THE COMPLETE CANONICAL PIECEWISE-LINEAR REPRESENTATION I. THE GEOMETRY OF THE DOMAIN SPACE

by

Claus Kahlert and Leon O. Chua

Memorandum No. UCB/ERL M90/26

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The Complete Canonical Piecewise-Linear Representation
I. The Geometry of the Domain Space† *

CLAUS KAHLERT AND LEON O. CHUA ‡

Abstract

Continuous piecewise-linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ are analyzed in terms of the dimension of their domain space and of degenerate k-th order intersections of region boundaries. The theory developed demonstrates how these two quantities are connected. Moreover, the *exact* number of independent parameters will be demonstrated for boundary configurations containing degenerate intersections of arbitrary orders.

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1. Introduction

Continuous piecewise-linear (PWL) models have proven very helpful in analyzing nonlinear circuits and other nonlinear problems. Since the domains of the functions describing such models are partitioned into *polyhedral regions* where the functions are linear throughout, all nonlinearities are localized in the *region boundaries.* This makes them much more amenable to analysis than virtually any other type of nonlinear functions. Moreover, since the composition of PWL functions (and the inverse, if it exists) is again piecewise-linear, unlike most other systems of functions, this class is closed under the above operations.

For a long time, the major obstacle to a broad and systematic application of PWL methods was the potentially huge number of parameters, i.e., the entries of the different Jacobians for the individual regions, and their redundancy resulting from continuity conditions. This problem was conceptually overcome by the notion of *canonical representations* [1]. These forms contain the *exact number of parametric degrees of freedom*, which is always (much) smaller than the number of entries in the Jacobians of a conventional representation (see the examples in [2]). The classical canonical representation of continuous PWL functions (since only continuous functions will be considered, continuity will henceforth be assumed implicitly) from $\mathbb{R}^n$ to $\mathbb{R}^m$:

$$\mathbf{F}(\mathbf{x}) = \mathbf{b} + \mathbf{B}\mathbf{x} + \sum_{i=1}^{\sigma} \mathbf{c}^i \left|\langle \alpha^i, \mathbf{x}\rangle - \beta^i\right| \tag{1}$$

introduced by Chua and Kang [1] assigns *one parameter vector,* $\mathbf{c}^i$, *to each boundary* (between regions) and includes thereby already a majority of PWL functions treated so far in practice. In particular, it covers *all* one-dimensional PWL functions. However, three region boundaries in $\mathbb{R}^2$ intersecting in one point were identified in [2] to cause a breakdown of this representation. Such intersections were called "degenerate" and a rank criterion for degeneracy was introduced there. Unfortunately, the criterion, as it stands, is only of value in $\mathbb{R}^2$. Below, we shall demonstrate how to extend it to higher dimensions and higher orders of intersections. Here we are going to call a boundary configuration, "degenerate of order k", if the PWL function possessing these boundaries cannot be described by a $(k-1)$-st order representation (see the formal definition below).

Recently [3], the problem of representing the threefold (and any multiple) intersection in $\mathbb{R}^2$, as well as its suspensions to $\mathbb{R}^n$, was overcome by the form

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \mathbf{b} + \mathbf{B}\mathbf{x} + \sum_{i=1}^{\sigma} \mathbf{c}^i \left|\langle \alpha^i, \mathbf{x}\rangle - \beta^i\right| + \\ &\sum_{j=1}^{\varrho} \sum_{k=3}^{\delta^j+2} \tilde{\mathbf{c}}^{jj_k} \Big\{ \Big| \left| a^j_{j_k j_1}(\langle \alpha^{j_1}, \mathbf{x}\rangle - \beta^{j_1})\right| + a^j_{j_k j_2}(\langle \alpha^{j_2}, \mathbf{x}\rangle - \beta^{j_2})\Big| - \\ &\qquad \Big| a^j_{j_k j_1}(\langle \alpha^{j_1}, \mathbf{x}\rangle - \beta^{j_1}) + \left| a^j_{j_k j_2}(\langle \alpha^{j_2}, \mathbf{x}\rangle - \beta^{j_2})\right| \Big| \Big\}\end{aligned} \tag{2}$$

which covers *all two-dimensional boundary configurations.* It adds *one parameter (vector),* $\tilde{\mathbf{c}}^{j_{j_k}}$*, for each degenerate intersection of boundaries* (strictly speaking, for each degree of degeneracy appearing, see [3]) to the representation. (See also [4] and the *Appendix* for a different approach describing a subclass of the functions presented in [3]). As a new feature, (2) possesses *nested absolute-value functions.*

However, in $\mathbb{R}^3$ and higher dimensional spaces, all canonical representations of PWL functions introduced so far are not closed under the operations mentioned above, i.e., in the general case, due to a lack of parametric degrees of freedom, the composition of two PWL functions cannot be characterized in terms of one of the currently known representations. The significance of functional compositions was demonstrated in the examples provided in [3]: In order to obtain the constitutive relation of a composite n-port circuit, one must often form the *composition* of the constitutive relations of its elements. If these elements are already piecewise-linear, the appearing absolute-value functions have to be *nested.* Some of these nestings can be rewritten in terms of simple (unnested) absolute-value functions, others cannot. In [3] we demonstrated that the latter type exhibits degenerate boundary intersections. Thus such intersections turn out to be generic and robust properties, which persist under perturbations of the parameters. Here our goal is to connect the geometric notion of degenerate intersections and the form of the functions possessing such a boundary structure for arbitrary orders of degeneracy.

For boundary configurations more complex than those shown in [3], where elements which are already described by (2), have to be combined, we have discovered that "degenerate intersections of degenerate boundary intersections" can occur in dimensions greater than two and the above representation (2) breaks down in such cases of higher degeneracy. This clearly demonstrates the necessity for a general framework which covers the nonlinearities emerging from intersections of all orders (not just intersections of region boundaries), and which properly handles the degeneracies appearing there. This will allow us to determine the effective parameters for arbitrary classes of PWL functions and characterize them in terms of these parameters.

2. The Geometry of Intersections

In this section we are going to demonstrate how the geometry of the region boundaries, specifically their intersections, can be characterized in a unified manner. Any PWL function

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m \quad ; \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) \tag{3}$$

is specified completely by a constant offset and a Jacobian for each region of the domain space. However, in order to maintain continuity, these quantities *cannot* be chosen completely independently for the different regions. It turns out (see for example [1]) that the exact behavior inside one arbitrary region and the jumps in the Jacobian (which, moreover, must have a special form, see below) along the region boundaries yield a sufficient description. In addition, all boundaries are linear manifolds (or hyperplanes). This fundamental property of boundaries, although frequently stated as an extra assumption, is an immediate consequence of continuity (see [3]). The configuration of boundaries can thus be seen as the *skeleton* for a class of functions, where the geometry gives all the structural information while the concrete behavior is determined by the values of the parameters.

In our following investigations of the fundamental structures, with very little loss of generality, only one intersection of the "highest order" appearing will be considered. This entails no real restriction since two distinct intersections "see" each other just linearly, unless they share a common structure, i.e., a lower order intersection (cf. [3]). Nevertheless, in analogy to what we saw in [3], this does not affect the representation and requires just a few modifications in the algorithm for finding the parameter vectors. For the time being, treating just one "k-th order intersection" will allow us to get rid of one index and the corresponding summation.

To avoid further clutter and without loss of generality, in the following offset constants, both in the domain and the range space of the function to be represented will be suppressed; they can always be included by a trivial translation in either of the two spaces. In other words, all nonlinear structures contain the origin and thus "subspaces" rather than "affine manifolds" will henceforth be assumed to be handled.

Since we are going to focus on the *nonlinear* aspects of piecewise-linear functions, let us introduce first the concept of *nonlinearity components*.
For every continuous PWL function $\mathbf{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, the domain space $\mathbb{R}^n$ can be decomposed *locally* into $\mathcal{N} \times \mathcal{L}$, where $\mathcal{L}$ and $\mathcal{N}$ are isomorphic to $\mathbb{R}^{n'}$ and $\mathbb{R}^{n''}$ ($n = n' + n''$), respectively. Inside $\mathcal{L}$ the function is assumed to be linear throughout, while $\mathcal{N}$ contains the nonlinearities; more precisely, for no vector $\mathbf{x} \in \mathcal{N}$ is the directional derivative $D_{\mathbf{x}}\mathbf{F}$ a global constant. Thus $\mathcal{N}$ is the *quotient space* relevant for our purposes. Since we are only interested in the nonlinear behavior of $\mathbf{F}$ (inside $\mathcal{N}$), the components from $\mathcal{L}$ will be suppressed and we shall frequently refer to $\mathcal{N}$ simply as *the* domain space. However, it should be stressed once more that this is a *local* decomposition. Eventually, all contributions coming from the different $\mathcal{N}$'s have to be combined, which effectively means a simple *superposition* and embedding into the original domain space, i.e., a summation over all the "local" representations at the level of functions.

The notion of nonlinearity components should shed some light into the obvious confusion found in the literature about the "dimensions of representations". Although any PWL function can be embedded into a space of arbitrarily high dimension, only inside its nonlinearity components we can find any behavior that cannot be represented by a simple linear mapping.

The dimension of $\mathcal{N}$ is tightly connected to the highest *order of intersection* appearing in a given boundary configuration. These fundamental structures are defined recursively as follows:

Def. 1 *A pre-boundary in the sense of Definition 6 of [3] (i.e., an $(n-1)$-dimensional subspace where at least part of it is a boundary) is said to be a first-order intersection, denoted by $S^{(1)}$. A subspace of dimension $n-k$ in $\mathbb{R}^n$ is a k-th order intersection $S^{(k)}$ if it is the intersection of two or more subspaces of type $S^{(k-1)}$, i.e.,*

$$S^{(k)} := \bigcap_{i=1}^{\geq 2} S^{(k-1),i} \quad . \tag{4}$$

■

Let us add a brief note concerning our notation: The order of an object will be indicated by an upper index in parenthesis, like in $S^{(k)}$. If necessary, a further upper index will provide a labeling of the object, like in $S^{(k-1),i}$. Lower indices will be reserved for vector or matrix components.

From *Definition* 1 it is clear that k is the codimension of the intersection, which for $k < n$ is spanned by the vectors $\gamma^{(k),1} \ldots \gamma^{(k),n-k}$ (for $k = n$, $\gamma^{(k)}$ is the null vector). However, the term "of order k" implies not just the codimension k, but also an intersection of different linear subspaces of order $k-1$. For example (see *Fig.*1), (a) as two boundaries ($S^{(1)}$'s) in $\mathbb{R}^n$ intersect, they form an $S^{(2)}$. This situation can be described by a space $\mathcal{N}$, isomorphic to $\mathbb{R}^2$. Consequently, (b) three planes in $\mathbb{R}^3$ ($S^{(1)}$'s) having three mutual (one-dimensional) intersections ($S^{(2)}$'s), possess one common point (the origin), which is an $S^{(3)}$. However, (c) a line intersecting a plane in $\mathbb{R}^3$ does not give rise to the above sort of intersection, because in our current context, a line can only occur as the intersection of two or more planes .

