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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

THE IDENTIFICATION OF PSEUDO-GRADIENT VECTOR FIELDS. †

Robert Lum AND Leon O. Chua. ††

Abstract

A vector field is called pseudo-gradient if it is either the composition of a matrix with a gradient vector field or under composition with a matrix becomes a gradient vector field. Of particular interest are those pseudo-gradient vector fields formed from the composition of a matrix with a gradient vector field. Such vector fields are especially amenable to construction as electronic circuits.

In this paper, the identification of such vector fields is completed for the cases when the matrix is either invertible, invertible symmetric, symmetric positive definite or diagonal positive definite. In the process of such identification, a decomposition of the original vector field as the composition of a matrix and a gradient vector field will ensue. The algorithm for identification is sufficiently deterministic to be fully implementable as part of a larger software package dealing with electronic circuits.

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†† The authors are with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA.

§0. Introduction.

In the paper[3], "Invariance properties of continuous piecewise-linear vector fields," several different types of vector fields were presented. For some of these types of vector fields necessary and sufficient conditions were imposed for their identification. However, one large and important class, pseudo-gradient piecewise-linear vector fields, had unanswered questions to whose resolution this paper is addressed.

The particular question that concerns this paper considers the decomposition of a piecewise-linear vector field as the composition of a matrix, either invertible, invertible symmetric, symmetric positive definite or diagonal positive definite, and a gradient piecewise-linear vector field. Thus, it is the identification of a psuedo-gradient vector field for which there is a decomposition with a matrix that is either invertible, invertible symmetric, symmetric positive definite or diagonal positive definite.

Resolution of the above question allows the quick and efficient identification of piecewise-linear vector fields whose electronic implementation is less complicated than the general piecewise-linear vector field but not as simple as the gradient piecewise-linear vector field.

§1. Definitions.

In this section the definitions of the different piecewise-linear vector fields are presented.

Definition 1.1. A continuous piecewise linear vector field ξ in n independent variables is given by

$$\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}\left|\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}-\gamma_j\right|$$

where $0<\alpha_{j1}^2+\ldots+\alpha_{jn}^2$, $0<\beta_{j1}^2+\ldots+\beta_{jn}^2$ for $j=1\ldots m$. Henceforth, continuous piecewise linear vector fields will be called vector fields.

Definition 1.2. The vector field ξ is a gradient vector field if there exists a function G such that

$$\begin{aligned}\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}&=\nabla G\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\\ &=\begin{bmatrix}\frac{\partial G}{\partial x_1}(x_1,\ldots,x_n)\\ \vdots\\ \frac{\partial G}{\partial x_n}(x_1,\ldots,x_n)\end{bmatrix}.\end{aligned}$$

Definition 1.3. A pseudo-gradient vector field ξ, is a vector field for which there exists a matrix $\mathbf{X}$ and gradient vector field ζ such that either $(\mathbf{X}\circ\xi)(\mathbf{x})=\zeta(\mathbf{x})$ or $\xi(\mathbf{x})=(\mathbf{X}\circ\zeta)(\mathbf{x})$.

Definition 1.4. Given a matrix A, define the set

$$\mathrm{Pg}(\mathbf{A}) = \{\mathbf{X} : \mathbf{XA} = \mathbf{A}^t\mathbf{X}^t\}.$$

The matrix $\mathbf{X}$ is such that $\mathbf{XA}$ is a symmetric matrix.

§2. Rejoiner to [3].

The following are results from [3] that will be used in this paper.

Lemma [3] 3.11. *Considering a matrix* $\mathbf{X}$ *written in the form of a* $n \times n$*-tuple*

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix}$$

there exists a finite set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \Re^{n \times n}$ *such that*

$$\mathrm{Pg}(\mathbf{A}) = \{t_1\mathbf{v}_1 + \ldots + t_p\mathbf{v}_p : t_1, \ldots, t_p \in \Re\}.$$

PROOF. To solve the equation $\mathbf{XA} = \mathbf{A}^t\mathbf{X}^t$ is the same as solving

$$\sum_{k=1}^{n} x_{1k}a_{k1} = \sum_{k=1}^{n} a_{k1}x_{1k}$$

$$\vdots$$

$$\sum_{k=1}^{n} x_{1k}a_{kn} = \sum_{k=1}^{n} a_{k1}x_{nk}$$

$$\vdots$$

$$\sum_{k=1}^{n} x_{nk}a_{k1} = \sum_{k=1}^{n} a_{kn}x_{1k}$$

$$\vdots$$

$$\sum_{k=1}^{n} x_{nk}a_{kn} = \sum_{k=1}^{n} a_{kn}x_{nk}$$

which can be rewritten in the form

$$\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} & \cdots & -a_{11} & \cdots & -a_{n1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ -a_{1n} & \cdots & -a_{nn} & \cdots & a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{X} \in \mathrm{Pg}(\mathbf{A})$ if and only if it solves the above equation. This means that

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix}$$

is in the kernel of the matrix in the right-handside of the above equation. By linear algebra, the kernel is a linear subspace of $\Re^{n\times n}$ which can written as the span of the linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$. Thus,

$$\mathrm{Pg}(\mathbf{A}) = \{t_1\mathbf{v}_1 + \ldots + t_p\mathbf{v}_p : t_1, \ldots, t_p \in \Re\}. \qquad \blacksquare$$

Lemma [3] 3.12. *Given two linear subspaces spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and $\mathbf{w}_1, \ldots, \mathbf{w}_q$ respectively, the intersection of the two subspaces is given by the span of some vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ with $r \leq p, q$.*

PROOF. Let $\mathbf{x} \in \{t_1\mathbf{v}_1 + \ldots + t_p\mathbf{v}_p : t_1, \ldots, t_p \in \Re\} \cap \{s_1\mathbf{w}_1 + \ldots + s_q\mathbf{w}_q : s_1, \ldots, s_q \in \Re\}$. Then

$$[\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q] \begin{bmatrix} t_1 \\ \vdots \\ t_p \\ -s_1 \\ \vdots \\ -s_q \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By linear algebra, the solution for $t_1, \ldots, t_p, -s_1, \ldots, -s_q$ is in the kernel of the matrix in the left of the above equality. Let the kernel be spanned by the vectors $\mathbf{y}^1, \ldots, \mathbf{y}^r$. Thus,

$$\begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} = \begin{bmatrix} y_1^1 & \cdots & y_1^r \\ \vdots & & \vdots \\ y_p^1 & \cdots & y_p^r \end{bmatrix} \begin{bmatrix} t_1' \\ \vdots \\ t_r' \end{bmatrix}$$

from which it follows that a spanning set of vectors for the intersection of the two subspaces is given by

$$[\mathbf{u}_1, \ldots, \mathbf{u}_r] = [\mathbf{v}_1, \ldots, \mathbf{v}_p] \begin{bmatrix} y_1^1 & \cdots & y_1^r \\ \vdots & & \vdots \\ y_p^1 & \cdots & y_p^r \end{bmatrix}.$$

Without loss of generality, the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ may be assumed to be independent and form a basis. The dimension of the intersection cannot exceed the dimension of the subspaces that it intersects, thus $r \leq p, q$. $\blacksquare$

Theorem [3] 3.13. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}\left|\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}-\gamma_j\right|.$$

There exists a matrix $\mathbf{X}$ *such that* $(\mathbf{X}\circ\xi)(\mathbf{x})$ *is a gradient vector field if and only if*

$$\mathbf{X}\in\mathrm{Pg}\left(\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\right)\bigcap\left(\bigcap_{j=1}^{m}\mathrm{Pg}\left(\begin{bmatrix}\alpha_{j1}\beta_{j1}&\dots&\alpha_{j1}\beta_{jn}\\ \vdots&&\vdots\\ \alpha_{jn}\beta_{j1}&\dots&\alpha_{jn}\beta_{jn}\end{bmatrix}\right)\right).$$

Proof. Assume that there exists a matrix $\mathbf{X}$ such that $(\mathbf{X}\circ\xi)(\mathbf{x})$ is a gradient vector field. As in the proof of theorem[3] 3.2, it is necessary and sufficient that

$$\mathbf{X}\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}$$

and

$$\mathbf{X}\begin{bmatrix}\alpha_{j1}\beta_{j1}&\dots&\alpha_{j1}\beta_{jn}\\ \vdots&&\vdots\\ \alpha_{jn}\beta_{j1}&\dots&\alpha_{jn}\beta_{jn}\end{bmatrix}$$

to be symmetric matrices for $(\mathbf{X}\circ\xi)(\mathbf{x})$ to be a gradient vector field. Thus

$$\mathbf{X}\in\mathrm{Pg}\left(\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\right)\bigcap\left(\bigcap_{j=1}^{m}\mathrm{Pg}\left(\begin{bmatrix}\alpha_{j1}\beta_{j1}&\dots&\alpha_{j1}\beta_{jn}\\ \vdots&&\vdots\\ \alpha_{jn}\beta_{j1}&\dots&\alpha_{jn}\beta_{jn}\end{bmatrix}\right)\right).\quad\blacksquare$$

§3. Auxiliary results.

