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Nonlinear Control via Approximate Input-Output Linearization: the Ball and Beam Example †

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Abstract. In this paper, we study approximate input-output linearization of SISO nonlinear systems which fail to have relative degree in the sense of Byrnes and Isidori. This work is in the same spirit as the earlier work of Krener on approximate full state linearization by state feedback. The general theory presented in this paper is motivated through it's application to a common undergraduate control laboratory experiment—the ball and beam system—where it is shown to be superior to the standard Jacobian linearization.

Keywords. Nonlinear control, input-output linearization, approximate linearization.

1 Introduction

The past few years have seen the maturation of the use of differential geometric techniques in understanding input-output and full state linearization of nonlinear systems, normal forms and zero dynamics. An elegant discussion of these results is in the work of Isidori [4]. The conditions for the existence of full state linearizable nonlinear systems or for that matter systems which are input-output linearizable are non-generic and it is of obvious interest to extend the results to situations where these conditions fail but do so only *slightly*. Such a program was begun by Krener in [6], who gave conditions for *approximate* full state linearization of nonlinear multi-input systems. In this paper we take this program one step forward by discussing approximate input-output linearization of single input single output systems which fail to have relative degree in the sense of Byrnes and Isidori [4]. Though in the same spirit as [6], it is different in detail in that the control objective is tracking: i.e., a prescribed output function is required to follow a given specific function of time. Such applications are prototypical in the flight control of aircraft where trajectory following rather than set point regulation are paramount to performance.

Approximate linearization of nonlinear systems has, of course, a lengthy history, starting with Jacobian linearizations and continuing with extended linearization [8] and pseudo-linearization [7]. Our approximate linearization is different in spirit in that it is specifically geared for tracking problems rather than the regulation problems that the extended or pseudo linearization techniques

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appear to be useful for. Also, our approximation is not an approximation by a linear system or family of linear systems but rather by a single input-output linearizable nonlinear system.

An outline of the paper is as follows: In Section 2, we start with an example drawn from undergraduate control laboratories, the ball and beam experiment, and use it to study the failure of exact input-output linearization and the latitude available in our proposed technique to do approximate input-output linearization. We also compare the linearizations with the Jacobian linearized system. In Section 3, we present the general method motivated by Section 2 to define *robust relative degree* and approximate input-output linearization of SISO systems. Section 4 has some concluding remarks.

2 The Ball and Beam Example

Consider a version of the familiar ball and beam experiment found in many undergraduate control laboratories (see Figure 1). In this setup, the beam is symmetric and is made to rotate in a vertical plane by applying a torque at the point of rotation (the center). Rather than have the ball roll on top of the beam as usual, we restrict the ball to frictionless sliding along the beam (as a bead along a wire). Note that this allows for complete rotations and arbitrary angular accelerations of the beam without the ball losing contact with the beam. We shall be interested in controlling the position of the ball along the beam. However, in contrast to the usual set-point problem, we would like the ball to track an arbitrary trajectory.

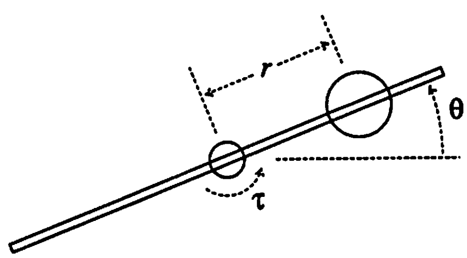

Figure 1: The ball and beam system.

In this section, we first derive the equations of motion for the ball and beam system. Then, we try to apply the techniques of *input-output linearization* and *full state linearization* to develop a control law for the system and demonstrate the shortcomings of these methods as they fail on this simple nonlinear system. Finally, we demonstrate a method of control law synthesis based on *approximate input-output linearization* and compare the performance of two control laws derived using differing approximations with that derived from the standard Jacobian approximation.

2.1 Dynamics

Consider the ball and beam system depicted in Figure 1. Let the moment of inertia of the beam be J, the mass of the ball be M, and the acceleration of gravity be G.

Choose, as generalized coordinates for this system, the angle, θ, of the beam and the position,

r, of the ball. Then, the Lagrangian equations of motion are given by

$$\begin{aligned} 0 &= \ddot{r} + G\sin\theta - r\dot{\theta}^2 \\ \tau &= (Mr^2 + J)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + MGr\cos\theta \end{aligned} \tag{2.1}$$

where τ is the torque applied to the beam and there is no force applied to the ball. Using the invertible transformation

$$\tau = 2Mr\dot{r}\dot{\theta} + MGr\cos\theta + (Mr^2 + J)u \tag{2.2}$$

to define a new input, u, the system can be written in state space form as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \underbrace{\begin{bmatrix} x_2 \\ x_1 x_4^2 - G\sin x_3 \\ x_4 \\ 0 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u \\ y &= \underbrace{x_1}_{h(x)} \end{aligned} \tag{2.3}$$

where $x = (x_1, x_2, x_3, x_4)^T := (r, \dot{r}, \theta, \dot{\theta})^T$ is the state and $y = h(x) := r$ is the *output* of the system (i.e., the variable that we want to control). Note that (2.2) is a *nonlinear* input transformation.

2.2 Exact Input-Output Linearization

We are interested in making the system output, $y(t)$, track a specified trajectory, $y_d(t)$, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.

To this end, we might try to *exactly* linearize the input-output response of the system. Following the usual procedure, we differentiate the output until the input appears:

$$\begin{aligned} y &= x_1, \\ \dot{y} &= x_2, \\ \ddot{y} &= x_1 x_4^2 - G\sin x_3, \\ y^{(3)} &= \underbrace{x_2 x_4^2 - G x_4 \cos x_3}_{b(x)} + \underbrace{2x_2 x_4}_{a(x)} u. \end{aligned} \tag{2.4}$$

At this point, if the coefficient of u ($a(x)$) were nonzero in the region of interest, we could use a control law of the form

$$u = \frac{1}{a(x)}[-b(x) + v] \tag{2.5}$$

to yield a linear input-output system described by

$$y^{(3)} = v. \tag{2.6}$$

Unfortunately, for the ball and beam, the control coefficient $a(x)$ is zero whenever the angular velocity $x_4 = \dot{\theta}$ or ball position $x_1 = r$ are zero. Therefore, the *relative degree* of the ball and beam system *is not well defined*! This is due to the fact that

$$L_g L_f^2 h(x) = 2x_1 x_4 \tag{2.7}$$

is neither nonzero at $x = 0$ (an equilibrium point of the undriven system) nor is it identically zero on a neighborhood of $x = 0$. This is a characteristic unique to *non*linear systems. Thus, when the system has nonzero angular velocity and nonzero ball position, the input acts one integrator sooner than when the angular velocity is zero.