Figure 1

In the following section, we shall be interested in a special kind of $S^{(k)}$'s, called *degenerate intersections of order k*, and denoted by $\bar{S}^{(k)}$. Their appearance will be directly connected to the terms in the canonical representation. The exact definition of degeneracy will, however, be postponed until we have gained sufficient motivation from the behavior of PWL functions.

Let us now look at a function's possible behavior on a boundary configuration containing a k-th order intersection. By definition, piecewise-linear functions behave linearly inside the different regions, i.e., they possess piecewise-constant Jacobians which change abruptly at the boundaries. Thus on every boundary $S^{(1),i}$ we define a jump (first-order) function for the Jacobian

$$\Delta^{(1)}\mathcal{J}^i : \quad S^{(1),i} \to \mathbb{R}^{m\times n} \quad ; \quad \mathbf{x} \mapsto \Delta^{(1)}\mathcal{J}^i(\mathbf{x}) \quad , \tag{5a}$$

which is piecewise-constant, i.e., it might switch where boundaries meet — at second-order intersections. Hence, as a generalization of (5a), on every intersection $S^{(k),i}$ a set (indicated by the index j, which labels the current α-vector, see below or [3]) of k-th order jump functions is defined

$$\Delta^{(k)}\mathcal{J}^{ij}: \quad S^{(k),i} \to \mathbb{R}^{m\times n} \quad ; \quad \mathbf{x} \mapsto \Delta^{(k)}\mathcal{J}^{ij}(\mathbf{x}) \quad , \tag{5b}$$

as the difference between the $(k-1)$-st order jump functions on the $S^{(k-1)}$'s which are involved in the pertinent $S^{(k)}$. These jump functions, due to the piecewise-linear nature of the functions to be represented, are all piecewise-constant, possessing values $\Delta^{(k)}\bar{\mathbf{J}}^{\nu}$ (where ν is any labeling, usually a multi-index indicating the $S^{(k-1)}$'s, the α-vectors involved and the segment of $S^{(k)}$ where this specific value appears).

This notion will become clear shortly, when we analyze the current state of the theory (as it is found in the literature) in the framework of k-th order intersections and the corresponding jump functions.

At this point we should include a further remark on the notation used. Three different symbols can be found here for a k-th order jump in the Jacobian on an intersection $S^{(k)}$: (a) The function $\Delta^{(k)}\mathcal{J}$, having as values at a point $\mathbf{x} \in S^{(k)}$ (b) the actual k-th order jumps $\Delta^{(k)}\bar{\mathbf{J}} = \Delta^{(k)}\mathcal{J}(\mathbf{x})$, which, due to the piecewise-constant character of $\Delta^{(k)}\mathcal{J}$, represent the jump on a whole portion of the k-th order intersection. Finally, (c) the symbol $\Delta^{(k)}\mathbf{J}$ for the jump characterizing the whole $S^{(k)}$ is needed to define a form of mean value, i.e., if no $(k+1)$-st order jump occurs on the intersection then $\Delta^{(k)}\mathcal{J}$ is constant and has the value $\Delta^{(k)}\mathbf{J}$. This third value is also called "characterizing" because it will yield the parameter for the canonical representation.

Let us now turn to a systematic treatment of boundary intersections and the functions found there: The simplest "PWL" function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a completely linear one. It contains no nonlinear structure at all, so it can be characterized by its Jacobian $\mathbf{J}$ at one arbitrary point, yielding $\Delta^{(0)}\mathcal{J} = \Delta^{(0)}\mathbf{J} = \Delta^{(0)}\bar{\mathbf{J}} = \mathbf{J}$. Thus the associated nonlinearity component $\mathcal{N}$ is of dimension zero.

As *two* objects of codimension zero (regions with Jacobians $\mathbf{J}$ and $\mathbf{J}'$, respectively) meet, a boundary (first-order intersection) emerges, which is associated to a (first-order) jump in the Jacobian, $\Delta^{(1)}\mathcal{J} = \Delta^{(1)}\mathbf{J} = \Delta^{(1)}\bar{\mathbf{J}} = \mathbf{J} - \mathbf{J}'$, appearing along this region boundary. Consequently such a boundary configuration requires $dim\mathcal{N} = 1$. In the present case, the boundary $S^{(1)}$ is completely from $\mathcal{L}$, i.e., it is of dimension zero in $\mathcal{N}$, hence the function $\Delta^{(1)}\mathcal{J}$ is merely a constant, showing no *internal structure*, thereby reflecting the "constant variation" property introduced in [2]. Note that in the first example $\Delta^{(0)}\mathcal{J}$ is constant while here it "picked up structure" from a first (uniform or constant) jump $\Delta^{(1)}\mathcal{J}$.

Going one step further, as region boundaries meet in a subspace of dimension $n-2$, they yield an intersection of *order two* and thus require $\mathcal{N}$ to be a two-dimensional space. This situation may allow us to impose a structure on the different $\Delta^{(1)}\mathcal{J}$ functions along the boundaries, which appears in the form of (constant) second-order jumps $\Delta^{(2)}\mathcal{J}$, located at the intersection $S^{(2)}$.

In higher dimensions ($dim\mathcal{N} = k \geq 3$) a similar pattern is maintained. As intersections of order $k-1$ (being lines in $\mathcal{N}$) meet in one point, an intersection $S^{(k)}$ emerges, which may allow the functions $\Delta^{(k-1)}\mathcal{J}$ to become non-constant by performing jumps $\Delta^{(k)}\bar{\mathbf{J}} = \Delta^{(k-1)}\mathcal{J}(\mathbf{x}) - \Delta^{(k-1)}\mathcal{J}(\mathbf{y})$, where $\mathbf{x}, \mathbf{y} \in S^{(k-1)}$ are on different sides of $S^{(k)}$, i.e, for one $\lambda \in (0,1)$ the point $(1-\lambda)\mathbf{x}+\lambda\mathbf{y}$ is from $S^{(k)}$. This will add new terms to the representation of the function $\mathbf{F}$.

These examples demonstrate the fundamental approach of the representation to be achieved. Instead of plainly stating the function's values in the different regions, a *hierarchy of differences* is developed, which is immediately reflected in the dimensions of the different intersections or "higher order boundaries" in $\mathcal{N}$. Due to the special functional form of the differences appearing, this will turn out to be a much more efficient procedure. The major problem to be addressed next will be to identify those cases where a non-zero $\Delta^{(k)}\mathcal{J}$ is possible.

At this point the impression could arise that the canonical representation to be developed here is similar to a sort of power series expansion. Although the notion introduced presents more and more details as higher order intersections are taken into account, every term appears linearly in the representation, that is, they are all at the same algebraic level. Thus there is no way of treating the higher orders as "perturbations". Hence the question on a representation is not of *quantitative* nature: How well can a function be approximated by a representation of a given order? PWL functions require a *qualitative* decision, whether a function can be described or not.

This means both an advantage and a disadvantage compared to approximations in the form of power series or other functions. The disadvantage is that no term can be dropped without affecting the function's behavior linearly *everywhere*. On the other hand, the (higher order) intersections of boundaries are found where the function "needs special attention". So it is always a good idea to treat these intersections carefully. If, however, for some particular problem the highest order intersection is outside the range of current consideration, then all the terms evolving from it will contribute just a constant Jacobian in the area of interest. This might help to identify simpler models, which may be more adequate for these situations.

3. Degenerate Intersections of Order k

In the previous *Section* the notion of k-th order jump functions and their interdependences was (by intention) not sharply specified, since it will turn out that on most $S^{(k)}$'s such a jump *cannot* occur, i.e., the functions $\Delta^{(k)}\mathcal{J}$ vanish and the $\Delta^{(k-1)}\mathcal{J}$'s remain unchanged . Thus here we are going to discuss, what sort of k-th order intersection allows a *non-zero* $\Delta^{(k)}\bar{\mathbf{J}}$ to appear, and how the different k-th order jumps *interact*. In this paper a prototype

k-th order degenerate intersection will be constructed which can eventually be used as a building block for higher order degenerate intersections, or as a basis for intersections of higher degeneracy.

Def. 2 *Every intersection of order k which allows a non-zero $\Delta^{(k)}\bar{\mathbf{J}}$ of a PWL function is called degenerate.* ■

As it stands, this definition of degeneracy appears not very constructive, nevertheless, it sets exactly the condition needed for the canonical representation and in the following we have to transform this into a geometrical notion of boundary intersections. First of all, however, let us quickly recall two fundamental properties to be used heavily further on: (I) Along any closed path inside an $S^{(k)}$ the jumps $\Delta^{(k+1)}\bar{\mathbf{J}}$ in the Jacobian have to sum up to zero. This is an immediate and obvious extension of the closed path property from [3]. (II) A jump in the Jacobian along a region boundary can always be represented as a dyad [5]. That is

$$\Delta^{(1)}\bar{\mathbf{J}}^{\nu} = \bar{\mathbf{c}}^{(1),\nu}\alpha^{\nu T} \quad . \tag{6a}$$

Here (as we used it before) $\Delta^{(1)}\bar{\mathbf{J}}^{\nu}$ is the current jump in the Jacobian associated with $\bar{\mathbf{c}}^{(1),\nu}$ where ν, as above, is a multi-index specifying the boundary and the section on this boundary. The parameter vector $\bar{\mathbf{c}}^{(1),\nu}$ is from the range space $\mathbb{R}^m$ and α^{ν} is a vector of the domain space, being normal to the section $S^{(1),\nu}$ of the region's boundary in question. This immediately demonstrates two things: (*a*) The number of linearly independent $\Delta^{(1)}\bar{\mathbf{J}}$'s, i.e., the dimension of the space of linear maps involved in the problem, is *dim* $\mathcal{N}$ (which shows that the space spanned by the $\Delta^{(1)}\bar{\mathbf{J}}$'s is isomorphic to $\mathcal{N}$). (*b*) Since a k-th order jump is the difference of two $\Delta^{(k-1)}\bar{\mathbf{J}}$'s, possessing the same α-vector, within one $(k-1)$-st order intersection, all the $\Delta^{(k)}\bar{\mathbf{J}}^{\nu}$ appearing have to have essentially the form (6a), i.e., we can write a k-th order jump as

$$\Delta^{(k)}\bar{\mathbf{J}}^{\nu} = \bar{\mathbf{c}}^{(k),\nu}\alpha^{\nu T} \quad . \tag{6b}$$

For the parameter vectors we apply the same convention as for the jumps in the Jacobian, i.e., a bar will be used for the current value while $\mathbf{c}^{(j),i}$ "characterizes" the whole intersection $S^{(j),i}$. This form will give us one key to the representation of functions which possess higher order intersections. Moreover, with the algebraic structure of all possible jumps in the Jacobian, no matter what order, now on hand, we have obtained an exact notion of the term parametric degrees of freedom. These are exactly those c-vectors which can be chosen freely for a function possessing a given boundary configuration. (This is a property of the function and thus of the geometry found in the *domain space*, hence it should not be confused with linear independence inside the *range space*. Since the dimension of the range space is not relevant for our present purposes, the term "parameter" always means a *parameter vector* from the range space.)

