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THE IDENTIFICATION OF GENERAL PSEUDO-GRADIENT VECTOR FIELDS. †

Robert Lum AND Leon O. Chua. ††

Abstract

A vector field is called pseudo-gradient if it is either the composition of a matrix with a gradient vector field or under composition with a matrix becomes a gradient vector field. This paper deals with those pseudo-gradient vector fields formed from the composition of a matrix with a gradient vector field. Such vector fields are especially important in the field of electrical engineering due to their comparative ease of implementation when compared to the general vector field.

In this paper, the identification of such vector fields is completed for the cases when the matrix is either invertible, invertible symmetric, symmetric positive definite or diagonal positive definite. In the process of such identification, a decomposition of the original vector field as the composition of a matrix and a gradient vector field will ensue.

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§0. Introduction.

It is a well known fact that not all vector fields are gradient vector fields. As vector fields, those that are gradient vector fields are much easier to implement as electronic circuits than those which are not of that form. As to those vector fields which are not gradient vector fields, one looks for particular structures that may help in modelling them as electronis circuits. One such class of vector fields that may be modelled with slightly more difficulty than gradient vector fields are those that are called pseudo-gradient.

The paper "The identification of pseudo-gradient vector fields," [4] addresses that identification problem for psuedo-gradient piecewise-linear vector fields. In a more general setting, this paper considers the identification of psuedo-gradient vector fields when the vector fields are C^1 differentiable. Note that piecewise-linear vector fields are not C^1 functions, although the techniques used are very similar in both cases, the approach used are fundamentally different. In the case of piecewise-linear vector fields, an explicit construction of the state function (the function whose derivative is the piecewise-linear vector field) is constructed, whereas for the case of C^1 vector fields the state function is implicitly proved to exist.

§1. Definitions.

This section will present the definition of gradient vector field and pseudo-gradient vector field. Examples of both these types of vector fields will be given. Note that not all vector fields are gradient vector fields, and that not all psuedo-gradient vector fields are gradient vector fields. However, the set of psudo-gradient vector fields is much more general than the set of gradient vector fields and includes the latter as a proper subset.

Definition 1.1. Let $f(\mathbf{x})$ be a C^1 vector valued function. The vector field given by $f(\mathbf{x})$ is a gradient vector field if and only if there is a C^2 real valued function $F(\mathbf{x})$ such that $f(\mathbf{x}) = \nabla F(\mathbf{x})$.

EXAMPLE 1.2. (Figure 1.) The vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is a gradient vector field since $f(\mathbf{x}) = \nabla F(\mathbf{x})$ where $F(\mathbf{x}) = x_1^2/2 + x_2^2/2$.

EXAMPLE 1.3. (Figure 2.) The vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

is not a gradient vector field. Assume to the contrary that a function $F(\mathbf{x})$ exists such that $f(\mathbf{x}) = \nabla F(\mathbf{x})$. Then

$$\frac{\partial F}{\partial x_1} = x_1$$
$$\frac{\partial F}{\partial x_2} = x_1.$$

As $F(\mathbf{x})$ is a C^2 function then

$$\begin{aligned}\frac{\partial^2 F}{\partial x_2 \partial x_1} &= 0 \\ &\neq 1 \\ &= \frac{\partial^2 F}{\partial x_1 \partial x_2}.\end{aligned}$$

By contradiction, $f(\mathbf{x})$ is not a gradient vector field.

Theorem 1.4. *Let $f(\mathbf{x})$ be a C^1 vector valued function defined on an open simply connected domain D. Then $f(\mathbf{x})$ is a gradient vector field if and only if the Jacobian matrix $\mathrm{D}f(\mathbf{x})$ is symmetric.*

Definition 1.5. Let $f(\mathbf{x})$ be a C^1 vector valued function. If there exists a matrix M and gradient vector field $g(\mathbf{x})$ such that either $f(\mathbf{x}) = \mathrm{M}g(\mathbf{x})$ or $\mathrm{M}f(\mathbf{x}) = g(\mathbf{x})$, then $f(\mathbf{x})$ is said to be a pseudo-gradient vector field.

Theorem 1.6. *Let*

$$\mathbf{X} = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}$$

be an $n \times n$ matrix and

$$f\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix}$$

be a C^1 vector valued function. Define the functions

$$G_{ij}(\mathbf{X}) = \sum_{k=1}^{n} \left(x_{ik} \frac{\partial}{\partial x_j} f_k(x_1, \dots, x_n) - x_{jk} \frac{\partial}{\partial x_i} f_k(x_1, \dots, x_n) \right)$$

for $1 \le i < j \le n$. Then $\mathrm{M}f(\mathbf{x})$ is a gradient vector field if and only if

$$G_{ij}(\mathbf{X}) = 0$$

for $1 \le i < j \le n$.

Proof. By theorem 1.4, $\mathbf{X}f(\mathbf{x})$ is a gradient vector field if and only if $\mathrm{D}\mathbf{X}f(\mathbf{x})$ is a symmetric matrix,

$$\begin{aligned} & [\mathrm{D}\mathbf{X}f(\mathbf{x})]_{ij} = [\mathrm{D}\mathbf{X}f(\mathbf{x})]_{ji} \\ \Rightarrow \quad & \sum_{k=1}^{n} x_{ik} \frac{\partial}{\partial x_j} f_k(x_1, \dots, x_n) = \sum_{k=1}^{n} x_{jk} \frac{\partial}{\partial x_i} f_k(x_1, \dots, x_n) \end{aligned}$$

for $1 \le i < j \le n$, thus

$$\begin{aligned} G_{ij}(\mathbf{X}) &= \sum_{k=1}^{n} \left(x_{ik} \frac{\partial}{\partial x_j} f_k(x_1, \dots, x_n) - x_{jk} \frac{\partial}{\partial x_i} f_k(x_1, \dots, x_n) \right) \\ &= 0 \end{aligned}$$

for $1 \le i < j \le n$. ∎

§2. Pseudo-gradient vector fields $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$ where $\mathbf{M}$ is invertible.

Given a vector field $f(\mathbf{x})$, this section will determine if there exists an invertible matrix $\mathbf{M}$ and gradient vector field $g(\mathbf{x})$ such that $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Since $\mathbf{M}$ is invertible, the problem is equivalent to finding an invertible matrix $\mathbf{X}$ such that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. Finding matrices $\mathbf{X}$ such that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field is done by application of theorem 1.6. There remains the problem to determine which, if any of the matrices $\mathbf{X}$ are invertible. If no such matrices exist then $f(\mathbf{x})$ may not be written as $\mathbf{M}g(\mathbf{x})$ with $\mathbf{M}$ invertible and $g(\mathbf{x})$ a gradient vector field. If however, an invertible $\mathbf{X}$ exists, then $\mathbf{M}$ can be set to $\mathbf{X}^{-1}$. Then $f(\mathbf{x}) = \mathbf{X}^{-1}(\mathbf{X}f(\mathbf{x}))$ is a decomposition of the desired form.

