AN ALGORITHM FOR OPTIMAL SLEWING OF FLEXIBLE STRUCTURES

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ABSTRACT

We present an optimal control algorithm which can be used for computing the accelerations required for optimal slewing of flexible structures subject to initial and final conditions, as well as constraints on the acceleration and the deformation of the structure during the slewing maneuver. The algorithm can be used to solve both fixed-time and free-time optimal control problems, with the dynamics described either by ordinary or partial differential equations. We illustrate the performance of the algorithm with computational examples.

† The research reported herein was sponsored in part by the National Science Foundation under grant ECS-8121149, the Air Force Office of Scientific Research grant AFOSR-83-0361, the Office of Naval Research under grant N00014-83-K-0602, the State of California MICRO Program and the General Electric Co.

1. INTRODUCTION

Controlled flexible structures are found in aerospace systems as well as in various earthbound mechanisms with flexible links. Control is used to ensure precise pointing/positioning, input following, and disturbance rejection. Since large slewing maneuvers are usually described by nonlinear dynamics, it is very difficult to control them effectively by feedback laws. Hence it seems meaningful to use open loop optimal control to bring a flexible structure close to the desired state and then use feedback control based on linearized models to ensure the final accuracy. Although coupled systems of ordinary differential equations are commonly used for modeling flexible structures, models consisting of linear partial differential equations coupled with nonlinear ordinary differential equations offer greater modeling precision and simplicity in identification. Because of this, we have developed an optimal control algorithm which can be used with either type of model.

The research literature on the optimal control of flexible structures parallels that on the optimal control of finite dimensional systems. Thus, in [Gib.1, Gib.2, Gib.3], Gibson presents a detailed solution of the linear quadratic regulator problem for structures with PDE dynamics, including conditions for the convergence of modal approximation schemes. In Junkins and Turner [Jun.1] (see also [Chu.1, Ben.1 Bur.1, Flo.1] for related results), we find one of the first methods for solving a class of open-loop optimal slewing problems involving a rotating structure described by a linear PDE coupled with a nonlinear ODE. They assume initial and final conditions on the state, but no *control or state-space* constraints. For fixed time problems, they use a quadratic performance criterion. They use the Rayleigh-Ritz method to approximate the original optimal control problem by an optimal control problem with *finite dimensional* dynamics. They solve this problem using the Pontryagin Maximum Principle and a fairly expensive, iterative homotopy technique [Sch.1] for solving the resulting nonlinear two-point boundary value problem. For *minimum-time* optimal slewing problems they assume *linear dynamics* and use Pontryagin Maximum Principle and Newton's method to solve the corresponding optimality conditions. More recently, Ben-Asher, Burns and Cliff [Ben.1] have used the method of assumed modes [Mei.1] to reduce the problem of slewing a beam in minimum time subject to a torque constraint, to an optimal control problem with finite dimensional dynamics and then solved it by fairly standard techniques. A serious shortcoming of the results in [Jun.1, Ben.1] is that they do not provide an analysis of the relation between the solutions they obtain by discretization and the solutions of the original problems. A totally different approach was taken by Araya [Ara.1], who used the Balakrishnan ε–technique [Bal.1], to decompose minimum-time slewing problems with PDE dynamics, proposed in the NASA SCOLE design challenge [Tay.1], and torque constraints into an infinite sequence of unconstrained, fixed time problems, whose solutions converge to the solution of the original optimal control problem. However, he did not propose a numerical technique for obtaining approximate solutions to these unconstrained problems.

In this paper we present an optimal control algorithm for solving both minimum-time and fixed-time slewing problems, with control and end-point inequality constraints, for flexible structures described either entirely by ODEs or by linear PDEs coupled with nonlinear ODEs. An important aspect of our algorithm is that it takes into account the approximations resulting from numerical integration. In [Kle.1] we find a discretization theory for the solution of *unconstrained* optimal control problems via the Armijo gradient method. Unfortunately, this theory cannot be extended to the solution of constrained optimal control problems via methods of feasible directions because it fails to ensure that the algorithm does not hang up near the boundary of the feasible region. Because of this, we devised a special phase I - phase II method of feasible directions (related to [Pol.3]) for solving *finite dimensional* optimization problems of the form $\min_{x \in X} \{ \psi^0(x) \mid \psi^j(x) \leq 0\,,\ j = 1, 2, \ldots, m \}$, with $X \subset \mathbb{R}^n$ a convex compact set, as well as a new precision refinement test.

Our optimal control algorithm consists of a *master algorithm* which (a) uses the finite element method together with a numerical method for integrating ODE's to construct a finite dimensional approximating problem P_q, (b) calls our new finite dimensional algorithm and proceeds solving P_q, and (c) when a discretization refinement test is satisfied, it arrests the solution of P_q, constructs a higher precision approximation P_{q+1} and repeats (b) using the last point for P_q as the first point for P_{q+1}.

As can be seen from the computational example in Section 3, the use of our algorithm can be extended to the solution of minimum-time and fixed-time slewing problems, with control and state space inequality constraints. State space constraints are accomodated by transcribing constraints of the form $\max_{t \in [0,T]} g(z^u(t)) \leq 0$ into the form $\int_0^T \max \{ 0, g(z^u(t)) \}\, dt \leq 0$. Provided the algorithm is initialized in the infeasible region and an algorithm steering parameter (to be defined later) is set to a low value, the iterates constructed by the algorithm approach a solution from *outside* the feasible region and the deleterious effects associated with such transcriptions do not materialize.

2. THE OPTIMAL CONTROL ALGORITHM

We are now ready to deal with optimal control problems. We will assume that the dynamics of our system are transcribable into the form:

$$\dot{z}(t) = Az(t) + h(z(t), u(t)), \quad t \in [0, T], \quad z(0) = z_0, \tag{2.1a}$$

where the *state vector* $z(t)$ is an element of a Hilbert space, **H**, so that (2.1a) can, in fact, be a *partial differential equation*, and the control $u(t) \in \mathbb{R}^p$ *is finite dimensional.*

Assumption 2.1: We will assume that

(i) A is an infinitesimal generator of a C_0 semigroup[1].

(ii) The operator $h(\cdot, \cdot)$ is continuously differentiable.

(iii) The control u is an element of the pre-Hilbert space $\mathbf{L}^m_{\infty,2}[0, T] \triangleq (\mathbf{L}^m_\infty[0, T], \|\cdot\|_2)$ with $\|u\|_2 \triangleq \left\{ \int_0^T \|u(t)\|^2 d\tau \right\}^{1/2}$.

(iv) We assume that (2.1a) has a weak solution, which will be denoted by $z^{u,T}(t)$. □

The open sets in the $\mathbf{L}^m_{\infty,2}[0, T]$ topology are those that are open in both the $\mathbf{L}^m_2[0, T]$ topology. We need the boundedness associated with the $\mathbf{L}_\infty$ topology. However $\mathbf{L}_\infty$ is too fine a topology; the $\mathbf{L}_2$ topology is sufficiently coarse for showing the differentiability properties that are needed for optimization. For notational convenience, we will denote the space $\mathbf{L}^m_{\infty,2}[0, 1]$ by $\mathbf{L}_{\infty,2}$.

Because unavoidable discretization techniques may lead to serious computational and theoretical difficulties (see [Cul.1, Cul.2]) in free time optimal control problems, we propose to scale the time to the interval [0 , 1], so that the final time T becomes a scale variable. This scaling changes (2.1a) to

$$\dot{z}(t) = TAz(t) + Th(z(t), u(t)), \quad t \in [0, 1], \quad z(0) = z_0. \tag{2.1b}$$

At this point, to simplify notation, we combine the two problem variables u and T into a single variable, by defining $x = (u, T)$. In addition to assuming that (2.1b) has a weak solution, which we will denote by $z^x(t)$, we will assume that the differential, $\delta z^x(t, \delta x)$, of this solution, where $\delta x = (\delta u, \delta T)$, is given by the weak solution of the linearized equation

[1] For a discussion of semigroup theory see [Bal.2] or [Paz.1]. When (2.1a) represents an ODE, we expect that $A = 0$ holds. We note in passing that PDEs derived using Lagrangian dynamics have the form (2.1a) and hence that (2.1a) represents a broad class of dynamical systems described either by ODEs or PDEs.

$$\delta \dot{z}(t) = T\left[A + \frac{\partial h(z^{u,T}(t), u(t))}{\partial z} \right] \delta z(t) + T \frac{\partial h(z^{u,T}(t), u(t))}{\partial u} \delta u(t)$$

$$+ [Az^x(t), u(t)) + h(z^x(t), u(t))]\delta T \ , \ t \in [0, 1] \ , \ \delta z(0) = 0 \ . \qquad (2.1c)$$

The following result was established in [Bak.1, Sec. 3.3]:

Proposition 2.1: Suppose that the function $h(\cdot, \cdot)$, in (2.1a, 2.1b) satisfies:

(i) For every pair of bounded sets $S \subset \mathbf{H}$ and $U \subset \mathbb{R}^p$, there exists a $K_{SU} < \infty$ such that for all $z, z^* \in S, u, u^* \in U$,

$$(a) \ \| h(z^*, u^*) - h(z, u) \| \le K_{SU} [\| z^* - z \| + \| u^* - u \|] \ ,$$

$$(b) \ \| \frac{\partial h}{\partial z}(z^*, u^*) - \frac{\partial h}{\partial z}(z, u) \| \le K_{SU} [\| z^* - z \| + \| u^* - u \|] \ ,$$

$$(c) \ \| \frac{\partial h}{\partial u}(z^*, u^*) - \frac{\partial h}{\partial u}(z, u) \| \le K_{SU} [\| z^* - z \| + \| u^* - u \|] \ ;$$

(ii) For every $0 < T < \infty$ and $u \in \mathbf{L}^m_{\infty,2}[0, T]$, (2.1a) has a mild solution, $z^x(\cdot)$, and for any bounded subset $U \subset \mathbb{R}^p$ and $T \in (0, \infty)$, there exists a $b_{U,T} < \infty$ such that if $u \in \mathbf{L}^m_{\infty,2}[0, T]$ is such that $u(t) \in U$ for all $t \in [0, T]$, then $\| z^x(t) \| < b_{U,T}$ for all and $t \in [0, T]$.

Under these assumptions, (2.1b) has a solution $\delta z^x(t, \delta x)$ which is the differential of $z^x(t)$ with respect to the control u and the time scaling parameter T. □

Now consider the optimal control problem with control and end point inequality constraints:

$$\min_{x \in \mathbf{L}_{\infty,2} \times \mathbb{R}} \{ g^0(z^x(1)) \mid g^j(z^x(1)) \le 0 \ , \ j = 1, 2, \dots, m \ ,$$

$$u(t) \in U \ \forall t \in [0, 1] \ , \ T \in [T_o, T_f] \} \ , \qquad (2.2a)$$

where $T_o > 0$ is assumed to be arbitrarily small and $T_f < \infty$. We assume that the set $U \subset \mathbb{R}^p$ is compact and that all the functions $g^j : \mathbf{H} \to \mathbb{R}$ are locally Lipschitz continuously differentiable.

If, for $j = 0, 1, \dots, m$, we define the functions $\psi^j : \mathbf{L}_{\infty,2} \times \mathbb{R} \to \mathbb{R}$ by $\psi^j(x) \triangleq g^j(z^x(1))$, we can rewrite (2.2a) in the more explicit form

$$\min \{ \psi^0(x) \mid \psi^j(x) \le 0 \ , \ j = 1, 2, \dots m \ , \ x \in \mathbf{X} \} \ , \qquad (2.2b)$$

where $\mathbf{X} \triangleq \mathbf{U} \times \mathbf{T}$, with $\mathbf{U} \triangleq \{ u \in \mathbf{L}_{\infty,2} \mid u(t) \in U \ \forall t \in [0, 1] \}$ and $\mathbf{T} \triangleq [T_o, T_f]$. We will use the norm $\|\cdot\|_{\mathbf{X}}$ on $\mathbf{X}$ defined by $\|x\|_{\mathbf{X}}^2 \triangleq \|u\|_2^2 + T^2$.

Now, let $\underline{m} \triangleq \{ 1, 2, \ldots, m \}$, let $\psi : \mathbf{L}_{\infty,2} \times \mathbb{R} \to \mathbb{R}$ be defined by $\psi(x) = \max_{j \in \underline{m}} \psi^j(x)$, and let $\psi_+(x) \triangleq \max \{ 0, \psi(x) \}$. Finally, with $\gamma > 0$ given, for any $\bar{x}$, $x \in \mathbf{X}$, let

$$F(x \mid \bar{x}) \triangleq \max \{ \psi^0(x) - \psi^0(\bar{x}) - \gamma\psi_+(\bar{x}), \psi(x) - \psi(\bar{x}) \}. \tag{2.2c}$$

Note that $F(\bar{x} \mid \bar{x}) = 0$.

