THE IDENTIFICATION OF PSEUDO-RECIPROCAL PIECEWISE-LINEAR VECTOR FIELDS

by

Robert Lum and Leon O. Chua

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Robert Lum AND Leon O. Chua. ††

Abstract

A vector field is called pseudo-reciprocal if it is either the composition of a matrix with a reciprocal piecewise-linear vector field or under composition with a matrix becomes a reciprocal piecewise-linear vector field. Of particular interest are those pseudo-reciprocal vector fields formed from the composition of a matrix with a reciprocal piecewise-linear vector field. Such vector fields are especially amenable to construction as electronic circuits.

In this paper, the identification of such vector fields is completed for the cases when the matrix is either invertible, invertible symmetric, symmetric positive definite or diagonal positive definite. In the process of such identification, a decomposition of the original vector field as the composition of a matrix and a reciprocal piecewise-linear vector field will ensue. The algorithm for identification is sufficiently deterministic to be fully implementable as part of a larger software package dealing with electronic circuits.

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§0. Introduction.

The particular question that concerns this paper considers the decomposition of a piecewise-linear vector field as the composition of a matrix, either invertible, invertible symmetric, symmetric positive definite or diagonal positive definite, and a reciprocal piecewise-linear vector field. Thus, it is the identification of a psuedo-reciprocal vector field for which there is a decomposition with a matrix that is either invertible, invertible symmetric, symmetric positive definite or diagonal positive definite.

Resolution of the above question allows the quick and efficient identification of piecewise-linear vector fields whose electronic implementation is less complicated than the general piecewise-linear vector field but not as simple as other types of piecewise-linear vector fields.

§1. The reciprocal vector field.

In this section the concept of a reciprocal vector is introduced. Sufficient and necessary conditions for their identification are then presented.

Definition 1.1. Let $\xi(\mathbf{x})$ be a vector field of the form

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|$$

where $0 < \alpha_{j1}^2 + \ldots + \alpha_{jn}^2$, $0 < \beta_{j1}^2 + \ldots + \beta_{jn}^2$ for $j = 1 \ldots m$. The signature of a point $\mathbf{x}$ is the m-tuple given by

$$\mathrm{sig}(\mathbf{x}) = (\mathrm{sgn}(\beta_1^t \mathbf{x} - \gamma_1), \ldots, \mathrm{sgn}(\beta_m^t \mathbf{x} - \gamma_m))$$

where $\mathrm{sgn}(x)$ is $-1, 0, 1$ depending on whether $x < 0, x = 0, 0 < x$ respectively. Define the sets

$$\mathrm{A}_{i_1,\ldots,i_m} = \{\mathbf{x} : \mathrm{sig}(\mathbf{x}) = (i_1, \ldots, i_m)\}$$

for $(i_1, \ldots, i_m) \in \{-1, 1\}^m$.

Lemma [3] 1.4. *For every* $\mathrm{A}_{i_1,\ldots,i_m}$ *there is a linear vector field* $\xi_{i_1,\ldots,i_m}$ *with*

$$\begin{aligned} \xi_{i_1,\ldots,i_m}|_{\mathrm{A}_{i_1,\ldots,i_m}}(\mathbf{x}) &= \xi|_{\mathrm{A}_{i_1,\ldots,i_m}}(\mathbf{x}) \\ &= \mathbf{M}_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_m}\mathbf{x} + \mathbf{d}_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_m}. \end{aligned}$$

Definition 1.2. A matrix is reciprocal of order (p, q) if and only if it has the form

$$\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ -\mathbf{C}^t & \mathbf{D} \end{bmatrix}$$

with $\mathbf{A}$ and $\mathbf{D}$ symmetric matrices of dimensions p and q respectively.

Definition 1.3. A vector field $\xi(\mathbf{x})$ is called a reciprocal vector field of order (p,q) if and only if $\mathbf{D}\xi(\mathbf{x})$ is a reciprocal matrix function of order (p,q).