The problem becomes one of solving

$$G_{ij}(\mathbf{X}) = 0$$

for $1 \leq i < j \leq n$ and

$$\det\mathbf{X} \neq 0.$$

The following examples will demonstrate the method outlines above.

EXAMPLE 2.1. (Figure 3.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{\sin x_1}(1 + 2x_2\cos x_1) \\ e^{\sin x_1} \end{bmatrix},$$

then

$$\begin{aligned} G_{12}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ &\quad x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ &= e^{\sin x_1}\cos x_1(2x_{11} - x_{22}) - e^{\sin x_1}(-2x_2\sin x_1 + \cos x_1 + 2x_2\cos^2 x_1)x_{21}. \end{aligned}$$

Thus $G_{12}(\mathbf{X}) = 0$ if and only if $x_{21} = 0, x_{22} = 2x_{11}$. Thus

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ 0 & 2x_{11} \end{bmatrix}.$$

If $\mathbf{X}$ is invertible then $2x_{11}^2 \neq 0$, i.e. $x_{11} \neq 0$. Let $x_{11} = 1$ and define $\mathbf{M} = \mathbf{X}^{-1}$, then

$$\begin{aligned} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & x_{12} \\ 0 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} 1 & x_{12} \\ 0 & 1 \end{bmatrix} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 2 & -x_{12} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{\sin x_1}(1 + 2x_2\cos x_1 + x_{12}) \\ 2e^{\sin x_1} \end{bmatrix} \end{aligned}$$

is a desired decomposition for the pseudo-gradient vector field $f(\mathbf{x})$.

EXAMPLE 2.2. (Figure 4.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{\sin x_1} + x_2 \\ x_2\sin x_1 \end{bmatrix},$$

then

$$\begin{aligned}G_{12}(\mathbf{X}) = x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ = x_{11} - x_{21}e^{\sin x_1}\cos x_1 + x_{12}\sin x_1 - x_{22}x_2\cos x_1.\end{aligned}$$

Thus $G_{12}(\mathbf{X}) = 0$ if and only if $x_{11} = x_{12} = x_{21} = x_{22} = 0$. It follows that there does not exist invertible matrix $\mathbf{M}$ and gradient vector field $g(\mathbf{x})$ such that $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

§3. Pseudo-gradient vector fields $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$ where M is symmetric invertible.

This section will determine if the vector field $f(\mathbf{x})$ can be written as $\mathbf{M}g(\mathbf{x})$ where the matrix $\mathbf{M}$ is symmetric invertible and $g(\mathbf{x})$ is a gradient vector field.

The solution to the identification problem can also be achieved by determining symmetric invertible matrices $\mathbf{X}$ such that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. By setting $\mathbf{M} = \mathbf{X}^{-1}$ then $f(\mathbf{x}) = \mathbf{M}(\mathbf{M}^{-1}f(\mathbf{x}))$ is a desired decomposition.

Determination of matrices $\mathbf{X}$ such that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field is achieved by application of theorem 1.6. By applying polynomial restrictions on the entries of the matrix $\mathbf{X}$, the matrix can be assured to be both invertible and symmetric.

Namely, solutions are required that satisfy the following set of conditions,

$$G_{ij}(\mathbf{X}) = 0$$

$$x_{ij} = x_{ji}$$

for $1 \leq i < j \leq n$,

$$\det\mathbf{X} \neq 0$$

and

$$x_{ij} = x_{ji}.$$

The first condition ensures that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. The second condition state that $\mathbf{X}$ is a symmetric matrix while the last condition considers only those matrices that are invertible. If no solution exists to the above set of conditions then a decomposition of the required form does not exist. A solution, if it exists, to the above set of conditions will determine a matrix $\mathbf{M} = \mathbf{X}^{-1}$ such that $f(\mathbf{x}) = \mathbf{M}(\mathbf{M}^{-1}f(\mathbf{x}))$ is a decomposition of the required form. The following examples will demonstrate the methodology involved.

EXAMPLE 3.1. (Figure 5.) Let

$$f\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\cos(e^{x_1}+x_2^2)(e^{x_1}+2x_1)\\ \cos(e^{x_1}+x_2^2)(e^{x_1}+4x_2)\end{bmatrix},$$

then

$$\begin{aligned}
G_{12}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\
&\quad x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\
&= (x_{11}+2x_{12})(2\cos(e^{x_1}+x_2^2) - 4x_2^2\sin(e^{x_1}+x_2^2)) - (x_{21}+x_{22})(\cos(e^{x_1}+x_2^2)e^{x_1} + \\
&\quad \sin(e^{x_1}+x_2^2)e^{2x_1}) - (x_{11}+x_{12}-x_{21}-2x_{22})e^{x_1}2x_2\sin(e^{x_1}+x_2^2).
\end{aligned}$$

A simultaneous solution to

$$G_{12}(\mathbf{X}) = 0$$

$$x_{12} = x_{21}$$

$$\det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \neq 0$$

is given by

$$\mathbf{X}(x_{11}) = x_{11}\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

for $x_{11} \neq 0$. Let $\mathbf{X}(x_{11})$ be the matrix corresponding to the choice of $x_{11} = 2$, with $\mathbf{M} = \mathbf{X}(2)^{-1}$ then

$$\begin{aligned}
f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \cos(e^{x_1}+x_2^2)e^{x_1} \\ \cos(e^{x_1}+x_2^2)2x_2 \end{bmatrix}.
\end{aligned}$$

EXAMPLE 3.2. (Figure 6.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1^2 + 4x_1x_2 + 4x_2 \\ 6x_1^2 + 6x_2 \end{bmatrix},$$

then

$$\begin{aligned}
G_{12}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\
&\quad x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\
&= 2x_2(2x_{11} - 7x_{21} - 6x_{22}) - 4x_2x_{21} + 2(2x_{11} + 3x_{12}).
\end{aligned}$$

Note that if $G_{12}(\mathbf{X}) = 0$ then

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{11}\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}.$$

It is immediate that there is no solution to

$$G_{12}(\mathbf{X}) = 0$$

$$x_{12} = x_{21}$$

$$\det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \neq 0$$

since the first and third conditions require that $x_{11} \neq 0$ from whence $x_{12} = -2/3x_{11} \neq 0 = x_{21}$ in violation of the third condition. It can be concluded that $f(\mathbf{x})$ may not be written as $\mathbf{M}g(\mathbf{x})$ with $\mathbf{M}$ symmetric invertible and $g(\mathbf{x})$ a gradient vector field.