It should be obvious that if $\hat{x}$ is a local minimizer for the problem (2.2b), then it must also be a local minimizer for the problem

$$\min_{x \in \mathbf{X}} F(x \mid \hat{x}). \tag{2.2c}$$

We can use this observation to obtain a simple first order optimality condition for (2.2b) (see [Pol.5] for an analogous development for the finite dimensional case), as follows.

For $j = 0, 1, 2, \ldots m$, and any $\bar{x}$, $x \in \mathbf{X}$, let

$$\hat{\psi}^j(x \mid \bar{x}) \triangleq g^j(z^{\bar{x}}(1)) + \langle \nabla g^j(z^{\bar{x}}(1)), \delta z^{\bar{x}}(1, x - \bar{x}) \rangle_{\mathbf{H}} + \tfrac{1}{2}\|x - \bar{x}\|_{\mathbf{X}}^2, \tag{2.3a}$$

where $\langle \cdot, \cdot \rangle_{\mathbf{H}}$ denotes the inner product on $\mathbf{H}$, and $\nabla g^j(z) \in \mathbf{H}^* = \mathbf{H}$ is defined by the formula for the Frechet differential of $g^j(\cdot)$: $dg^j(z, \delta z) = \langle \nabla g^j(z), \delta z \rangle_{\mathbf{H}}$. Thus, $\hat{\psi}^j(x \mid \bar{x})$ is a first order, quadratic approximation to the value $\psi^j(x)$. Similarly, we get a first order, quadratic approximation to the value $F(x \mid \bar{x})$:

$$\hat{F}(x \mid \bar{x}) \triangleq \max_{j \in \underline{m}} \max \{ \hat{\psi}^0(x \mid \bar{x}) - \psi^0(\bar{x}) - \gamma\psi_+(\bar{x}), \hat{\psi}^j(x \mid \bar{x}) - \psi_+(\bar{x}) \}. \tag{2.3b}$$

Next we define an optimality function $\theta : \mathbf{X} \to \mathbb{R}$ and a search direction function $\xi : \mathbf{X} \to \mathbf{X}$ by

$$\theta(x) \triangleq \min_{x' \in \mathbf{X}} \hat{F}(x' \mid x), \tag{2.4a}$$

$$\xi(x) \triangleq \arg \min_{x' \in \mathbf{X}} \hat{F}(x' \mid x). \tag{2.4b}$$

The following theorem is a straightforward extension of a result in finite dimensional optimization (see [Pol.5]). For a proof see [Bak.2].

Theorem 2.1 : (a) The functions $\theta(\cdot)$ and $\xi(\cdot)$ are both continuous. (b) If $\hat{x} \in \mathbf{X}$ is a local minimizer for (2.1b), then

$$\theta(\hat{x}) = 0 . \tag{2.5}$$

□

Our first observation is that expressions such as $\langle \nabla g^j(z^{\bar{x}}(1)) , \delta z^{\bar{x}}(1 , x - \bar{x}) \rangle_{\mathbf{H}}$ can be computed using adjoints. Our second observation is that the numerical solution of ordinary or partial differential equations requires discretization of at least one variable and hence that we cannot utilize an analogue of a nonlinear programming algorithm in solving (2.2b) without addressing this source of difficulty. To ensure that our final implementable algorithm has the desired convergence properties, we must use discretizations in the solution of the ODEs or PDEs which guarantee that the resulting family of approximating optimal control problems is *consistent* with respect to the requirements of the convergence theorem which we will present at the end of this section.

To make matters concrete, we may assume that **H** is the space of r–times differentiable functions from $[0 , 1]$ into $\mathbb{R}^p$. Next we introduce a set of 2^{q_s} orthogonal spline functions $\{ \zeta^i_{q_s}(\cdot) \}_{i=1}^{2^{q_s}} \subset \mathbf{H}$, where $q_s \in \mathbb{N}$, for "spatial" discretization[2], write $z^x(t)(s)$ in the form

$$z^x(t)(s) = \sum_{i=1}^{2^{q_s}} \zeta^i_{q_s}(s)\omega^i_{q_s}(t)(s) . \tag{2.6a}$$

and compute the projection Πz_0 of z_0 onto the subspace of **H** spanned by the splines. Let $\omega = (\omega^1_{q_s} , \dots , \omega^{2^{q_s}}_{q_s})$, and let Z_{q_s} be a matrix with columns $\zeta^i_{q_s}$, $i = 1 , 2 , \dots , 2^{q_s}$. Then (2.6a) can be written in the shorter form

$$z^x(t)(s) = Z_{q_s}(s)\omega(t , x) . \tag{2.6b}$$

On the subspace spanned by the splines, our dynamics have the form

$$Z_{q_s}(s)\dot{\omega}_{q_s}(t , x) = Th(Z_{q_s}(s)\omega_{q_s}(t , x) , u(t)) , \tag{2.6c}$$

$$Z_{q_s}(s)\omega(0 , x) = \Pi z_o(s) \ , \ \ \forall s \in [0 , 1] . \tag{2.6d}$$

Next we use the orthogonality of the splines to set up the differential equations for the functions $\omega^i_{q_s}(t , x)$:

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[2] For many dynamical systems, a system of second order PDEs coupled with ODEs is a more "natural" description than (2.1a). In that case all calculations are carried out with the original dynamics. As a result, since the weak form of a solution is used, it is often possible to use splines that are only $r/2$-times differentiable, which results in considerable computational simplification (in the example in Section 3, we use standard Hermit cubic splines). Then Newmark's method [New.1] is used for temporal discretization. See [Str.1] for details.

$$\dot{\omega}^i_{q_s}(t\,,x) = \int_0^1 \langle\, \zeta^i_{q_s}(s)\,, Th(Z_{q_s}(s)\omega(t\,,x)\,, u(t))\rangle_{\mathbf{H}}\, ds\ ,\quad i = 1\,,2\,,3\,,\dots,2^{q_s}\ , \tag{2.6e}$$

$$\omega^i_{q_s}(0\,,x) = \int_0^1 \langle\, \zeta^i_{q_s}(s)\,, \Pi z_o(s)\rangle_{\mathbf{H}}\, ds\ ,\quad i = 1\,,2\,,3\,,\dots,2^{q_s}\ . \tag{2.6f}$$

Then (2.6b,c) can be written as a first order vector differential equation

$$\dot{\omega}(t) = Y(\omega(t)\,, u(t))\ ,\quad \forall\, t \in [0\,,1]\,,\quad \omega(0) = \omega_0\ , \tag{2.7a}$$

in which the function $Y(\cdot\,,\cdot)$ is defined by (2.6e). Finally, we discretize the normalized time interval $[0\,,1]$ into 2^{q_t} equal intervals, set $\Delta_{q_t} \triangleq 1/2^{q_t}$, and replace (2.7a) by the difference equation resulting from the use of the Euler method of integration:[3]

$$\omega((k+1)\Delta_{q_t}) = \omega(k\Delta_{q_t}) + F(\omega(k\Delta_{q_t})\,, u(k\Delta_{q_t}\,, T))\,,\ k = 0\,,1\,,2\,,\dots,2^{q_t}-1\,,\ \omega(0) = \omega_0\,. \tag{2.7b}$$

Let $p_{q_t} : [0\,,\infty) \to \{\,0\,,1\,\}$ be defined by

$$p_{q_t}(t) \triangleq \begin{cases} 1 \text{ for } t \in [0\,,\Delta_{q_t}] \\ 0 \text{ for } t > \Delta_{q_t} \end{cases}\ , \tag{2.8a}$$

let $\mathbf{U}_{q_t} \subset \mathbf{U}$ be the set of controls which are constant over our time grid, i.e., if $u(t) \in \mathbf{U}_{q_t}$, then for a sequence of vectors $\{\,u_k\,\}_{k=0}^{2^{q_t}-1} \subset U$,

$$u(t) = \sum_{k=0}^{2^{q_t}-1} u_k p_{q_t}(t - k\Delta_{q_t})\,. \tag{2.8b}$$

It is easy to see that the sets $\mathbf{U}_q$ are nested, i.e., if $q < r$ then $\mathbf{U}_q \subset \mathbf{U}_r$. Next, let $\omega^x_{q_t}(k\Delta_{q_t})$, $k = 0\,,1\,,\dots,2^{q_t}-1$, denote the solution of (2.7b) corresponding to a control in $\mathbf{U}_{q_t}$ and $T \in \mathbf{T}$. Then we see that for $t \in [0\,,1]$, our numerical integration procedure yields the following approximation to the actual solution $z^x(t)$:

$$z^x_{q_s\,,q_t}(t) \triangleq \sum_{k=0}^{2^{q_t}-1} Z_{q_s}\omega^x_{q_t}(k\Delta_{q_t}))p_{q_t}(t - k\Delta_{q_t})\,. \tag{2.8c}$$

Hence, for $x \in \mathbf{X}_{q_t} \triangleq \mathbf{U}_{q_t} \times \mathbf{T}$, the use of our numerical integration procedure yields the following approximating values to $\psi^j(x)$:

[3] Euler's method results in the simplest exposition. In fact, any method which is first order or better may be used provided care is taken to ensure that the resulting discretizations satisfy our assumptions.

$$\psi^j_{q_s,q_t}(x) \triangleq g^j(z^{x}_{q_s,q_t}(1)) , \; j = 0,1,2,\ldots,m . \tag{2.9}$$

Next, for x , $\bar{x} \in \mathbf{X}_{q_s}$ and $\delta x = (\delta u \, , \delta T) = x - \bar{x}$, the sensitivities of the difference equation (2.7b) to the perturbation δx, in the control and scale factor are given by the solution $\delta\omega^{\bar{x}}_{q_t}(k\Delta_{q_t} \, , \delta x)$, of the linearized difference equation:

$$\delta\omega((k+1)\Delta_{q_t}) = \delta\omega(k\Delta_{q_t}) + \frac{\partial F(\omega(k\Delta_{q_t}) \, , \bar{u}(k\Delta_{q_t}) \, , \bar{T})}{\partial\omega}\delta\omega(k\Delta_{q_t}) + \frac{\partial F(\omega(k\Delta_{q_t}) \, , \bar{u}(k\Delta_{q_t}) \, , \bar{T})}{\partial u}\delta u_k$$

$$+ F(\omega(k\Delta_{q_t}) \, , \bar{u}(k\Delta_{q_t}) \, , \bar{T})\delta T \, , \; k = 0,1,2,\ldots,2^{q_t} \, , \; \delta\omega(0) = \omega_0 . \tag{2.10a}$$

Clearly, $\delta\omega^{\bar{x}}_{q_t}(k\Delta_{q_t} \, , \delta x)$ can be used to define the function

$$\delta z^{\bar{x}}_{q_s,q_t}(t \, , \delta x) \triangleq \sum_{k=0}^{2^{q_t}-1} Z_{q_s}\delta\omega^{\bar{x}}_{q_t}(k\Delta_{q_t} \, , \delta x))p_{q_t}(t - k\Delta_{q_t}) . \tag{2.10b}$$

Hence, given any x , $\bar{x} \in \mathbf{X}_{q_s}$, we define the first order, quadratic approximation to $\psi^j_{q_s,q_t}(x)$ by

$$\hat{\psi}^j_{q_s,q_t}(x \mid \bar{x}) \triangleq g^j(z^{\bar{x}}_{q_s,q_t}(1)) + \langle \nabla g^j(z^{\bar{x}}_{q_s,q_t}(1) \, , \delta z^{\bar{x}}_{q_s,q_t}(1 \, , x - \bar{x}) \rangle_{\mathbf{H}}$$

$$+ \tfrac{1}{2}\| x - \bar{x} \|^2_{\mathbf{X}} \, , \; j = 0,1,2,\ldots,m . \tag{2.11}$$

In turn, these definitions lead to the following analogs of (2.3b), (2.4a) and (2.4b), respectively. For any x , $\bar{x} \in \mathbf{X}_{q_t}$,

$$\hat{F}_{q_s,q_t}(x \mid \bar{x}) \triangleq \max_{j \in \underline{m}} \max \{ \hat{\psi}^0_{q_s,q_t}(x \mid \bar{x}) - \psi^0_{q_s,q_t}(\bar{x}) - \gamma\psi_{q_t+}(\bar{x}) \, ,$$

$$\hat{\psi}^j_{q_s,q_t}(x \mid \bar{x}) - \psi_{q_s,q_t+}(\bar{x}) \} \, , \tag{2.12a}$$

where $\psi_{q_t,q_s+}(\bar{x}) \triangleq \max \{ 0 \, , \psi_{q_t,q_s}(\bar{x}) \}$, with $\psi_{q_t,q_s}(\bar{x}) \triangleq \max_{j \in \bar{m}} \psi^j_{q_t,q_s}(\bar{x})$,

$$\theta_{q_s,q_t}(\bar{x}) \triangleq \min_{x \in \mathbf{X}_{q}} \hat{F}_{q_s,q_t}(x \mid \bar{x}) , \tag{2.12b}$$

$$\xi_{q_s,q_t}(\bar{x}) \triangleq \arg \min_{x \in \mathbf{X}_{q}} \hat{F}_{q_s,q_t}(x \mid \bar{x}) . \tag{2.12c}$$

To complete our analysis we need the following assumption which can be expected to be satisfied in most practical situations:

Assumption 2.2: There exists a $K_z < \infty$ and a monotone decreasing function $e : \mathbb{N} \to \mathbb{R}$, such that $e(q) > 0$ for all $q \in \mathbb{N}$ and $e(q) \to 0$ as $q \to \infty$, such that for any positive integer q, if $\min\{q_s, q_t\} > q$, then for all $\bar{x}$, $x \in \mathbf{X}_{q_t}$,

$$\| z^x(1) - z^x_{q_s, q_t}(1) \|_\infty \le K_z e(q) , \tag{2.12d}$$

$$\| \delta z^x(1, x - \bar{x}) - z^x_{q_s, q_t}(1, x - \bar{x}) \|_\infty \le K_z e(q) . \tag{2.12e}$$

□

Our algorithm does not require knowledge of the constant K_z, but it does require knowledge of $e(\cdot)$. However, when Hermite splines (with $2^{q_s - 1}$ mesh points) are used for spatial discretization and Newmark's method (with 2^{q_t} nodes) is used for time integration, one can usually set $e(q) = 2^q$.