Theorem 1.4. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\begin{bmatrix}b_{1\,1}&\dots&b_{1\,n}\\ \vdots&&\vdots\\ b_{n\,1}&\dots&b_{n\,n}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{j=1}^{m}\begin{bmatrix}\alpha_{j\,1}\\ \vdots\\ \alpha_{j\,n}\end{bmatrix}\left|\begin{bmatrix}\beta_{j\,1}\\ \vdots\\ \beta_{j\,n}\end{bmatrix}^t\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}-\gamma_j\right|.$$

The vector field ξ is a reciprocal system if and only if there exists $1\le p\le n$ such that

$$\begin{bmatrix}b_{1\,1}&\dots&b_{1\,n}\\ \vdots&&\vdots\\ b_{n\,1}&\dots&b_{n\,n}\end{bmatrix}$$

is reciprocal of order $(p,n-p)$ and

$$\begin{bmatrix}\alpha_{j\,1}\\ \vdots\\ \alpha_{j\,p}\\ \alpha_{j\,p+1}\\ \vdots\\ \alpha_{j\,n}\end{bmatrix}=k_j\begin{bmatrix}\beta_{j\,1}\\ \vdots\\ \beta_{j\,p}\\ -\beta_{j\,p+1}\\ \vdots\\ -\beta_{j\,n}\end{bmatrix}$$

for $1\le j\le m$.

Proof. Assume that $\xi(\mathbf{x})$ is reciprocal with the respective matrices being of fixed order $(p,n-p)$. For given signatures

$$(i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m),$$

$$(i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m)$$

consider the linear vector fields

$$\xi_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m},$$

$$\xi_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}.$$

By definition of a reciprocal vector field it happens that the matrices

$$\mathbf{M}_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m}$$

and

$$\mathbf{M}_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$$

corresponding to these two linear vector fields are reciprocal of order $(p,n-p)$. In particular, the matrix

$$\mathbf{M}_{i_1,\dots,i_{j-1},i_j,i_{j+1},\dots,i_m}-\mathbf{M}_{i_1,\dots,i_{j-1},-i_j,i_{j+1},\dots,i_m}$$

is reciprocal of order $(p, n-p)$ from which it follows that

$$2\begin{bmatrix} \alpha_{j\ 1}\beta_{j\ 1} & \cdots & \alpha_{j\ 1}\beta_{j\ p} & \alpha_{j\ 1}\beta_{j\ p+1} & \cdots & \alpha_{j\ 1}\beta_{j\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{j\ p}\beta_{j\ 1} & \cdots & \alpha_{j\ p}\beta_{j\ p} & \alpha_{j\ p}\beta_{j\ p+1} & \cdots & \alpha_{j\ p}\beta_{j\ n} \\ \alpha_{j\ p+1}\beta_{j\ 1} & \cdots & \alpha_{j\ p+1}\beta_{j\ p} & \alpha_{j\ p+1}\beta_{j\ p+1} & \cdots & \alpha_{j\ p+1}\beta_{j\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{j\ n}\beta_{j\ 1} & \cdots & \alpha_{j\ n}\beta_{j\ p} & \alpha_{j\ n}\beta_{j\ p+1} & \cdots & \alpha_{j\ n}\beta_{j\ n} \end{bmatrix}$$

is reciprocal. Thus $\alpha_{j\ k}\beta_{j\ l} = \alpha_{j\ l}\beta_{j\ k}$ for $l, k = 1, \dots p$. As $0 < \beta_{j\ 1}^2 + \dots + \beta_{j\ n}^2$ then $\beta_{j\ l} \neq 0$ for some $1 \leq l \leq n$. Assume that $\beta_{j\ 1} \neq 0$ for $1 \leq j \leq n$, (this assumption is to simplify the proof) then $\alpha_{j\ k} = (\alpha_{j\ 1}/\beta_{j\ 1})\beta_{j\ k}$. Let $k_j = (\alpha_{j\ 1}/\beta_{j\ 1})$, then

$$\begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ p} \end{bmatrix} = k_j \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ p} \end{bmatrix}.$$