§4. Pseudo-gradient vector fields $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$ where $\mathbf{M}$ is symmetric positive definite. An algorithm will be presented to determine if a given vector field may be rewritten in the form $\mathbf{M}g(\mathbf{x})$ where the matrix $\mathbf{M}$ is symmetric positive definite and the function $g(\mathbf{x})$ is a gradient vector field. The algorithm works by observing that if such a decomposition exists then there exists a symmetric postive definite matrix $\mathbf{X}$ such that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. Conversely, if the matrix $\mathbf{X}$ symmetric definite positive exists then a decomposition of the required form can be constructed. It may also be noted that if such a matrix $\mathbf{X}$ cannot be found then a decomposition of the desired form does not exist.

Theorem 1.6 will determine those matrices $\mathbf{X}$ such that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. Restrictions on the elements of $\mathbf{X}$ create a set of conditions that, if met, will determine that the matrix $\mathbf{X}$ is symmetric positive definite.

Effectively, one looks for solutions to the following set of conditions,

$$G_{ij}(\mathbf{X}) = 0$$

$$x_{ij} = x_{ji}$$

for $1 \le i < j \le n$ and

$$\det \begin{bmatrix} x_{11} & \dots & x_{1i} \\ \vdots & & \vdots \\ x_{i1} & \dots & x_{ii} \end{bmatrix} > 0$$

for $1 \le i \le n$. The first condition ensures that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. The second condition states that $\mathbf{X}$ is a symmetric matrix while that last condition restricts the symmetric matrices under consideration tothose that are positive definite. If such matrices $\mathbf{X}$ exist, then by setting $\mathbf{M} = \mathbf{X}^{-1}$ a decompostion $f(\mathbf{x}) = \mathbf{M}(\mathbf{M}^{-1}f(\mathbf{x}))$ is obtained.

The following examples should illustrate the algorithm.

EXAMPLE 4.1. (Figure 7.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8x_1^3x_2^4 + 4x_1^4x_2^3 + 4x_1x_2^2 + 2x_1^2x_2 + 2x_2 + x_1 \\ 4x_1^3x_2^4 + 16x_1^4x_2^3 + 2x_1x_2^2 + 8x_1^2x_2 + x_2 + 4x_1 \end{bmatrix},$$

then

$$\begin{aligned} G_{12}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ &\quad x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ &= (16x_1^3x_2^3 + 4x_1x_2 + 1)(2x_{11} - x_{21} + x_{12} - 4x_{22}) + 2x_1^2(6x_1^2x_2^2 + 1)(x_{11} + 4x_{12}) \\ &\quad - 2x_2^2(6x_1^2x_2^2 + 1)(2x_{21} + x_{22}). \end{aligned}$$

A simultaneous solution to

$$G_{12}(\mathbf{X}) = 0$$

$$x_{12} = x_{21}$$

$$\det[x_{11}] = x_{11} > 0$$

$$\det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{11}x_{22} - x_{12}x_{21} > 0$$

is given by

$$\mathbf{X}(x_{11}) = x_{11}\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

for $x_{11} > 0$. Let $x_{11} = 4/7$, then a desired decomposition is given by setting $\mathbf{M} = \mathbf{X}(4/7)^{-1}$, thus

$$\begin{aligned} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}^{-1} \left(\begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4x_1^3x_2^4 + 2x_1x_2^2 + x_2 \\ 4x_1^4x_2^3 + 2x_1^2x_2 + x_1 \end{bmatrix}. \end{aligned}$$

Example 4.2. (Figure 8.) Let

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{x_1x_2^2}(2x_1x_2 - x_2^2) \\ e^{x_1x_2^2}(x_2^2 + 8x_1x_2) \end{bmatrix},$$

then

$$\begin{aligned} G_{12}(\mathbf{X}) = {} & x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{21}\frac{\partial}{\partial x_1}f_1(\mathbf{x}) + \\ & x_{12}\frac{\partial}{\partial x_2}f_2(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\ = {} & e^{x_1x_2^2}2x_1(1 + 2x_1x_2^2)(x_{11} + 4x_{12}) + e^{x_1x_2^2}2x_2(1 + x_1x_2^2)(-x_{11} - x_{21} + x_{12} - 4x_{22}) + \\ & e^{x_1x_2^2}x_2^4(x_{21} - x_{22}). \end{aligned}$$

A simultanoeus solution to

$$\begin{aligned} G_{12}(\mathbf{X}) &= 0 \\ x_{12} &= x_{21} \\ \det[x_{11}] &= x_{11} > 0 \\ \det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} &= x_{11}x_{22} - x_{12}x_{21} > 0 \end{aligned}$$

requires that

$$\begin{aligned} x_{11} + 4x_{12} &= 0 \\ x_{21} - x_{22} &= 0 \\ x_{12} &= x_{21} \\ x_{11} &> 0 \\ x_{11}x_{22} - x_{12}x_{21} &> 0. \end{aligned}$$

The first four conditions determine the matrix

$$\mathbf{X}(x_{11}) = x_{11}\begin{bmatrix} 1 & -4 \\ -4 & 4 \end{bmatrix}$$

with $x_{11} > 0$. However, this matrix has determinant $-20x_{11}^2 < 0$, violating the last condition. Thus, $f(\mathbf{x})$ does not have a decomposition of the required form.

§5. Pseudo-gradient vector fields $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$ where M is diagonal positive definite.

This final section will consider the decompostion of a vector field $f(\mathbf{x})$ as $\mathbf{M}g(\mathbf{x})$ where the matrix $\mathbf{M}$ is diagonal positive definite and the vector field $g(\mathbf{x})$ ia a gradient vector field. Since the matrix $\mathbf{M}$ is diagonal, it may be written as $\mathbf{M} = \Lambda(m_{11}, \ldots, m_{nn})$ where $m_{11}, \ldots, m_{nn}$ are its diagonal elements.

The identification problem is the same as finding a diagonal positive definite matrix $\mathbf{X} = \Lambda(x_{11}, \ldots, x_{nn})$ such that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. Once this is achieved, by setting $\mathbf{M} = \mathbf{X}^{-1}$, a decomposition of the required form is $f(\mathbf{x}) = \mathbf{M}(\mathbf{M}^{-1}f(\mathbf{x}))$. If the vector field $f(\mathbf{x})$ does not have a decomposition of the required form then the matrix $\mathbf{X}$ does not exist.