Proposition 2.2 : There exists a $K < \infty$ such that for any positive integer q, if $\min\{q_s, q_t\} > q$, then for all $\bar{x}$, $x \in \mathbf{X}_{q_t}$,

$$| \psi^j(x) - \psi^j_{q_s, q_t}(x) | \le K e(q) , \quad j = 0, 1, 2, \dots, m , \tag{2.13a}$$

$$| \hat{\psi}^j(x \mid \bar{x}) - \hat{\psi}^j_{q_s, q_t}(x \mid \bar{x}) | \le K e(q) , \quad j = 0, 1, 2, \dots, m , \tag{2.13b}$$

$$| \theta(x) - \theta_{q_s, q_t}(x) | \le K e(q) , \tag{2.13c}$$

$$\| \xi(x) - \xi_{q_s, q_t}(x) \|^2_{\mathbf{X}} \le K e(q) . \tag{2.13d}$$

Proof: Since $\mathbf{X}_{q_t}$ is bounded and the functions $g^j(\cdot)$ are locally Lipschitz continuously differentiable, there exists a Lipschitz constant $L < \infty$ such that for $j = 0, 1, \dots m$,

$$| g^j(z(1)) - g^j(z'(1)) | \le L \| z'(1) - z(1) \|_s \tag{2.14a}$$

$$\| \nabla g^j(z(1)) - \nabla g^j(z'(1)) \| \le L \| z'(1) - z(1) \|_s , \tag{2.14b}$$

for all z, $z' \in \mathbf{H}$ that we need to deal with.

Next, by construction, if $q_t > q$, then $\mathbf{X}_q \subset \mathbf{X}_{q_t}$, and hence, if $\min\{q_s, q_t\} > q$, and $x \in \mathbf{X}_q$, it follows that $x \in \mathbf{X}_{q_t}$ and hence, from (2.14a) and (2.12d), that

$$| g^j(z^x(1)) - g^j(z^x_{q_s, q_t}(1)) | \le L \| z^x(1) - z^x_{q_s, q_t}(1) \|_s \le L K_z e(q) . \tag{2.14c}$$

Hence (2.13a) follows. A similar argument leads to (2.13b)

Next, for any $x\,,\,\bar{x} \in \mathbf{X}$, let $\tilde{\psi}^0(x \mid \bar{x}) \triangleq \hat{\psi}^0(x \mid \bar{x}) - \psi^0(\bar{x}) - \gamma\psi_+(\bar{x})$, and, for $j \in \underline{m}$, let $\tilde{\psi}(x \mid \bar{x}) \triangleq \hat{\psi}(x \mid \bar{x}) - \psi_+(\bar{x})$. Similarly, for any $q_s\,,q_t \in \mathbb{N}_+$, and any $x\,,\,\bar{x} \in \mathbf{X}$, let $\tilde{\psi}^0_{q_s\,,q_t}(x \mid \bar{x}) \triangleq \hat{\psi}^0_{q_s\,,q_t}(x \mid \bar{x}) - \psi^0_{q_s\,,q_t}(\bar{x}) - \gamma\psi_{q_s\,,q_t\,+}(\bar{x})$, and, for $j \in \underline{m}$, let $\tilde{\psi}^j_{q_s\,,q_t}(x \mid \bar{x}) \triangleq \hat{\psi}^j_{q_s\,,q_t}(x \mid \bar{x}) - \psi_{q_s\,,q_t+}(\bar{x})$. Then (2.13b) obviously holds with $\hat{\psi}^j(x \mid \bar{x})$ replaced by $\tilde{\psi}^j(x \mid \bar{x})$ and with $\hat{\psi}^j_{q_s\,,q_t}(x \mid \bar{x})$ replaced by $\tilde{\psi}^j_{q_s\,,q_t}(x \mid \bar{x})$. Let $\bar{m} \triangleq \{\,0\,,1\,,2\,,\ldots\,,m\,\}$. Then we see that $\theta(\bar{x}) = \min\limits_{x \in \mathbf{X}} \max\limits_{j \in \bar{m}} \tilde{\psi}^j(x \mid \bar{x})$ and $\theta_{q_s\,,q_t}(\bar{x}) = \min\limits_{x \in \mathbf{X}} \max\limits_{j \in \bar{m}} \tilde{\psi}^j_{q_s\,,q_t}(x \mid \bar{x})$. Now by definition of $\xi(\cdot)$,

$$\theta(\bar{x}) = \max_{j \in \bar{m}} \tilde{\psi}^j(\xi(\bar{x}) \mid \bar{x})\,. \tag{2.14d}$$

Hence we must have that

$$\theta_{q_s\,,q_t}(\bar{x}) \leq \max_{j \in \bar{m}} \tilde{\psi}^j_{q_s\,,q_t}(\xi(\bar{x}) \mid \bar{x})$$

$$\leq \max_{j \in \bar{m}} \tilde{\psi}^j(\xi(\bar{x}) \mid \bar{x}) + Le(q) = \theta(\bar{x}) + Le(q)\,. \tag{2.14e}$$

We complete the proof of (2.13c), by interchanging $\theta(\cdot)$ and $\theta_q(\cdot)$ above.

Finally we turn to (2.13d). For every $\bar{x}\,,\,x \in \mathbf{X}$, let $\tilde{\psi}(x \mid \bar{x}) \triangleq \max\limits_{j \in \bar{m}} \tilde{\psi}^j(x \mid \bar{x})$, and let $\tilde{\psi}_{q_s\,,q_t}(x \mid \bar{x}) \triangleq \max\limits_{j \in \bar{m}} \tilde{\psi}^j_{q_s\,,q_t}(x \mid \bar{x})$. Then, because these functions are maxima of quadratics with unit second derivatives, it follows that for all $x \in \mathbf{X}$,

$$\tilde{\psi}(x \mid \bar{x}) - \tilde{\psi}(\xi(\bar{x}) \mid \bar{x}) \geq \tfrac{1}{2}\|x - \xi(\bar{x})\|^2_{\mathbf{X}}\,. \tag{2.14f}$$

Hence for all $q_s\,,q_t \in \mathbb{N}_+$,

$$\tilde{\psi}(\xi_{q_s\,,q_t}(\bar{x}) \mid \bar{x}) - \tilde{\psi}(\xi(\bar{x}) \mid \bar{x}) \geq \tfrac{1}{2}\|\xi_{q_s\,,q_t}(\bar{x}) - \xi(\bar{x})\|^2_{\mathbf{X}}\,. \tag{2.14g}$$

Making use of (2.13b), (2.13c), we now obtain from (2.14g) that if $\min\{\,q_s\,,q_t\,\} > q$ then

$$Ke(q) \geq Ke(q) + \tilde{\psi}_{q_s\,,q_t}(\xi_{q_s\,,q_t}(\bar{x}) \mid \bar{x}) - \tilde{\psi}_{q_s\,,q_t}(\xi(\bar{x}) \mid \bar{x}) \geq \tfrac{1}{2}\|\xi_{q_s\,,q_t}(\bar{x}) - \xi(\bar{x})\|^2_{\mathbf{X}}\,, \tag{2.14h}$$

which proves that (2.13d) must hold, with the constant K suitably redefined. □

With these developments out of the way, we can now state our implementable optimal control algorithm. A close examination will show that the algorithm below consists of a master algorithm which constructs a finite dimensional optimal control problem P_q, in which the spatial and temporal discretizations are coordinated, and defined on $\in \mathbf{U}_{q_t} \times \mathbf{T}$, and then calls our new modification of the

Polak-Trahan-Mayne phase I - phase II method of feasible directions [Pol.3] to proceed solving this problem until a discretization refinement test indicates that the discretization must be refined.

Algorithm 2.1 :

Parameters : $q \in \mathbb{N}, \varepsilon, \delta \in (0,1), \gamma > 1, \alpha, \beta \in (0,1), \sigma_t, \sigma_s \in \mathbb{N}$.

Data : A vector coefficient sequence $u_0 = (u_0^0, \ldots, u^{2^q-1}) \in \mathbb{R}^{p2^q}$, defining the control $u_0(t)$ via (2.8b), with $q_t = \sigma_t q$, and a scaling parameter $T_0 \in [T_o, T_f]$.

Step 0 : Set $i = 0$.

Step 1 : Set $q_t = \sigma_t q$, $q_s = \sigma_s q$ and compute the the *optimality function* value $\theta_i = \theta_{q_s, q_t}(x_i)$, and the corresponding *search direction* $\eta_i \triangleq \xi_{q_s, q_t}(x_i) - x_i$, defined by (2.12b) and (2.12c).

Step 2 : Compute the *step size* λ_i (the rule depends on the sign of $\psi_{q_s, q_t}(x_i)$):

If $\psi_{q_s, q_t}(x_i) > 0$, compute

$$k_i = \arg\max \{ \beta^k \mid k \in \mathbb{N}, \ \psi_{q_s, q_t}(x_i + \beta^k \eta_i) - \psi_{q_s, q_t}(x_i) \le \beta^k \alpha \theta_i \} . \tag{2.15a}$$

$$\text{If} \quad \psi_{q_s, q_t}(x_i + \beta^{k_i} \eta_i) - \psi_{q_s, q_t}(x_i) > -\varepsilon e(q)^\delta , \tag{2.15b}$$

replace q by $q + 1$, and go to Step 1.

Else set $\lambda_i = \beta^{k_i}$ and go to Step 2.

If $\psi_{q_s, q_t}(x_i) \le 0$, compute

$$k_i = \arg\max \{ \beta^k \mid k \in \mathbb{N}, \ F_{q_s, q_t}(x_i + \beta^k \eta_i \mid x_i) \le \beta^k \alpha \theta_i \} . \tag{2.15c}$$

$$\text{If} \quad F_{q_s, q_t}(x_i + \beta^{k_i} \eta_i \mid x_i) > -\varepsilon e(q)^\delta , \tag{2.15d}$$

replace q by $q + 1$, and go to Step 1.

Else set $\lambda_i = \beta^{k_i}$ and go to Step 2.

Step 3 : Replace x_i by $x_i + \lambda_i \eta_i$, set $i = i + 1$ and go to Step 1. □

A few remarks are in order here. First, the parameter γ can be used the to control the emphasis the algorithm places on iterates becoming feasible. Thus, when problems such as those in Section 3 are being solved, one should set γ to a low value, so as to approach a solution from outside the feasible region. Second, it is shown in [Bak.1, Sec. 7.3] that the Polyak-Levitin constrained Newton method [Lev.1], can be used for computing both θ_i and η_i.

Theorem 2.2 : Suppose that for every $x \in \mathbf{X}$ such that $\psi(x) > 0$, $\theta(x) < 0$. If Algorithm 2.1 jams up, cycling between Step 1 and Step 2, at a point x_k, then x_k satisfies the first order condition $\psi(x_k) \le 0$ and $\theta(x_k) = 0$. If Algorithm 2.1 constructs an infinite sequence $\{ x_i \}_{i=0}^{\infty}$, then every accumulation point $\hat{x}$ of $\{ x_i \}_{i=0}^{\infty}$ satisfies the first order optimality condition, for (2.2b), $\psi(\hat{x}) \le 0$, $\theta(\hat{x}) = 0$.

Proof: First we make an observation. Suppose that $\hat{x} \in \mathbf{X}$ is such that $\theta(\hat{x}) < 0$. Then it is straightforward to show that there is a neighborhood B of $\hat{x}$ and a $\hat{k} \in \mathbb{N}$ such that if $x \in B$ and $\psi(x) > 0$, then

$$\psi(x + \beta^{\hat{k}} \eta(x)) - \psi(x) < \beta^{\hat{k}} \alpha\theta(x) \le \tfrac{1}{2}\beta^{\hat{k}} \alpha\theta(\hat{x}) \ , \tag{2.16a}$$

and if $x \in B$ and $\psi(x) < 0$, then

$$F(x + \beta^{\hat{k}} \eta(x) \mid x) < \beta^{\hat{k}} \alpha\theta(x) \le \tfrac{1}{2}\beta^{\hat{k}} \alpha\theta(\hat{x}) \, . \tag{2.16b}$$

(a) It now follows from Proposition 2.2 that if $\hat{x}$ is such that $\theta(\hat{x}) < 0$, then the test (2.15b) or (2.15d) will fail for a finite value of q, and hence Algorithm 2.1 cannot jam up at such a point.