Substituting into the above matrix gives

$$2\begin{bmatrix} k_j\beta_{j\ 1}\beta_{j\ 1} & \cdots & k_j\beta_{j\ 1}\beta_{j\ p} & k_j\beta_{j\ 1}\beta_{j\ p+1} & \cdots & k_j\beta_{j\ 1}\beta_{j\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ k_j\beta_{j\ p}\beta_{j\ 1} & \cdots & k_j\beta_{j\ p}\beta_{j\ p} & k_j\beta_{j\ p}\beta_{j\ p+1} & \cdots & k_j\beta_{j\ p}\beta_{j\ n} \\ \alpha_{j\ p+1}\beta_{j\ 1} & \cdots & \alpha_{j\ p+1}\beta_{j\ p} & \alpha_{j\ p+1}\beta_{j\ p+1} & \cdots & \alpha_{j\ p+1}\beta_{j\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{j\ n}\beta_{j\ 1} & \cdots & \alpha_{j\ n}\beta_{j\ p} & \alpha_{j\ n}\beta_{j\ p+1} & \cdots & \alpha_{j\ n}\beta_{j\ n} \end{bmatrix}.$$

Thus

$$\begin{bmatrix} k_j\beta_{j\ 1}\beta_{j\ p+1} & \cdots & k_j\beta_{j\ 1}\beta_{j\ n} \\ \vdots & & \vdots \\ k_j\beta_{j\ p}\beta_{j\ p+1} & \cdots & k_j\beta_{j\ p}\beta_{j\ n} \end{bmatrix} = \begin{bmatrix} -\alpha_{j\ p+1}\beta_{j\ 1} & \cdots & -\alpha_{j\ n}\beta_{j\ 1} \\ \vdots & & \vdots \\ -\alpha_{j\ p+1}\beta_{j\ p} & \cdots & -\alpha_{j\ n}\beta_{j\ p} \end{bmatrix}.$$

Thus, equating the first rows of both matrices,

$$\begin{bmatrix} \alpha_{j\ p+1} \\ \vdots \\ \alpha_{j\ n} \end{bmatrix} = -k_j \begin{bmatrix} \beta_{j\ p+1} \\ \vdots \\ \beta_{j\ n} \end{bmatrix}.$$

Now consider the linear vector field $\xi_{1,\dots,1}$. The matrix $\mathrm{M}_{1,\dots,1}$ corresponding to this linear vector field is reciprocal, thus

$$\begin{bmatrix} b_{1\ 1} & \cdots & b_{1\ n} \\ \vdots & & \vdots \\ b_{n\ 1} & \cdots & b_{n\ n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{2n} \end{bmatrix} + \sum_{j=1}^{m}$$

$$\begin{bmatrix} k_j\beta_{j\ 1}\beta_{j\ 1} & \cdots & k_j\beta_{j\ 1}\beta_{j\ p} & k_j\beta_{j\ 1}\beta_{j\ p+1} & \cdots & k_j\beta_{j\ 1}\beta_{j\ 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ k_j\beta_{j\ p}\beta_{j\ 1} & \cdots & k_j\beta_{j\ p}\beta_{j\ p} & k_j\beta_{j\ p}\beta_{j\ p+1} & \cdots & k_j\beta_{j\ p}\beta_{j\ n} \\ -k_j\beta_{j\ p+1}\beta_{j\ 1} & \cdots & -k_j\beta_{j\ p+1}\beta_{j\ p} & -k_j\beta_{j\ p+1}\beta_{j\ p+1} & \cdots & -k_j\beta_{j\ p+1}\beta_{j\ n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -k_j\beta_{j\ n}\beta_{j\ 1} & \cdots & -k_j\beta_{j\ n}\beta_{j\ p} & -k_j\beta_{j\ n}\beta_{j\ p+1} & \cdots & -k_j\beta_{j\ n}\beta_{j\ n} \end{bmatrix}$$

is reciprocal of order $(p, n-p)$ from which it follows that

$$\begin{bmatrix} b_{1\,1} & \dots & b_{1\,n} \\ \vdots & & \vdots \\ b_{n\,1} & \dots & b_{n\,n} \end{bmatrix}$$

is a reciprocal matrix of order $(p, n-p)$.