Thus we conclude the *exact* input-output linearization does not provide a methodology for designing a trajectory tracking controller.

2.3 Full State Linearization

Next we try our hand at fully linearizing the state of this system, that is to say, find a set of coordinates and a feedback law such that the input-to-state behavior of the transformed system is linear. The necessary and sufficient conditions for this were given by Jakubczyk and Respondek [5] and, independently, by Hunt, Su, and Meyer [3].

First we check the dimension of the *controllability distribution*,

$$\text{span}\left\{g \;\; ad_f g \;\; \cdots \;\; ad_f^{n-1} g\right\} \tag{2.8}$$

where $ad_f^i g$ denotes the iterated *Lie bracket* $[f, [f, \cdots [f, g] \cdots]]$. Since, the matrix

$$Q(x) = \begin{bmatrix} 0 & 0 & 2x_1x_4 & 4x_2x_4 + G\cos x_3 \\ 0 & -2x_1x_4 & -2x_2x_4 - G\cos x_3 & -4x_1x_4^3 + 3Gx_4\cos x_3 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.9}$$

has full rank at $x = 0$ ($\det Q(0) = G^2$), it follows that the ball and beam is locally controllable.

The second requirement is not generic. It is required that the distribution

$$\text{span}\left\{g \;\; ad_f g \;\; \cdots \;\; ad_f^{n-2} g\right\} \tag{2.10}$$

be involutive, that is, the Lie bracket of any two vector fields in the distribution should also be contained in the distribution.

Checking the brackets for the ball and beam we find that

$$[g, ad_f^2 g] = (2x_1 \quad -2x_2 \quad 0 \quad 0)^T \tag{2.11}$$

does not lie within the span of the first three columns (vector fields) of (2.9).

Failing this condition, we see that it is not possible to fully linearize the ball and beam system.

2.4 Approximate Input-Output Linearization

In this section, we see that, by appropriate choice of vector fields *close* to the system vector fields, we can design a feedback control law to achieve bounded error output tracking. The control law will, in fact, be the *exact* output tracking control law for an *approximate system* defined by these vector fields.

Ideally, we would like to find a state feedback control law, $u(x) = \alpha(x) + \beta(x)v$, that would transform the ball and beam system into a linear system of the of the form $y^{(4)} = v$. Then,

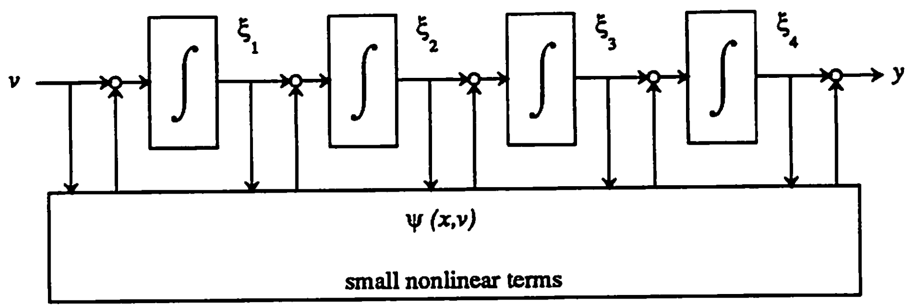

Figure 2: Approximate input-output linearization: a chain of intergrators perturbed by small nonlinear terms.

the system could be made to track an arbitrary (C^4) trajectory, $y_d(t)$, asymptotically by using a tracking control law of the form

$$v = y_d^{(4)}(t) + \alpha_3(y_d^{(3)}(t) - y^{(3)}(x)) + \alpha_2(\ddot{y}_d(t) - \ddot{y}(x)) + \alpha_1(\dot{y}_d(t) - \dot{y}(x)) + \alpha_0(y_d(t) - y(x)) \quad (2.12)$$

where $s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$ is a Hurwitz polynomial. Note that y, $\dot{y}$, etc., are all functions of the state x.

Unfortunately, due to the presence of the centrifugal term $r\dot{\theta}^2 = x_1 x_4^2$, the input-output response of the ball and beam cannot be exactly linearized. Here we try to find an input-output linearizable system that is *close* to the true system. We present two such approximations for the ball and beam system. In each case, we will design a nonlinear change of (state) coordinates, $\xi = \phi(x)$, and a state dependent feedback, $u(x,v) = \alpha(x) + \beta(x)v$, to make the system look like a chain of integrators (i.e., Brunovsky canonical form) perturbed by small higher order terms, $\psi(x,v)$, as depicted in Figure 2. We also compare the performance of these designs to a *linear* controller based on the standard Jacobian approximation to the system.

We then build an approximate tracking control law by designing u so that

$$v = y_d^{(4)}(t) + \alpha_3(y_d^{(3)}(t) - \phi_4(x)) + \alpha_2(\ddot{y}_d(t) - \phi_3(x)) + \alpha_1(\dot{y}_d(t) - \phi_2(x)) + \alpha_0(y_d(t) - \phi_1(x)) \quad (2.13)$$

making the error system into an exponentially stable linear system perturbed by small nonlinear terms.

For each approximation, we present simulation results depicting (a) the output error, $y_d(t) - \phi_1(x(t))$, (b) the neglected nonlinearity, $\psi(x,u)$, (c) the angle of the beam, $\theta(t) = x_3(t)$, and (d) the position of the ball, $r(t) = x_1(t)$, for a desired trajectory of $y_d(t) = R\cos \pi t/30$, with $R = 5$, 10, and 15.

Approximation 1

Let $\xi_1 = \phi_1(x) = h(x)$. Then, along the system trajectories, we have

$$
\begin{aligned}
\dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\phi_2(x)} \\
\dot{\xi}_2 &= \underbrace{-G\sin x_3}_{\xi_3=\phi_3(x)} + \underbrace{x_1 x_4^2}_{\psi_2(x)} \\
\dot{\xi}_3 &= \underbrace{-Gx_4\cos x_3}_{\xi_4=\phi_4(x)} \\
\dot{\xi}_4 &= \underbrace{Gx_4^2\sin x_3}_{b(x)} + \underbrace{(-G\cos x_3)}_{a(x)}u
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 + \psi_2(x) \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= b(x) + a(x)u =: v(x,u).
\end{aligned}
\tag{2.14}
$$

In this case, the approximate system is obtained by a simple modification of the f vector field (i.e., by neglecting $\psi_2(\cdot)$).