(b) Next we will show that if we denote by q_i the value of q used in the construction of x_{i+1}, then $q_i \to \infty$ as $i \to \infty$. Since $q_s = \sigma_s q$ and $q_t = \sigma_t q$, we may simplify our notation in Algorithm 2.1 and replace the subscript q_s , q_t by the subscript q.

For the sake of contradiction, suppose that $q_i = q^*$ for all $i \ge i^* < \infty$, and suppose, without loss of generality that $\psi_{q^*}(x_{i^*}) > 0$. Then it follows from (2.15b) that

$$\psi_{q^*}(x_{i+1}) - \psi_{q^*}(x_i) \le -\varepsilon e (q^*)^{\delta} \tag{2.16c}$$

for all $i \ge i^*$ such that $\psi_{q^*}(x_i) > 0$. It follows from this and (2.15c) that there must exist an $i_1 \ge i^*$ such that $\psi_{q^*}(x_i) \le 0$ for all $i \ge i_1$. Furthermore, by (2.15d), for all $i \ge i_1$,

$$\psi^0_{q^*}(x_{i+1}) - \psi^0_{q^*}(x_i) \le -\varepsilon e (q^*)^{\delta} \, . \tag{2.16d}$$

Since $\mathbf{X}$ is bounded, this is clearly impossible. Hence we conclude that $q_i \to \infty$ as $i \to \infty$.

(c) Now suppose that $\hat{x}$ is an accumulation point of $\{ x_i \}_{i=0}^{\infty}$ and that $\theta(\hat{x}) < 0$. It then follows from Proposition 2.2 and (2.16a), (2.16b) that there exists a neighborhood B of $\hat{x}$ and an i_2 such that for all $i \ge i_2$ such that $x_i \in B$ (of which there must exist an infinite number) with $\psi_{q_i}(x_i) > 0$,

$$\psi(x_{i+1}) - \psi(x_i) \le 2Ke(q_i) + \psi_{q_i}(x_{i+1}) - \psi_{q_i}(x_i) < \beta^{\hat{k}}\alpha\theta(\hat{x})/4 + 2Ke(q_i) \le \beta^{\hat{k}}\alpha\theta(\hat{x})/8 < 0 , \qquad (2.16e)$$

and for all $i \ge i_2$ such that $x_i \in B$ with $\psi_{q_i}(x_i) \le 0$,

$$F(x_{i+1} \mid x_i) \le 2Ke(q_i) + F_{q_i}(x_{i+1} \mid x_i) \le 2Ke(q_i) + \beta^{\hat{k}}\alpha\theta(\hat{x})/4 \le \beta^{\hat{k}}\alpha\theta(\hat{x})/8 < 0 . \qquad (2.16f)$$

We need to consider two cases. First suppose that $\psi_{q_i}(x_i) > 0$ for all but a finite number of i. It then follows from (2.15b) and (2.13a) that

$$\psi(x_{i+1}) - \psi(x_i) \le 2Ke(q_i) - \frac{\varepsilon}{2^{\delta q_i}} = -e(q_i)\left[\ \varepsilon e(q_i)^{(1-\delta)} - 2K\ \right] . \qquad (2.16g)$$

Since $q_i \to \infty$ as $i \to \infty$, we conclude from (2.16g) that there exists an i_3 such that the sequence $\{\psi(x_i)\}_{i=i_3}^{\infty}$ is monotone decreasing. It now follows from (2.16e) that $\psi(x_i) \to -\infty$ as $i \to \infty$ which is clearly impossible.

Hence we must consider the second possibility, i.e., that there is an infinite subset I of the positive integers such that for all $i \in I$, $\psi_{q_i}(x_i) \le 0$. Now suppose that $i \in I$, then, making use of (2.15d) and (2.13a) we obtain that

$$\psi_{q_{i+1}}(x_{i+1}) \le 2Ke(q_i) - \varepsilon e(q_i) = -e(q_i)\left[\ \varepsilon e(q_i)^{(1-\delta)} - 2K\ \right] . \qquad (2.16h)$$

Since $q_i \to \infty$ as $i \to \infty$, we conclude from (2.16h) that there exists an i_4 such for all $i \ge i_4$, $\psi_{q_i}(x_i) \le 0$. It now follows from (2.15d) that for all $i \ge i_4$,

$$\psi^0(x_{i+1}) - \psi^0(x_i) \le 2Ke(q_i) - \varepsilon e(q_i) = -e(q_i)\left[\ \varepsilon e(q_i)^{(1-\delta)} - 2K\ \right] . \qquad (2.16i)$$

Next,it follows from (2.16f)that for all $i \ge i_4$ such that $x_i \in B$,

$$\psi^0(x_{i+1}) - \psi^0(x_i) \le 2Ke(q_i) + \psi^0_{q_i}(x_{i+1}) - \psi^0_{q_i}(x_i)$$

$$< \beta^{\hat{k}}\alpha\theta(\hat{x})/4 + 2Ke(q_i) \le \beta^{\hat{k}}\alpha\theta(\hat{x})/8 < 0 . \qquad (2.16j)$$

Since (2.16i) together with (2.16j) imply that $\psi^0(x_i) \to -\infty$ as $i \to \infty$, which is impossible, we have a contradiction and our proof is complete. □

We recall that optimal control problems, such as (2.2a) do not necessarily have solutions in U. Similarly, the sequence of controls $u_i(t)$ constructed by Algorithm 2.1 need not have accumulation points in U. This difficulty can be resolved by showing that the conclusions of Theorem 2.2 are valid in the space of relaxed controls (see [Bak.2]). Alternatively, one may resort to arguments involving infimizing sequences, as in [Pol.4].

3. COMPUTATIONAL EXPERIMENTS

We will now describe three computational experiments involving the slewing motion of hollow aluminum tube depicted in Figure 1. The tube is one meter long, has a cross sectional radius of 1.0 cm, and a thickness of 1.6 mm. Attached to one end of the tube is a mass of 1 kg, and attached to the other end is a shaft connected to a motor. The model below was obtained by neglecting small non-linear terms, the coupling between the flexural and extensional vibrations, and by assuming that the acceleration can be controlled, instead of assigning a mass to the shaft and assuming that the torque is controlled. These simplifications were introduced to to reduce the computational burded of solving the optimal control problem. Our aim is to determine the accelaration necessary to rotate the tube and bring it to rest. The maximum accelaration produced by the motor is 5 rads/sec^2. The equations of motion determined by application of the standard Euler-Bernoulli tube with Kelvin-Voigt visco-elastic damping are:

$$mw_{tt}(t\,,x) + CIw_{txxxx}(t\,,x) + EIw_{xxxx}(t\,,x) - m\Omega^2(t)w(t\,,x) = -\frac{m}{1+m/3}u(t)x\,,$$

$$t \in [0\,,T]\,,\quad x \in [0\,,1]\,, \tag{3.1a}$$

with boundary conditions:

$$w(t\,,0) = 0\,,\quad w_x(t\,,0) = 0\,,\quad CIw_{txx}(t\,,1) + EIw_{xx}(t\,,1) = 0\,,\quad t \in [0\,,T]\,, \tag{3.1b}$$

$$M[\Omega^2(t)w(t\,,1) - w_{tt}(t\,,1) - u(t)] + CIw_{txxx}(t\,,1) + EIw_{xxx}(t\,,1) = 0\,,\quad t \in [0\,,T] \tag{3.1c}$$

$$\Theta_t(t) = \Omega(t)\,,\quad t \in [0\,,T]\,, \tag{3.1d}$$

$$\Omega_t(t) = u(t)\,,\quad t \in [0\,,T]\,, \tag{3.1e}$$

where $w(t\,,x)$ is the displacement of the tube from the *shadow tube* (which remains undeformed during the motion) due to bending as a function of time and distance along the tube; $u(t)$ is the acceleration produced by the motor, and $\Omega(t)$ is the resulting angular velocity (in radians per second). We will denote by $\Theta(t)$ the angular displacement of the rigid body (in radians). The values for the parameters in (3.1a) - (3.1c) are: $m = .2815$ kg/m, $C = 6.89\times10^7$ pascals/sec., $E = 6.89\times10^9$ pascals, $I = 1.005\times10^{-8}m^4$, $M = 1.00$ kg. These values are from the CRC Handbook of Material Science. The tube is very lightly damped (0.1 per cent).

When time is normalized to lie in the interval [0 , 1], the dynamics become:

$$m w_{tt}(t\ ,x) + TCI w_{txxxx}(t\ ,x) + T^2 EI w_{xxxx}(t\ ,x) - T^2 m\,\Omega^2(t) w(t\ ,x) \;=\; -T^2 \frac{m}{1+m/3} u(t)x\ ,$$

$$t \in [0\,,1]\,,\quad x \in [0\,,1]\,, \tag{3.2a}$$

with boundary conditions:

$$w(t\ ,0) \;=\; 0\,,\quad w_x(t\ ,0) \;=\; 0\,,\quad CI w_{txx}(t\ ,1) + TEI w_{xx}(t\ ,1) \;=\; 0\,,\quad t \in [0\,,1]\,, \tag{3.2b}$$

$$M\,[T^2\Omega^2(t) w(t\ ,1) - w_{tt}(t\ ,1) - T^2 u(t)] + TCI w_{txxx}(t\ ,1) + T^2 EI w_{xxx}(t\ ,1) \;=\; 0\,,\quad t \in [0\,,1]\ , \tag{3.2c}$$

$$\Theta_t(t) \;=\; T\,\Omega(T)\,,\quad t \in [0\,,1]\,, \tag{3.2d}$$

$$\Omega_t(t) \;=\; Tu(t)\quad t \in [0\,,1]\,. \tag{3.2e}$$

In [Bak.1] these dynamics were transcribed into the standard form (2.1a) as follows. Let $z(t) \in X \triangleq L_2([0\ ,\ 1]) \times \mathbb{R}$, and $F : X \times \mathbb{R}^2 \to X$ be defined by

$$z(t) \triangleq \begin{bmatrix} w(t\ ,x) \\ w(1\ ,x) \end{bmatrix}\ ,\quad F(z(t)\,,\alpha(t)\,,\Omega(t)) \triangleq \begin{bmatrix} \Omega^2(t) w(t\ ,x) - \alpha(t)x \\ \Omega^2(t) w(t\ ,1) - \alpha(t) \end{bmatrix}\ . \tag{3.2f}$$

We define A with domain of A, $D(A)$:

$$D(A) \triangleq \{\bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mid \frac{\partial^4}{\partial x^4} w_1 \in L_2([0\,,1])\,,$$

$$w_1(0) = \frac{\partial}{\partial x} w_1(0) = \frac{\partial^2}{\partial x^2} w_1(1) = 0\,, w_1(1) = w_2\,\}\ , \tag{3.2g}$$

$$A : D(A) \to X \text{ is such that } A \begin{bmatrix} w(t\ ,x) \\ w(t\ ,1) \end{bmatrix} \;=\; \begin{bmatrix} \dfrac{\mathrm{EI}}{\mathrm{m}} \dfrac{\partial^4}{\partial x^4} w(t\ ,\ x) \\ \dfrac{\mathrm{EI}}{\mathrm{M}} \dfrac{\partial^3}{\partial x^3} w(t\ ,\ 1) \end{bmatrix}\ . \tag{3.2h}$$

Referring to Appendix II in [Bak.1], we see that A is a generator of a compact semigroup and that F is a bounded operator.

Our computational experiments involve the following three optimal slewing problems:

P_1: Minimize the time required to rotate the tube 45^o, from rest[4] to rest, subject to the given acceleration constraint.

P_2: Minimize the total energy required to rotate the tube 45^o, from rest to rest, subject to the given acceleration constraint and the maneuver time not exceeding a given bound.

P_3: Minimize the time required to rotate the tube 45^o, from rest to rest, subject to the given acceleration constraint and an upper bound on the potential energy due to deformation of the tube throughout the entire maneuver.

To express the above problems P_1, P_2, and P_3 in the form (2.10b), we make the following definitions: We will be making use of the following functions. First, let T denote the final time. Then we define

$$\psi^1(u\ , T) \triangleq T\ . \tag{3.3}$$

The input energy is defined as the integral of the square of the input; hence we define

$$\psi^2(u\ , T) \triangleq \int_0^1 u(t)^2 dt\ . \tag{3.4}$$

Next we define

$$\psi^3(u\ , T) \triangleq (\Theta(1) - \pi/4)^2 \tag{3.5}$$

to be the square of the angular error at the final time.