Conversely, assume that

$$\begin{bmatrix} b_{1\,1} & \dots & b_{1\,n} \\ \vdots & & \vdots \\ b_{n\,1} & \dots & b_{n\,n} \end{bmatrix}$$

is reciprocal of order $(p, n-p)$ and

$$\begin{bmatrix} \alpha_{j\,1} \\ \vdots \\ \alpha_{j\,p} \\ \alpha_{j\,p+1} \\ \vdots \\ \alpha_{j\,n} \end{bmatrix} = k_j \begin{bmatrix} \beta_{j\,1} \\ \vdots \\ \beta_{j\,p} \\ -\beta_{j\,p+1} \\ \vdots \\ -\beta_{j\,n} \end{bmatrix}$$

for $1 \le j \le m$. Note that the matrices

$$\begin{bmatrix} \alpha_{j\,1}\beta_{j\,1} & \dots & \alpha_{j\,1}\beta_{j\,p} & \alpha_{j\,1}\beta_{j\,p+1} & \dots & \alpha_{j\,1}\beta_{j\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{j\,p}\beta_{j\,1} & \dots & \alpha_{j\,p}\beta_{j\,p} & \alpha_{j\,p}\beta_{j\,p+1} & \dots & \alpha_{j\,p}\beta_{j\,n} \\ \alpha_{j\,p+1}\beta_{j\,1} & \dots & \alpha_{j\,p+1}\beta_{j\,p} & \alpha_{j\,p+1}\beta_{j\,p+1} & \dots & \alpha_{j\,p+1}\beta_{j\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{j\,n}\beta_{j\,1} & \dots & \alpha_{j\,n}\beta_{j\,p} & \alpha_{j\,n}\beta_{j\,p+1} & \dots & \alpha_{j\,n}\beta_{j\,n} \end{bmatrix} =$$

$$\begin{bmatrix} k_j\beta_{j\,1}\beta_{j\,1} & \dots & k_j\beta_{j\,1}\beta_{j\,p} & k_j\beta_{j\,1}\beta_{j\,p+1} & \dots & k_j\beta_{j\,1}\beta_{j\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ k_j\beta_{j\,p}\beta_{j\,1} & \dots & k_j\beta_{j\,p}\beta_{j\,p} & k_j\beta_{j\,p}\beta_{j\,p+1} & \dots & k_j\beta_{j\,p}\beta_{j\,n} \\ -k_j\beta_{j\,p+1}\beta_{j\,1} & \dots & -k_j\beta_{j\,p+1}\beta_{j\,p} & -k_j\beta_{j\,p+1}\beta_{j\,p+1} & \dots & -k_j\beta_{j\,p+1}\beta_{j\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -k_j\beta_{j\,n}\beta_{j\,1} & \dots & -k_j\beta_{j\,n}\beta_{j\,p} & -k_j\beta_{j\,n}\beta_{j\,p+1} & \dots & -k_j\beta_{j\,n}\beta_{j\,n} \end{bmatrix}$$

are reciprocal of order $(p, n-p)$. Then for the linear vector fields $\xi_{i_1,\dots,i_m}$ the matrices $\mathbf{M}_{i_1,\dots,i_m}$,

$$\begin{bmatrix} b_{1\,1} + \sum_{j=1}^m i_j\alpha_{j\,1}\beta_{j\,1} & \dots & b_{1\,n} + \sum_{j=1}^m i_j\alpha_{j\,1}\beta_{j\,n} \\ \vdots & & \vdots \\ b_{n\,1} + \sum_{j=1}^m i_j\alpha_{j\,n}\beta_{j\,1} & \dots & b_{n\,n} + \sum_{j=1}^m i_j\alpha_{j\,n}\beta_{j\,n} \end{bmatrix}$$

are reciprocal of order $(p, n-p)$. Thus $\xi(\mathbf{x})$ is a reciprocal vector field of order $(p, n-p)$. ∎

EXAMPLE 1.5. (Figure 1.) A reciprocal vector field in $\Re^2$ of order $(1,1)$ is that given by

$$\xi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}\left|\begin{bmatrix} 3 \\ 1 \end{bmatrix}^t\begin{bmatrix} x \\ y \end{bmatrix} - 1\right| + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\left|\begin{bmatrix} 1 \\ 2 \end{bmatrix}^t\begin{bmatrix} x \\ y \end{bmatrix} - 1\right|.$$

EXAMPLE 1.6. A reciprocal vector field in $\Re^4$ of order $(2,2)$ is that given by

$$\xi\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 & -4 & 4 & -1 \\ 3 & -2 & -1 & 5 \\ 1 & 4 & -1 & 3 \\ 4 & 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ -8 \\ -10 \end{bmatrix}\left|\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix}^t\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right| + \begin{bmatrix} 15 \\ 12 \\ -9 \\ -6 \end{bmatrix}\left|\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}^t\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} - 2\right|.$$

§2. The pseudo-reciprocal vector field.