Theorem 5.1 gives conditions on the elements of the diagonal matrix $\mathbf{X} = \Lambda(x_{11}, \ldots, x_{nn})$ to ensure that $\mathbf{X}f(\mathbf{x})$ is a gradient vector field. Of these diagonal matrices, one searches for those that are positive definite, this entails the consideration of those matrices for which $x_{11}, \ldots, x_{nn} > 0$. Examples follow to illustrate the technique.

Theorem 5.1. *For $1 \le i < j \le n$ define the functions*

$$H_{ij}(\mathbf{X}) = x_{ii}\frac{\partial}{\partial x_j} f_i(x_1, \ldots, x_n) - x_{jj}\frac{\partial}{\partial x_i} f_j(x_1, \ldots, x_n).$$

The diagonal matrices $\mathbf{X} = \Lambda(x_{11}, \ldots, xnn)$ satisfy $\mathbf{X}f(\mathbf{x})$ a gradient vector field if and only if

$$H_{ij}(\mathbf{X}) = 0$$

for $1 \le i < j \le n$.

Proof. The vector field $\mathbf{X}f(\mathbf{x})$ is a gradient vector field if and only if $\mathbf{D}\mathbf{X}f(\mathbf{x})$ is a symmetric matrix,

$$[\mathbf{D}\mathbf{X}f(\mathbf{x})]_{ij} = [\mathbf{D}\mathbf{X}f(\mathbf{x})]_{ji}$$

$$\Rightarrow \qquad x_{ii}\frac{\partial}{\partial j} f_i(x_1, \ldots, x_n) = x_{jj}\frac{\partial}{\partial i} f_j(x_1, \ldots, x_n)$$

for $1 \le i < j \le n$, thus

$$\begin{aligned} H_{ij}(\mathbf{X}) &= x_{ii}\frac{\partial}{\partial x_j} f_i(x_1, \ldots, x_n) - x_{jj}\frac{\partial}{\partial x_i} f_j(x_1, \ldots, x_n) \\ &= 0 \end{aligned}$$

for $1 \le i < j \le n$. ∎

Example 5.2. Let

$$f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2e^{x_2}\sin(x_1e^{x_2}) \\ -4x_1e^{x_2}\sin(x_1e^{x_2}) + 4e^{x_2}x_3 \\ e^{x_2} \end{bmatrix},$$

then

$$
\begin{aligned}
H_{12}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\
&= -2e^{2x_2}\left(x_1\cos(x_1e^{x_2}) + \sin(x_1e^{x_2})\right)(x_{11} - 2x_{22}) \\
H_{13}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_3}f_1(\mathbf{x}) - x_{33}\frac{\partial}{\partial x_1}f_3(\mathbf{x}) \\
&= 0 \\
H_{23}(\mathbf{X}) &= x_{22}\frac{\partial}{\partial x_3}f_2(\mathbf{x}) - x_{33}\frac{\partial}{\partial x_2}f_3(\mathbf{x}) \\
&= e^{x_2}(4x_{22} - x_{33}).
\end{aligned}
$$

A solution is required for

$$
\begin{aligned}
H_{12}(\mathbf{X}) &= -2e^{2x_2}\left(x_1\cos(x_1e^{x_2}) + \sin(x_1e^{x_2})\right)(x_{11} - 2x_{22}) = 0 \\
H_{13}(\mathbf{X}) &= 0 \\
H_{23}(\mathbf{X}) &= e^{x_2}(4x_{22} - x_{33}) = 0
\end{aligned}
$$

$$x_{11}, x_{22}, x_{33} > 0.$$

Solutions are given by

$$\mathbf{X}(x_{11} = x_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

for $x_{11} > 0$. Let $x_{11} = 1/2$, setting $\mathbf{M} = \mathbf{X}(1/2)^{-1}$ then

$$
\begin{aligned}
f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -e^{x_2}\sin(x_1e^{x_2}) \\ -x_1e^{x_2}\sin(x_1e^{x_2}) + e^{x_2}x_3 \\ e^{x_2} \end{bmatrix}.
\end{aligned}
$$

EXAMPLE 5.3. Let

$$f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} e^{x_1} \\ 2e^{x_2} \\ 3e^{x_1x_2x_3} \end{bmatrix},$$

then

$$
\begin{aligned}
H_{12}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_2}f_1(\mathbf{x}) - x_{22}\frac{\partial}{\partial x_1}f_2(\mathbf{x}) \\
&= 0 \\
H_{13}(\mathbf{X}) &= x_{11}\frac{\partial}{\partial x_3}f_1(\mathbf{x}) - x_{33}\frac{\partial}{\partial x_1}f_3(\mathbf{x}) \\
&= -3x_1x_2x_3e^{x_1x_2x_3}x_{33} \\
H_{23}(\mathbf{X}) &= x_{22}\frac{\partial}{\partial x_3}f_2(\mathbf{x}) - x_{33}\frac{\partial}{\partial x_2}f_3(\mathbf{x}) \\
&= -3x_1x_3e^{x_1x_2x_3}x_{33}.
\end{aligned}
$$

A solution is required for

$$H_{12}(\mathbf{X}) = 0$$

$$H_{13}(\mathbf{X}) = -3x_1x_2x_3e^{x_1x_2x_3}x_{33} = 0$$

$$H_{23}(\mathbf{X}) = -3x_1x_3e^{x_1x_2x_3}x_{33} = 0$$

$$x_{11}, x_{22}, x_{33} > 0.$$

By the first three set of conditions, $x_{33} = 0$. This is in violation of the third set of conditions for a diagonal positive definite matrix $\mathbf{X}$ with $\mathbf{X}f(\mathbf{x})$ a gradient vector field. It can be concluded that a decomposition of $f(\mathbf{x})$ as $\mathbf{M}g(\mathbf{x})$ where $\mathbf{M}$ is diagonal positve definite and $g(\mathbf{x})$ a gradient vector field does not exist.

References.

[1] Chua L.O. and Deng A., "Canonical piecewise-linear modeling." *IEEE Transactions on Circuits and Systems.*, vol.33, pp.511-525, May 1986.

[2] Chua L.O. and Deng A., "Canonical piecewise-linear representations." *IEEE Transactions on Circuits and Systems.*, vol.35, pp.101-111, January 1988.

[3] Lum R., and Chua L.O., "Invariance properties of continuous piecewise-linear vector fields." Electronics Research Laboratory No. UCB/ERL M90/38 3 May 1990.

[4] Lum R., and Chua L.O., "The identification of pseudo-gradient vector fields." Electronics Research Laboratory No. UCB/ERL M90/85.