Next, turning to the "from rest to rest" requirement, rigid body energy at final time is proportional to the square of the angular velocity. Hence we define

$$\psi^4(u\ , T) \triangleq \Omega(1)^2\ . \tag{3.6}$$

The kinetic energy due to vibration of the tube at time t is given by

$$K(t\ , u) \triangleq \frac{m}{2}\int_0^1 w_t(t\ , x)^2 dx\ , \tag{3.7}$$

and the potential energy due to deformation of the tube at time t is given by

[4] We say that *the tube is at rest* when the total energy of the tube is zero. This energy is composed of the energy due to rigid body motion and energy due to vibration and deformation.

$$P(t, u) \triangleq \frac{EI}{2}\int_0^1 w_{xx}(t, x)^2 dx \,. \tag{3.8}$$

Hence we define

$$\psi^5(u, T) \triangleq K(1, u), \tag{3.9}$$

$$\psi^6(u, T) \triangleq P(1, u). \tag{3.10}$$

The tube is at rest if $\psi^4(u, T) = \psi^5(u, T) = \psi^6(u, T) = 0$.

For problem $\mathbf{P}_3$, we require that the potential energy due to the tube deformation be within a specified range throughout the entire maneuver. This constraint has the form $P(t, u) \le f(t)$ for all $t \in [0, 1]$, where $f(\cdot)$ is a given positive bound function. This is a *state-space constraint* which we have elected to replace by the equivalent requirement $\psi^7(u, T) \le 0$, where

$$\psi^7(u, T) \triangleq \int_0^1 [\max\{P(t, u) - f(t), 0\}]^2 dt \,. \tag{3.10}$$

Since $P(t, u)$ is continuous, $\psi^7(u, T) = 0$ if and only if $P(t, u) \le f(t)$ for all $t \in [0, T]$. Transformations such as (3.10) must be used with great care because for any feasible pair (u, T), $\psi^7(u, T) = 0$ and $\nabla\psi^7(u, T) = 0$, and hence $\theta(u, T) = 0$, which causes our algorithm to stop up at such a pair. However, the problems caused by this violation can be circumvented by initializing the algorithm with an infeasible point, keeping the parameter γ small, in Algorithm 2.1, and introducing an ε into the function definitions, as shown below.

It can be shown that the functions $\psi^j : \mathbf{U} \times T \to \mathbb{R}$ are continuously differentiable (in the L_∞ topology) in u and t for all $j \in \{1, 2, \ldots, 7\}$. To conform with the format of problem (2.10b), we relax each of the equality constraints by a small amount. The relaxation can be be chosen to be sufficiently small so as not to matter from a practical point of view. The three problems now acquire the following mathematical form[5], where $\mathbf{U} = \{u \in L_\infty[0, 1] \mid |u(t)| \le 1 \ \forall t \in [0, 1]\}$ and $\mathbf{T} = [T_0, T_f]$, with $T_0 > 0$ very small and $T_f < \infty$ very large.

[5] Note that we find it convenient at this point to abandon the convention that the cost function is $\psi^0(\cdot, \cdot)$ as well as the linear numbering of the constraints.

$$\mathbf{P}_1: \ \min\{\psi^1(u,T) \mid \psi^3(u,T)-\varepsilon \le 0,\ \psi^4(u,T)-\varepsilon \le 0,\ \psi^5(u,T)-\varepsilon \le 0,\ \psi^6(u,T)-\varepsilon \le 0,\ (u,T) \in \mathbf{U}\times\mathbf{T}\}. \tag{3.11a}$$

$$\mathbf{P}_2: \ \min\{\psi^2(u,T) \mid \psi^1(u,T)-T_f \le 0,\ \psi^3(u,T)-\varepsilon \le 0,\ \psi^4(u,T)-\varepsilon \le 0,\ \psi^5(u,T)-\varepsilon \le 0,\ \psi^6(u,T)-\varepsilon \le 0,\ (u,T) \in \mathbf{U}\times\mathbf{T}\} \tag{3.11b}$$

$$\mathbf{P}_3: \ \min\{\psi^1(u,T) \mid \psi^3(u,T)-\varepsilon \le 0,\ \psi^4(u,T)-\varepsilon \le 0,\ \psi^5(u,T)-\varepsilon \le 0,\ \psi^6(u,T)-\varepsilon \le 0,\ \psi^7(u,T)-\varepsilon \le 0,\ (u,T) \in \mathbf{U}\times\mathbf{T}\}. \tag{3.11c}$$

In our experiments, we set $\varepsilon = 10^{-4}$. Thus, with this relaxation, we are requiring that the final value of the angle Θ be in the interval $[45^o - 0.5^o, 45^o + 0.5^o]$. We assume that because of model simplifications and other inevitable modelling errors, a linear feedback system would be used to assure final pointing accuracy.

In the computational experiments reported in this paper, the term $\Omega^2(t)$ was neglected in equation (3.1a) - (3.1c). Similar results were obtained in computational experiments in which the term $\Omega^2(t)$ was kept. We used a cubic Hermit spline implementation of the Finite Element Method for spatial discretization and Newmark's *beta*–method for temporal discretization of both responses and sensitivities[6]. This approach is quite stable and gives accurate simulations. The results of our experiments are shown in Figs. 2 -11.

Problem $\mathbf{P}_1$:

For simplicity, we choose the zero function as initial control and 2 for an initial value for the maneuver time. Figure 2 is a graph of the control after 150 iterations. The number of time steps is 256 and the number of finite elements is 48. Figure 3a is a graph of $\psi_{q_s,q_t}(u,T)$ as a function of the iteration number. Figure 3b shows $\psi_{q_s,q_t}(u,T)$ for the first 15 iterations. The initial discretization is 32 time steps and 6 finite elements. The discretization is refined at iterations 67, 99, and 123. After precision refinement, algorithm finds a a control $u \in \mathbf{U}_{q_t}$ and final time $T \in \mathbf{T}$ such that $\psi_{q_s,q_t}(u,T) < 0$ in only a few additional iterations. Note that each time precision of discretization was increased, the value of $\psi_{q_s,q_t}(u_i,T_i)$ increases. This is due to improvement in the accuracy of the evaluation of the partial differential equation. This increase in constraint violation $\psi_{q_s,q_t}(u_i,T_i)$

[6] See [Bak.1, Chap. 8] for implementation details.

decreases each time the discretization is increased and we can show that in the limit the increase is zero. Figure 4 is the graph of the cost as a function of iteration number. Figure 5 is the graph of $w(t\ ,1)$, the displacement of the tip of the tube, from the *shadow tube*, as a function of time. There is a maximum displacement of the tip of about 5 mm. This is within the range of validity of the Euler-Bernoulli model. The tip displacement is large between 0.36 seconds and 0.437 seconds. Figure 6 is a profile of the tube deformation, $w(t\ ,x)$ (see Figure 1), during this interval. The total time for the entire maneuver is 0.7886 seconds.

Problem P_2:

Formulating the slewing problem as a minimum time problem has two drawbacks. First, the solution to the problem is a bang-bang control (Figure 2). Bang-bang controls may be undesirable because they may cause damage to the equipment. Furthermore, bang-bang controls tend to excite the high frequency modes of the system. High frequency modes are less well modeled by system (3.1a) - (3.1c), and it is therefore best not to excite them. Second, the simple minimum time formulation does not take into account the amount of energy expended in performing the maneuver. In certain applications, the total energy available may be limited, while the total time of the slewing motion is less critical. Fortunately, both of the problems arising from minimum time control can be mitigated by reformulating the problem. We minimize the total input energy while constraining the final time to be less that a specified amount. Figure 7 is the graph of the control produced by minimizing the total input energy while constraining the final time to be less than 0.800 seconds. The resulting final time is 0.800 seconds. This is an increase of only 1.4 percent in the final time. The control has become much smoother, and the total energy is reduced from 19.15 to 15.72, a reduction of 18 percent. Figure 8 is the graph of the control for final time being 1.00 second. This is an increase of 27 percent in time over the minimum time case, but the total energy is reduced to 7.27, a decrease of 62 percent.

Problem P_3:

In Figure 9, curve A is the graph of the potential energy of the tube as a function of time for the control generated in solving the minimum time problem $\mathbf{P_1}$. In problem $\mathbf{P_3}$, we have the additional requirement to keep the potential energy, which is a measure of the total tube deformation, below the parabola (B) for all time. Figure 10 shows the minimum-time bang-bang control for problem $\mathbf{P_3}$. The optimal final time for this case is 0.8177 seconds, an increase of 3.7 percent over the solution of problem $\mathbf{P_1}$. Figure 11 shows the potential energy curve for the optimal control (Figure 10).

4. CONCLUSION

We have presented an implementable optimal control algorithm which can be used for solving optimal slewing problems with control and state space constraints, and either ODE or PDE dynamics. Our computational results show that the algorithm is effective in solving reasonably difficult problems.

5. ACKNOWLEDGEMENT

We wish to thank Profs. Loc Quoc Vu and Robert Taylor for their assistance in developing the model for the example; Prof. Faryar Jabbari for useful discussions on unbounded operators in PDE models; and the anonymous referees for pointing out some shortcomings in the first version of our paper.

6. REFERENCES

[Ara.1] Araya-Schulz, R., *Control of a Large Space Structure Using a Distributed Parameter* Model, Ph.D. Dissertation, University of California, Los Angeles, 1986.

[Arm.1] Armijo, L., "Minimization of functions having Lipschitz continuous first partial derivatives", *Pacific Journal of Mathematics* Vol 16, pp 1-3, 1966.

[Bak.1] Baker, T. E., "Algorithms for Optimal Control of Systems Described by Partial and Ordinary Differential Equations", (Ph. D. Dissertation, University of California, Berkeley, June 1988) Memo No. UCB/ERL M88/45, *University of California, Berkeley, Electronics Research Laboratory* 21 June, 1988.

[Bak.2] Baker, T. E., and E. Polak, "On the Optimal Control of Systems Described by Evolution Equations", *University of California, Electronics Research Laboratory*, Memo UCB/ERL M89/113, 27 September 1989.

[Bal.1] Balakrishnan, A. V., "A Computational Approach to the Maximum Principle", *J. Comput. System Sci.*, Vol. 5, pp. 163-191, 1971. ip "[Bal.2]" 8 Balakrishnan, A. V., *Applied Functional Analysis*, Springer-Verlag, 1981.

[Ben.1] Ben-Asher, J., Burns, J. A. and Cliff, E. M., "Time Optimal Slewing of Flexible Spacecraft", Proc. *26th IEEE Conference on Decision and Control*, pp. 524-528, 1987.

[Bur.1] Burns, J. A., Miller, R. E., and Cliff, E. M., "Control of a Viscoelastic Shaft with Attached

Tip Mass", Proc. *26th Conference on Decision and Control,* pp. 997-999, 1987.

[Ber.1] Berge, C., *Topological Spaces* Macmillan Co., N.Y., 1963.

[Chu.1] Chun, H. M., "Large-Angle Slewing Maneuvers for Flexible Spacecraft", Ph. D. Dissertation, M.I.T., 1986.

[Cul.1] Cullum, J., "Discrete Approximations to Continuous Optimal Control Problems," *Siam J. Control,* Vol. 7, No. 1, February, 1969.

[Cul.2] Cullum, J., "An Explicit Procedure for Discretizing Continuous Optimal Control Problems," *JOTA,* Vol. 8, pp. 15-34.

[Flo.1] Floyd, M. A., Brown, M. E., Turner, J. D. and Vandervelde, W. E., "Implementation of a Minimum Time and Fuel On/Off Thruster Control System for Flexible Spacecraft," *Journal of Astronautical Sciences,* To Appear.

[Gib.1] Gibson, J. S., "An Analysis of Optimal Modal Regulation: Convergence and Stability," *SIAM J. Control and Optimization,* Vol. 19, No. 5, September 1981.

[Gib.2] Gibson, J.S., Mingori, D. L., Adamian, A., and Jabbari, F., "Approximation of Optimal Infinite Dimensional Compensators for Flexible Structures," Proc. *Workshop on Identification and Control of Flexible Space Structures,* Vol. II, pp. 201-218, April 1985.

[Gib.3] Gibson, J. S., "The Riccati Integral Equations for Optimal Control Problems on Hilbert Spaces", *SIAM J. Control and Optimization,* Vol. 17, No. 4, July 1979.

[Jun.1] Junkins, J. L. and Turner, J. D., *Optimal Spacecraft Rotational Maneuvers,* Studies in Astronautics, 3, Elsevier, Amsterdam, 1986.

[Hua.1] Huard, P., "Programmation Mathematique Convexe," *Rev. Francaise Inf. Rech. Oper.,* Vol. 7, pp. 43-59, 1968.

[Mei.1] Meirovitch, L., *Computational Methods in Structural Dynamics,* The Netherlands: Sijthoff and Noordhoff Co., 1980.