In order to find the independent parameters, the crucial properties to be worked out first are the *interdependences* of the different $\Delta^{(k)}\bar{\mathbf{J}}$'s appearing on the $\bar{S}^{(k-1)}$'s involved in an $\bar{S}^{(k)}$. Taking into account the restrictions being imposed on the different $\Delta^{(k)}\mathcal{J}$

functions by the continuity of the overall function $\mathbf{F}$, will yield the number of parametric degrees of freedom. Since here our goal is to construct the simplest k-th order degenerate intersections, each of them should eventually turn out to contribute exactly one parameter vector $\mathbf{c}^{(k)}$.

In this light, we may now analyze again the above examples:

In one dimension ($dim\,\mathcal{N} = 1$, where all vectors from the regions are normal to the boundary spanned by the null-vector) *two* Jacobians ($\Delta^{(0)}\bar{\mathbf{J}}$'s) meet at an intersection of order-*one* to yield a $\Delta^{(1)}\bar{\mathbf{J}}$. This automatically makes *every* boundary a degenerate intersection $\bar{S}^{(1)}$ of order-one and degeneracy-one (i.e., a degeneracy of degree-one). Thus it is no surprise that the representation in [1] could be found without a concept of degeneracy. (Here we are confronted with two different concepts within the notion of degeneracy, the *order* and the *degree*. While the order was defined above, a general notion of the *degree of degeneracy* will be elaborated in a forthcoming publication, for second order intersections it was defined in [3]. For the time being, we are only interested in degeneracies of degree-one, meaning that it will contribute the minimal number of parameters to the representation.)

Figure 2

For *two* dimensions, it was shown in [2] that *three* boundaries (with three different α's) intersecting in one point, form a degenerate boundary configuration $\bar{S}^{(2)}$. In this case the different α-vectors involved and with them the $\Delta^{(1)}\bar{\mathbf{J}}$ are no longer linearly independent (see *Fig.* 2) and thus the consistency of the jump in the Jacobian along a boundary breaks down. This notion is discussed in detail in [3].

Heading towards three dimensions, the boundary configuration depicted in *Fig.* 3*a* (although characterized as being "degenerate" by the criterion in [2]) shows no degenerate second-order intersections, i.e., it can be described by (1). *Fig.* 3*b* possesses two degenerate second-order intersections, which makes it accessible to (2), while the boundary configuration in *Fig.* 3*c* has a degenerate intersection of order *three* and is thus beyond the capability of the currently known representations.

For the example of *Fig* 3*a*, it is, on the one hand, obvious that the four α-vectors of the boundaries are linearly dependent (cf. [2], *p.* 104). On the other hand, the six $S^{(2)}$'s are formed by two $S^{(1)}$'s each, so none of them is degenerate. For the two other boundary configurations, the properties mentioned can be demonstrated by performing some bookkeeping along closed paths on both sides of the $S^{(3)}$ around every $\bar{S}^{(2)}$ appearing (indicated by the arrows in *Figs.* 3*a*, *b*). Thereby several conditions on the observed jumps ($\Delta^{(1)}\bar{\mathbf{J}}$, which, via *Eq.* 6, yield parameter vectors $\bar{\mathbf{c}}^{(1)}$) are obtained. Eventually, a rank analysis can be performed for the resulting set of linear equations. In the case of *Fig.* 3*b*, 14 different $\bar{\mathbf{c}}^{(1)}$'s are found, and of the eight conditions on them, five turn out to be independent. Therefore, the effective number of free parameters for this problem is $14-5 = 9$. Looking at the constituents of $S^{(3)}$, we find six $\mathbf{c}^{(1)}$ (characterizing the six region boundaries) and three $\mathbf{c}^{(2)}$-vectors (one coming from the degeneracy-one $\bar{S}^{(2)}$ and the two others emerging from the second-order intersection of degeneracy-two; for a definition of the degree of degeneracy in 2-D see [3]). So this boundary configuration is a *non-interacting* superposition of two $\bar{S}^{(2)}$'s with $\Delta^{(2)}\mathcal{J}$ functions which are constant throughout the second-order intersections.

Figure 3

For the final case (*Fig.* 3c), 24 $\bar{\mathbf{c}}^{(1)}$'s appear together with 16 conditions on them, emerging from eight closed paths. Here a rank of 13 is obtained for the coefficient matrix of the $\bar{\mathbf{c}}^{(1)}$'s, which means that 11 free parameters are required to describe a general function with this geometry of boundaries. Up to second-order, six $\mathbf{c}^{(1)}$ and four $\mathbf{c}^{(2)}$ parameter vectors are available. This demonstrates the necessity of one additional parameter of type $\mathbf{c}^{(3)}$.

Proceeding to higher dimensions, the number of $\Delta^{(1)}\bar{\mathbf{J}}$'s (or $\bar{\mathbf{c}}^{(1)}$'s) involved grows rapidly, so the described "brute force" approach reaches its limits very fast. Thus we are going to introduce a characterization of intersections, which does not go back all the way to the level of region boundaries but rather involves just the previous two levels of (degenerate) intersections. To this end, the promised prototypic degenerate k-th order intersection will be constructed. Since it will turn out to be the most fundamental boundary configuration for a given order, it is said to be *minimal*.

Def. 3 *A degenerate intersection of order k is called minimal if it is formed by the smallest possible number of $\bar{S}^{(j)}$'s for all $1 \leq j < k$.* ∎

For this type of degenerate intersection, the number of lower order intersection follows a strict law, which is stated in the following lemma.

Lemma 1 *A minimal degenerate intersection $\bar{S}^{(k)}$ ($k \geq 3$) is formed by $k+1$ minimal degenerate intersections of order $k-1$, being the intersections of $\frac{k\cdot(k+1)}{2}$ minimal $\bar{S}^{(k-2)}$'s.* ∎

Here the number of intersections of order $k-2$ is very crucial since in three and more dimensions the number of $\bar{S}^{(k-1)}$'s does not specify the number of intersections of the next lower order involved. This difficulty, having to deal simultaneously with three orders, also to some extend makes clear why the older canonical representations (in one and two dimensions) could not immediately lead to the general form.

Proof First, from the above examples we know that a region boundary is a minimal degenerate intersection of order one, *three* region boundaries form a minimal degenerate intersection of order two (see [3]), and *four* $\bar{S}^{(2)}$'s on six boundaries are required for an $\bar{S}^{(3)}$, which will also turn out to be minimal.

In order to obtain the general structure, we show that each $\bar{S}^{(k-2)}$ involved in $\bar{S}^{(k)}$ carries exactly two $\bar{S}^{(k-1)}$. This will immediately yield the number of $\bar{S}^{(k-1)}$'s and $\bar{S}^{(k-2)}$'s appearing in a minimal degenerate boundary configuration.

Assume first that the $(k-2)$-nd order degenerate intersection $\bar{S}^{(k-2),i^*}$ (being a plane in $\mathcal{N}$) possesses just one $(k-1)$-st order degenerate intersection, say $\bar{S}^{(k-1),j^*}$, which has the $(k-1)$-st order jumps $\Delta^{(k-1)}\bar{\mathfrak{J}}^{j_1^*}$ and $\Delta^{(k-1)}\bar{\mathfrak{J}}^{j_2^*}$, to the left and the right of $S^{(k)}$, respectively. Since $\bar{S}^{(k-2),i^*}$ is two-dimensional, we can "encircle" the point $S^{(k)}$ by a closed path Γ inside this plane, which yields $\Delta^{(k-1)}\bar{\mathfrak{J}}^{j_1^*} - \Delta^{(k-1)}\bar{\mathfrak{J}}^{j_2^*} = 0$ (see *Fig.* 4*a*). From this it immediately follows that $\Delta^{(k-1)}\mathcal{J}^{j^*}$ is constant throughout. Thus $\bar{S}^{(k-2),i^*}$ does not contribute to a $\Delta^{(k)}\bar{\mathfrak{J}}$, thereby implying that the boundary configuration is either not minimal or even not degenerate.

Figure 4

With *two* $\bar{S}^{(k-1)}$ on $\bar{S}^{(k-2),i^*}$ (*Fig.*4*b*), *four* $\Delta^{(k-1)}\bar{\mathfrak{J}}^{i_j^*}$ $(j = 1 \ldots 4)$ appear, which right now have to obey the closed path property (for every boundary, i.e., α-vector, involved in $\bar{S}^{(k-2),i^*}$). Hence three independent $\Delta^{(k-1)}\bar{\mathfrak{J}}$ are found, which means that indeed a k-th order jump in the Jacobian can occur.

Next let us explicitly construct a k-th order minimal degenerate intersection. (For a concrete visualization of the discussion in 3-D please cf. *Fig.* 3*c*.) To this end we assume *exactly two* $(k-1)$-st order degenerate intersections (lines) $\bar{S}^{(k-1),i}$ $(i = 1, 2)$ on the plane $\bar{S}^{(k-2),1}$. Then besides $\bar{S}^{(k-2),1}$ another $(k-1)$ $(k-2)$-nd order intersections, say $\bar{S}^{(k-2),j}$ $(j = 2 \ \ldots \ k)$ and $\bar{S}^{(k-2),j}$ $(j = k+1 \ \ldots \ 2k-1)$, contribute to either of the two $\bar{S}^{(k-1)}$. These $2\cdot(k-1)$ planes possess $(k-1)^2$ mutual, non-degenerate intersections. Let us pick one of these $S^{(k-1)}$'s, the intersection of $\bar{S}^{(k-2),2}$ and $\bar{S}^{(k-2),k+1}$. In order to make it the degenerate intersection $\bar{S}^{(k-1),3}$, another $k-2$ new $\bar{S}^{(k-2)}$'s have to be added, which can be chosen to "connect" $\bar{S}^{(k-1),3}$ with $k-2$ of the previously constructed $(k-1)^2$ intersections of order $k-1$ which now involve three $\bar{S}^{(k-2)}$ each. This step terminates the process in the three-dimensional case.