[5] Parker T.S. and Chua L.O., "Practical numerical algorithms for chaotic systems." Springer-Verlag, New York, 1989.

Figure captions.

Figure 1. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This vector field is also a gradient vector field.

Figure 2. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}.$$

This vector field is not a gradient vector field.

Figure 3. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{\sin x_1}(1 + 2x_2 \cos x_1) \\ e^{\sin x_1} \end{bmatrix}.$$

This vector field may be written as the product of an invertible matrix $\mathbf{M}$ and a gradient vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 4. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{\sin x_1} + x_2 \\ x_2 \sin x_1 \end{bmatrix}.$$

This vector field may not be written as the product of an invertible matrix $\mathbf{M}$ and a gradient vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 5. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(e^{x_1} + x_2^2)(e^{x_1} + 2x_1) \\ \cos(e^{x_1} + x_2^2)(e^{x_1} + 4x_2) \end{bmatrix}.$$

This vector field may be written as the product of an invertible symmetric matrix $\mathbf{M}$ and a gradient vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 6. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1^2 + 4x_1x_2 + 4x_2 \\ 6x_1^2 + 6x_2 \end{bmatrix}.$$

This vector field may not be written as the product of an invertible symmetric matrix $\mathbf{M}$ and a gradient vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 7. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8x_1^3x_2^4 + 4x_1^4x_2^3 + 4x_1x_2^2 + 2x_1^2x_2 + 2x_2 + x_1 \\ 4x_1^3x_2^4 + 16x_1^4x_2^3 + 2x_1x_2^2 + 8x_1^2x_2 + x_2 + 4x_1 \end{bmatrix}.$$

This vector field may be written as the product of a symmetric positive definite matrix $\mathbf{M}$ and a gradient vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

Figure 8. This is the phase portrait corresponding to the vector field given by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{x_1x_2^2}(2x_1x_2 - x_2^2) \\ e^{x_1x_2^2}(x_2^2 + 8x_1x_2) \end{bmatrix}.$$

This vector field may not be written as the product of a symmetric positive definite matrix $\mathbf{M}$ and a gradient vector field $g(\mathbf{x})$ as $f(\mathbf{x}) = \mathbf{M}g(\mathbf{x})$.