[NAS.1] Proceedings, *First NASA / DOD CSI Technology Conference,* OMNI International Hotel, Norfolk, Virginia November, 1986

[Lev.1] Levitin, E. S., and Polyak, B. T., "Constrained Minimization Methods", *USSR Comp. Math. and Math. Phys.,* Vol.6, No.5, pp 27-29, 1966.

[New.1] Newmark, N. M., "A Method of Computation for Structural Dynamics," *American Society of Civil Engineers*, Vol. 127, pp. 1406-1435, 1962.

[Paz.1] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[Pol.1] Polak, E., *Computational Methods in Optimization: A Unified Approach*, Academic Press, 329 pages, 1971.

[Pol.2] Polak, E. and Mayne, D. Q., "A Feasible Directions Algorithm for Control Problems with Control and Terminal Inequality Constraints," *IEEE Trans. on Automatic Control*, Vol. AC-22, pp. 741-751, 1977.

[Pol.3] Polak, E., Trahan, R. and Mayne, D. Q., "Combined Phase I - Phase II Methods of Feasible Directions," *Math. Programming*, Vol. 17, No. 1, pp. 32-61, 1979.

[Pol.4] Polak, E. and Wardi, Y. Y., "A Study of Minimizing Sequences," *SIAM J. Contr. and Optim.*, Vol. 22, No. 4, pp. 599-609, 1984.

[Pol.5] Polak, E., "On the Mathematical Foundations of Nondifferentiable Optimization in Engineering Design", *SIAM Review*, pp. 21-91, March 1987.

[Sch.1] Schmidt, W. F., "Adaptive Step Size Selection for Use with the Continuation Method", *International Journal for Numerical Methods in Engineering*, Vol. 12, 1978.

[Sin.1] Sing G., Kabamba, P. T., and McLamroch, N. H. "Planar, time-Optimal Rest to Rest Slewing Maneuvers of Flexible Spacecraft", *J. Guidance, Control, Dyn.*, Vol.12, pp 71-81, 1989.

[Str.1] Strang, G. and Fix, G., *An Analysis of the Finite Element Method*, Prentice Hall, Englewood Cliffs, N. J., 1973.

[Tay.1] Taylor, L. W. and Balakrishnan, A. V., "A Mathematical Problem and a Spacecraft Control Laboratory Experiment (SCOLE) Used to Evaluate Control Laws for Flexible Spacecraft" *NASA/IEEE Design Challenge*, January 1984. (Revised June and December 1984)