The following are some auxiliary results needed to ensure that the alogrithms to be presented can indeed be implemented in a deterministic fashion. Unlike existence proofs where it is sufficient to demonstrate validity of a claim, constructive proofs are much more useful in the design and implementation of functional algorithms.

The first two results deal with properties of polynomials while the rest deal with symmetric matrices.

Definition 3.1. Let $\alpha \in (\mathrm{IN} \cup \{0\})^k$, then

$$|\alpha| = \sum_{i=1}^{k} \alpha_i$$

and

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_k^{\alpha_k}.$$

Proposition 3.2. *Let*

$$f(x_1) = \sum_{i=0}^{r} c_i x_1^i$$

be a polynomial in the variable x_1 of degree r with $c_r \neq 0$. There exists $y_1 \neq 0$ such that $f(y_1) \neq 0$.

PROOF. If $0 = r$ then let $y_1 = 1$. In this case, $f(y_1) = c_0 \neq 0$. Assume that $1 \leq r$, then

$$f(x_1) = c_r x_1^r + \sum_{i=0}^{r-1} c_i x_1^i.$$

It may be assumed that $c_r > 0$, otherwise consider $-f(x_1)$ instead of $f(x_1)$.

Let

$$\begin{aligned} M &= \max\{|c_i| : i = 0, \ldots, r-1\} \\ y_1 &= \max\{1, \frac{Mr+1}{c_r}\}. \end{aligned}$$

Then

$$\begin{aligned} f(y_1) &= c_r y_1^r + \sum_{i=0}^{r-1} c_i y_1^i \\ &\geq c_r y_1^r - M \sum_{i=0}^{r-1} y_1^i \\ &\geq c_r y_1^r - M r y_1^{r-1} \\ &= y_1^{r-1}(c_r y_1 - M r) \\ &> 0. \end{aligned}$$

∎

Proposition 3.3. *Let*

$$f(x_1,\ldots,x_k)=\sum_{i=0}^{r}\left(\sum_{\substack{|\alpha|=r\\ \alpha\in(\mathbb{N}\cup\{0\})^k}} c_\alpha \mathbf{x}^\alpha\right)$$

be a polynomial in k variables of degree r with $c_\alpha \neq 0$ for some $|\alpha| = r$. There exists $y_1,\ldots,y_k \neq 0$ such that $f(y_1,\ldots,y_k)\neq 0$.

Proof. If $0 = r$ then let $y_1 = \ldots = y_k = 1$. In this case $f(1,\ldots,1) = c_{(0,\ldots,0)} \neq 0$. Assume that $1 \le r$, then

$$f(x_1,\ldots,x_k)=\sum_{|\alpha|=r} c_\alpha \mathbf{x}^\alpha+\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i} c_\alpha \mathbf{x}^\alpha\right).$$

Consider the nontrivial homogeneous polynomial given by

$$g(x_1,\ldots,x_k)=\sum_{|\alpha|=r} c_\alpha \mathbf{x}^\alpha.$$

Define new variables $y_1,\ldots,y_k$ by $y_i = x_1^{(r+1)^{i-1}}$. Then $g(y_1,\ldots,y_k) = h(x_1)$ is a polynomial in x_1 of order at most $r(r+1)^{k-1}$. By proposition 3.2 there is a value $y_1 \neq 0$ such that $h(y_1)\neq 0$. Then the values $y_1,\ldots,y_1^{(r+1)^{k-1}} \neq 0$ satisfy $g(y_1,\ldots,y_1^{(r+1)^{k-1}}) = K \neq 0$. It may be assumed that $K > 0$, otherwise consider $-f(x_1,\ldots,x_k)$ instead of $f(x_1,\ldots,x_k)$.

Let

$$M=\max\{|c_\alpha| : |\alpha| = 0,\ldots,r-1\}$$

$$\epsilon=\max\{|y_i| : i = 1,\ldots,k\}$$

and choose

$$\lambda=\max\left\{1,\frac{1}{\epsilon},\frac{1}{K}\left(M\frac{(r+k-2)!}{(k-1)!}r\epsilon^{r-1}+1\right)\right\}.$$

Then

$$\begin{aligned}
f(\lambda y_1,\ldots,\lambda y_k) &= \sum_{|\alpha|=r} c_\alpha(\lambda\mathbf{y})^\alpha+\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i} c_\alpha(\lambda\mathbf{y})^\alpha\right)\\
&= \lambda^r\sum_{|\alpha|=r} c_\alpha\mathbf{y}^\alpha+\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i} c_\alpha\lambda^i\mathbf{y}^\alpha\right)\\
&\geq \lambda^r K - M\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i}\lambda^i\mathbf{y}^\alpha\right)\\
&\geq \lambda^r K - M\sum_{i=0}^{r-1}\binom{i+k-1}{k-1}\lambda^i\epsilon^i\\
&\geq \lambda^r K - M\frac{(r+k-2)!}{(k-1)!}r\lambda^{r-1}\epsilon^{r-1}\\
&= \lambda^{r-1}\left(\lambda K - M\frac{(r+k-2)!}{(k-1)!}r\epsilon^{r-1}\right)\\
&> 0.
\end{aligned}$$

∎

Proposition 3.4. *Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$ be a basis for a linear manifolds of matrices. An element*

$$\sum_{i=0}^{q} x_i \mathbf{w}_i$$

is symmetric if and only if

$$\sum_{i=0}^{q} x_i(\mathbf{w}_i - \mathbf{w}_i^t) = 0.$$

PROOF. An element

$$\sum_{i=0}^{q} x_i \mathbf{w}_i$$

is symmetric if and only if

$$\begin{aligned}\sum_{i=0}^{q} x_i \mathbf{w}_i &= \left(\sum_{i=0}^{q} x_i \mathbf{w}_i\right)^t \\ &= \sum_{i=0}^{q} x_i \mathbf{w}_i^t\end{aligned}$$

∎

Proposition 3.5. *Let the given symmetric matrix be*

$$\mathbf{Y}_n = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix}.$$

Define the symmetric submatrices

$$\mathbf{Y}_n = \begin{bmatrix} y_{11} & \cdots & y_{1i} \\ \vdots & & \vdots \\ y_{i1} & \cdots & y_{ii} \end{bmatrix}$$

for $1 \leq i \leq n$. The matrix $\mathbf{Y}_n$ is postive definite if and only if $\det(\mathbf{Y}_i) > 0$ *for $1 \leq i \leq n$.*

PROOF. Assume that $\det(\mathbf{Y}_i) \leq 0$ for some $1 \leq i \leq n$. As $\det(\mathbf{Y}_i)$ is the product of the eigenvalues of the $\mathbf{Y}_i$ then one of the eigenvalues of $\mathbf{Y}_i$ is nonpositive. Let λ be this eigenvalue and $\mathbf{x} \neq (0, \ldots, 0)$ be the associated eigenvector. Then

$$\begin{aligned}\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}^t \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} &= \begin{bmatrix} x_1 \\ \vdots \\ x_i \end{bmatrix}^t \begin{bmatrix} y_{11} & \cdots & y_{1i} \\ \vdots & & \vdots \\ y_{i1} & \cdots & y_{ii} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \end{bmatrix} \\ &= \lambda(x_1^2 + \ldots, x_i^2) \\ &\leq 0.\end{aligned}$$

Thus, if $\mathbf{Y}_n$ is positive definite then $\det(\mathbf{Y}_i) > 0$ for $1 \leq i \leq n$.

Conversely, assume that $\det(\mathbf{Y}_i) > 0$ for $1 \le i \le n$. The proof will be by induction on n. If $n = 1$ then $\mathbf{Y}_1 = [y_{11}]$ is a matrix with $0 < y_{11}$. It is clear in this case that $\mathbf{Y}_1$ is positive definite. Assume that $1 < n$ and the proposition is true for $n_0 < n$. Observe for $1 < i \le n$,

$$\begin{bmatrix} y_{11} & \cdots & y_{1i} \\ \vdots & & \vdots \\ y_{i1} & \cdots & y_{ii} \end{bmatrix} = \begin{bmatrix} 1 & y_{21}/y_{11} & \cdots & y_{i1}/y_{11} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}^t$$

$$\begin{bmatrix} y_{11} & 0 & \cdots & 0 \\ 0 & y_{22} - y_{21}y_{21}/y_{11} & \cdots & y_{2i} - y_{21}y_{i1}/y_{11} \\ \vdots & \vdots & & \vdots \\ 0 & y_{i2} - y_{i1}y_{21}/y_{11} & \cdots & y_{ii} - y_{i1}y_{i1}/y_{11} \end{bmatrix} \begin{bmatrix} 1 & y_{21}/y_{11} & \cdots & y_{i1}/y_{11} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Define

$$\mathbf{Z}_{i-1} = \begin{bmatrix} y_{22} - y_{21}y_{21}/y_{11} & \cdots & y_{2i} - y_{21}y_{i1}/y_{11} \\ \vdots & & \vdots \\ y_{i2} - y_{i1}y_{21}/y_{11} & \cdots & y_{ii} - y_{i1}y_{i1}/y_{11} \end{bmatrix}.$$