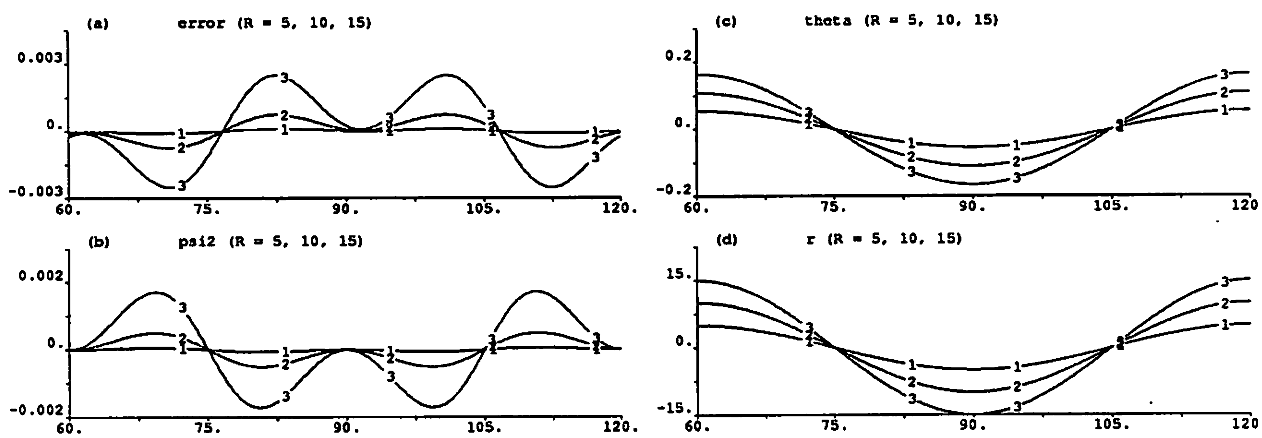

Figure 3: Simulation results for $y_d(t) = R\cos\pi t/30$ using the first approximation ((a) $e = y_d - \phi_1$, (b) ψ_2, (c) θ, (d) r)

The simulation results in Figure 3 show that the closed loop system provides good tracking. Notice that the tracking error increases in a nonlinear fashion as the amplitude of the desired trajectory increases. This is expected since the approximation error term $\psi_2(x)$ is a nonlinear function of the state. A good *a priori* estimate of the mismatch of the approximate system for a desired trajectory can be calculated using $\psi(\Phi^{-1}(y_d, \dot{y}_d, \ddot{y}_d, y_d^{(3)}))$ where $\Phi^{-1} : \xi \mapsto x$ is the inverse of the coordinate transformation. This in turn may be a useful way to define a class of trajectories that the system can track with small error.

Approximation 2

Again, let $\xi_1 = \phi_1(x) = h(x)$. Then, along the system trajectories, we have

$$\begin{aligned}
\dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\phi_2(x)} \\
\dot{\xi}_2 &= \underbrace{-G\sin x_3 + x_1 x_4^2}_{\xi_3=\phi_3(x)} \\
\dot{\xi}_3 &= \underbrace{-Gx_4\cos x_3 + x_2 x_4^2}_{\xi_4=\phi_4(x)} + \underbrace{2x_2x_4u}_{\psi_3(x,u)} \\
\dot{\xi}_4 &= \underbrace{x_1x_4^4}_{b(x)} + \underbrace{(-G\cos x_3 + 2x_2x_4)}_{a(x)}u
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 + \psi_3(x) \\
\dot{\xi}_4 &= b(x) + a(x)u =: v(x,u).
\end{aligned}
\tag{2.15}$$

This time the approximate system is obtained by modifying the g vector field in a more subtle way. Pulling back the modified g vector field (obtained by neglecting $\psi_3(x,u)$) to the original x coordinates (using u as input) we get

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{2Gx_1x_4\cos x_3 - 4x_1x_2x_4^2}{G(G\cos^2 x_3 - 2x_2x_4\cos x_3 - x_1x_4^2\sin x_3)} \\ \frac{2x_1x_4^2\sin x_3}{G\cos^2 x_3 - 2x_2x_4\cos x_3 - x_1x_4^2\sin x_3} \end{bmatrix}}_{\Delta g(x)}. \tag{2.16}$$

The system with g modified in this manner is input-output linearizable and is an approximation to the original system since Δg is small for small angular velocity, $\dot{\theta} = x_4$.

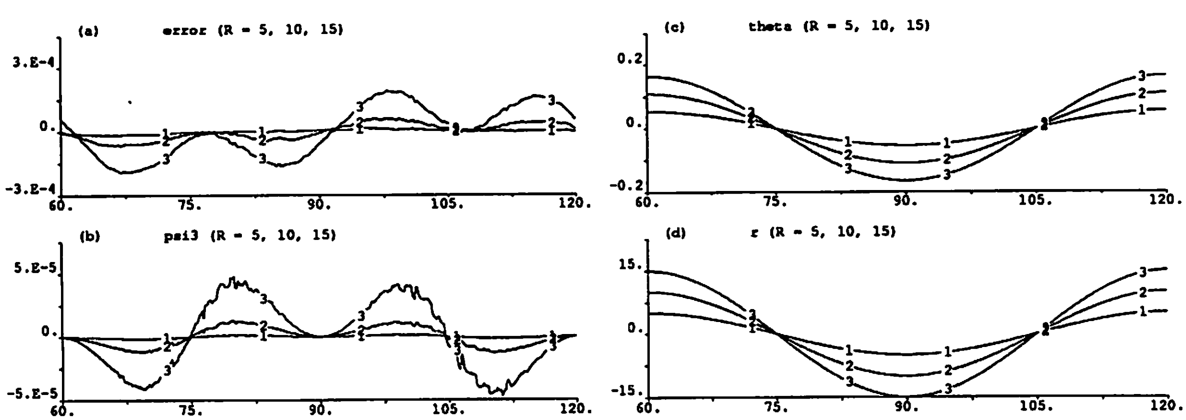

Figure 4: Simulation results for $y_d(t) = R\cos\pi t/30$ using the second approximation ((a) $e = y_d - \phi_1$, (b) ψ_3, (c) θ, (d) r)

The simulation results in Figure 4 show that the tracking error is substantially less than that obtained by the first design.