For $k \geq 4$ we have to continue, since we need *two* $\bar{S}^{(k-1)}$'s on every $\bar{S}^{(k-2)}$, $\bar{S}^{(k-1),4}$ has to be constructed. To this end let us pick one of the previously constructed threefold intersections. To make it degenerate, $k-3$ new planes must be added, leaving us with $k-3$ fourfold (for $k \geq 5$ non-degenerate) intersections. This scheme continues until $2+k-1 = k+1$ $\bar{S}^{(k-1)}$'s are introduced. Thus, in the end the number of $\bar{S}^{(k-2)}$'s employed is

$$1 + 2\cdot(k-1) + \underbrace{\frac{(k-1)\cdot(k-2)}{2}}_{\sum_{i=1}^{k-2} i} = \underbrace{\frac{k\cdot(k+1)}{2}}_{\sum_{i=1}^{k} i} \quad . \tag{7}$$

This is a very appealing result since on the one hand it tells us that in order to obtain a k-th order degenerate boundary configuration, $k+1 = 1 + dim\,\mathcal{N}$ one-dimensional $\bar{S}^{(k-1)}$'s have to contribute and that every "pair" of $\bar{S}^{(k-1)}$'s spans an $\bar{S}^{(k-2)}$, thereby leading to a completely symmetric, maximally connected situation. A symbolic representation of these properties is depicted in *Fig.* 5. Another remarkable property of this type of boundary configuration is that the $\gamma^{(k-1)}$-vectors (defining the $\bar{S}^{(k-1)}$'s) form a minimal set of vectors which both span the whole space $\mathcal{N}$ and are linearly dependent. This looks very similar to the rank criterion for degeneracy in [2]. However, without a concept of higher order degeneracies, the roles of (what we now call) $\bar{S}^{(1)}$ and $\bar{S}^{(k-1)}$, which coincide in $\mathbb{R}^2$, were mixed up. Thus, although stated for $\mathbb{R}^n$, this criterion gives the desired answer only in $\mathbb{R}^2$.

Figure 5

If we follow the above scheme to the next lower order of intersections, in every $\bar{S}^{(k-2)}$, $k-1$ $\bar{S}^{(k-3)}$'s intersect, with each of them being capable of participating in *three* $(k-2)$-nd order intersections, etc. This immediately yields the number of l-th order degenerate intersections involved in a k-th order boundary configuration:

$$N_k(l) := (k+1) \times \frac{k}{2} \times \frac{k-1}{3} \times \ \ldots \ \times \frac{l+2}{k-l} = \prod_{j=1}^{k-l} \frac{k+2-j}{j} \quad . \tag{8}$$

Eventually, as we are going to count the boundaries (i.e, the α-vectors) involved in a minimal intersection, their number turns out to be

$$N_k := N_k(1) = \frac{k \cdot (k+1)}{2} \quad . \tag{8a}$$

This again is a very interesting result, which will be fully appreciated later, when the representation of a function is constructed explicitly [6]. Here let us just mention that $N_k = N_{k-1} + k$, this is the number of α-vectors involved in *one* $\bar{S}^{(k-1)}$ plus $\dim \mathcal{N}$, which means that the basis spanning $\mathcal{N}$ can be obtained in a natural manner from the α-vectors of those region boundaries which have to be added in order to obtain an $S^{(k)}$ from a single $(k-1)$-st order degenerate intersection (see *Fig.* 6).

Figure 6

Finally, to complete the proof, we have to demonstrate that an $\bar{S}^{(k-2)}$ containing more than two $\bar{S}^{(k-1)}$'s leads to a boundary configuration with more than $k+1$ degenerate intersections of order $k-1$. To this end, let us assume r (> 2) $\bar{S}^{(k-1),i_j^*}$ $(j = 1 \ldots r)$ in $\bar{S}^{(k-2),i^*}$. Since the planes intersecting in, say $\bar{S}^{(k-1),i_{j^*}^*}$, have no further mutual intersections, each of them has to carry one more $(k-1)$-st order intersection. Thus the number of $\bar{S}^{(k-1)}$'s in such a boundary configuration is at least $r + (k-1)$, which is greater than the minimal number obtained above. ∎

As an immediate consequence of our construction of a minimal k-th order intersection, we can state the following lemma:

Lemma 2 *In a minimal $\bar{S}^{(k)}$ every α-vector is either considered a basis vector of $\mathcal{N}$ or it is a linear combination of two basis vectors.* ∎

This property will turn out to be of fundamental importance for the concrete representation of PWL functions since it tells us that the scheme of substituting a vector by an expression of *two* basis vectors, as developed in [3], is also valid for higher order intersections, which is not at all obvious.

Proof Of the N_k α-vectors in $\bar{S}^{(k)}$ let us pick the k vectors α^i $(i = N_{k-1}+1 \ldots N_k)$ as a basis (see above). Let us further assume that the boundaries $\bar{S}^{(1),1} \ldots \bar{S}^{(1),N_{k-1}}$ form an $\bar{S}^{(k-1)}$, say $\bar{S}^{(k-1),1}$. Due to the symmetry of the boundary configuration this can be done without loss of generality.

Now the k $\bar{S}^{(1)}$'s belonging to the basis vectors have $\frac{k(k-1)}{2} = N_{k-1}$ mutual intersections of type $S^{(2)}$. In order to make them degenerate second-order intersections, each of the N_{k-1} boundaries of $\bar{S}^{(k-1),1}$ also has to participate in one of them. Then the overall number of degenerate second-order intersections is $N_{k-1}(2) + N_{k-1} = N_k(2)$ (this formula is a special case of an identity which will be stated and proved below), as is required for $\bar{S}^{(k)}$.

Hence we found that N_{k-1} of the $N_k(2)$ second-order degenerate intersections which contribute to $\bar{S}^{(k)}$ involve in each case *two* basis vectors and one α from $\bar{S}^{(k-1),1}$. Since the three α-vectors in an $\bar{S}^{(2)}$ are linearly dependent, each of the α^i $(i = 1 \ldots N_{k-1})$ can indeed be written as a linear combination of two basis vectors. ∎

Now that the two fundamental lemmas dealing with the geometry of degenerate intersections have been proved, we are ready to investigate the crucial property of the functional representation — the number of parametric degrees of freedom.

4. The Number of Effective Parameters

With the $\Delta^{(k)}\mathcal{J}$ functions introduced above, the structure of the canonical representation is already obtained, albeit, on a rather abstract level. A concrete formulation will be presented in [6]. The major problem we are facing right now is the bookkeeping of the interdependences of the various $\Delta^{(k)}\mathcal{J}$'s present, which are reflected in the side conditions guaranteeing continuity. Eventually two properties will emerge from the canonical representation: (a) It is a closed form with all the side conditions being absorbed, i.e., they are properties of the representation with no need to be stated explicitly. And (b) all parametric degrees of freedom are realized, i.e., the representation contains the correct number of parameters and all of them are independent.

In order to put this into an exact formulation, we present:

Lemma 3 *Every minimal degenerate intersection $\bar{S}^{(k)}$ adds exactly one parametric degree of freedom to a superposition of its constituents.* ∎

This lemma provides the key to the concrete representation because it allows us to construct a k-th order jump along one arbitrary $\bar{S}^{(k-1)}$, thereby assigning the free parameter. Then all the other jumps appearing have to be automatically correct.

Proof For *first* and *second*-order degenerate intersections the property stated in *Lemma* 3 is demonstrated in [1] and [3], respectively, by the representations and their completeness in one and two dimensions, respectively. For an arbitrary $\bar{S}^{(k)}$ we use induction, assuming that the property holds for all $\bar{S}^{(j)}$ $(j = 1 \ldots k-1)$ involved.

To prove that an intersection $\bar{S}^{(k)}$ possesses exactly one parametric degree of freedom, we follow the construction of a minimal k-th order degenerate intersection (and utilize the symbols) introduced in the proof of *Lemma* 1.

The plane $\bar{S}^{(k-2),1}$ (see *Fig.* 4*b*) possesses four segments, separated by the two intersections $\bar{S}^{(k-1),i}$ $(i = 1,2)$, on which we find four $(k-2)$-nd order jumps $\Delta^{(k-2)}\bar{\mathbf{J}}^{1_j}$ $(j = 1 \ldots 4)$. The differences between these jumps are the four $(k-1)$-st order jumps $\Delta^{(k-1)}\bar{\mathbf{J}}^{1_l}$ and $\Delta^{(k-1)}\bar{\mathbf{J}}^{2_l}$ $(l = 1,2)$. All of these jumps should carry an additional index, referring to their α-vector (cf. *Eq.* 6 above). However, since we assume *Lemma* 3 to hold for all intersections of order smaller than k, we can pick any α contributing to a specific intersection and characterize the intersection by the "jump in this direction"; all jumps with respect to the other α's can then be obtained due to the lemma. Thus the α-index will be suppressed and the jump functions used are considered to be generic.

Due to the closed path property, three out of the four $\Delta^{(k-1)}\bar{\mathbf{J}}$'s can be chosen freely. So we pick the characterizing parameters $\Delta^{(k-1)}\mathbf{J}^1$, $\Delta^{(k-1)}\mathbf{J}^2$, and $\Delta^{(k)}\mathbf{J}$, which gives us $\Delta^{(k-1)}\bar{\mathbf{J}}^{1_{1/2}} = \Delta^{(k-1)}\mathbf{J}^1 \pm \frac{1}{2}\Delta^{(k)}\mathbf{J}$ and $\Delta^{(k-1)}\bar{\mathbf{J}}^{2_{1/2}} = \Delta^{(k-1)}\mathbf{J}^2 \pm \frac{1}{2}\Delta^{(k)}\mathbf{J}$. Thus $\Delta^{(k)}\mathbf{J}$, via (6), contains the parameter vector characterizing $\bar{S}^{(k)}$.