As $y_{11}\det(\mathbf{Z}_{i-1}) = \det(\mathbf{Y}_i)$ then $\det(\mathbf{Z}_{i-1}) = \det(\mathbf{Y}_i)/y_{11}$. Thus $\det(\mathbf{Z}_{i-1}) > 0$ for $i-1 = 1, \ldots, n-1$. By induction, the matrix

$$\mathbf{Z}_{n-1} = \begin{bmatrix} y_{22} - y_{21}y_{21}/y_{11} & \cdots & y_{2n} - y_{21}y_{n1}/y_{11} \\ \vdots & & \vdots \\ y_{n2} - y_{n1}y_{21}/y_{11} & \cdots & y_{ii} - y_{n1}y_{n1}/y_{11} \end{bmatrix}$$

is positive definite. Thus

$$\sum_{i=2}^{n}\sum_{j=2}^{n}(y_{ij} - y_{i1}y_{j1}/y_{11})x_i x_j > 0$$

for $(x_2, \ldots, x_n) \ne (0, \ldots, 0)$ and

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^t \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \end{bmatrix}^t \begin{bmatrix} 1 & y_{21}/y_{11} & \cdots & y_{n1}/y_{11} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}^t$$

$$\begin{bmatrix} y_{11} & 0 & \cdots & 0 \\ 0 & y_{22} - y_{21}y_{21}/y_{11} & \cdots & y_{2n} - y_{21}y_{n1}/y_{11} \\ \vdots & \vdots & & \vdots \\ 0 & y_{n2} - y_{n1}y_{21}/y_{11} & \cdots & y_{nn} - y_{n1}y_{n1}/y_{11} \end{bmatrix}$$

$$\begin{bmatrix} 1 & y_{21}/y_{11} & \cdots & y_{i1}/y_{11} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \end{bmatrix}$$

$$= y_{11}\left(x_1 + \sum_{i=2}^{n} y_{i1}/y_{11}x_i\right)^2 + \sum_{i=2}^{n}\sum_{j=2}^{n}(y_{ij} - y_{i1}y_{j1})x_i x_j$$

$$> 0$$

for $(x_1, \ldots, x_n) \ne (0, \ldots, 0)$. Thus $\mathbf{Y}_n$ is positive definite. ∎

§4. The pseudo-gradient vector field $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ an invertible matrix.

If ξ is a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is an invertible matrix then $\mathbf{X}^{-1} \circ \xi$ is a gradient vector field. Thus, a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ invertible, has an invertible matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field. Conversely, if there does not exist an invertible matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ an invertible matrix and ζ a gradient vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form.

Let the vector field ξ be given by

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|,$$

then an algorithm to determine the existence of invertible matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a gradient vector field is given by the following sequence of steps:

Step 1: Let $S = \{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{Pg}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\right).$$

Step 2: For i=1 to m repeat the steps 2.1 through to 2.3.

Step 2.1: Let $T = \{\mathbf{v}_1^i, \dots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \dots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathrm{Pg}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \dots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \dots & \alpha_{jn}\beta_{jn} \end{bmatrix}\right).$$

Step 2.2: Let $R = \{\mathbf{w}_1^i, \dots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \dots, \mathbf{w}_{q_i}^i\}$ form a basis for span$(S) \cap$ span(T).

Step 2.3: Let $S = R$.

Step 3: Form the matrix

$$\mathbf{Y}(x_1, \dots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

and let $f(x_1, \dots, x_{q_m})$ be the polynomial given by

$$f(x_1, \dots, x_{q_m}) = \det \mathbf{Y}(x_1, \dots, x_{q_m}).$$

Step 4: Determine if $f(x_1, \dots, x_{q_m})$ is identically the zero function. If it is then go to step 5 else choose values for $x_1, \dots, x_{q_m}$ such that $f(x_1, \dots, x_{q_m}) \neq 0$ and go to step 6.

Step 5: In this case, all matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field are non-invertible. Thus, there do not exist invertible matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is invertible and ζ is a gradient vector field.

Step 6: In this case, there exists a set of values $x_1, \ldots, x_{q_m}$ such that the matrix

$$\mathbf{Y}(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is invertible and $\mathbf{Y} \circ \xi$ is a gradient vector field. Thus ξ can be written in the form $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ with $\mathbf{Y}^{-1}$ invertible and $\mathbf{Y} \circ \xi$ a gradient vector field.

EXAMPLE 4.1. (Figure 1.) This example will demonstrate a case where the desired decomposition does not exist. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 0 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 \right| + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left| \begin{bmatrix} 0 \\ 1 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \right| .$$

Step 1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix} .$$

This means solving the set of linear equations given by

$$0 = 0$$

$$x_{11} = 0$$

$$0 = x_{11}$$

$$x_{21} = x_{21}$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

A basis for S is given by the vectors

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} .$$

Step 2: Since $m = 2$ then steps 2.1 to 2.3 need only be used twice.

Step 2.1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix} .$$

This means solving the set of linear equations given by

$$x_{12} = x_{12}$$

$$0 = x_{22}$$

$$x_{22} = 0$$

$$0 = 0$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 2.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Step 2.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Step 2.1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}.$$

This means solving the set of linear equations given by

$$0 = 0$$

$$x_{11} + x_{12} = 0$$

$$0 = x_{11} + x_{12}$$

$$x_{21} + x_{22} = x_{21} + x_{22}$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 2.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the span of the vector

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vector

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Step 2.3: Let S be the span of the vector

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Step 3: The matrix $\mathbf{Y}(x_1)$ is given by

$$\mathbf{Y}(x_1) = x_1 \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

and the function $f(x_1)$ is given by $f(x_1) = \det(\mathbf{Y}(x_1)) = 0$.

Step 4: It is clear that $f(x_1)$ is identically the zero function.

Step 5: It can be concluded that ξ may not be decomposed as the composition of an invertible matrix $\mathbf{X}$ and a gradient vector field ζ.

EXAMPLE 4.2. (Figure 2.) This example will demonstrate a case where a desired decomposition exists. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 0 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \right| + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \left| \begin{bmatrix} 0 \\ 1 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 \right|.$$

Step 1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}.$$

This means solving the set of linear equations given by

$$x_{12} = x_{12}$$

$$x_{11} = x_{22}$$

$$x_{22} = x_{11}$$

$$x_{21} = x_{21}$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for S is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 2: Since $m = 2$ then steps 2.1 to 2.3 need only be used twice.

Step 2.1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}.$$

This means solving the set of linear equations given by

$$x_{11} = x_{11}$$

$$0 = x_{21}$$

$$x_{21} = 0$$

$$0 = 0$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 2.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Step 2.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Step 2.1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}.$$

This means solving the set of linear equations given by

$$\begin{aligned} 0 &= 0 \\ -x_{11} + x_{12} &= 0 \\ 0 &= -x_{11} + x_{12} \\ -x_{21} + x_{22} &= -x_{21} + x_{22} \end{aligned}$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 2.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

which is the span of the vector

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vector

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Step 2.3: Let S be the span of the vector

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Step 3: The matrix $\mathbf{Y}(x_1)$ is given by

$$\mathbf{Y}(x_1) = x_1 \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

and the function $f(x_1)$ is given by $f(x_1) = \det(\mathbf{Y}(x_1)) = x_1^2$.

Step 4: It is clear that $f(x_1)$ is not identically the zero function. By proposition 3.2, a value of $x_1 = 1$ satisfies $f(x_1) \neq 0$.

Step 6: It can be concluded that ξ may be decomposed as the composition of an invertible matrix $\mathbf{X}$ and a gradient vector field ζ as

$$\begin{aligned} \xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \circ \xi \right) \\ &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \circ \left(\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 0 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \right| + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \left| \begin{bmatrix} 0 \\ 1 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 \right| \right). \end{aligned}$$

§5. The pseudo-gradient vector field $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ an invertible symmetric matrix.

If ξ is a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is an invertible symmetric matrix then $\mathbf{X}^{-1} \circ \xi$ is a gradient vector field. Thus, a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ invertible symmetric, has an invertible symmetric matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field. Conversely, if there does not exist an invertible symmetric matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ an invertible symmetric matrix and ζ a gradient vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form. However, if there does not exist an invertible symmetric matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field there may still exist invertible matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a gradient vector field.

Let the vector field ξ be given by

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|,$$

then an algorithm to determine the existence of invertible symmetric matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ gradient vector fields is given by the following sequence of steps:

Step 1: Let $S = \{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{Pg}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \right).$$

Step 2: For i=1 to m repeat the steps 2.1 through to 2.3.

