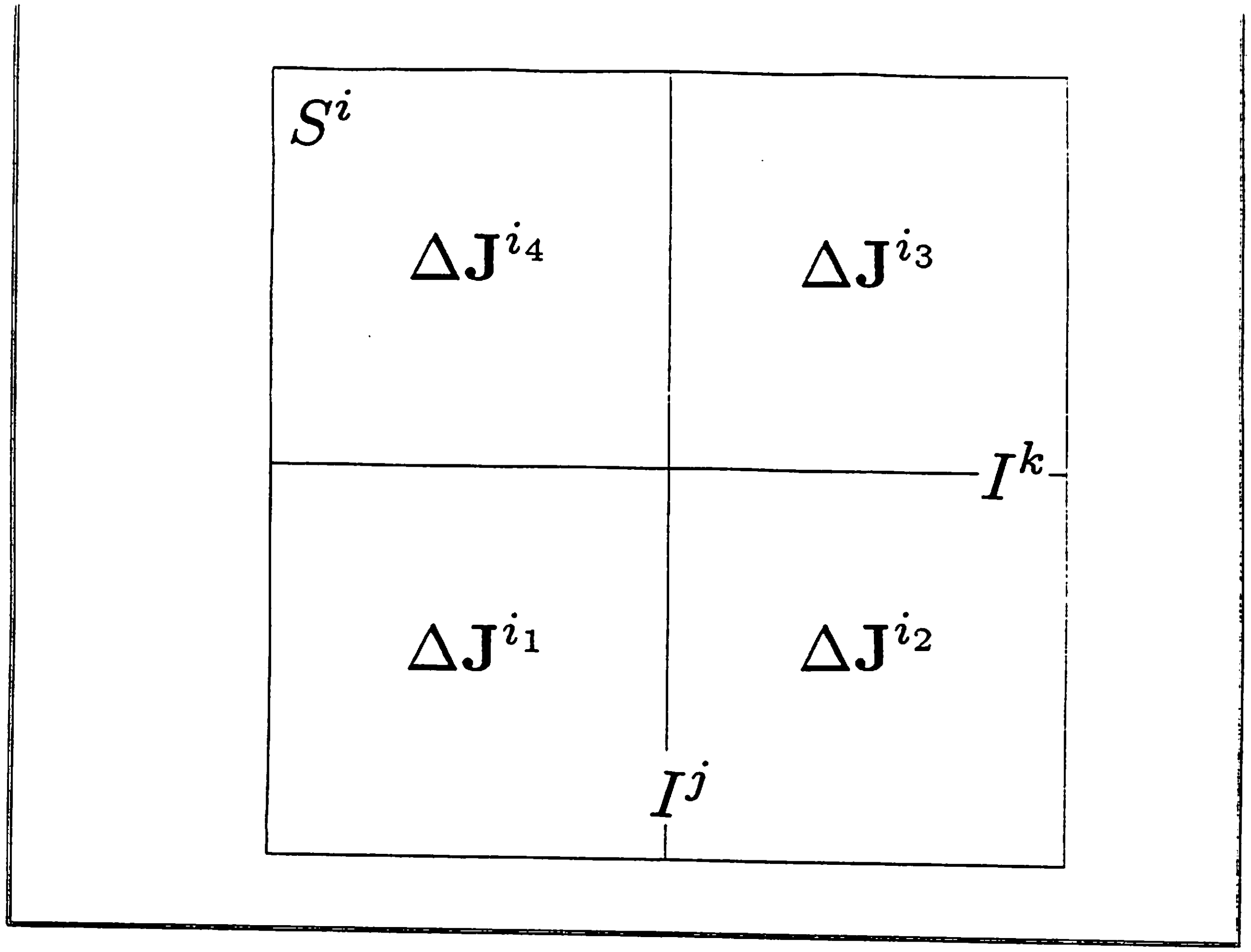


Figure 6 (*b*)



S^i
$\Delta\mathbf{J}^{i_4}$
$\Delta\mathbf{J}^{i_3}$
I^k
$\Delta\mathbf{J}^{i_1}$
$\Delta\mathbf{J}^{i_2}$
I^j

Figure 6 (c)