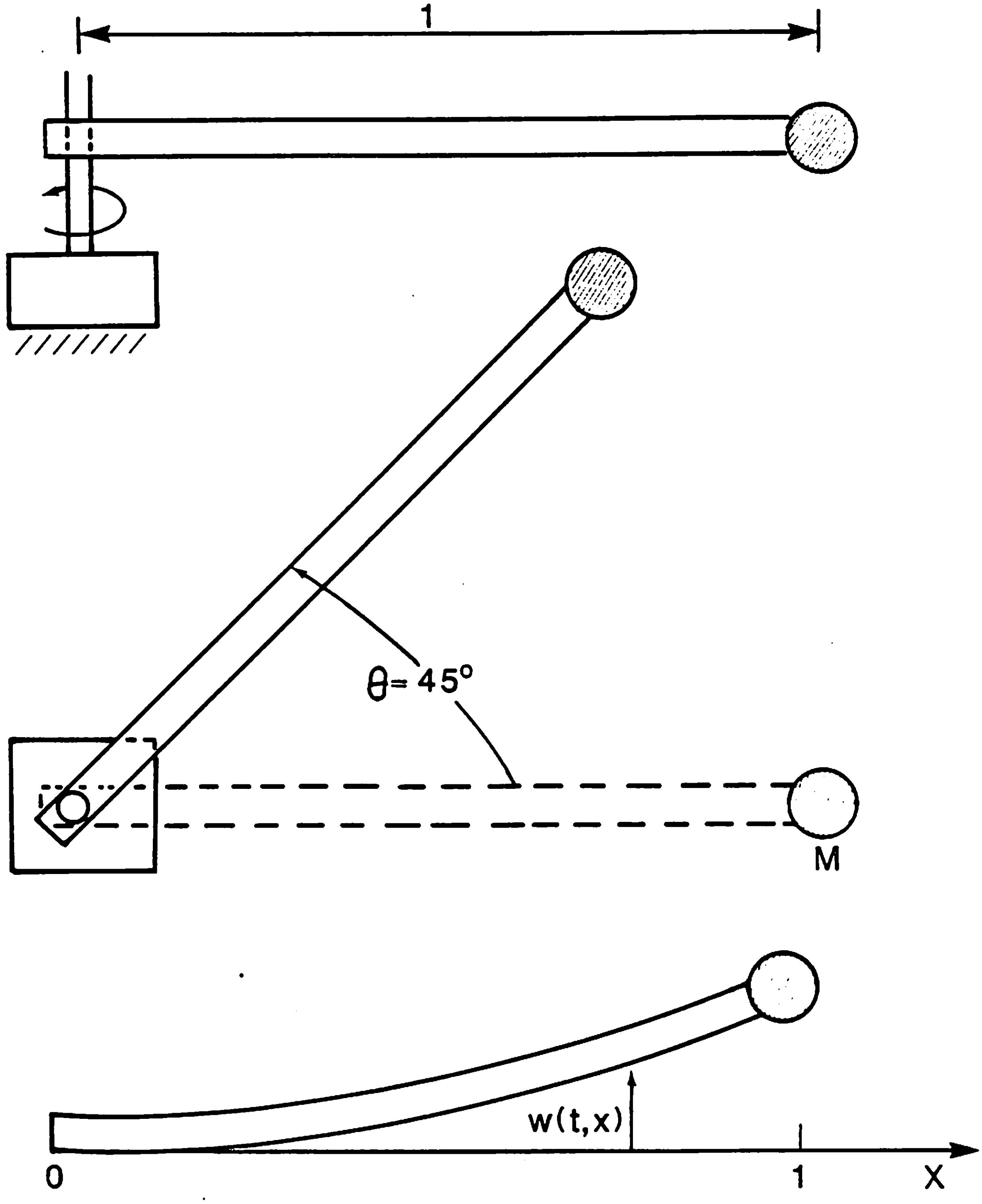

Figure 1. Configuration of Slewing Experiment

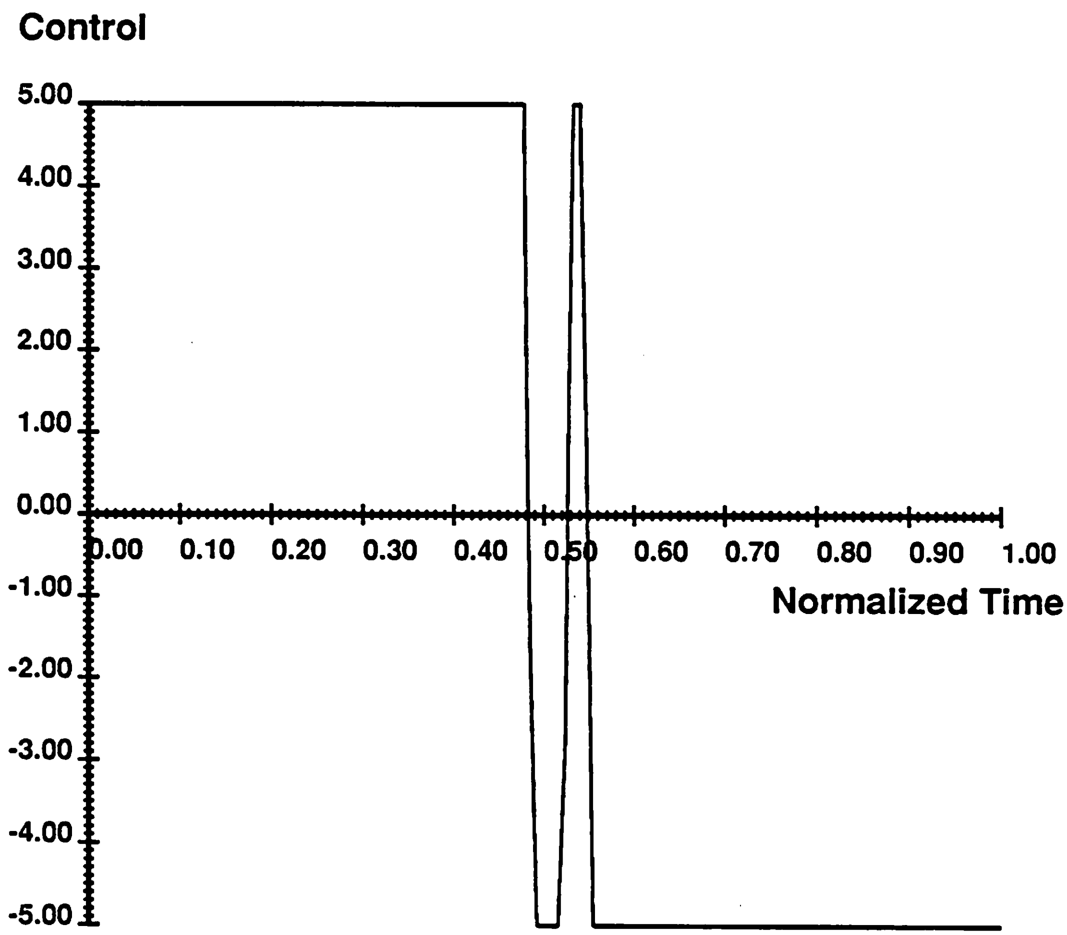

Figure 2. Problem 1: Optimal Control

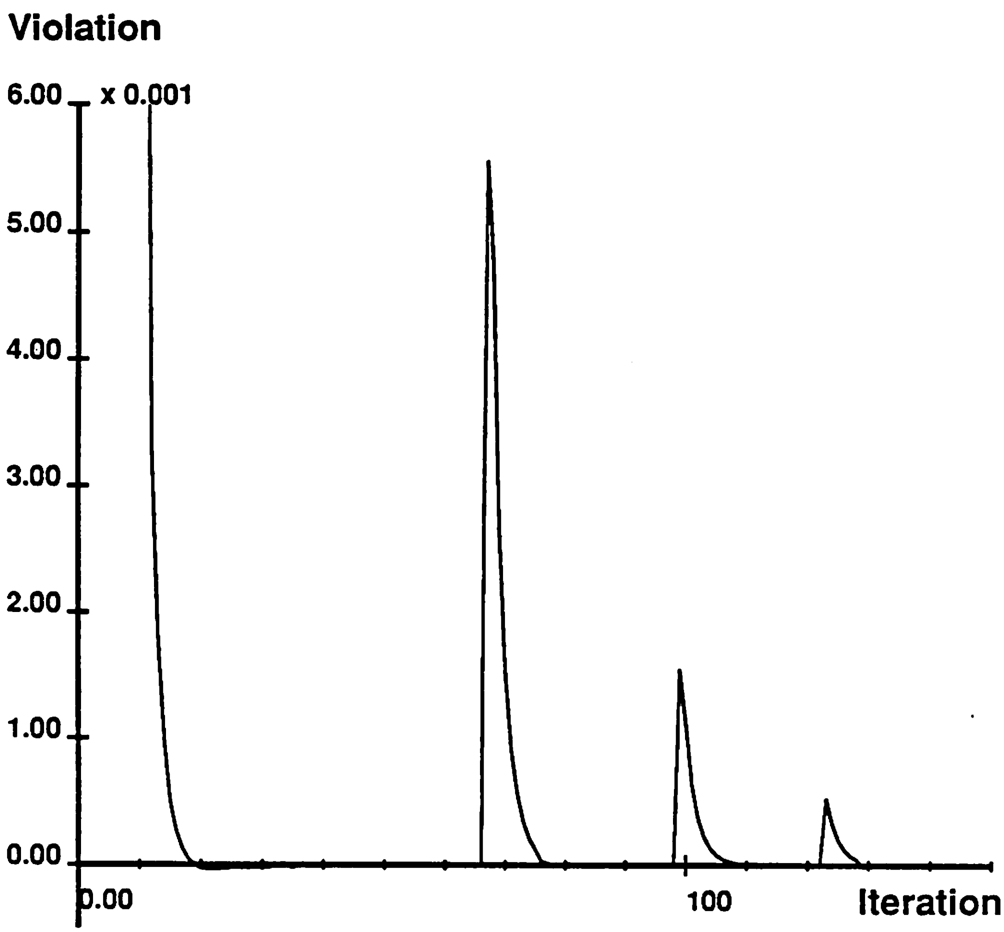

Figure 3a. Problem 1: Constraint Violation v/s Iteration

Figure 3b. Problem 1: Constraint Violation v/s Iteration

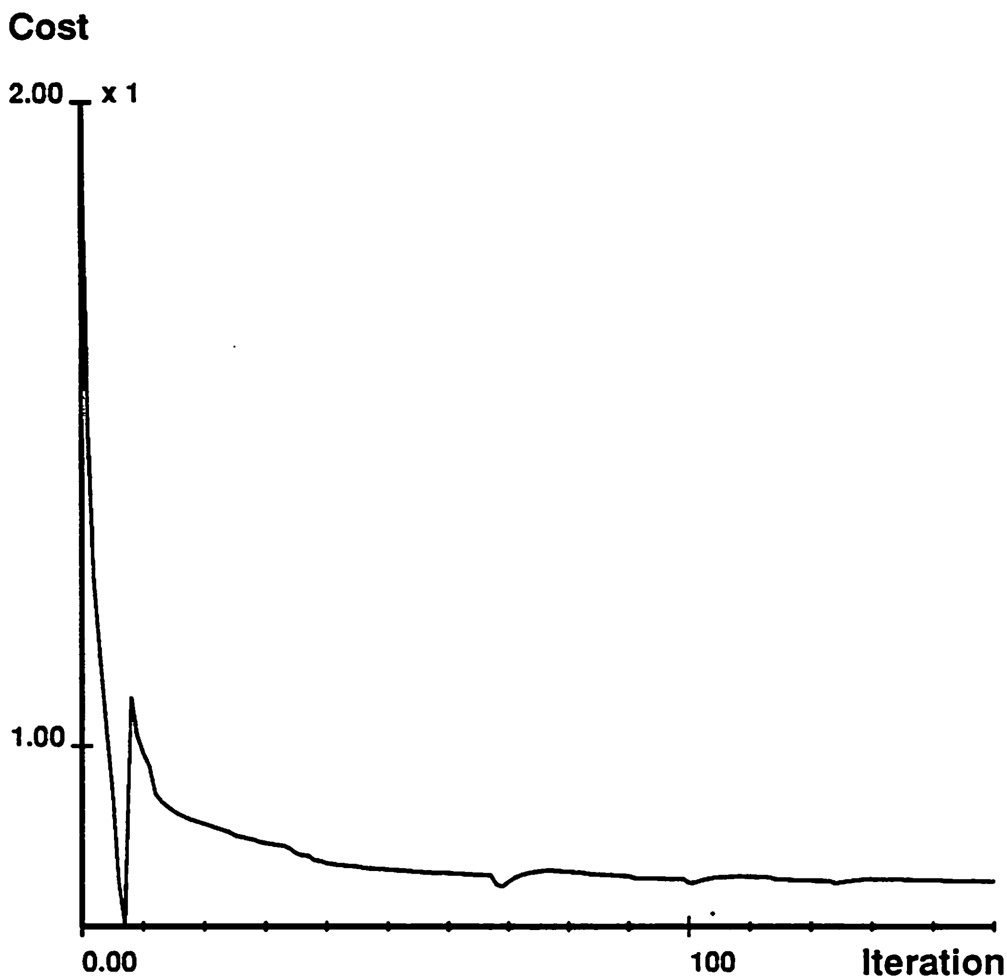

Figure 4. Problem 1: Cost v/s Iteration

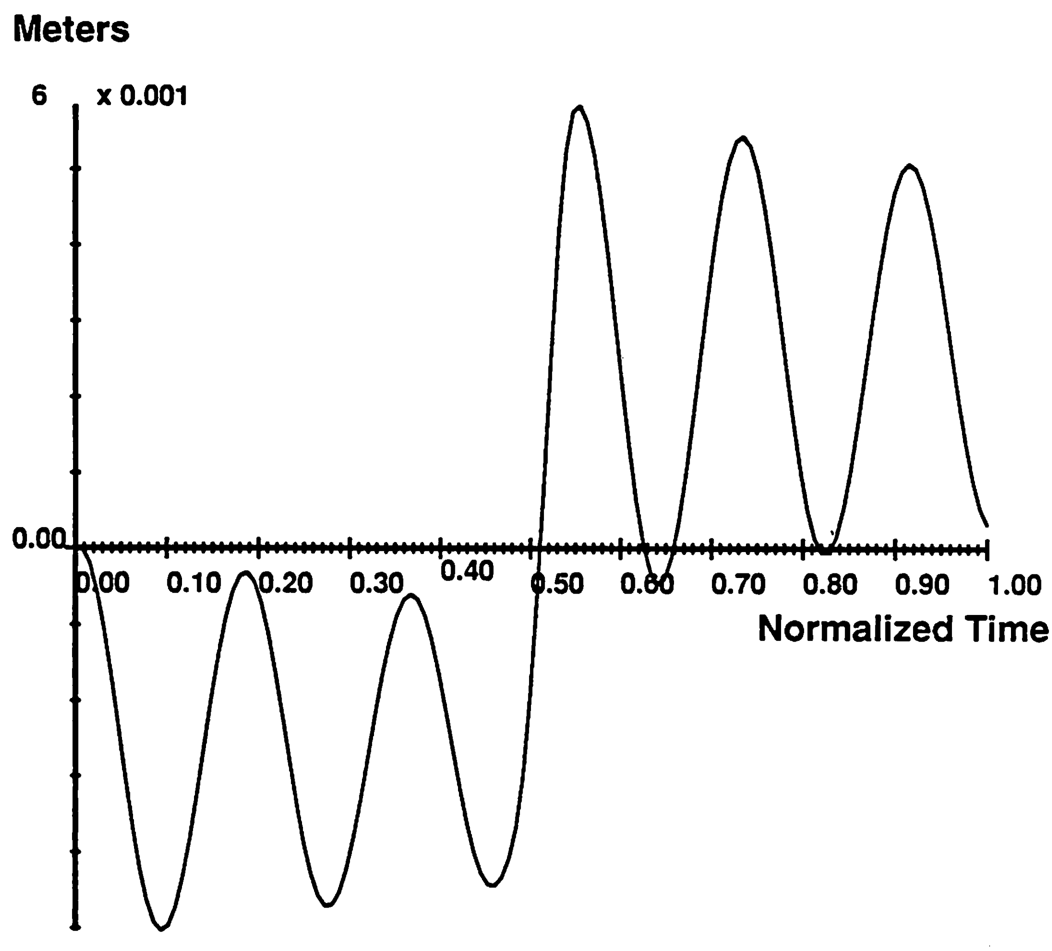

Figure 5. Displacement of Tip of Tube