Step 2.1: Let $T = \{\mathbf{v}_1^i, \dots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \dots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathrm{Pg}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \dots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \dots & \alpha_{jn}\beta_{jn} \end{bmatrix} \right).$$

Step 2.2: Let $R = \{\mathbf{w}_1^i, \dots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \dots, \mathbf{w}_{q_i}^i\}$ form a basis for $\mathrm{span}(S) \cap \mathrm{span}(T)$.

Step 2.3: Let $S = R$.

Step 3: From the equation

$$\sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m - (\mathbf{w}_i^m)^t) = 0$$

determine a set of independent variables $x_1, \dots, x_k$ and dependent variables $x_{k+1}, \dots, x_{q_m}$. Form the matrix

$$\mathbf{Y}(x_1, \dots, x_k) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

and let $f(x_1,\ldots,x_k)$ be the polynomial given by

$$f(x_1,\ldots,x_k) = \det\mathbf{Y}(x_1,\ldots,x_k).$$

Step 4: Determine if $f(x_1,\ldots,x_k)$ is identically the zero function. If it is then go to step 5 else choose values for $x_1,\ldots,x_k$ such that $f(x_1,\ldots,x_k) \neq 0$ and go to step 6.

Step 5: In this case, all symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y}\circ\xi$ is a gradient vector field are non-invertible. Thus, there do not exist invertible symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y}\circ\xi$ is a gradient vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X}\circ\zeta$ where $\mathbf{X}$ is invertible symmetric and ζ is a gradient vector field.

Step 6: In this case, there exists a set of values $x_1,\ldots,x_k$ such that the matrix

$$\mathbf{Y}(x_1,\ldots,x_{q_m}) = \sum_{i=1}^{q_m} x_i\mathbf{w}_i^m$$

is invertible symmetric and $\mathbf{Y}\circ\xi$ is a gradient vector field. Thus ξ can be written in the form $\xi = \mathbf{Y}^{-1}\circ(\mathbf{Y}\circ\xi)$ with $\mathbf{Y}^{-1}$ invertible symmetric and $\mathbf{Y}\circ\xi$ a gradient vector field.

EXAMPLE 5.1. (Figure 2.) This example will demonstrate a case where a vector field ξ can be decomposed as $\xi = \mathbf{X}\circ\zeta$ where the matrix $\mathbf{X}$ cannot be invertible symmetric but may be invertible. Let the vector field be given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}\left|\begin{bmatrix}1\\0\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}+1\right| + \begin{bmatrix}-1\\1\end{bmatrix}\left|\begin{bmatrix}0\\1\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}-1\right|.$$

This is the same vector field as in example 4.2. All the steps are identical until the end of step 2.3, at which point a basis for S is given by

$$\left\{\begin{bmatrix}-1\\-1\\0\\-1\end{bmatrix}\right\}.$$

Step 3: The equation

$$x_1\left(\begin{bmatrix}-1&-1\\0&-1\end{bmatrix} - \begin{bmatrix}-1&-1\\0&-1\end{bmatrix}^t\right) = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

determines that

$$x_1\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

from which $x_1 = 0$. Thus,

$$\mathbf{Y}(x_1) = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

and $f(x_1) = \det(\mathbf{Y}(x_1)) = 0$.

Step 4: It is clear that $f(x_1)$ is identically the zero function.

Step 5: It can be concluded that ξ may not be decomposed as the composition of an invertible symmetric matrix $\mathbf{X}$ and a gradient vector field ζ.

EXAMPLE 5.2. (Figure 3.) This example will demonstrate a case where a vector field ξ can be decomposed as $\xi = \mathbf{X} \circ \zeta$ where the matrix $\mathbf{X}$ is invertible symmetric. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \right|.$$

Step 1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}.$$

This means solving the set of linear equations given by

$$x_{11} + 3x_{12} = x_{11} + 3x_{12}$$

$$2x_{11} + x_{12} = x_{21} + 3x_{22}$$

$$x_{21} + 3x_{22} = 2x_{11} + x_{12}$$

$$2x_{21} + x_{22} = 2x_{21} + x_{22}$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & -3 \\ -2 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for S is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Step 2: Since $m = 1$ then steps 2.1 to 2.3 need only be used once.

Step 2.1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}.$$

This means solving the set of linear equations given by

$$5x_{11} + 5x_{12} = 5x_{11} + 5x_{12}$$

$$10x_{11} + 10x_{12} = 5x_{21} + 5x_{22}$$

$$5x_{21} + 5x_{22} = 10x_{11} + 10x_{12}$$

$$10x_{21} + 10x_{22} = 10x_{21} + 10x_{22}$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 10 & 10 & -5 & -5 \\ -10 & -10 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Step 2.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 1 \\ -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} -5 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

Step 2.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} -5 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

Step 3: The equation

$$x_1 \left(\begin{bmatrix} -5 & 4 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} -5 & 4 \\ 0 & -2 \end{bmatrix}^t \right) + x_2 \left(\begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}^t \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

determines that

$$x_1 \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from which $x_2 = -2x_1$. Thus,

$$\begin{aligned} \mathbf{Y}(x_1) &= x_1 \begin{bmatrix} -5 & 4 \\ 0 & -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix} \\ &= x_1 \begin{bmatrix} -3 & 4 \\ 4 & -2 \end{bmatrix} \end{aligned}$$

and $f(x_1) = \det(\mathbf{Y}(x_1)) = -10x_1^2$.

Step 4: It is clear that $f(x_1)$ is not identically the zero function. By proposition 3.2, a value of $x_1 = 1$ satisfies $f(x_1) \neq 0$.

Step 6: It can be concluded that ξ may be decomposed as the composition of an invertible symmetric matrix $\mathbf{X}$ and a gradient vector field ζ as

$$\xi\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 4 & -2 \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} -3 & 4 \\ 4 & -2 \end{bmatrix} \circ \xi\right)$$
$$= \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.3 \end{bmatrix} \circ \left(\begin{bmatrix} 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}\left|\begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3\right|\right).$$

§6. The pseudo-gradient vector field $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ a symmetric positive definite matrix.

If ξ is a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is a symmetric positive definite matrix then $\mathbf{X}^{-1} \circ \xi$ is a gradient vector field. Thus, a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ symmetric positive definite, has a symmetric positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field. Conversely, if there does not exist a symmetric positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ a symmetric positive definite matrix and ζ a gradient vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form. However, if there does not exist a symmetric positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field there may still exist invertible symmetric matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a gradient vector field.

Theorem 6.1. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|.$$

There exists a symmetric positive definite matrix $\mathbf{X}$ *such that* $(\mathbf{X} \circ \xi)(\mathbf{x})$ *is a gradient vector field if and only if*

$$\mathbf{X} \in \operatorname{Pg}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\right) \bigcap \left(\bigcap_{j=1}^{m} \operatorname{Pg}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \dots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \dots & \alpha_{jn}\beta_{jn} \end{bmatrix}\right)\right),$$

$\mathbf{X}$ *is symmetric and for* $i = 1, \dots, n$,

$$0 < \det\begin{bmatrix} x_{11} & \dots & x_{1i} \\ \vdots & & \vdots \\ x_{i1} & \dots & x_{ii} \end{bmatrix}.$$

PROOF. Immediate from theorem[3] 3.13 and proposition 3.5. ∎

Let the vector field ξ be given by

$$\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}\left|\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}-\gamma_j\right|,$$

then an algorithm to determine the existence of symmetric positive definite matrices $\mathbf{Y}$ with $\mathbf{Y}\circ\xi$ gradient vector fields is given by the following sequence of steps:

Step 1: Let $S=\{\mathbf{w}_1^0,\dots,\mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0,\dots,\mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{Pg}\left(\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\right).$$

Step 2: For i=1 to m repeat the steps 2.1 through to 2.3.

Step 2.1: Let $T=\{\mathbf{v}_1^i,\dots,\mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i,\dots,\mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathrm{Pg}\left(\begin{bmatrix}\alpha_{j1}\beta_{j1}&\dots&\alpha_{j1}\beta_{jn}\\ \vdots&&\vdots\\ \alpha_{jn}\beta_{j1}&\dots&\alpha_{jn}\beta_{jn}\end{bmatrix}\right).$$

Step 2.2: Let $R=\{\mathbf{w}_1^i,\dots,\mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i,\dots,\mathbf{w}_{q_i}^i\}$ form a basis for span$(S)\cap$ span(T).

Step 2.3: Let $S=R$.

Step 3: From the equation

$$\sum_{i=1}^{q_m}x_i(\mathbf{w}_i^m-(\mathbf{w}_i^m)^t)=0$$

determine a set of independent variables $x_1,\dots,x_k$ and dependent variables $x_{k+1},\dots,x_{q_m}$. Form the matrix

$$\mathbf{Y}_n(x_1,\dots,x_k)=\sum_{i=1}^{q_m}x_i\mathbf{w}_i^m.$$

Define the matrices

$$\mathbf{Y}_i(x_1,\dots,x_k)=\begin{bmatrix}\mathbf{Y}_n(x_1,\dots,x_k)_{11}&\dots&\mathbf{Y}_n(x_1,\dots,x_k)_{1i}\\ \vdots&&\vdots\\ \mathbf{Y}_n(x_1,\dots,x_k)_{i1}&\dots&\mathbf{Y}_n(x_1,\dots,x_k)_{ii}\end{bmatrix}$$

and let $f_i(x_1,\dots,x_k)$ be the polynomial given by

$$f_i(x_1,\dots,x_k)=\det\mathbf{Y}_i(x_1,\dots,x_k)$$

for $1\le i\le n$.