Jacobian Approximation

To provide a basis for comparison, we calculate a linear control law based on the Jacobian approximation. Previously, we used the invertible nonlinear transformation of (2.2) to simplify the form of $\dot{x}_4$. Since we are only allowed *linear* functions in the control, we must work directly with the original input τ and the true angular acceleration $\ddot{\theta} = \dot{x}_4$ given by

$$\dot{x}_4 = \frac{-MGx_1 \cos x_3 - 2Mx_1x_2x_4}{Mx_1^2 + J} + \frac{1}{Mx_1^2 + J}\tau \,. \tag{2.17}$$

We will linearize about $x = 0$, $\tau = 0$. Since the output is a *linear* function of the state, we begin with $\xi_1 = \phi_1(x) = h(x)$. Then, along the system trajectories, we have

$$\begin{aligned}
\dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\phi_2(x)} \\
\dot{\xi}_2 &= \underbrace{-Gx_3}_{\xi_3=\phi_3(x)} + \underbrace{x_1x_4^2 + G(x_3 - \sin x_3)}_{\psi_2(x)} \\
\dot{\xi}_3 &= \underbrace{-Gx_4}_{\xi_4=\phi_4(x)} \\
\dot{\xi}_4 &= \underbrace{\frac{MG^2}{J}x_1}_{b(x)} + \underbrace{\frac{-G}{J}}_{a(x)}\tau + \underbrace{\frac{MG^2x_1\cos x_3 + 2MGx_1x_2x_4}{Mx_1^2 + J} - \frac{MG^2x_1}{J} + \left(\frac{G}{J} - \frac{G}{Mx_1^2 + J}\right)\tau}_{\psi_4(x,\tau)}
\end{aligned} \tag{2.18}$$

The Jacobian approximation is, of course, obtained by replacing the f vector field by its linear approximation and the g vector field by its constant approximation.

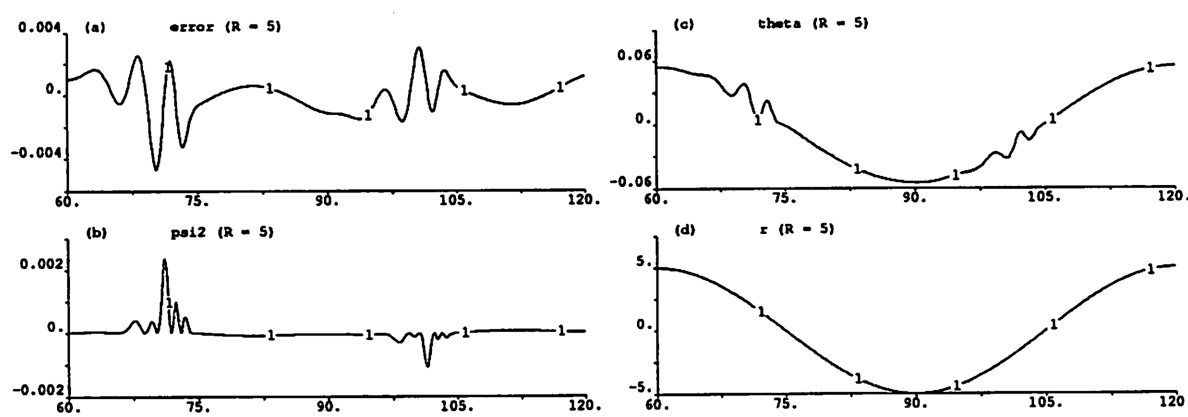

Figure 5: Simulation results for $y_d(t) = R\cos \pi t/30$ using the Jacobian approximation ((a) $e = y_d - \phi_1$, (b) ψ_3, (c) θ, (d) r)

Figure 5 shows the simulation results from the Jacobian approximation. Unfortunately, the control system with the linear controller is not stable for R greater than about 7.

The following table provides a direct comparison of the error $e = y_d - \phi_1$ for the three approximations:

R	Approximation 1	Approximation 2	Jacobian Approximation
5	$\pm 9.6 \cdot 10^{-5}$	$\pm 1.5 \cdot 10^{-5}$	$-4.7 \cdot 10^{-3}$ $+3.0 \cdot 10^{-3}$
10	$\pm 7.5 \cdot 10^{-4}$	$\pm 6.5 \cdot 10^{-5}$	*unstable*
15	$\pm 2.5 \cdot 10^{-3}$	$\pm 1.9 \cdot 10^{-4}$	*unstable*

Note that Approximation 2 provides better tracking for this class of inputs by about an order of magnitude over Approximation 1. Due to the large excursions from the origin, the Jacobian Approximation is no longer a good approximation so the system goes unstable. Of course, the other approximations will eventually go unstable as R becomes large.

In the next section, we will see that these approximations belong to a large class of approximations that provide the model to design stable closed loop control laws for approximate output tracking.

3 Theory for Approximate Linearization

In this section, we will consider single-input single-output systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3.1}$$

where $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}$, f and g are smooth vector fields on $\mathbf{R}^n$ (i.e., $f(x) \in T_x\mathbf{R}^n = \mathbf{R}^n$, $x \in \mathbf{R}^n$), and $h : \mathbf{R}^n \to \mathbf{R}$ is a smooth function (smooth is understood to mean as differentiable as needed). We *assume* that $x = 0$ is an equilibrium point of the undriven system, i.e., $f(0) = 0$.