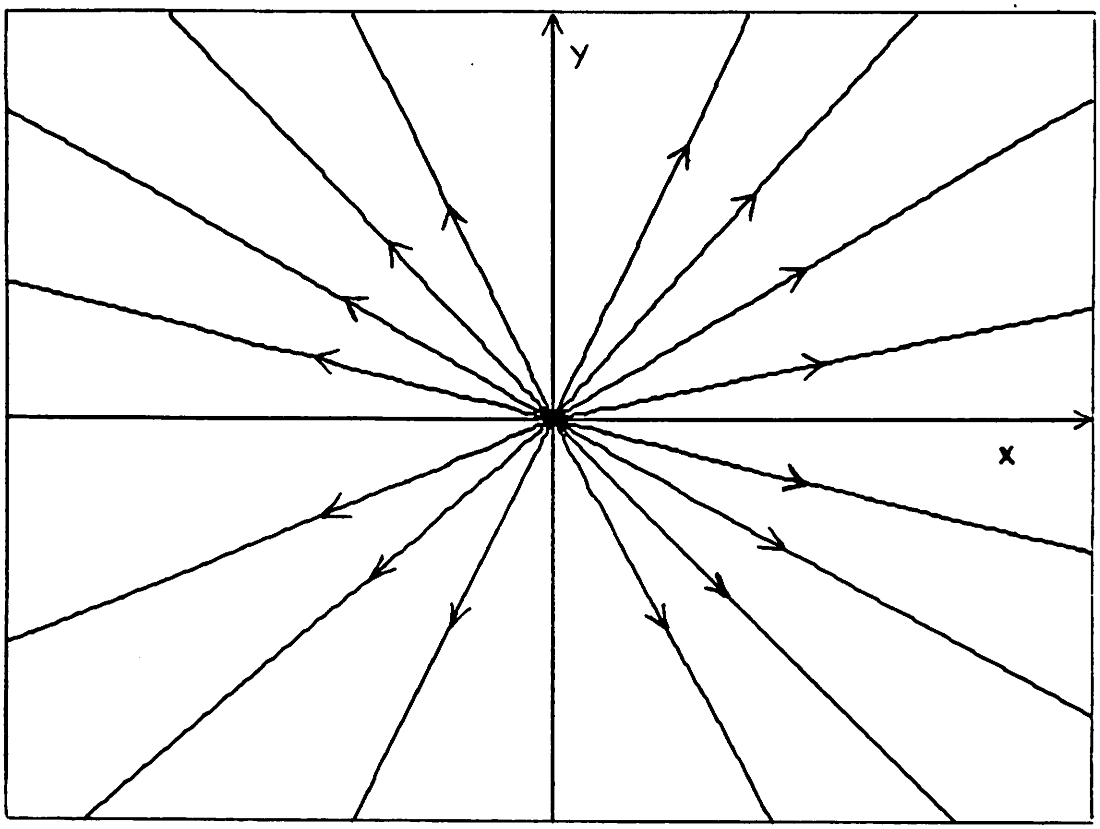

Figure 1

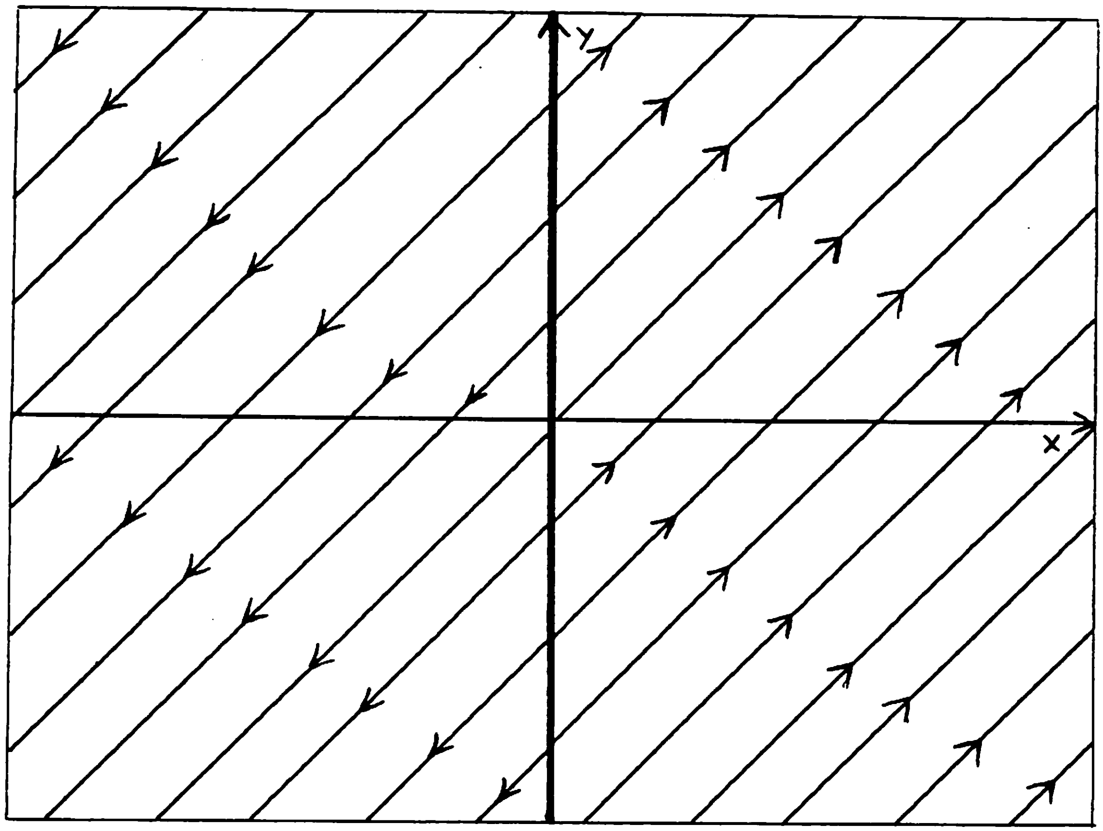

Figure 2

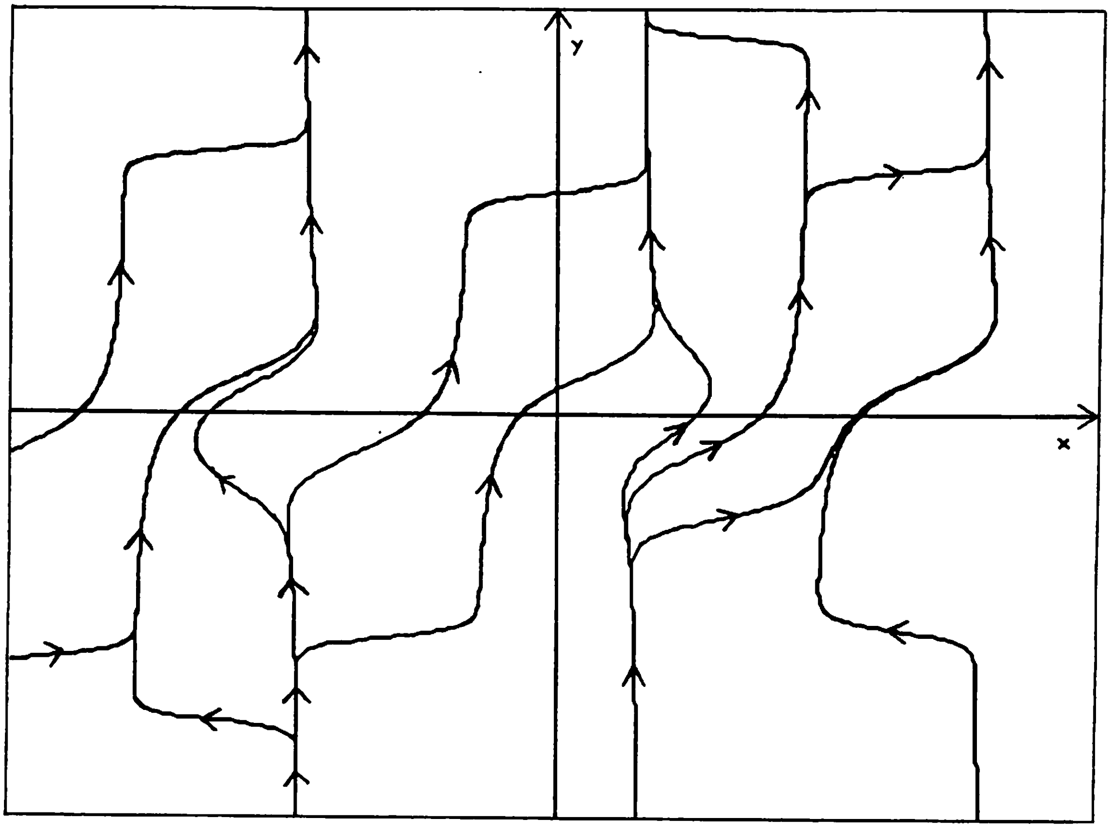