Step 4: Determine if there exist values $x_1,\dots,x_k$ such that the following set of inequalities hold simultaneously,

$$f_1(x_1,\dots,x_k)>0$$
$$\vdots$$
$$f_n(x_1,\dots,x_k)>0.$$

If such values do not exist then go to step 5 else go to step 6.

Step 5: In this case, all symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field are either non-invertible or invertible and not positive definite. Thus, there do not exist symmetric positive definite matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is symmetric positive definite and ζ is a gradient vector field.

Step 6: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\mathbf{Y}_n(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is symmetric positive definite and $\mathbf{Y}_n \circ \xi$ is a gradient vector field. Thus ξ can be written in the form $\xi = \mathbf{Y}_n^{-1} \circ (\mathbf{Y}_n \circ \xi)$ with $\mathbf{Y}_n^{-1}$ symmetric positive definite and $\mathbf{Y}_n \circ \xi$ a gradient vector field.

EXAMPLE 6.1. (Figure 3.) This example will demonstrate a case where a vector field ξ can be decomposed as $\xi = \mathbf{X} \circ \zeta$ where the matrix $\mathbf{X}$ cannot be symmetric positive definite but may be invertible symmetric. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \right|.$$

This is the same vector field as in example 5.2. All the steps are identical until the end of step 2.3, at which point a basis for S is given by

$$\left\{ \begin{bmatrix} -5 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

Step 3: The equation

$$x_1 \left(\begin{bmatrix} -5 & 4 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} -5 & 4 \\ 0 & -2 \end{bmatrix}^t \right) + x_2 \left(\begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}^t \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

determines that

$$x_1 \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from which $x_2 = -2x_1$. Thus,

$$\begin{aligned} \mathbf{Y}_2(x_1) &= x_1 \begin{bmatrix} -5 & 4 \\ 0 & -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix} \\ &= x_1 \begin{bmatrix} -3 & 4 \\ 4 & -2 \end{bmatrix} \\ \mathbf{Y}_1(x_1) &= x_1 [-3] \end{aligned}$$

and

$$f_2(x_1) = \det(\mathbf{Y}_2(x_1)) = -10x_1^2$$

$$f_1(x_1) = \det(\mathbf{Y}_1(x_1)) = -3x_1.$$

Step 4: It is clear that the inequalities

$$-10x_1^2 > 0$$

$$-3x_1 > 0$$

cannot be satisfied simultaneously.

Step 5: It can be concluded that ξ may not be decomposed as the composition of a symmetric positive definite matrix $\mathbf{X}$ and a gradient vector field ζ.

EXAMPLE 6.2. (Figure 4.) This example will demonstrate a case where a vector field ξ can be decomposed as $\xi = \mathbf{X} \circ \zeta$ where the matrix $\mathbf{X}$ is symmetric positive definite. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \right| .$$

Step 1: A basis for S is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 2: Since $m = 1$ then steps 2.1 to 2.3 need only be used once.

Step 2.1: By lemma[3] 3.11, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 7 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}.$$

This means solving the set of linear equations given by

$$3x_{11} + 7x_{12} = 3x_{11} + 7x_{12}$$

$$.6x_{11} + 14x_{12} = 3x_{21} + 7x_{22}$$

$$3x_{21} + 7x_{22} = 6x_{11} + 14x_{12}$$

$$6x_{21} + 14x_{22} = 6x_{21} + 14x_{22}$$

which is the same as finding the kernel of the matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & 14 & -3 & -7 \\ -6 & -14 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} -7 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \\ 6 \end{bmatrix} \right\}.$$

Step 2.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -7 & 1 & 7 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 6 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} 7 \\ -3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 0 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} -7 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \\ 6 \end{bmatrix} \right\}.$$

Step 2.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} -7 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \\ 6 \end{bmatrix} \right\}.$$

Step 3: The equation

$$x_1 \left(\begin{bmatrix} -7 & 3 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -7 & 0 \\ 3 & 0 \end{bmatrix}^t \right) + x_2 \left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}^t \right) + x_3 \left(\begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}^t \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

determines that

$$x_1 \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from which $x_2 = 3/2x_1$. Thus,

$$\begin{aligned} \mathbf{Y}_2(x_1, x_3) &= x_1 \begin{bmatrix} -7 & 3 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{11}{2}x_1 + 7x_3 & 3x_1 \\ 3x_1 & 6x_3 \end{bmatrix} \\ \mathbf{Y}_1(x_1, x_3) &= \begin{bmatrix} -\frac{11}{2}x_1 + 7x_3 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} f_2(x_1) &= \det(\mathbf{Y}_2(x_1)) = -9x_1^2 + 42x_3^2 - 33x_1x_3 \\ f_1(x_1) &= \det(\mathbf{Y}_1(x_1)) = -\frac{11}{2}x_1 + 7x_3. \end{aligned}$$

Step 4: It is clear that the inequalities

$$\begin{aligned} -9x_1^2 + 42x_3^2 - 33x_1x_3 &> 0 \\ -\frac{11}{2}x_1 + 7x_3 &> 0 \end{aligned}$$

can be satisfied simultaneously by $(x_1, x_3) = (0, 1)$.

Step 6: It can be concluded that ξ may be decomposed as the composition of a symmetric positive definite matrix $\mathbf{X}$ and a gradient vector field ζ as

$$\begin{aligned} \xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix} \circ \xi \right) \\ &= \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \circ \left(\begin{bmatrix} 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 21 \\ 42 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \right| \right). \end{aligned}$$

§7. The pseudo-gradient vector field $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ a diagonal positive definite matrix.

If ξ is a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is a diagonal positive definite matrix then $\mathbf{X}^{-1} \circ \xi$ is a gradient vector field. Thus, a pseudo-gradient vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ diagonal positive definite, has a diagonal positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field. Conversely, if there does not exist a diagonal positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ a diagonal positive definite matrix and ζ a gradient vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form. However, if there does not exist a diagonal positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field there may still exist symmetric positive definite matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a gradient vector field.

Definition 7.1. Define the set

$$\mathsf{C}\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}, \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}\right) = \left\{\begin{bmatrix}d_1\\ \vdots\\ d_n\end{bmatrix} : \exists\lambda \in \Re \ni \begin{bmatrix}d_1\alpha_1\\ \vdots\\ d_n\alpha_n\end{bmatrix} = \lambda \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}\right\}.$$

Lemma 7.2. *There exists vectors such that*

$$\mathsf{C}\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}, \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}\right) = \left\{\sum_{i=1}^{k} t_i \begin{bmatrix}c_{i1}\\ \vdots\\ c_{in}\end{bmatrix} : t_i \in \Re\right\}.$$

PROOF. It is required to solve the equations

$$\begin{aligned} d_1\alpha_1 &= \lambda\beta_1 \\ &\vdots \\ d_n\alpha_n &= \lambda\beta_n. \end{aligned}$$

If there exist $\alpha_i = 0$ with $\beta_i \neq 0$ then $0 = \lambda\beta_i$ from which $\lambda = 0$ and the above equations reduce to

$$\begin{aligned} d_1\alpha_1 &= 0 \\ &\vdots \\ d_n\alpha_n &= 0. \end{aligned}$$

Let $\mathbf{e}_i$ denote the i-th coordinate vector. If $\alpha_{i_1}, \ldots, \alpha_{i_j} \neq 0$ and $\alpha_{i_{j+1}}, \ldots, \alpha_{i_n} = 0$ then

$$\mathsf{C}\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}, \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}\right) = \left\{\sum_{k=j+1}^{n} t_{i_k}\mathbf{e}_{i_k} : t_{i_k} \in \Re\right\}.$$

If it happens that whenever $\alpha_i = 0$ that $\beta_i = 0$ then consider $\alpha_{i_1}, \ldots, \alpha_{i_j} \neq 0, \alpha_{i_{j+1}}, \ldots, \alpha_{i_n} = 0$. The equations reduce to

$$\begin{aligned} d_{i_1}\alpha_{i_1} &= \lambda\beta_{i_1} \\ &\vdots \\ d_{i_j}\alpha_{i_j} &= \lambda\beta_{i_j}. \end{aligned}$$

Thus

$$C\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix},\begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}\right)=\left\{\lambda\left(\sum_{k=1}^{j}\frac{\beta_{i_k}}{\alpha_{i_k}}\mathbf{e}_{i_k}\right)+\sum_{k=j+1}^{n}t_{i_k}\mathbf{e}_{i_k}:\lambda,t_{i_k}\in\Re\right\}. \qquad \blacksquare$$