If the control objective is tracking, the input-output linearization proceeds as follows: differentiate the output repeatedly until the input appears for the first time on the right hand side. Thus, we obtain

$$\begin{aligned} \dot{y} &= L_f h(x) \,, \\ \ddot{y} &= L_f^2 h(x) \,, \\ &\vdots \\ y^{(\gamma)} &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x) u \,. \end{aligned} \tag{3.2}$$

Here, $L_f h(x)$ stands for the Lie derivative of $h(x)$ along f, $L_f^2 h(x)$ stands for $L_f(L_f h)(x)$ and so on. It follows that in (3.2) above, that

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{\gamma-2} h(x) \equiv 0 \quad \text{for } x \in U \tag{3.3}$$

where U is an open neighborhood of the origin. In the event that $L_g L_f^{\gamma-1} h(x) \neq 0$ for $x \in U$, the system is said to have *relative degree* γ and the control law

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)} \left[-L_f^\gamma h(x) + v \right] \tag{3.4}$$

linearizes the system from v to y. However, it may happen that $L_g L_f^{\gamma-1} h(x) = 0$ at $x = 0$ but is not identically zero in a neighborhood U of $x = 0$, i.e., $L_g L_f^{\gamma-1} h(x)$ is a function which is of order $O(x)$ rather than $O(1)$. Then, the relative degree of the system is not well defined and the input-output linearizing control law of (3.4) is no longer valid.

(In the sequel we will use the O notation. Recall that a function $\delta(x)$ is said to be $O(x)^n$ if

$$\lim_{|x| \to 0} \frac{|\delta(x)|}{|x|^n} \text{ exists and is } \neq 0.$$

Also, functions which are $O(x)^0$ are referred to as $O(1)$. By abuse of notation, we will also use the notation $O(x,u)^2$ to mean functions of x,u which are sums of terms of $O(x)^2$, $O(xu)$ or $O(u)^2$. Similarly for $O(x,u)^\rho$.)

Failing this, we seek a set of functions of the state, $\phi_i(x)$, $i = 1,\ldots,\gamma$, that approximate the output and its derivatives in a special way. The integer γ will be determined during the approximation process.

Since our control objective is tracking, the first function, $\phi_1(x)$, should approximate the output function, that is

$$h(x) = \phi_1(x) + \psi_0(x,u) \tag{3.5}$$

where $\psi_0(x,u)$ is $O(x,u)^2$ (actually, ψ_0 does not depend on u, but for consistency below we include it). Differentiating $\phi_1(x)$ along the system trajectories we get

$$\dot{\phi}_1(x) = L_f h(x) + L_g h(x) u. \tag{3.6}$$

If $L_g h(x)$ is $O(x)$ or of higher order, we cannot effectively control the system at this level so we neglect it (and a small part of $L_f h(x)$ if we so desire) in our choice of $\phi_2(x)$:

$$L_{f+gu}\phi_1(x) = \phi_2(x) + \psi_1(x,u) \tag{3.7}$$

where $\psi(x,u)$ is $O(x,u)^2$. We continue this procedure with

$$L_{f+gu}\phi_i(x) = \phi_{i+1}(x) + \psi_i(x,u) \tag{3.8}$$

until at some step, say γ, the control term, $L_g\phi_\gamma(x)$, is $O(1)$, that is,

$$L_{f+gu}\phi_\gamma(x) = b(x) + a(x)u \tag{3.9}$$

where $a(x)$ is $O(1)$. Using this procedure, it looks like we have found an approximate system of relative degree γ. This motivates the following definition:

Definition 3.1 *We say that a nonlinear system (3.1) has a* robust relative degree *of γ about $x = 0$ if there exists smooth functions $\phi_i(x)$, $i = 1, \ldots, \gamma$, such that*

$$\begin{aligned}
h(x) &= \phi_1(x) + \psi_0(x,u) \\
L_{f+gu}\phi_i(x) &= \phi_{i+1}(x) + \psi_i(x,u) \qquad i = 1,\ldots,\gamma-1 \\
L_{f+gu}\phi_\gamma(x) &= b(x) + a(x)u + \psi_\gamma(x,u)
\end{aligned} \tag{3.10}$$

where the functions $\psi_i(x,u)$, $i = 0, \ldots, \gamma$, are $O(x,u)^2$ and $a(x)$ is $O(1)$.

Remarks

- In equation (3.10) above, the ψ_i have the form

$$\begin{aligned}
\psi_0(x,u) &= \psi_0^1(x)\ , \\
\psi_i(x,u) &= \psi_i^1(x) + \psi_i^2(x)u, \qquad i = 1,\ldots,\gamma-1
\end{aligned} \tag{3.11}$$

  where $\psi_i^1(x)$ is $O(x)^2$ and $\psi_i^2(x)$ is $O(x)$.

- There is considerable latitude in the definition of the $\phi_i(x)$ since each $\psi_i^1(x)$ may be chosen in a number of ways as long as it is $O(x)^2$.

We now characterize the robust relative degree. First, define the Jacobian linearized version of the system(3.1) about $x = 0$, $u = 0$ to be

$$\begin{aligned} \dot{z} &= Az + bu \\ y &= cz \end{aligned} \tag{3.12}$$

with $A = Df(0)$, $b = g(0)$, and $c = dh(0)$. Then, we have

Theorem 3.1 *The robust relative degree of the nonlinear system (3.1) is equal to the relative degree of the Jacobian linearized system (3.12) and so is well defined.*

Proof: For $i = 1, \ldots, \gamma - 1$, we have

$$\begin{aligned} L_{f+gu}\phi_i &= L_f\phi_i + L_g\phi_i u \\ &= \phi_{i+1} + \psi_i^1 + \psi_i^2 u \end{aligned} \tag{3.13}$$

so that

$$\begin{aligned} \phi_{i+1}(x) &= L_f\phi_i(x) - \psi_i^1(x) , \\ \psi_i^2(x) &= L_g\phi_i(x) . \end{aligned} \tag{3.14}$$

Also, since $\psi_i^1(x)$ is $O(x)^2$, we have, for $i = 1, \ldots, \gamma - 1$,

$$d\psi_i^1(0) = 0 . \tag{3.15}$$

Using this and the fact that $f(0) = 0$, the differentials of the functions ϕ_i are given by

$$\begin{aligned} d\phi_1(0) &= dh(0) - d\psi_0^1(0) \\ &= c - 0, \\ d\phi_2(0) &= dL_f\phi_1(0) - d\psi_1^1(0) \\ &= d^2\phi_1(0) \cdot f(0) + d\phi_1(0) \cdot Df(0) - 0 \\ &= 0 + cA, \\ &\vdots \\ d\phi_\gamma(0) &= cA^{\gamma-1} . \end{aligned} \tag{3.16}$$

Calculating the control coefficients, we find

$$\begin{aligned} \psi_1^2(0) &= d\phi_1(0) \cdot g(0) \\ &= cb, \\ \psi_2^2(0) &= cAb, \\ &\vdots \\ \psi_{\gamma-1}^2(0) &= cA^{\gamma-2}b, \\ a(0) &= cA^{\gamma-1}b. \end{aligned} \tag{3.17}$$