Now it is an easy task to complete the proof and demonstrate that no more parametric degrees of freedom emerge from a minimal boundary configuration. For this purpose let us pick the plane $\bar{S}^{(k-2),2}$ (and any convenient α-vector), carrying $\bar{S}^{(k-1),1}$ and $\bar{S}^{(k-1),3}$. Here we find again three parameters to be filled in; with two of them, $\Delta^{(k-1)}\mathbf{J}^1$ and $\Delta^{(k)}\mathbf{J}$, already set in the previous step of the construction, only one, $\Delta^{(k-1)}\mathbf{J}^3$, is left for the characterization of $\bar{S}^{(k-1),3}$. Following the path of the construction of a minimal boundary configuration, we can access all the remaining $k-1$ $\bar{S}^{(k-1)}$'s in a plane where one $(k-1)$-st order intersection is already characterized by its $\Delta^{(k-1)}\mathbf{J}$ and $\Delta^{(k)}\mathbf{J}$. This shows that, compared to a non-degenerate boundary configuration containing $k+1$ different $\bar{S}^{(k-1)}$'s, exactly one new parametric degree of freedom, in the form of $\Delta^{(k)}\mathbf{J}$, emerges. ∎

This is again a very appealing result, since it tells us that, no matter how complicated the geometry of a k-th order intersection is, it contributes *exactly one* term to the representation. This term describes the *interaction* of the contributing $(k-1)$-st order intersections and its weight is controlled by a parameter vector from the range space. Our goal in [6] will be to write the above discussed minimal boundary configurations as a genuine term of the highest order appearing and a *superposition* of functions describing its constituents, which themselves are formed similarly:

$$\mathbf{F}(\mathbf{x}) = \sum_{j=1}^{k} \sum_{i=1}^{N_k(j)} \mathbf{c}^{(j),i} \cdot F^{(j),i} \quad . \tag{9}$$

Here the scalar functions $F^{(j),i}$ contain the geometric information while the parameter vectors $\mathbf{c}^{(j),i}$ determine the values of the function. Moreover, for a given j^* all the $F^{(j^*),i}$ are of the "same type", i.e., they can be transformed into each other by linear maps. For example, the generic first-order function is $F^{(1)} = |\langle\alpha, \mathbf{x}\rangle|$ while for the generic second-order term we found $F^{(2)} = ||\langle\alpha^1, \mathbf{x}\rangle| + \langle\alpha^2, \mathbf{x}\rangle| - |\langle\alpha^1, \mathbf{x}\rangle + |\langle\alpha^2, \mathbf{x}\rangle||$.

As a corollary of *Lemmas* 1 and 3 we now present the following result which yields the *total* number of parametric degrees of freedom for a minimal boundary configuration:

Lemma 4 *The total number of free parameters for a minimal degenerate boundary configuration of order k is*

$$T_k = \sum_{j=1}^{k} N_k(j) \quad . \tag{10}$$

This yields the recursive formula

$$T_1 = 1 \quad ; \quad T_k = 2T_{k-1} + k \qquad (k > 1) \tag{11}$$

■

Proof The first statement is an immediate consequence of *Lemma* 3. In a minimal boundary configuration of order k, each order j of degeneracy contributes exactly $N_k(j)$ parameters.

For the second part we have to utilize an identity coming from *Eq.* 8, where a special case was applied above already:

$$N_k(l) = N_{k-1}(l) + N_{k-1}(l-1) \qquad (2 \leq l \leq k-1) \quad . \tag{12}$$

This can be seen from

$$\begin{aligned}
N_{k-1}(l) + N_{k-1}(l-1) &= \prod_{j=1}^{k-1-l} \frac{k+1-j}{j} + \prod_{j=1}^{k-l} \frac{k+1-j}{j} \\
&= \prod_{j=1}^{k-1-l} \frac{k+1-j}{j} \cdot \left(1 + \frac{1+l}{k-l}\right) \\
&= (k+1) \cdot \left(\prod_{j=1}^{k-1-l} \frac{k+1-j}{j} \right) \cdot \frac{1}{k-l} \qquad (13) \\
&= (k+1) \cdot \prod_{j=1}^{k-1-l} \frac{k+1-j}{j+1} \\
&= N_k(l) \quad .
\end{aligned}$$

Now it is an easy task to evaluate T_k directly in terms of T_{k-1} by utilizing (12), in order to decompose the sum over $N_k(j)$ into two components:

$$\begin{aligned}
T_k &= N_k + \sum_{j=2}^{k-1} N_k(j) \ + \ 1 \\
&= \left(N_{k-1} + \sum_{j=2}^{k-1} N_{k-1}(j) \right) \ + \ \left(1 + \sum_{j=1}^{k-2} N_{k-1}(j) \right) \ + \ k \qquad (14) \\
&= 2 \cdot T_{k-1} + k \quad .
\end{aligned}$$

This concludes the proof of *Lemma* 4. ∎

*Lemma*4 makes clear how fast (stronger than exponentially) the number of parameters is growing with the order of degeneracy. This behavior is mainly due to the numerous degenerate intersections of intermediate orders appearing in such a boundary configuration. Nevertheless, the number of the most prominent degenerate intersections, those of orders one and $k-1$, grow just quadratically and linearly, respectively, with the order of the highest degenerate intersection.

5. Discussion

An analysis of the *nonlinearity component* $\mathcal{N}$, the subspace relevant for the nonlinear behavior of a PWL function, turns out to be a fruitful concept. First of all, it allows us to extract those directions in space where a function behaves just linearly. Thereby it turns out that, at least conceptually, it is not necessary to formulate a function's representation in a space bigger than the nonlinearity component.

Since the dimension of $\mathcal{N}$ is directly connected to the highest order of intersections appearing, thereby rendering the latter object a point in $\mathcal{N}$, it also yields an upper bound for the order of functional representations to be employed. The exact order, however, is only accessible after a more detailed analysis, which requires the determination the *degeneracies* of the boundary configuration (non-degenerate intersections can be desribed as simple superpositions of their constituents). Here we introduced a general concept of degeneracy for the intersections of region boundaries and linked it to the parametric degrees of freedom of PWL functions, which is the main contribution of the present work.

For a special type of boundary configurations, *minimal degenerate intersections*, the number of parameters for every order of intersection could be obtained. These configurations play a prominent role: On the one hand, they allow us to distinguish between a degenerate intersection of order k and a mere superposition of lower order intersections. On the other hand, we shall show in a forthcoming paper how all other k-th order intersections can be derived from the minimal one.

Finally, we would like to mention that the characterization of boundaries introduced here also indicates the form of the general representation of PWL functions, with one term representing each degenerate intersection. As mentioned already, this problem will be treated in [6], where the major problem will be to find a "pure representation" for each order.

Appendix

Recently, an extension of the canonical representation from [1] was proposed in [4], using an approach which is different from the one introduced in [3]: The idea elaborated there is to replace some of the hyperplanes by "piecewise-affine planes". This results in a representation of the form

$$\begin{aligned}\mathbf{f}(\mathbf{x}) = \mathbf{s} + \mathbf{J}\mathbf{x} + \sum_{i=1}^{l} \mathbf{b}_i \left|\langle \alpha_i, \mathbf{x}\rangle + \beta_i\right| + \\ \sum_{j=1}^{p} \mathbf{c}_j \left|\delta_j + \langle \gamma_j, \mathbf{x}\rangle + \sum_{i=1}^{l} d_{ij} \left|\langle \alpha_i, \mathbf{x}\rangle + \beta_i\right|\right| \quad .\end{aligned} \tag{$A1$}$$

For the different symbols and their meaning cf. [4]. Let us just mention that the first sum ranges over l hyperplanes while the second one covers the p piecewise-affine planes, having their breakpoints at hyperplanes.

In order to put this result into perspective and to avoid further confusion about the class of functions described by this representation, we show how $(A1)$ can be derived from *Eq.* 2 of [3] by imposing *three* conditions on a second-order representation: (a) All the degeneracies appearing are either of degree-zero or one ($\forall j \in \{\texttt{node labels}\}\ \delta^j \in \{0,1\}$). For any node j^* with $\delta^{j^*} = 1$, the parameter vectors of the first and second-order terms have to be linked by (b) $\mathbf{c}^{(1),j_2^*} \pm a_{j_1^* j_2^*} \mathbf{c}^{(2),j^*} = 0$ and (c) $\mathbf{c}^{(1),j_3^*} \pm a_{j_1^* j_3^*} \mathbf{c}^{(2),j^*} = 0$. The effect of the latter two conditions was visualized in the examples of [3] in the form of "terminating boundaries". The actual sign applying in the conditions depends on the sign convention employed (see [3]) and the orientation of the two branches of a piecewise-affine plane which meet at the hyperplane defined by $\langle \alpha^{j_1^*}, \mathbf{x}\rangle = \beta^{j_1^*}$ (note the different sign conventions for the β's in [4] and [3]).

Figure 7

Let us now look at one intersection as depicted in *Fig.* 7 and show how to obtain $(A1)$ from a second-order representation. This boundary configuration can be chosen without loss of generality since both the representations $(A1)$ and (2) in [3] are linear superpositions of nodes, and, besides the simple intersection of two hyperplanes, the depicted situation is the only one that can appear in $(A1)$. As usual, constant shifts are neglected. The three pre-boundaries in *Fig.* 7 already reflect condition (a). Next, let us transform conditions (b) and (c) into a form appropriate for $(A1)$. Since this representation requires

$$\alpha^2 = \frac{1}{2}\left(\alpha^2 + \alpha^3\right) + d\alpha^1 \tag{$A2a$}$$

and

$$\alpha^3 = \frac{1}{2}\left(\alpha^2 + \alpha^3\right) - d\alpha^1 \tag{$A2b$}$$

we have to rescale the α-vectors, which yields a condition on the expansion coefficients a_{1j}:

$$\alpha^1 = a_{12}\alpha^2 + a_{13}\alpha^3 = \frac{1}{2d}\alpha^2 + \frac{1}{2d}\alpha^3 \quad . \tag{A3}$$

This gives us the new set of conditions: (b') $a_{12} = -a_{13} = \frac{1}{2d} =: a$ and (c') $a\mathbf{c}^{(2)} = \mathbf{c}^{(1),2} = \mathbf{c}^{(1),3} =: \mathbf{c}^{(1)}$. Actually, the α-vectors could be scaled to make both a and d equal to unity. However, since this cannot be achieved for all nodes simultaneously, for the sake of generality, we omit it. Now it is just a matter of some algebra to plug the conditions (a), (b'), and (c') into *Eq.* 2 of [3] and show that

$$\begin{aligned}
&\mathbf{c}^{(1)}\langle\alpha^2+\alpha^3,\mathbf{x}\rangle + \\
&\mathbf{c}^{(1),1}\left|\langle\alpha^1,\mathbf{x}\rangle\right| + \mathbf{c}^{(1)}\left(\left|\langle\alpha^2,\mathbf{x}\rangle\right| + \left|\langle\alpha^3,\mathbf{x}\rangle\right|\right) + \\
&\frac{\mathbf{c}^{(1)}}{a}\left\{\Big|\left|a\langle\alpha^2,\mathbf{x}\rangle\right| - a\langle\alpha^3,\mathbf{x}\rangle\Big| - \Big|a\langle\alpha^2,\mathbf{x}\rangle + \left|a\langle\alpha^3,\mathbf{x}\rangle\right|\Big|\right\} \\
&\quad = \mathbf{c}^{(1),1}\left|\langle\alpha^1,\mathbf{x}\rangle\right| + 2\mathbf{c}^{(1)}\left|\frac{1}{2}\langle\alpha^2+\alpha^3,\mathbf{x}\rangle + d\left|\langle\alpha^1,\mathbf{x}\rangle\right|\right| \quad .
\end{aligned} \tag{A4}$$

(The readers are invited to verify this equation.) The right-hand side of ($A4$) obviously has the required form of ($A1$) with the constant and linear term being omitted.