Definition 7.3. Define the set

$$D\left(\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\right)=\left\{\begin{bmatrix}d_1\\ \vdots\\ d_n\end{bmatrix}:\begin{bmatrix}d_1b_{11}&\dots&d_1b_{1n}\\ \vdots&&\vdots\\ d_nb_{n1}&\dots&d_nb_{nn}\end{bmatrix}=\begin{bmatrix}d_1b_{11}&\dots&d_1b_{1n}\\ \vdots&&\vdots\\ d_nb_{n1}&\dots&d_nb_{nn}\end{bmatrix}^t\right\}.$$

Lemma 7.4. *There exists vectors such that*

$$D\left(\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\right)=\left\{\sum_{i=1}^{k}t_i\begin{bmatrix}d_{i1}\\ \vdots\\ d_{in}\end{bmatrix}:t_i\in\Re\right\}.$$

Proof. It is required to solve the equations

$$\begin{aligned}\begin{bmatrix}d_1b_{11}&\dots&d_1b_{1n}\\ \vdots&&\vdots\\ d_nb_{n1}&\dots&d_nb_{nn}\end{bmatrix}&=\begin{bmatrix}d_1b_{11}&\dots&d_1b_{1n}\\ \vdots&&\vdots\\ d_nb_{n1}&\dots&d_nb_{nn}\end{bmatrix}^t\\ &=\begin{bmatrix}d_1b_{11}&\dots&d_nb_{n1}\\ \vdots&&\vdots\\ d_1b_{1n}&\dots&d_nb_{nn}\end{bmatrix},\end{aligned}$$

i.e. $d_ib_{ij}=d_jb_{ji}$ or,

$$\begin{bmatrix}0&0&\dots&0\\ b_{12}&-b_{21}&\dots&0\\ \vdots&\vdots&&\vdots\\ b_{1n}&0&\dots&-b_{n1}\\ \vdots&\vdots&&\vdots\\ -b_{1n}&0&\dots&b_{n1}\\ 0&-b_{2n}&\dots&b_{n2}\\ \vdots&\vdots&&\vdots\\ 0&0&\dots&0\end{bmatrix}\begin{bmatrix}d_1\\ \vdots\\ d_n\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ o\end{bmatrix}.$$

By linear algebra theory there are vectors such that

$$D\left(\begin{bmatrix}b_{11}&\dots&b_{1n}\\ \vdots&&\vdots\\ b_{n1}&\dots&b_{nn}\end{bmatrix}\right)=\left\{\sum_{i=1}^{k}t_i\begin{bmatrix}d_{i1}\\ \vdots\\ d_{in}\end{bmatrix}:t_i\in\Re\right\}. \qquad \blacksquare$$

Definition 7.5. Given $d_1,\dots,d_n\in\Re$ then

$$\Lambda(d_1,\dots,d_n)=\begin{bmatrix}d_1&0&\dots&0\\ 0&d_2&\dots&0\\ \vdots&\vdots&&\vdots\\ 0&0&\dots&d_n\end{bmatrix}.$$

Theorem 7.6. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|.$$

There exists a diagonal positive definite matrix $\Lambda(d_1,\dots,d_n)$ such that $(\Lambda(d_1,\dots,d_n)\circ\xi)(\mathbf{x})$ is a gradient vector field if and only if

$$\Lambda(d_1,\dots,d_n)\in \mathrm{D}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\right)\cap\left(\bigcap_{j=1}^{m}\mathrm{C}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix},\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}\right)\right),$$

and $0 < d_i$ for $i = 1,\dots,n$.

Proof. Assume that there exists a matrix $\Lambda(d_1,\dots,d_n)$ such that $(\Lambda(d_1,\dots,d_n)\circ\xi)(\mathbf{x})$ is a gradient vector field. As in the proof of theorem[3] 3.2, it is necessary and sufficient that

$$\begin{bmatrix} d_1b_{11} & \dots & d_1b_{1n} \\ \vdots & & \vdots \\ d_nb_{n1} & \dots & d_nb_{nn} \end{bmatrix}$$

and

$$\begin{bmatrix} d_1\alpha_{j1} \\ \vdots \\ d_n\alpha_{jn} \end{bmatrix} = \lambda_j\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}$$

for $(\Lambda(d_1,\dots,d_n)\circ\xi)(\mathbf{x})$ to be a gradient vector field. Thus, by lemmas 7.2 and 7.4,

$$\Lambda(d_1,\dots,d_n)\in \mathrm{D}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\right)\cap\left(\bigcap_{j=1}^{m}\mathrm{C}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix},\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}\right)\right).$$

The condition that $0 < d_i$ is necessary and sufficient for $\Lambda(d_1,\dots,d_n)$ to be a diagonal positive definite matrix. ∎

Let the vector field ξ be given by

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|,$$

then an algorithm to determine the existence of diagonal positive definite matrices $\mathbf{Y}$ with $\mathbf{Y}\circ\xi$ gradient vector fields is given by the following sequence of steps:

Step 1: Let $S = \{\mathbf{w}_1^0,\dots,\mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0,\dots,\mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{D}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\right).$$

Step 2: For i=1 to m repeat the steps 2.1 through to 2.3.

Step 2.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathbf{C}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}, \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}\right).$$

Step 2.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for $\operatorname{span}(S) \cap \operatorname{span}(T)$.

Step 2.3: Let $S = R$.

Step 3: Determine if there exist values $x_1, \ldots, x_{q_m}$ such that the following set of inequalities hold simultaneously,

$$x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 > 0$$
$$\vdots$$
$$x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n > 0.$$

If such values do not exist then go to step 4 else go to step 5.

Step 4: In this case, all diagonal matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field are either non-invertible or invertible and not positive definite. Thus, there do not exist diagonal positive definite matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a gradient vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is diagonal positive definite and ζ is a gradient vector field.

Step 5: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\Lambda_(y_1, \ldots, y_n)$$

with

$$y_j = \sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m)_j$$

is diagonal positive definite and $\Lambda(y_1, \ldots, y_n) \circ \xi$ is a gradient vector field. Thus ξ can be written in the form $\xi = \Lambda(y_1, \ldots, y_n)^{-1} \circ (\Lambda(y_1, \ldots, y_n) \circ \xi)$ with $\Lambda(y_1, \ldots, y_n)^{-1}$ diagonal positive definite and $\Lambda(y_1, \ldots, y_n) \circ \xi$ a gradient vector field.

Note that if there exists a solution $y_1, \ldots, y_{q_m}$ to

$$x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 = \epsilon_1 > 0$$
$$\vdots$$
$$x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n = \epsilon_n > 0$$

then there exists a solution $y_1', \ldots, y_{q_m}'$ to

$$x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 \geq 1$$
$$\vdots$$
$$x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n \geq 1$$

by scaling the original values $y_1, \ldots, y_{q_m}$ with a sufficiently large constant. Decompose the variables $x_1, \ldots, x_{q_m}$ as $x_i = x_i^1 - x_i^2$ for $i = 1, \ldots, q_m$. It then follows that $y_1^1 = y_1', \ldots, y_{q_m}^1 = y_{q_m}', y_1^2 = 0, \ldots, y_{q_m}^2 = 0$ is an optimal solution to the linear programming problem of

$$\text{minimise} \sum_{i=1}^{n} v_i$$

subject to

$$\begin{aligned}
x_1^1(\mathbf{w}_1^m)_1 - x_1^2(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}^1(\mathbf{w}_{q_m}^m)_1 - x_{q_m}^2(\mathbf{w}_{q_m}^m)_1 + v_1 - u_1 &= 1 \\
&\vdots \\
x_1^1(\mathbf{w}_1^m)_n - x_1^2(\mathbf{w}_1^m)_n + \ldots + x_{q_m}^1(\mathbf{w}_{q_m}^m)_n - x_{q_m}^2(\mathbf{w}_{q_m}^m)_1 + v_n - u_n &= 1 \\
x_1^1, x_1^2, \ldots, x_{q_m}^1, x_{q_m}^2 &\geq 0 \\
u_1, \ldots, u_n, v_1, \ldots, v_n &\geq 0.
\end{aligned}$$

Conversely, given an optimal solution to the above linear programming problem, if $0 \leq u_i - v_i$ for $i = 1, \ldots, n$ then $x_i = x_i^1 - x_i^2$ is a solution to the original problem

$$\begin{aligned}
x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &> 0 \\
&\vdots \\
x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &> 0.
\end{aligned}$$

EXAMPLE 7.1. (Figure 4.) This example will demonstrate a case where a vector field ξ can be decomposed as $\xi = \mathbf{X} \circ \zeta$ where the matrix $\mathbf{X}$ is diagonal positive definite. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3 \right|.$$

Step 1: A basis for S is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Step 2: Since $m = 1$ then steps 2.1 to 2.3 need only be used once.