Since $\psi_i^2(0) = 0$ and $a(0) \neq 0$, it follows that

$$\begin{aligned} cb = cAb = \cdots = cA^{\gamma-2}b = 0, \\ cA^{\gamma-1}b \neq 0. \end{aligned} \tag{3.18}$$

Thus, γ, the robust relative degree of (3.1), is equal to the relative degree of the Jacobian linearized system (3.12). From this, it is easy to see that γ is independent of the choice of the neglected functions $\psi_i(x, u)$ of order $O(x, u)^2$ and is therefore well defined. □

An immediate corollary of this theorem is

Corollary 3.2 *The approximate relative degree of a nonlinear system (3.1) is* invariant *under a state dependent change of control coordinates of the form*

$$u(x, v) = \alpha(x) + \beta(x)v \tag{3.19}$$

where α and β are smooth functions and $\alpha(0) = 0$ while $\beta(0) \neq 0$.

In order to show that this procedure produces an approximation of the true system, we need to show that the functions $\phi_i(\cdot)$ can be used as part of a (local) nonlinear change of coordinates. To this end, we prove:

Proposition 3.3 *Suppose that the nonlinear system (3.1) has approximate relative degree γ. Then the functions $\phi_i(\cdot)$, $i = 1, \ldots, \gamma$, are independent in a neighborhood of the origin.*

Proof: Since the $\phi_i(\cdot)$ are smooth, it is sufficient to check that the *constant* $\gamma \times n$ matrix

$$D\phi(0) = \begin{bmatrix} d\phi_1(0) \\ d\phi_2(0) \\ \vdots \\ d\phi_\gamma(0) \end{bmatrix} = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{\gamma-1} \end{bmatrix} \tag{3.20}$$

(from (3.16)) has full rank. If we multiply $D\phi(0)$ on the right by the $n \times \gamma$ matrix

$$\left[A^{\gamma-1}b \ A^{\gamma-2}b \ \cdots \ b \right] \tag{3.21}$$

we get the nonsingular $\gamma \times \gamma$ matrix

$$\begin{bmatrix} a(0) & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & a(0) \end{bmatrix} \tag{3.22}$$

where '$*$' denotes possibly nonzero entries. This shows that $D\phi(0)$ has a rank of γ and the proposition is proved. □

With the γ independent functions, $\phi_i(\cdot)$, in hand, we can, by the Frobenius theorem, complete the nonlinear change of coordinates with a set of functions, $\eta_i(x)$, $i = 1, \ldots, n-\gamma$, such that

$$L_g \eta_i(x) = 0 \qquad x \in U. \tag{3.23}$$

Defining new coordinates, (ξ, η), by

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_\gamma \\ \eta_1 \\ \vdots \\ \eta_{n-\gamma} \end{bmatrix} := \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_\gamma(x) \\ \eta_1(x) \\ \vdots \\ \eta_{n-\gamma}(x) \end{bmatrix}, \tag{3.24}$$

we can rewrite the true system (3.1) as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \psi_1(x,u) \\ &\vdots \\ \dot{\xi}_{\gamma-1} &= \xi_\gamma + \psi_{\gamma-1}(x,u) \\ \dot{\xi}_\gamma &= b(\xi,\eta) + a(\xi,\eta)u \\ \dot{\eta} &= q(\xi,\eta) \\ \\ y &= \xi_1 + \psi_0(x,u) \end{aligned} \tag{3.25}$$

where $q(\xi, \eta)$ is $L_f \eta$ expressed in (ξ, η) coordinates.

Note that the form (3.25) is a generalization of the standard normal form of Byrnes and Isidori [4,1] with the extra terms $\psi_i(x,u)$, $i = 0, \ldots, \gamma$ of $O(x,u)^2$. Thus the control law

$$u = \frac{1}{a(\xi,\eta)}\left[-b(\xi,\eta) + v\right] \tag{3.26}$$

approximately linearizes the system (3.1) from the input v to the output y up to terms of $O(x,u)^2$.

If the robust relative degree of the system (3.1) is $\gamma = n$, then the system (3.1) is almost completely linearizable from input to state as well (since there will be no η state variables). This situation was investigated by Krener [6] who showed that the system

$$\dot{x} = f(x) + g(x)u \tag{3.27}$$

with no output *explicitly* defined was linearizable to terms of $O(x,u)^\rho$ iff the distribution

$$\text{span}\left\{g \ ad_f g \ \cdots \ ad_f^{n-1} g\right\} \ \text{ has rank } n \tag{3.28}$$

and the distribution

$$\text{span}\left\{g \ ad_f g \ \cdots \ ad_f^{n-2} g\right\} \ \text{ is order } \rho \text{ involutive,} \tag{3.29}$$

i.e., has a basis, up to terms of $O(x)^\rho$, which is involutive up to terms of $O(x)^\rho$. Equivalently, conditions (3.28) and (3.29) guarantee (through a version of the Frobenius theorem with remainder [6]) the existence of an *output* function $h(x)$ with respect to which the system (3.27) has *robust relative degree n* and further that the remainder functions $\psi_i(x,u)$ are $O(x,u)^\rho$. Our development differs somewhat from that in [6] in that we are given a specific output function $y = h(x)$ and a tracking objective for this output. However, there is a happy confluence of our results and those of Krener for the ball and beam example of the previous section where it may be verified that the condition of (3.29) is satisfied for $\rho = 3$ and further more the desired output function $h(x)$ is in fact an order $\rho = 3$ integral manifold of the distribution of that equation. Consequently the ball and beam can be input-output and state space linearized up to terms of order 3.