Figure 3

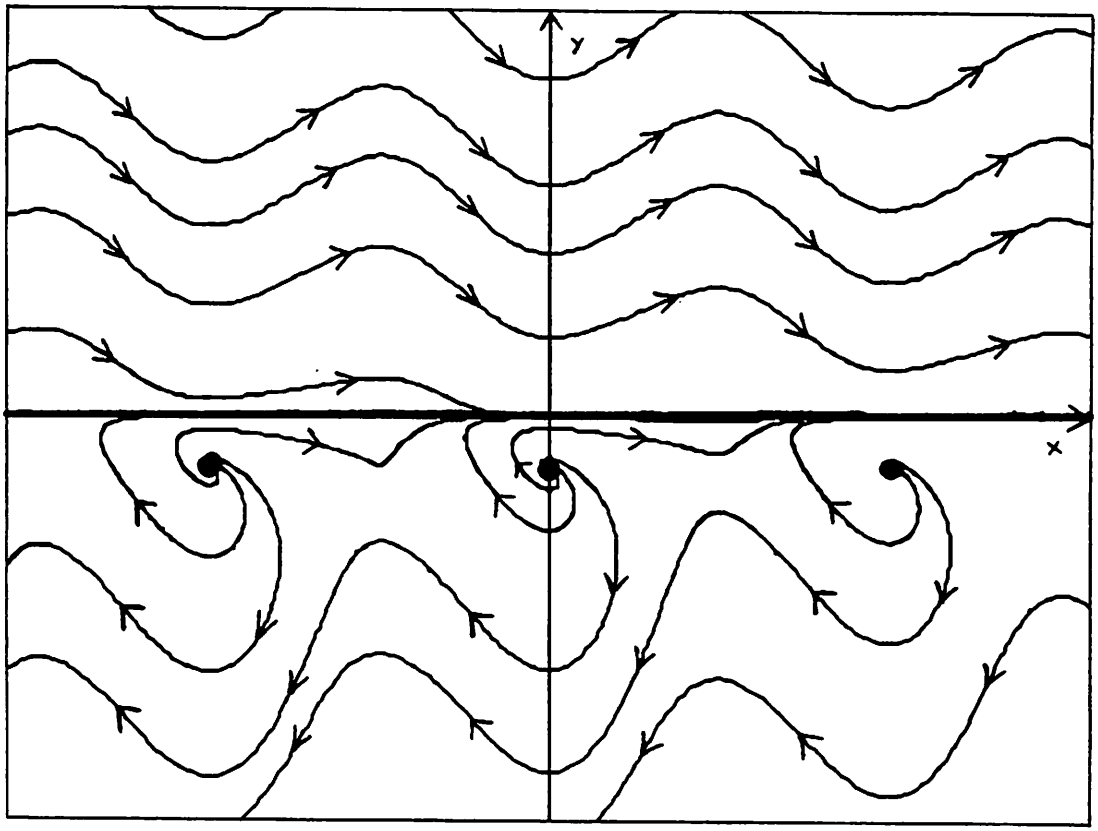

Figure 4

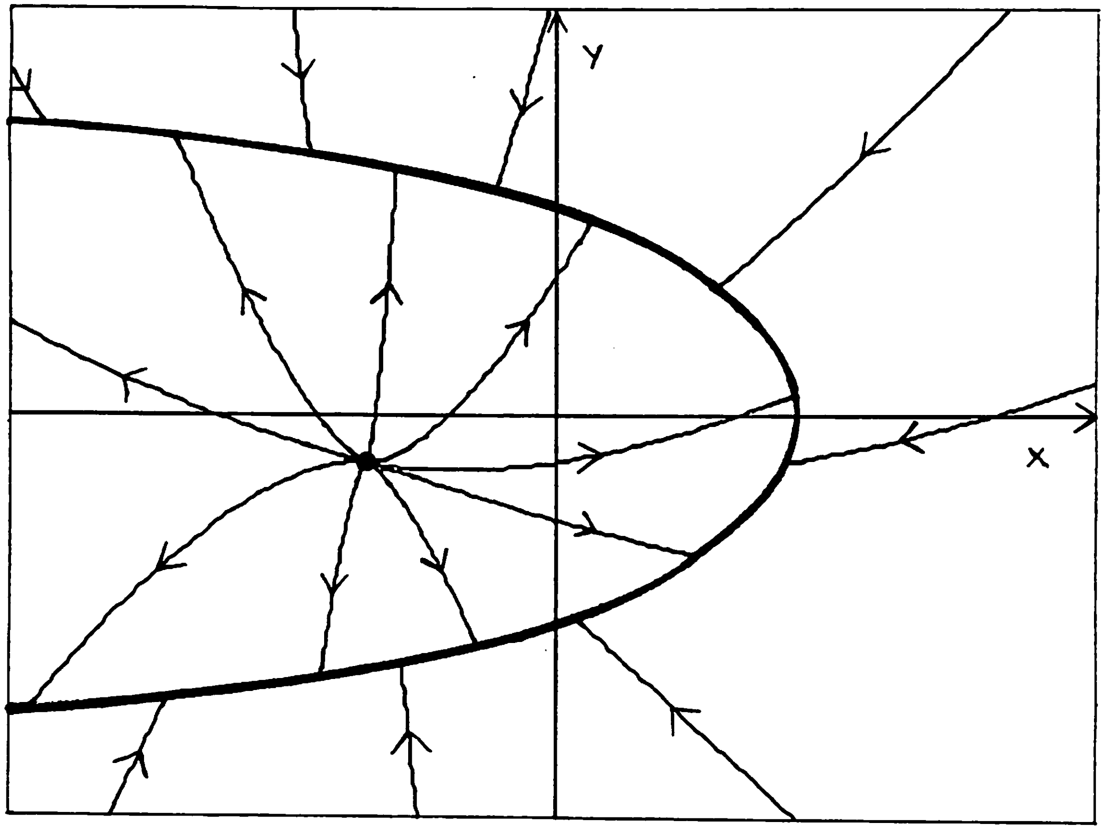

Figure 5