Step 2.1: By lemma 7.2 a basis for the T is given by the vector

$$\left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{7} \end{bmatrix} \right\}.$$

Step 2.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{7} \end{bmatrix}.$$

which is the span of the vector

$$\left\{ \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{7} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vector

$$\left\{\begin{bmatrix} \frac{1}{3} \\ \frac{2}{7} \end{bmatrix}\right\}.$$

Step 2.3: Let S be the span of the vector

$$\left\{\begin{bmatrix} \frac{1}{3} \\ \frac{2}{7} \end{bmatrix}\right\}.$$

Step 3: The equations

$$\frac{1}{3}x_1 > 0$$
$$\frac{2}{7}x_1 > 0$$

can be satisfied simultaneously with $x_1 = 21$.

Step 5: It can be concluded that ξ may be decomposed as the composition of a diagonal positive definite matrix $\Lambda(7,6)$ and a gradient vector field ζ as

$$\xi\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix} \circ \xi\right)$$
$$= \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \circ \left(\begin{bmatrix} 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 21 \\ 42 \end{bmatrix}\left|\begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3\right|\right).$$

References.

[1] Chua L.O. and Deng A., "Canonical piecewise-linear modeling." *IEEE Transactions on Circuits and Systems.*, vol.33, pp.511-525, May 1986.

[2] Chua L.O. and Deng A., "Canonical piecewise-linear representations." *IEEE Transactions on Circuits and Systems.*, vol.35, pp.101-111, January 1988.

[3] Lum R., and Chua L.O., "Invariance properties of continuous piecewise-linear vector fields." Electronics Research Laboratory No. UCB/ERL M90/38 3 May 1990.

[4] Parker T.S. and Chua L.O., "Practical numerical algorithms for chaotic systems." Springer-Verlag, New York, 1989.

Figure captions.

Figure 1. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\left|\begin{bmatrix}1\\0\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}-1\right|+\begin{bmatrix}1\\1\end{bmatrix}\left|\begin{bmatrix}0\\1\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}+1\right|.$$

Figure 2. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\left|\begin{bmatrix}1\\0\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}+1\right|+\begin{bmatrix}-1\\1\end{bmatrix}\left|\begin{bmatrix}0\\1\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}-1\right|.$$

Figure 3. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}+\begin{bmatrix}1&2\\3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}5\\5\end{bmatrix}\left|\begin{bmatrix}1\\2\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}+3\right|.$$

Figure 4. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}3\\7\end{bmatrix}\left|\begin{bmatrix}1\\2\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}+3\right|.$$