Besides being limited to intersections of degeneracy one, two more major properties of the representation from [4] become clear from *Eq.* $A4$. The first one is the reduction of the parametric degrees of freedom from *four* to *two*, which is an immediate consequence of the conditions (b) and (c), and thus, besides the flexibility in the boundaries gained, it does not possess more parametric degrees of freedom than a non-degenerate intersection of two boundaries. Moreover, ($A1$) does not share the symmetry properties of the representation introduced in [3] (which will turn out to be fundamental also for the higher order representations [6]). So the final sum, with the nested absolute values, is in fact a superposition of terms of all orders appearing (up to second-order), which means that a modification there cannot be isolated in general, but might require "corrections" in other terms of the representation.

References

[1] L.O. Chua and S.M. Kang, "Section-wise piecewise-linear functions: Canonical representation, properties, and applications," *Proceedings of the IEEE* vol. **65**, pp. 915–929, 1977.

[2] L.O. Chua and A.-C. Deng, "Canonical piecewise-linear representation," *IEEE Trans. Circuits Syst.*, vol. **35**, pp. 101–111, 1988.

[3] C. Kahlert and L.O. Chua, "A generalized canonical piecewise-linear representation," *IEEE Trans. Circuits Syst.*, vol. **37**, in press, 1990.

[4] C. Güzelis and I.C. Göknar, "A canonical representation for piecewise-affine maps and its applications to circuit analysis," preprint, submitted to *IEEE Trans. Circuits Syst.*, 1989.

[5] T. Ohutsuki, T. Fujisawa, and S. Kumagai, "Existence theorems and a solution algorithm for piecewise-linear resistor networks," *SIAM J. Math. Anal.*, vol. **8**, pp. 69–99, 1977.

[6] C. Kahlert and L.O. Chua, "The complete canonical piecewise-linear representation, II. The functional form," *IEEE Trans. Circuits Syst.*, to be published.

Captions

Figure 1 (a) Two lines in $\mathbb{R}^2$, being themselves first-order intersections, meet in a second-order intersection, $S^{(2)}$. (b) A boundary configuration with three (non-parallel) planes in $\mathbb{R}^3$ possesses three second-order intersections. If they coincide in one point, a third-order intersection appears. (c) However, a line intersecting a plane is not a valid boundary configuration.

Figure 2 With three boundaries meeting in one point in $\mathbb{R}^2$, the α-vectors and with them the jumps in the Jacobian become linearly dependent. This causes the breakdown of the consistent variation property, making the intersection degenerate of order two.

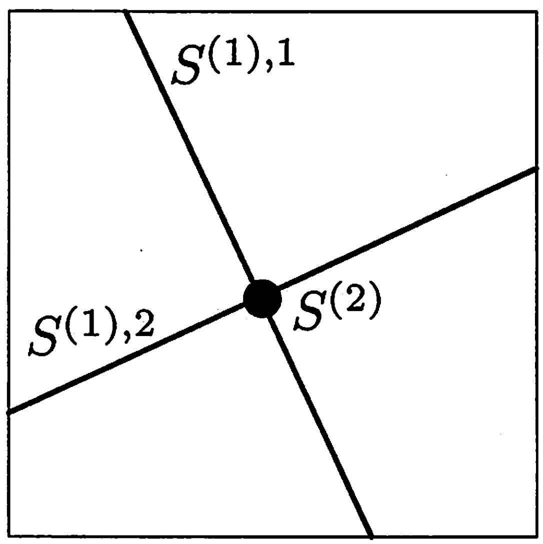
Figure 3 Three examples of third-order intersections at the origin. (a) In spite of a third-order intersection, no degeneracies (except the boundaries, being of first-order) appear. Hence this boundary configuration can be described by *Eq.* 1. (b) A superposition of two degenerate second-order intersections, of first and second-degree, respectively. This situation is fully described by *Eq.* 2. (c) Four degenerate second-order intersections, $\bar{S}^{(2)}$'s, are required to render the origin a degenerate third-order intersection. This lets us gain one additional parametric degree of freedom.

Figure 4 Representation of the situation on a selected $\bar{S}^{(k-2)}$. (a) With just one $\bar{S}^{(k-1)}$ residing on the plane ($\bar{S}^{(k-2)}$), from the closed path property, it becomes obvious that the two $(k-1)$-st order jumps in the Jacobian on both sides of $S^{(k)}$ have to coincide. (b) Having two $\bar{S}^{(k-1)}$ available, allows four $\Delta^{(k-1)}\mathbf{J}$'s with up to three (via the closed path property) independent values. This eventually permits non-uniform $\Delta^{(k-1)}\mathcal{J}$ functions and thereby a k-th order jump in the Jacobian.

Figure 5 A symbolic representations of second (a), third (b), and fourth (c) order minimal degenerate intersections. The $(k-1)$-st order degenerate intersections are represented by the corners while the $\bar{S}^{(k-2)}$'s are visualized as edges. This demonstrates the close relation of polygons in the plane and the higher order degenerate intersections, stressing again the co-space properties.

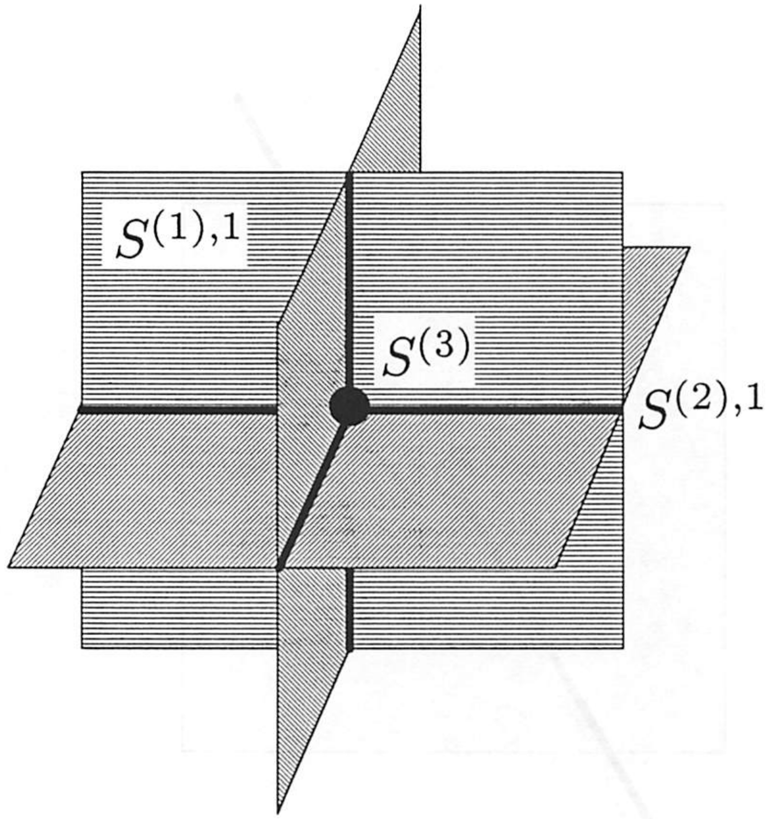
Figure 6 The boundary configuration of *Fig.* 3c with the boundaries grouped. The first set (light gray) are those $\bar{S}^{(1)}$'s forming a degenerate second-order intersection. The three other boundries are the "additional" ones, which are necessary to obtain a $\bar{S}^{(3)}$. Their corresponding α-vectors form a basis of $\mathcal{N}$.

Figure 7 The only non-trivial boundary configuration described by [4]. Since two of the boundaries terminate, the origin is a true second-order intersection. However, these terminations are mandatory, so the representation ($A1$) does not possess more parametric degrees of freedom than a boundary configuration of *two* boundaries, i.e., two (*three* emerging from first-order contributions *plus one* from the second-order intersection *minus two* due to side conditions).



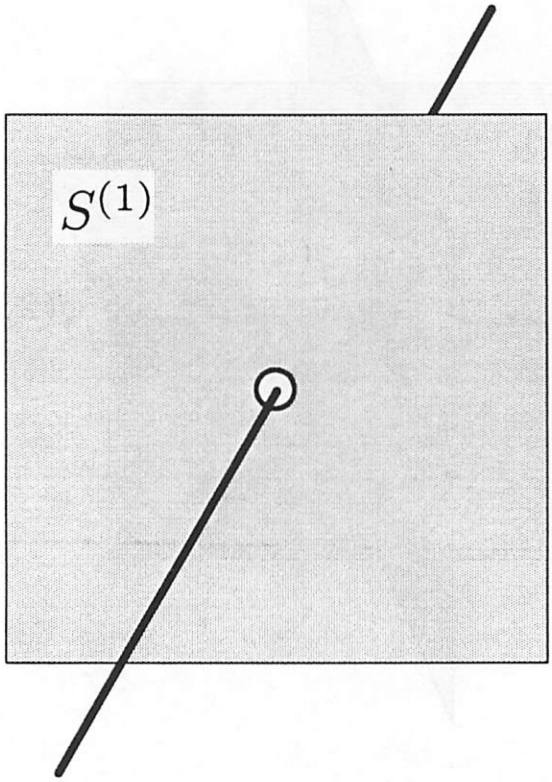
(a)

Figure 1



(b)

Figure 1



(c)