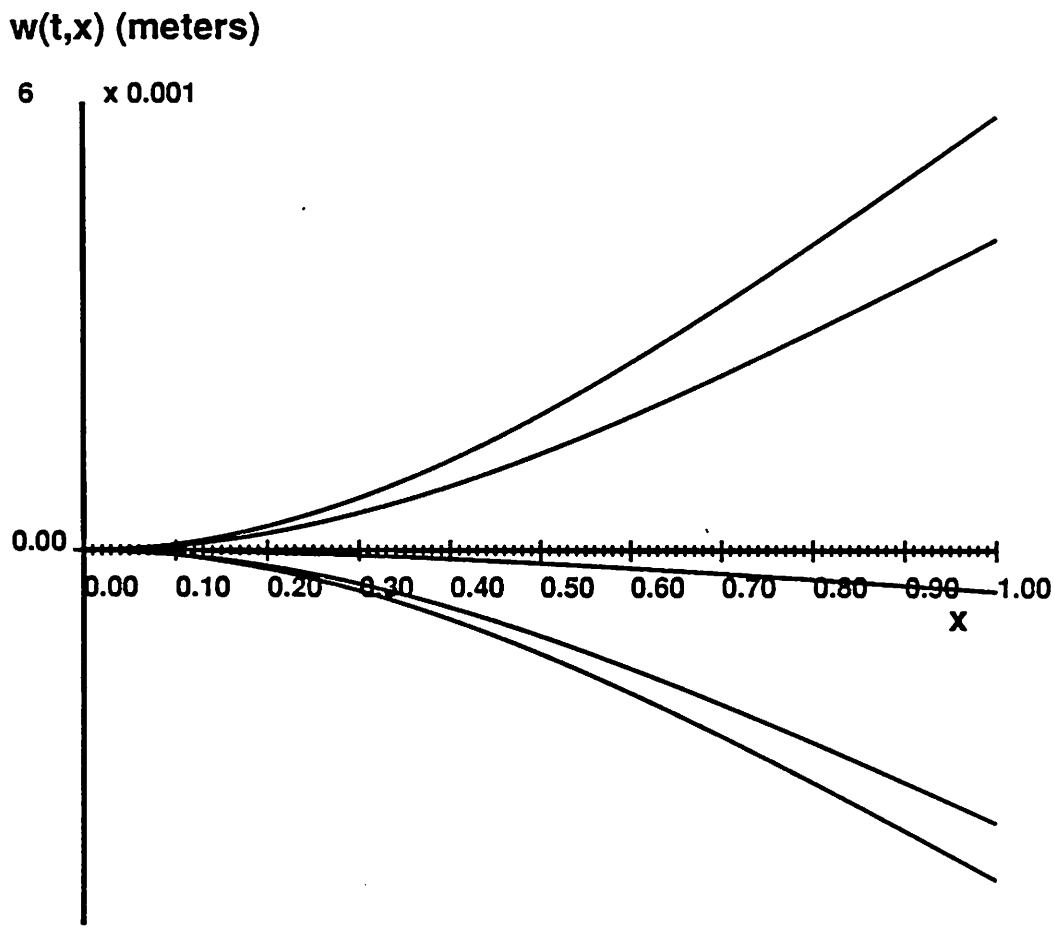

Figure 6. Variation of Beam Profile with Time

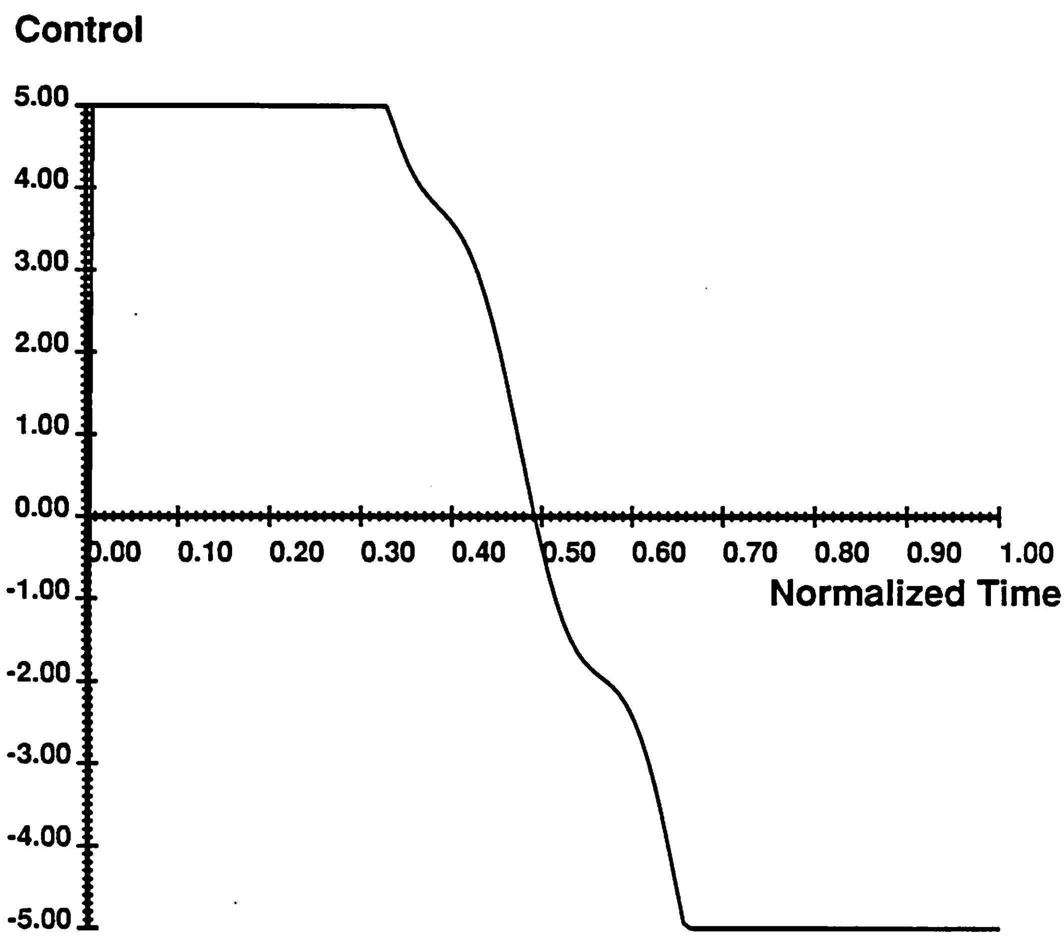

Figure 7. Problem 2: Time of Maneuver = 0.800 seconds

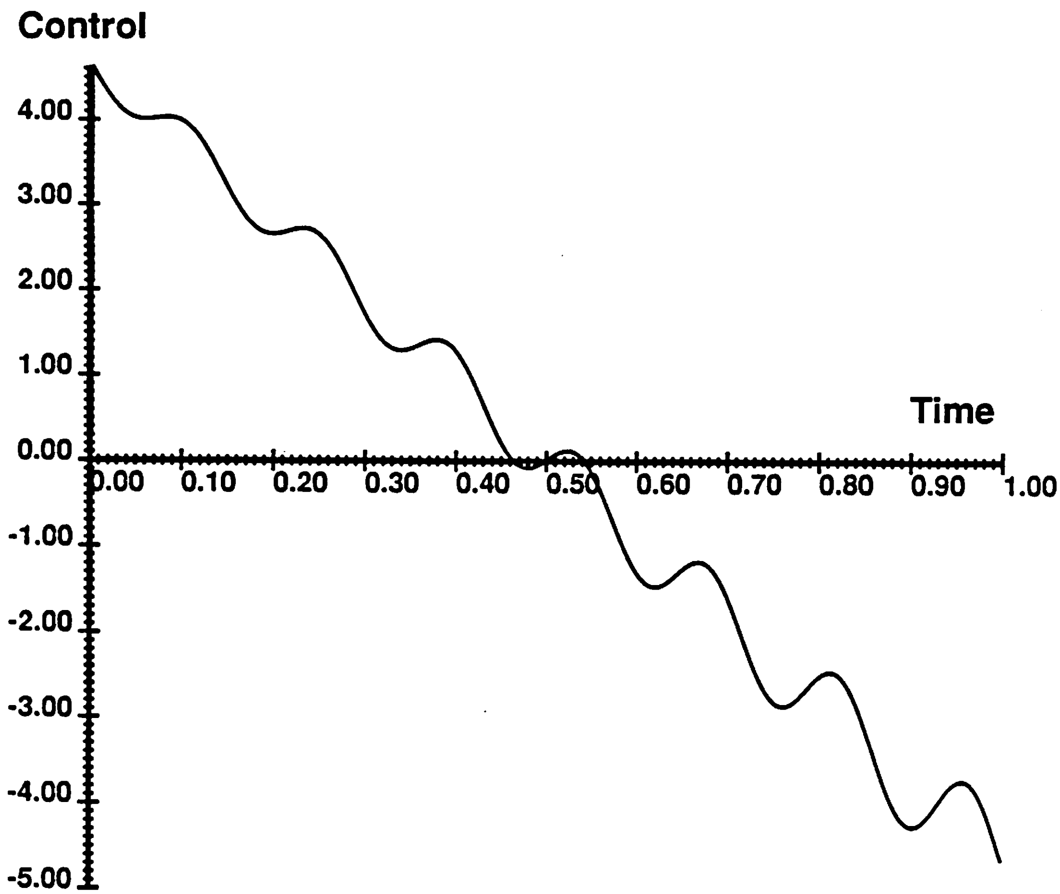

Figure 8. Problem 2: Time of Maneuver = 1.000 seconds

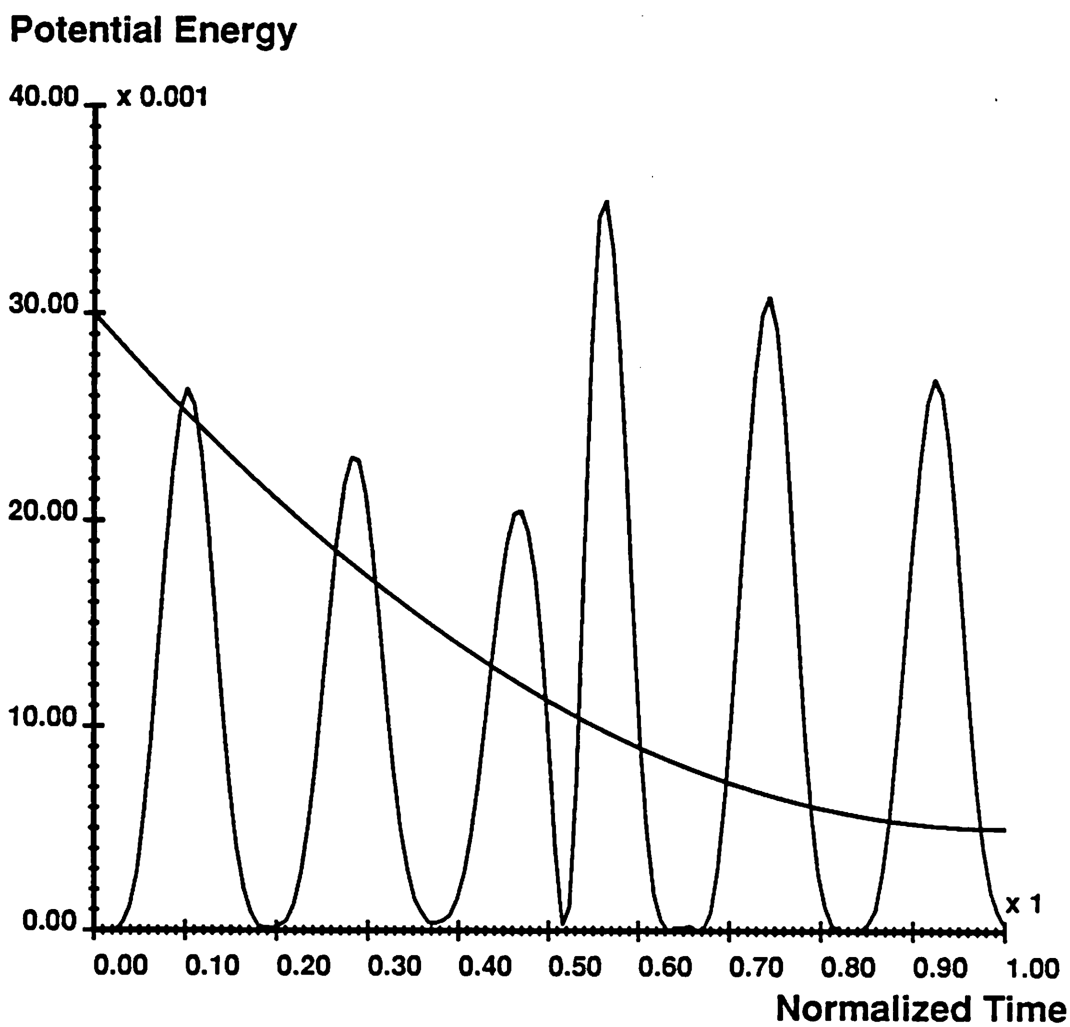

Figure 9. Problem 1 Potential Energy

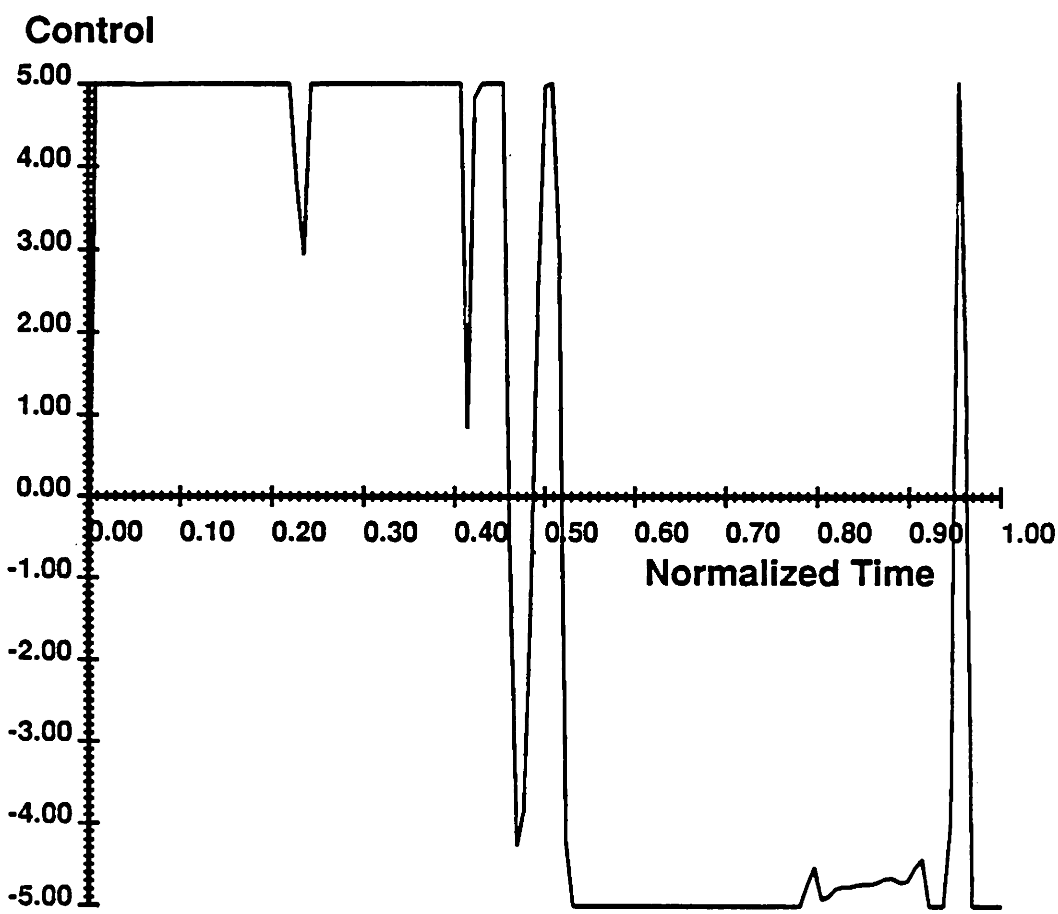

Figure 10. Optimal Control for Problem 3

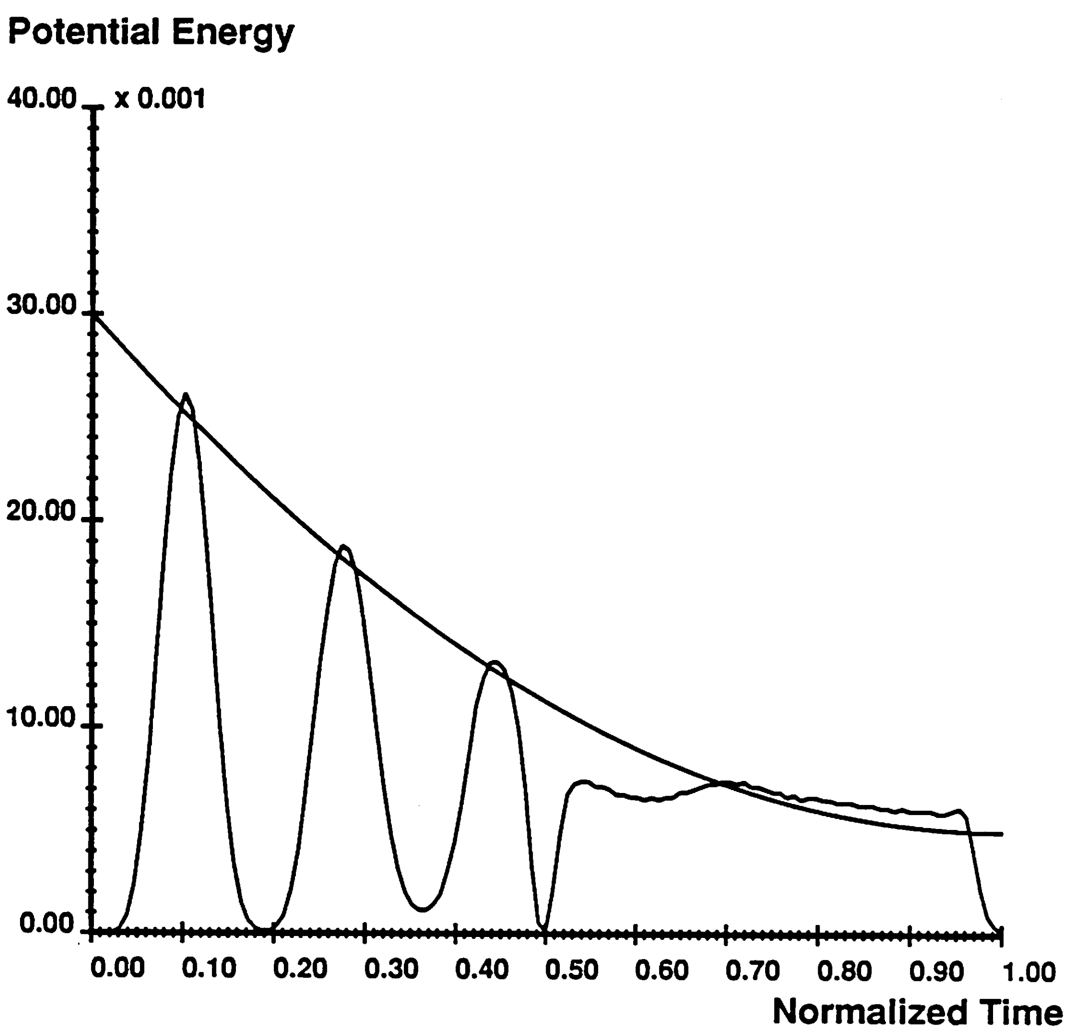

Figure 11. Problem 3: Potential Energy