As was remarked after Definition 3.1, there is a great deal of latitude in the choice of the functions $\psi_i^1(x)$, $i = 0,\ldots,\gamma-1$, so long as they are $O(x)^2$. To improve the quality of the approximation, one may insist on choosing these terms to be $O(x)^\rho$ for some $\rho \geq 2$. There is less latitude in the choice of the functions $\psi_i^2(x)$. They must be neglected if they are $O(x)$ or higher and not neglected if they are $O(1)$ (this determines γ). We cannot in general guarantee that an approximation of $O(x,u)^\rho$ for $\rho > 2$ can be found. At this level of generality, it is difficult to give analytically rigorous design guidelines for the choice of the functions $\psi_i^1(x)$. However, from the ball and beam example of section 2, it would appear that it is advantageous to have the $\psi_i^1(x)$ be identically zero for as long (as large an i) as possible. *We conjecture that the larger the value of the first i at which either $\psi_i^1(x)$ or $\psi_i^2(x)$ are nonzero, the better the approximation.*

It is also important to note the distinction between the *nonlinear* feedback control law (3.26) which approximately linearizes the system (3.25) and the *linear* feedback control law obtained from the Jacobian linearization of the original system (3.1) given by

$$u = \frac{1}{cA^{\gamma-1}b}\left[-cA^\gamma x + v\right] , \tag{3.30}$$

though, as we have shown in the proof of Theorem 3.1, they agree up to first order at $x = 0$ since $cA^{\gamma-1}b = a(0)$ and $cA^\gamma = dL_f\phi_\gamma(0) = dh(0)$. It is also useful to note that the control law (3.26) is the *exact* input-output linearizing control law for the *approximate system*

$$\begin{array}{rcl}
\dot{\xi}_1 & = & \xi_2 \\
 & \vdots & \\
\dot{\xi}_{\gamma-1} & = & \xi_\gamma \\
\dot{\xi}_\gamma & = & b(\xi,\eta) + a(\xi,\eta)u \\
\dot{\eta} & = & q(\xi,\eta) \\
 & & \\
y & = & \xi_1
\end{array} \tag{3.31}$$

In general, we can only guarantee the existence of control laws of the form (3.26) that approximately linearize the system up to terms of $O(x,u)^2$—the Jacobian law of (3.30) is such a law. In specific applications, we see that the control law (3.26) may produce better approximations (the ball and beam of section 2 was linearized up to terms of $O(x,u)^3$). Furthermore, the resulting approximations may be valid on larger domains than the Jacobian linearization (also seen in the ball and beam example). We try to make this notion precise by studying the properties enjoyed by the approximately linearized system (3.1), (3.26) on a parameterized family of *operating envelopes*) defined as:

Definition 3.2 *We call $U_\epsilon \subset \mathbf{R}^n$, $\epsilon > 0$, a family of* operating envelopes *provided that*

$$U_\delta \subset U_\epsilon \text{ whenever } \delta < \epsilon \tag{3.32}$$

and

$$\sup\{\delta : B_\delta \subset U_\epsilon\} = \epsilon \tag{3.33}$$

where B_δ is a ball of radius δ centered at the origin.

Remarks

- It is not necessary that each U_ϵ be bounded (or compact) although this might be useful in some cases.
- Since the largest ball that fits in U_ϵ is B_ϵ, the set U_ϵ must get smaller in at least one direction as ϵ is decreased.

The functions $\psi_i(x,u)$ that are omitted in the approximation are of $O(x,u)^2$ in a neighborhood of the origin. However, if we are interested in extending the approximation to (larger) regions, say of the form of U_ϵ, we will need the following definition:

Definition 3.3 *A function $\psi : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is said to be* uniformly higher order *on $U_\epsilon \times B_\sigma \subset \mathbf{R}^n \times \mathbf{R}$, $\epsilon > 0$, if, for some $\sigma > 0$, there exists a monotone increasing function of ϵ, K_ϵ such that*

$$|\psi(x,u)| \leq \epsilon K_\epsilon(|x| + |u|) \quad \text{for } x \in U_\epsilon,\ |u| \leq \sigma. \tag{3.34}$$

Remarks

- If $\psi(x,u)$ is uniformly higher order on $U_\epsilon \times B_\sigma$ then it is $O(x,u)^2$.
- This definition is a refinement of the condition that $\psi(x,u)$ be $O(x,u)^2$ in as much as it does not allow for terms of the form $O(u)^2$.

Now, return to the original problem. If the approximate system (3.31) is exponentially minimum phase and the error term is uniformly higher order on $U_\epsilon \times B_\sigma$, we may use the stable tracking control law for the approximate system given by

$$u = \frac{1}{a(\xi,\eta)}\left[-b(\xi,\eta) + y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - \xi_\gamma) + \cdots + \alpha_0(y_d - \xi_1)\right] \tag{3.35}$$

(with $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \cdots + \alpha_0$ a Hurwitz polynomial). We can now prove the following result:

Theorem 3.4 *Let U_ϵ, $\epsilon > 0$, be a family of operating envelopes and suppose that*

- *the zero dynamics of the approximate system (3.31) (i.e., $\dot\eta = q(0,\eta)$) are* exponentially stable *and q is Lipschitz in ξ and η on $\Phi(U_\epsilon)$ for each ϵ and*
- *the functions $\psi_i(x,u)$ are* uniformly higher order *on $U_\epsilon \times B_\sigma$.*

Then, for ϵ sufficiently small and desired trajectories with sufficiently small derivatives (y_d, $\dot{y}_d$, ..., $y^{(\gamma)}$), the states of the closed loop system (3.1), (3.35) will remain bounded and the tracking error will be $O(\epsilon)$.

Proof: Define the trajectory error, $e \in \mathbf{R}^\gamma$, to be

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\gamma \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\gamma \end{bmatrix} - \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(\gamma-1)} \end{bmatrix}. \tag{3.36}$$

Then, the closed loop system (3.1), (3.35) (equivalently, (3.25), (3.35)) may be expressed as

$$\begin{aligned} \begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{\gamma-1} \\ \dot{e}_\gamma \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{\gamma-1} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_{\gamma-1} \\ e_\gamma \end{bmatrix} + \begin{bmatrix} \psi_1(x, u(x, \bar{y}_d)) \\ \vdots \\ \psi_{\gamma-1}(x, u(x, \bar{y}_d)) \\ \psi_\gamma(x, u(x, \bar{y}_d)) \end{bmatrix} \\ \dot{\eta} &= q(\xi, \eta) \end{aligned} \tag{3.37}$$

or, compactly,

$$\begin{aligned} \dot{e} &= Ae + \psi(x, u(x, \bar{y}_d)) \\ \dot{\eta} &= q(\xi, \eta) \end{aligned} \tag{3.38}$$

where $\bar{y}_d := (y_d, \dot{y}_d, \ldots, y_d^{(\gamma)})$. Since the zero dynamics are exponentially stable, a converse Lyapunov theorem implies the existence of a Lyapunov function (see, e.g., [2]) $V_2(\eta)$ for the system

$$\dot{\eta} = q(0, \eta) \tag{3.39}$$

satisfying

$$\begin{aligned} k_1|\eta|^2 \leq V_2(\eta) \leq k_2|\eta|^2 \\ \frac{\partial V_2}{\partial \eta} q(0, \eta) \leq -k_3|\eta|^2 \\ \left|\frac{\partial V_2}{\partial \eta}\right| \leq k_4|\eta| \end{aligned} \tag{3.40}$$

for some positive constants k_1, k_2, k_3, and k_4.