Figure 1

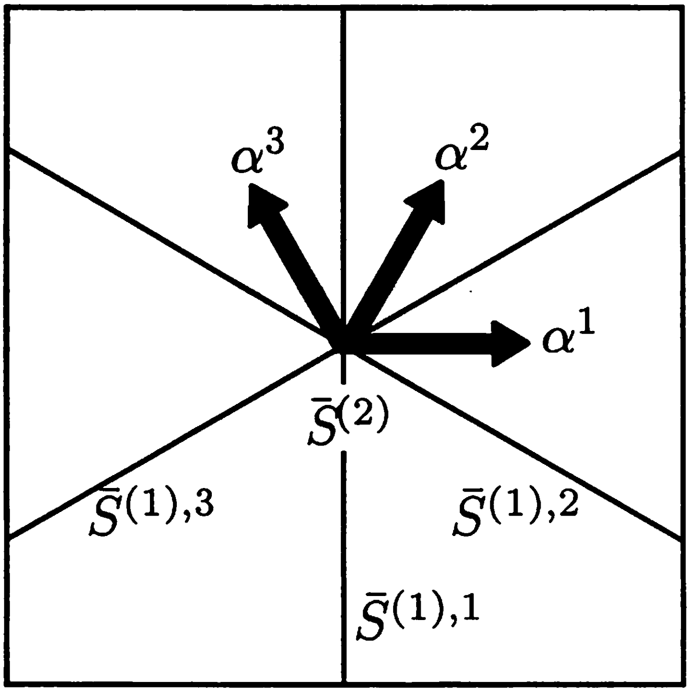
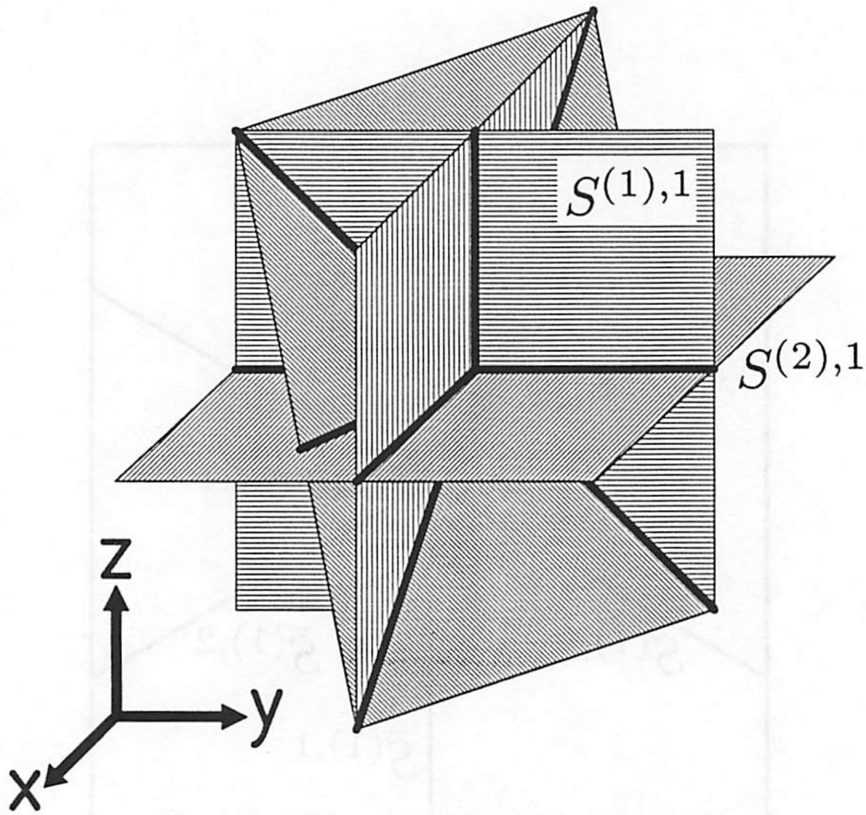
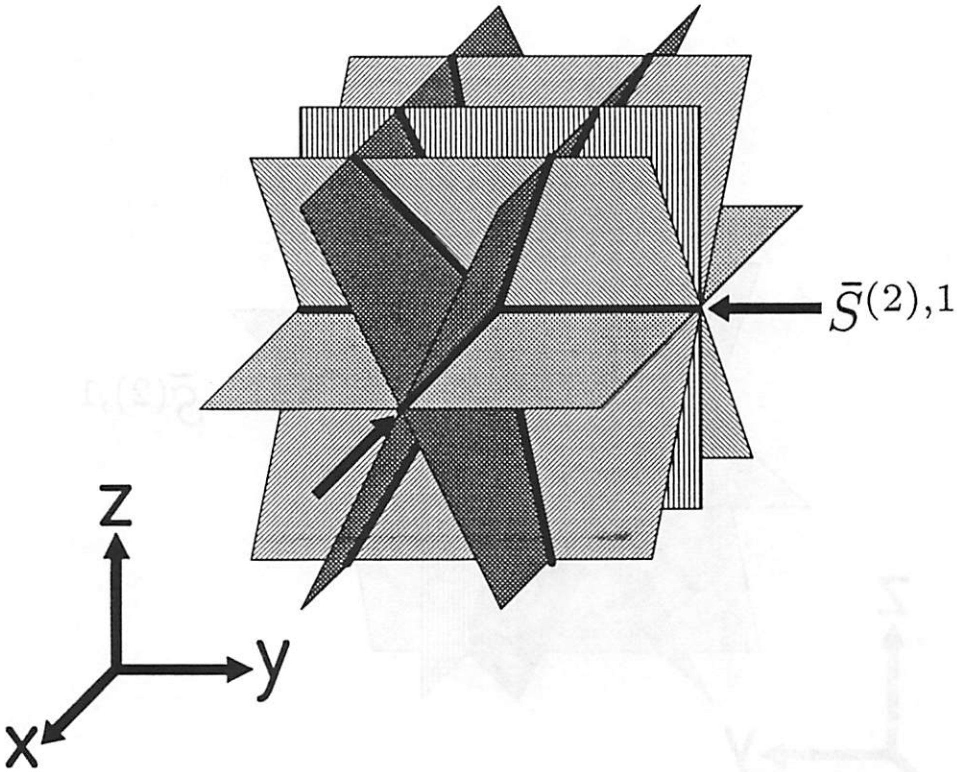


Figure 2



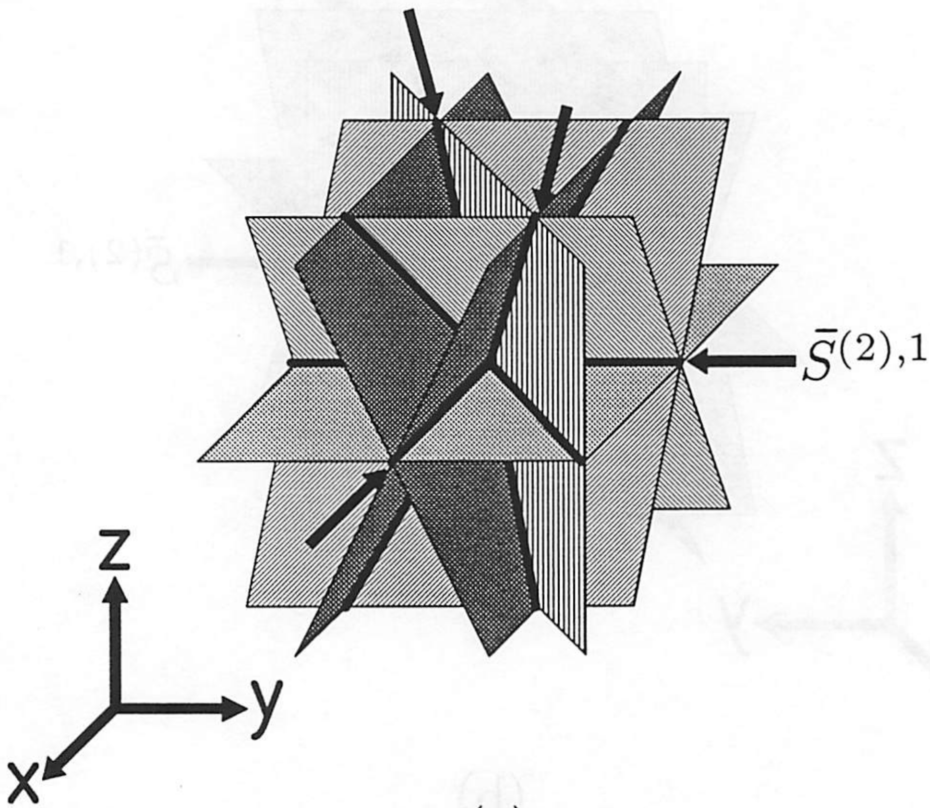
(a)

Figure 3



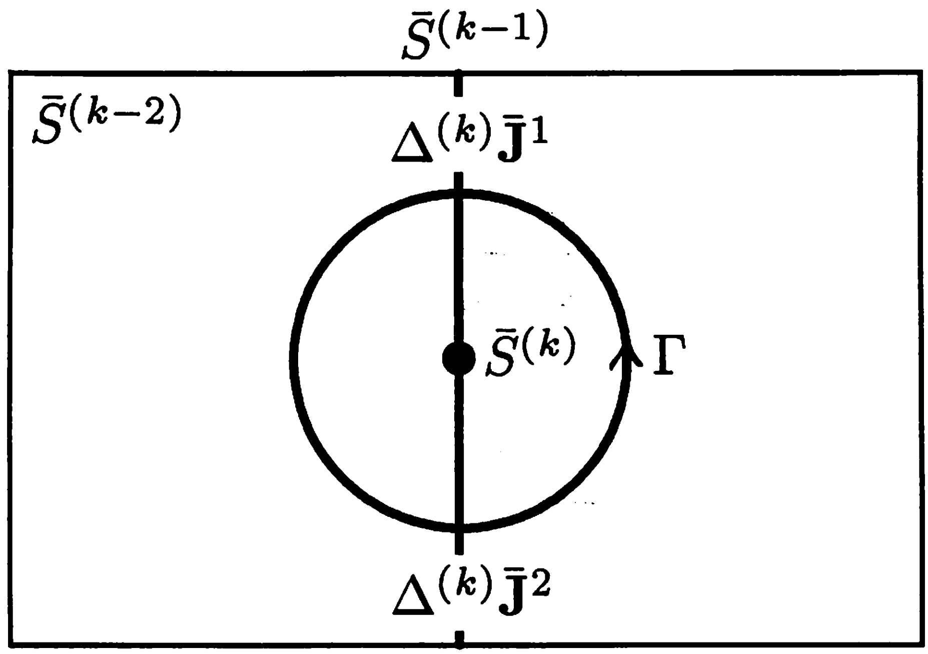
(b)