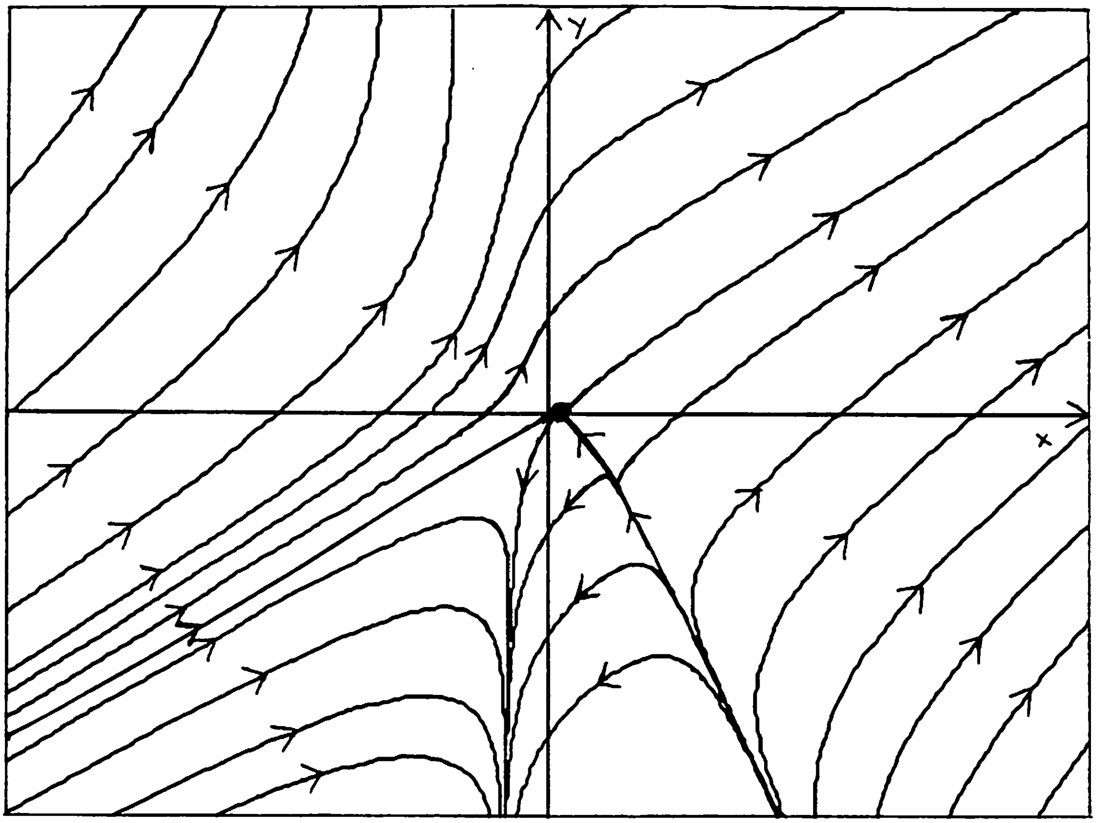

Figure 6

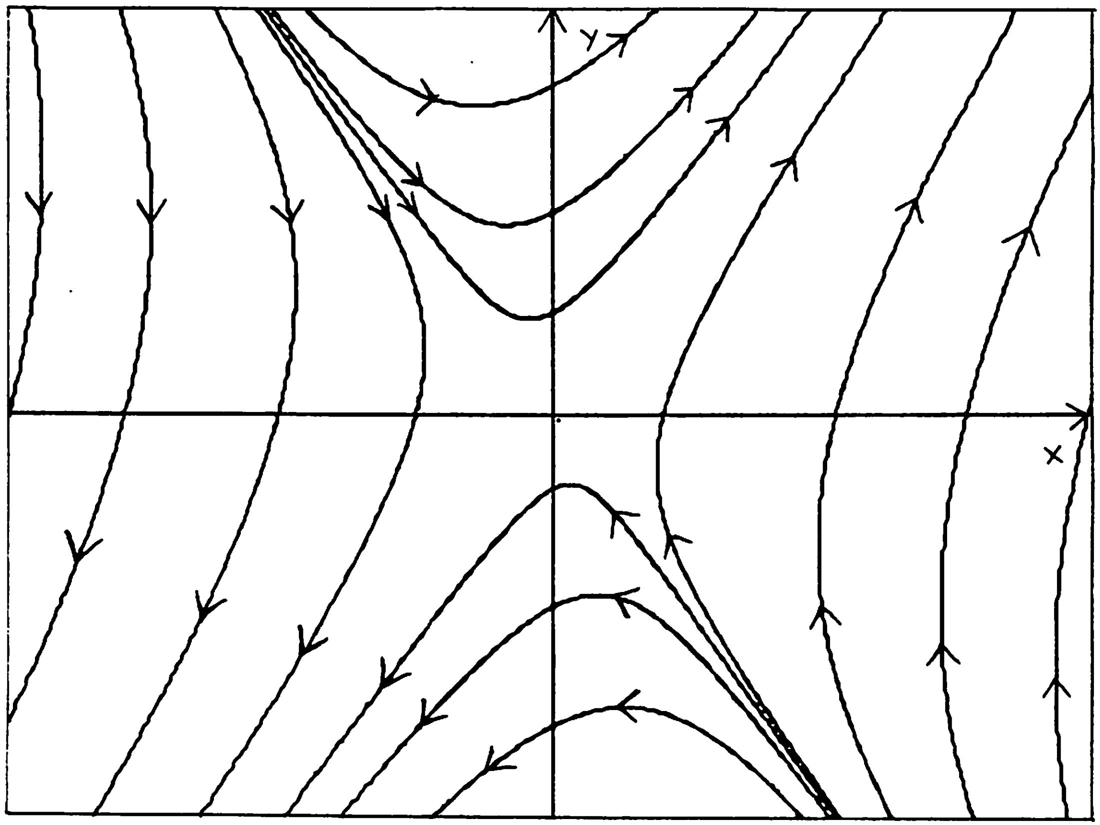

Figure 7

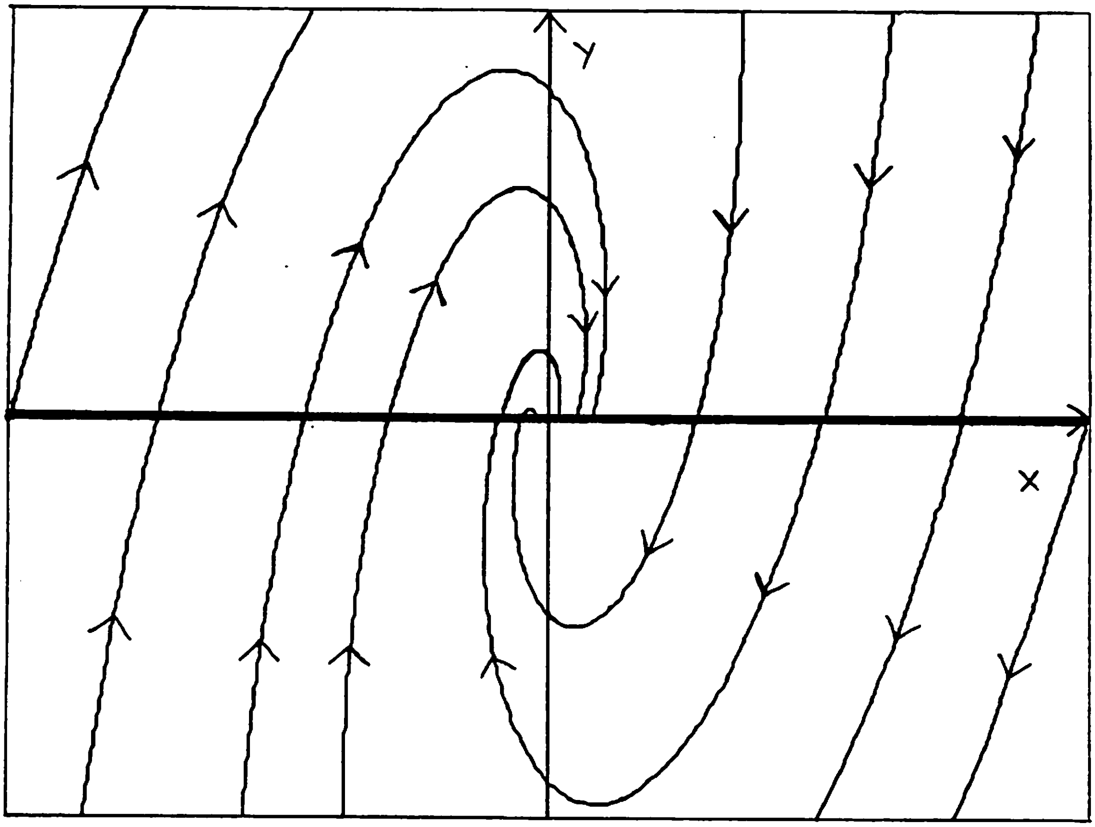

Figure 8