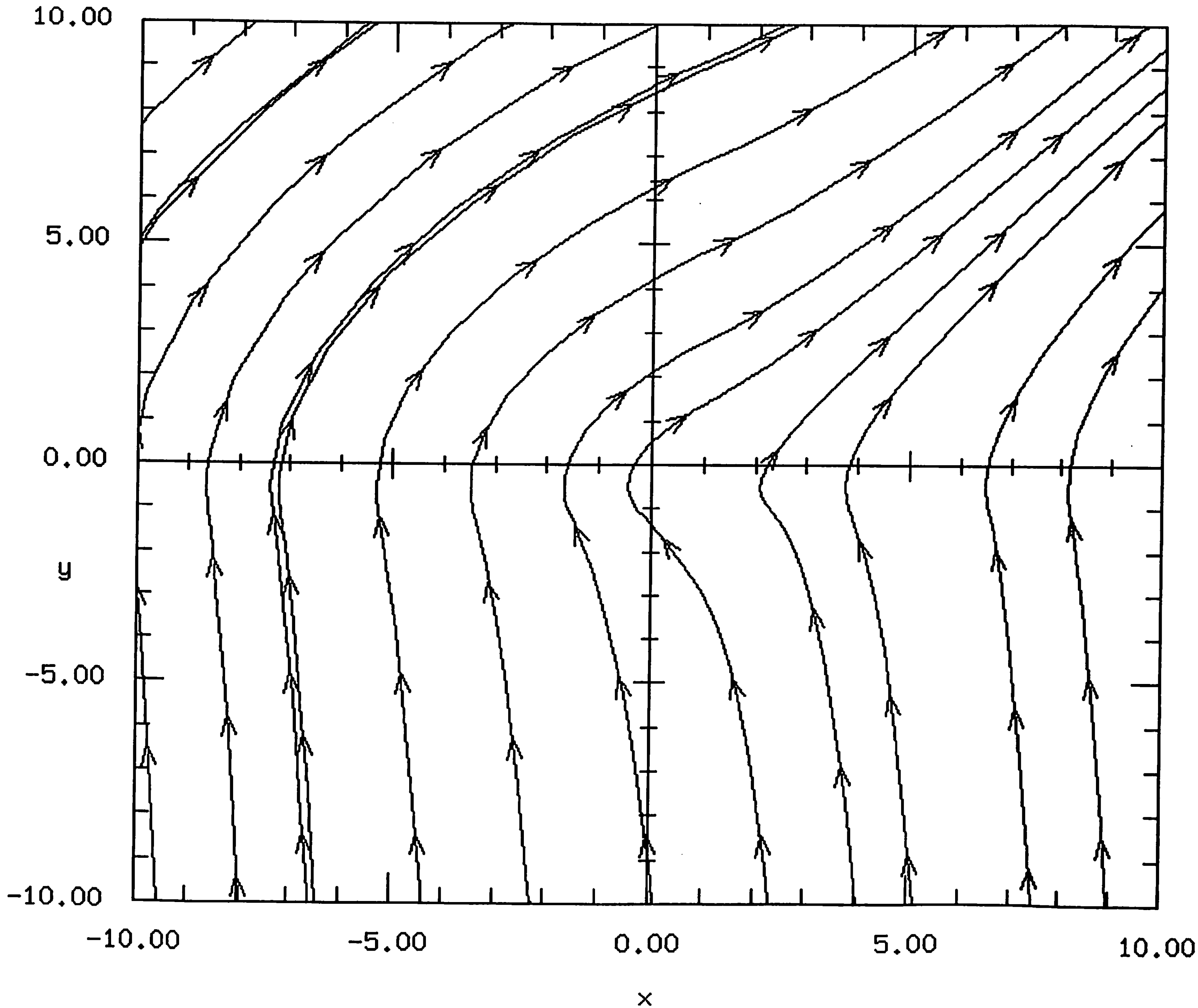

Figure 1

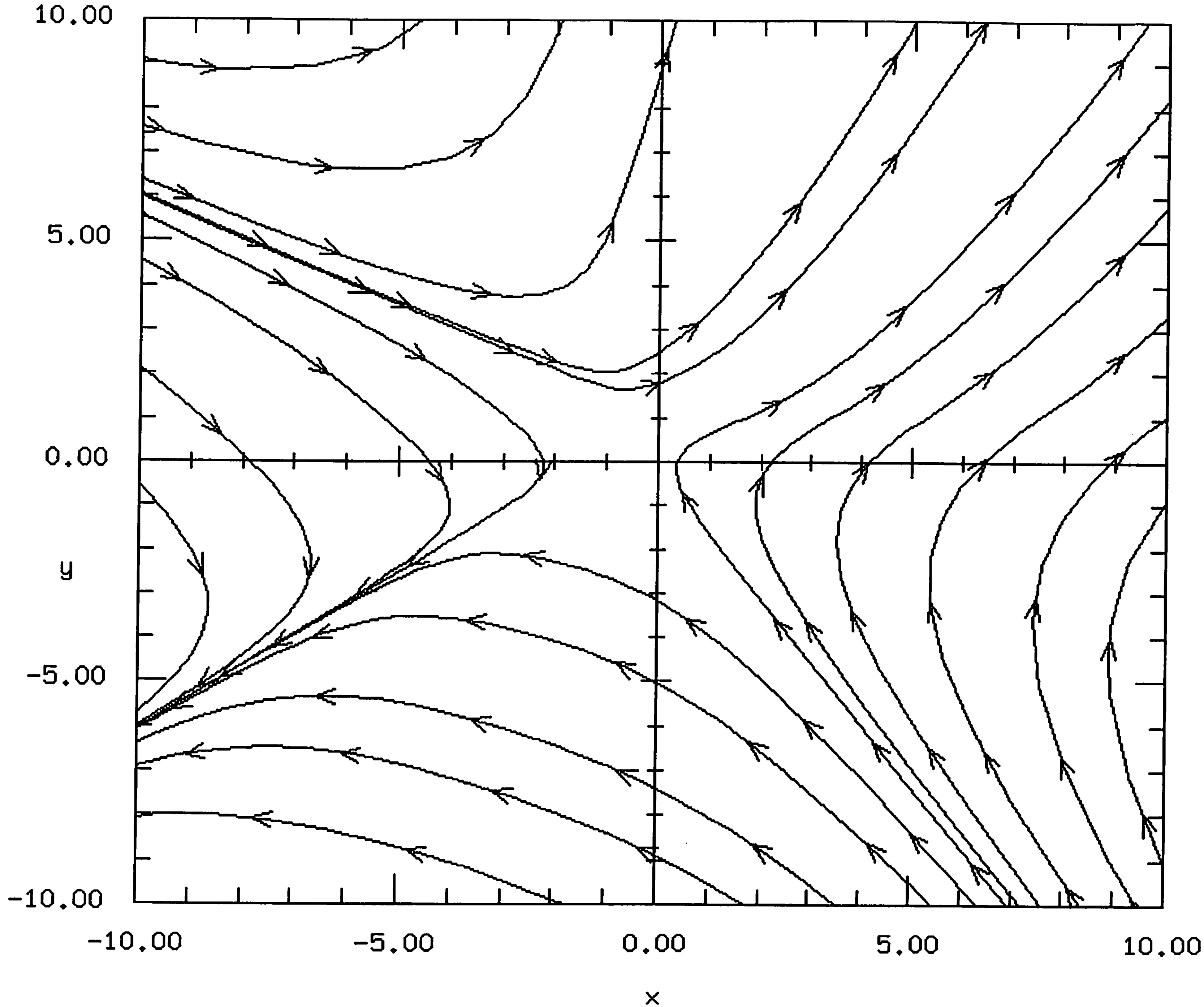

Figure 2

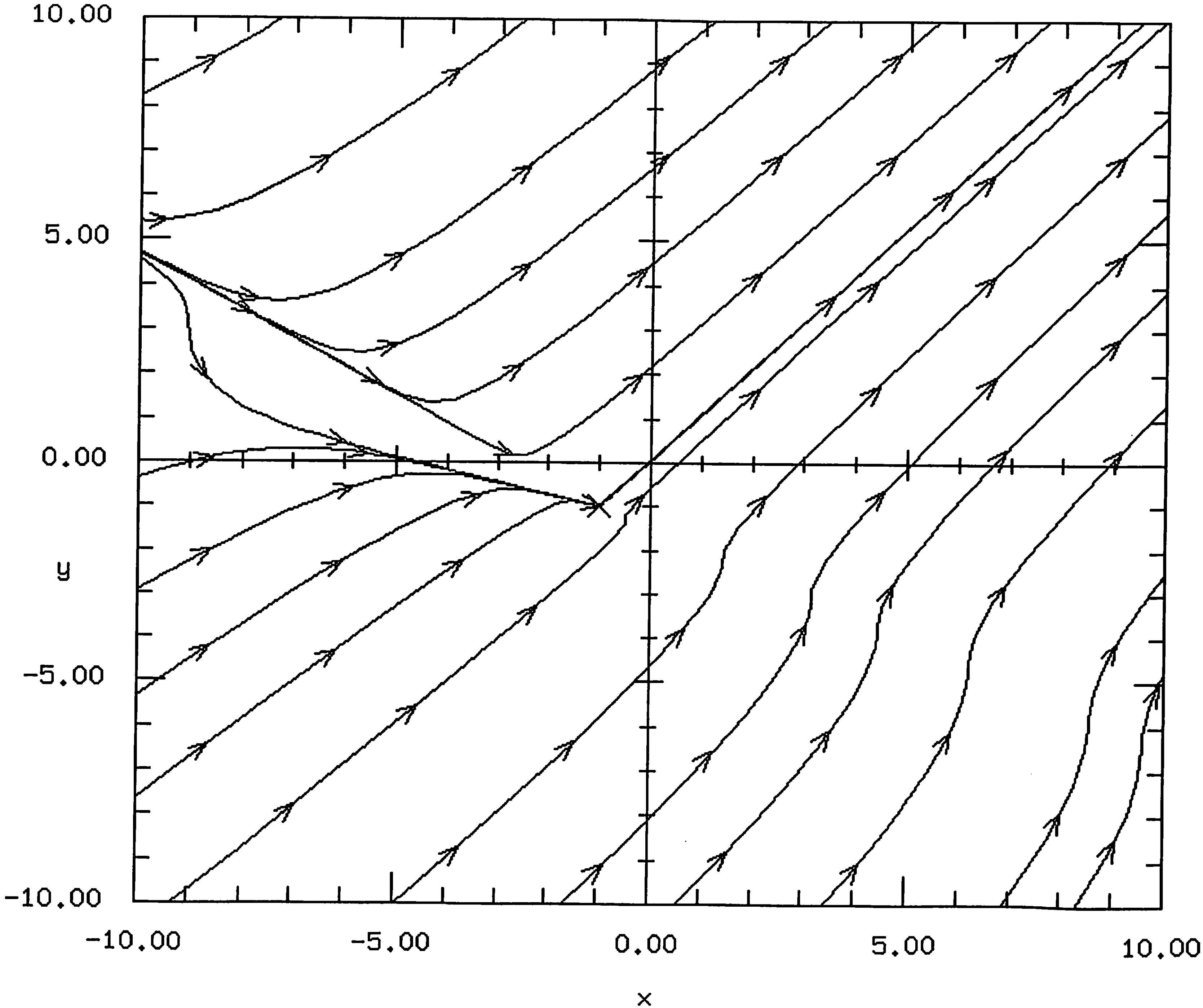

Figure 3

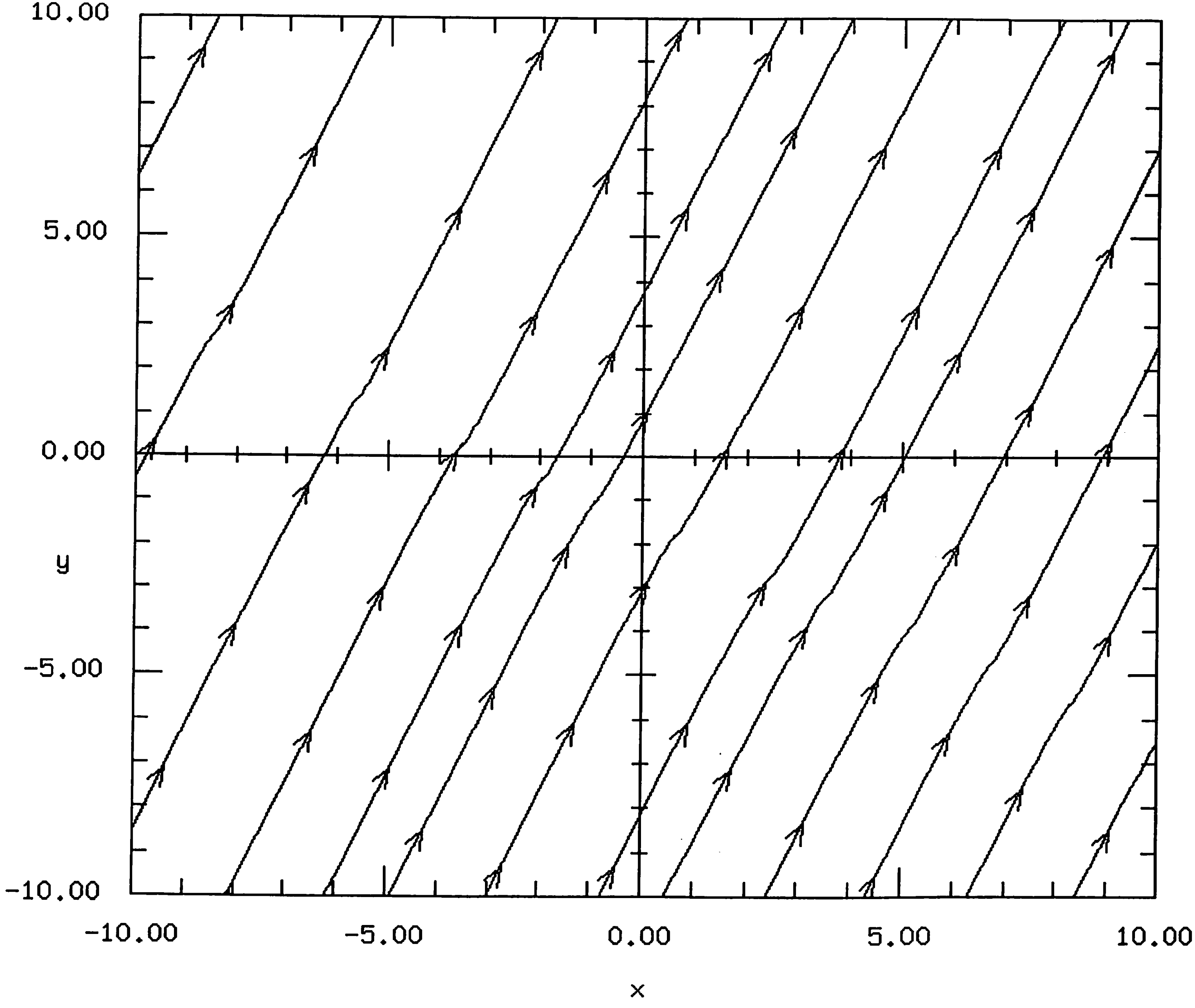

Figure 4