Figure 3



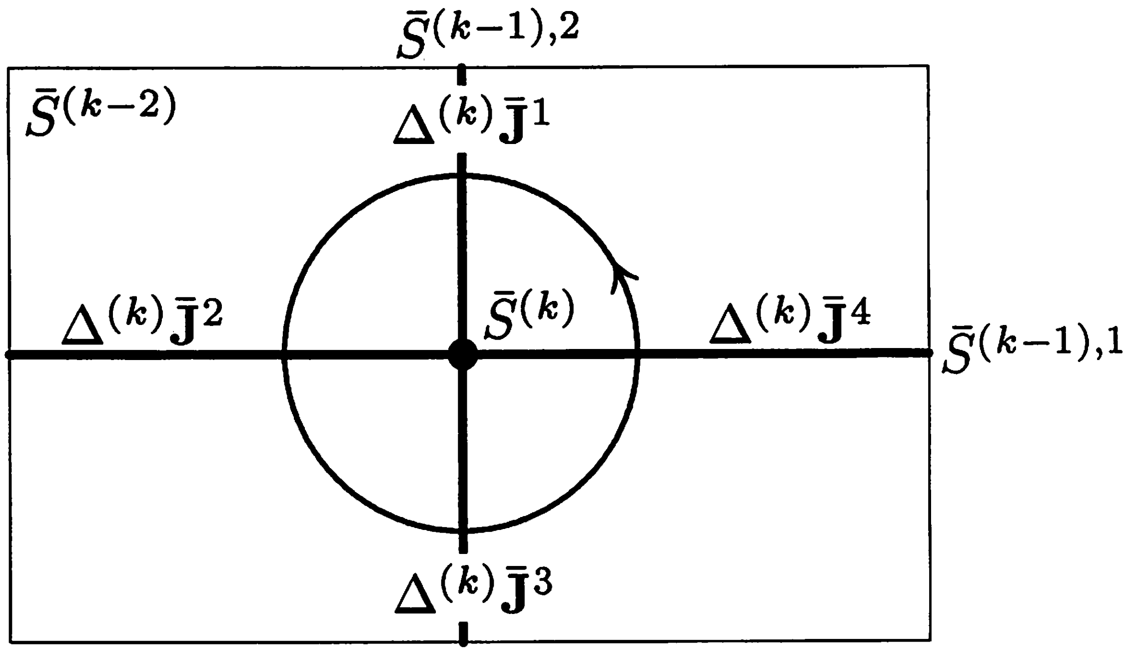
(c)

Figure 3


$\bar{S}^{(k-1)}$
$\bar{S}^{(k-2)}$
$\Delta^{(k)}\bar{\mathbf{J}}^1$
$\bar{S}^{(k)}$
Γ
$\Delta^{(k)}\bar{\mathbf{J}}^2$

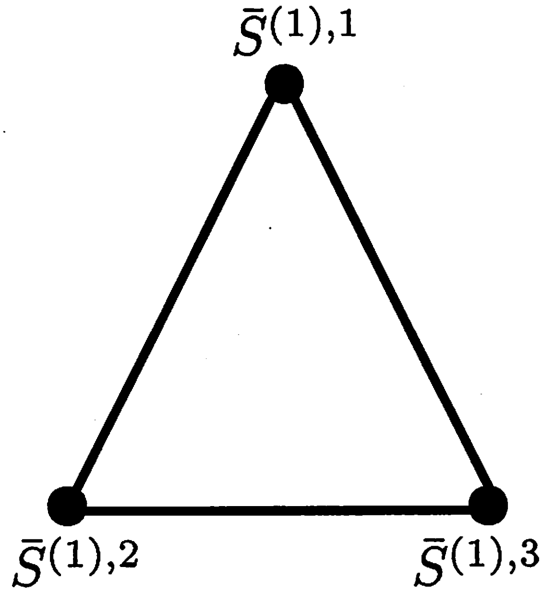
(a)

Figure 4


$\bar{S}^{(k-1),2}$
$\bar{S}^{(k-2)}$
$\Delta^{(k)}\bar{\mathbf{J}}^1$
$\Delta^{(k)}\bar{\mathbf{J}}^2$
$\bar{S}^{(k)}$
$\Delta^{(k)}\bar{\mathbf{J}}^4$
$\bar{S}^{(k-1),1}$
$\Delta^{(k)}\bar{\mathbf{J}}^3$

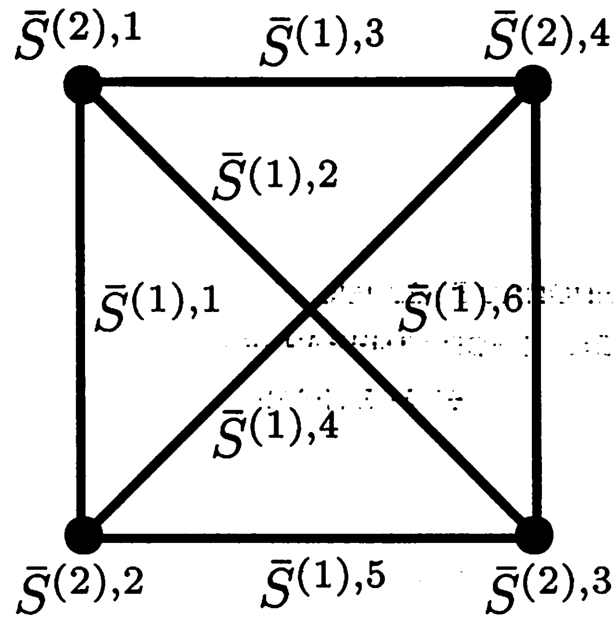
(b)

Figure 4



(a)

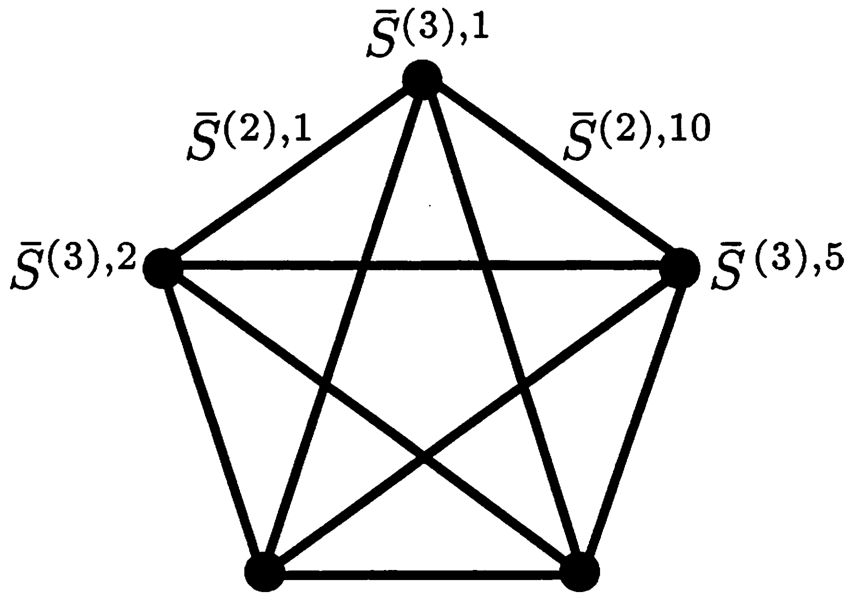
Figure 5



$\bar{S}^{(2),1}$
$\bar{S}^{(1),3}$
$\bar{S}^{(2),4}$
$\bar{S}^{(1),2}$
$\bar{S}^{(1),1}$
$\bar{S}^{(1),6}$
$\bar{S}^{(1),4}$
$\bar{S}^{(2),2}$
$\bar{S}^{(1),5}$
$\bar{S}^{(2),3}$

(b)

Figure 5



(c)

Figure 5

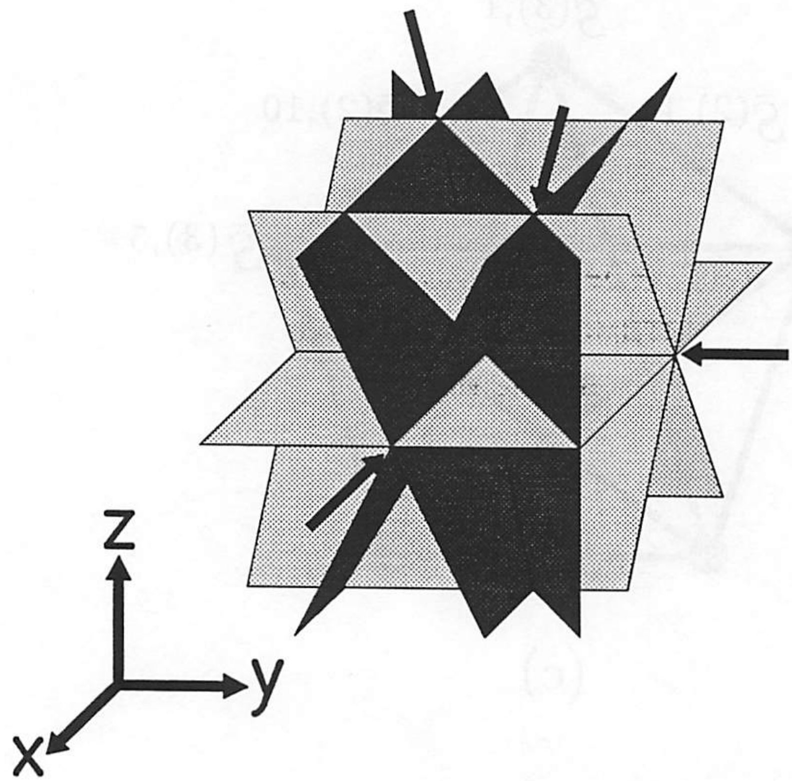


Figure 6

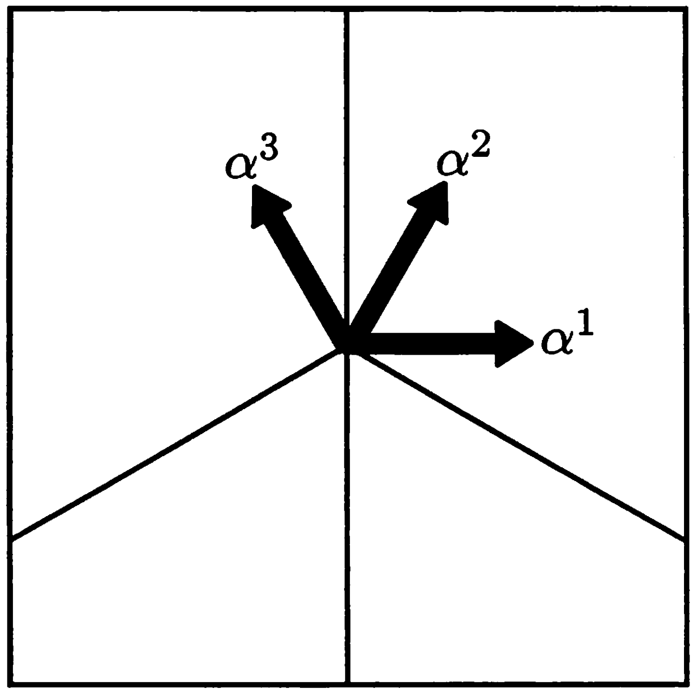


Figure 7