We first show that e and η are bounded. To this end, consider as Lyapunov function for the error system (3.38)

$$V(e, \eta) = e^T Pe + \mu V_2(\eta) \tag{3.41}$$

where $P > 0$ is chosen so that

$$A^T P + PA = -I \tag{3.42}$$

(possible since $\dot{e} = Ae$ is stable) and μ is a positive constant to be determined later.

Note that, by assumption, y_d and its first γ derivatives are bounded,

$$|\xi| \leq |e| + b_d \text{ and } |y^{(\gamma)}| \leq b_d, \tag{3.43}$$

the function, $q(\xi, \eta)$ is Lipschitz

$$|q(\xi^1, \eta^1) - q(\xi^2, \eta^2)| \leq l_q(|\xi^1 - \xi^2| + |\eta^1 - \eta^2|), \tag{3.44}$$

the function, $\psi(x, u)$, is uniformly higher order with respect to $U_\epsilon \times B_\sigma$ and $u(x, \bar{y}_d)$ locally Lipschitz in its arguments with $u(0, 0) = 0$,

$$|2P\psi(x, u(x, \bar{y}_d))| \leq \epsilon K_\epsilon l_u(|x| + b_d) \quad (x, u) \in U_\epsilon \times B_\sigma, \tag{3.45}$$

and x is a local diffeomorphism of (ξ, η),

$$|x| \leq l_x(|\xi| + |\eta|). \tag{3.46}$$

Using these bounds and the properties of $V_2(\cdot)$, we have

$$\begin{aligned} \tfrac{\partial V_2}{\partial \eta} q(\xi, \eta) &= \tfrac{\partial V_2}{\partial \eta} q(0, \eta) + \tfrac{\partial V_2}{\partial \eta}(q(\xi, \eta) - q(0, \eta)) \\ &\leq -k_3|\eta|^2 + k_4 l_q |\eta|(|e| + b_d). \end{aligned} \tag{3.47}$$

Taking the derivative of $V(\cdot, \cdot)$ along the trajectories of (3.38), we find, for $(x, u) \in U_\epsilon \times B_\sigma$,

$$\begin{aligned} \dot{V} &= -|e|^2 + 2e^T P\psi(x, u(x, \bar{y}_d)) + \mu \tfrac{\partial V_2}{\partial \eta} q(\xi, \eta) \\ &\leq -|e|^2 + \epsilon|e| K_\epsilon l_x(|e| + b_d + |\eta|) + \mu(-k_3|\eta|^2 + k_4 l_q |\eta|(|e| + b_d)) \\ &\leq -(\tfrac{|e|}{2} - \epsilon K_\epsilon l_x b_d)^2 + (\epsilon K_\epsilon l_x b_d)^2 \\ &\quad -(\tfrac{|e|}{2} - (\epsilon K_\epsilon l_x + \mu k_4 l_q)|\eta|)^2 + (\epsilon K_\epsilon l_x + \mu k_4 l_q)^2 |\eta|^2 \\ &\quad -\mu k_3(\tfrac{|\eta|}{2} - \tfrac{k_4 l_q b_d}{k_3})^2 + \mu \tfrac{(k_4 l_q b_d)^2}{k_3} \\ &\quad -(\tfrac{1}{2} - \epsilon K_\epsilon l_x)|e|^2 - \tfrac{3}{4}\mu k_3 |\eta|^2 \\ &\leq -(\tfrac{1}{2} - \epsilon K_\epsilon l_x)|e|^2 - (\tfrac{3}{4}\mu k_3 - (\epsilon K_\epsilon l_x + \mu k_4 l_q)^2)|\eta|^2 \\ &\quad +(\epsilon K_\epsilon l_x b_d)^2 + \mu \tfrac{(k_4 l_q b_d)^2}{k_3}. \end{aligned} \tag{3.48}$$

Define

$$\mu_0 = \frac{k_3}{4(K_\epsilon l_x + k_4 l_q)^2}. \tag{3.49}$$

Then, for all $\mu \leq \mu_0$ and all $\epsilon \leq min(\mu, \frac{1}{4K_\epsilon l_x})$, we have

$$\dot{V} \leq -\frac{|e|^2}{4} - \frac{\mu k_3 |\eta|^2}{2} + \frac{\mu(k_4 l_q b_d)^2}{k_3} + (\epsilon K_\epsilon l_x b_d)^2. \tag{3.50}$$

Thus, $\dot{V} < 0$ whenever $|\eta|$ or $|e|$ is large which implies that $|\eta|$ and $|e|$ and, hence, $|\xi|$ and $|x|$, are bounded. The above analysis is valid for $(x, u) \in U_\epsilon \times B_\sigma$. Indeed, by choosing b_d sufficiently small and appropriate initial conditions, we can guarantee that the state remains in U_ϵ and the input is bounded by σ. Using this fact, we may abuse notation and write the function $\psi(x, u(x, \bar{y}_d))$ as $\epsilon\psi(t)$ and note that

$$\dot{e} = Ae + \epsilon\psi(t) \tag{3.51}$$

is an exponentially stable linear system driven by an order ϵ input. Thus, we conclude that the tracking error will be $O(\epsilon)$. □

4 Conclusion

In this paper, we have presented an approach for the approximate input-output linearization of nonlinear systems, particularly those for which relative degree is not well defined. We saw that there is in fact a great deal of freedom in the selection of the approximation. We have seen that, by designing a tracking controller based on the approximating system, we can achieve tracking of reasonable trajectories with small error. The approximating system is a nonlinear system, with the difference that it is input-output linearizable by state feedback. We have shown some properties of the accuracy of the approximation and in the context of the ball and beam example shown it to be far superior to the Jacobian approximation. Future research in this area will include developing methods to effectively search among the prospective approximate systems and to evaluate their accuracy.

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