Definition 2.1. A pseudo-reciprocal vector field ξ is a vector field for which there exists a matrix $\mathbf{X}$ and reciprocal vector field ζ such that either $(\mathbf{X} \circ \xi)(\mathbf{x}) = \zeta(\mathbf{x})$ or $\xi(\mathbf{x}) = (\mathbf{X} \circ \zeta)(\mathbf{x})$.

Definition 2.2. Given a matrix $\mathbf{A}$, define the set

$$\mathrm{Pi}(\mathbf{A}, p) = \{\mathbf{X} : \mathbf{XA} \text{ is reciprocal of order } (p, \dim \mathbf{A} - p)\}.$$

The matrix $\mathbf{X}$ is such that $\mathbf{XA}$ is a reciprocal matrix of order $(p, \ \dim \mathbf{A} - p)$.

Lemma 2.3. *Considering a matrix* $\mathbf{X}$ *written in the form of a* $n \times n$*-tuple*

$$\begin{bmatrix} x_{1\,1} \\ \vdots \\ x_{1\,n} \\ \vdots \\ x_{n\,1} \\ \vdots \\ x_{n\,n} \end{bmatrix}$$

there exists a finite set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \Re^{n \times n}$ *such that*

$$\mathrm{Pi}(\mathbf{A}, p) = \{t_1 \mathbf{v}_1 + \ldots + t_s \mathbf{v}_s : t_1, \ldots, t_s \in \Re\}.$$

PROOF. If $\mathbf{XA}$ is reciprocal of order $(p, n-p)$ then

$$\sum_{k=1}^{n} x_{i\,k} a_{k\,j} = \sum_{k=1}^{n} x_{j\,k} a_{k\,i}$$

for $1 \le i, j \le p$,

$$\sum_{k=1}^{n} x_{i\,k} a_{k\,j} = -\sum_{k=1}^{n} x_{j\,k} a_{k\,i}$$

for $1 \le i \le p, \ p+1 \le j \le n$,

$$\sum_{k=1}^{n} x_{i\,k} a_{k\,j} = -\sum_{k=1}^{n} x_{j\,k} a_{k\,i}$$

for $p+1 \le i \le n, \ 1 \le j \le p$, and

$$\sum_{k=1}^{n} x_{i\,k} a_{k\,j} = \sum_{k=1}^{n} x_{j\,k} a_{k\,i}$$

for $p+1 \le i, j \le n$.

The solution to the above equalities can be written as the matrix equation

$$\begin{bmatrix} c_{1\,1} & \cdots & c_{1\,n\times n} \\ \vdots & & \vdots \\ c_{n\times n\,1} & \cdots & c_{n\times n\,n\times n} \end{bmatrix} \begin{bmatrix} x_{1\,1} \\ \vdots \\ x_{1\,n} \\ \vdots \\ x_{n\,1} \\ \vdots \\ x_{n\,n} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{X} \in \mathrm{Pi}(\mathbf{A}, p)$ if and only if it solves the above equation. This means that

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{n1} \\ \vdots \\ x_{nn} \end{bmatrix}$$

is in the kernel of the matrix in the left-handside of the above equation. By linear algebra, the kernel is a linear subspace of $\Re^{n\times n}$ which can written as the span of the linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$. Thus,

$$\mathrm{Pi}(\mathbf{A}, p) = \{t_1\mathbf{v}_1 + \ldots + t_s\mathbf{v}_s : t_1, \ldots, t_s \in \Re\}. \qquad \blacksquare$$

Lemma [3] 3.12. *Given two linear subspaces spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and $\mathbf{w}_1, \ldots, \mathbf{w}_q$ respectively, the intersection of the two subspaces is given by the span of some vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ with $r \leq p, q$.*

Theorem 2.4. *Let ξ be a vector field of the form*

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left\| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|.$$

There exists a matrix $\mathbf{X}$ such that $(\mathbf{X} \circ \xi)(\mathbf{x})$ is a reciprocal vector field of order $(p, n-p)$ if and only if

$$\mathbf{X} \in \mathrm{Pi}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, p\right) \bigcap \left(\bigcap_{j=1}^{m} \mathrm{Pi}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}, p\right)\right).$$

PROOF. Assume that there exists a matrix $\mathbf{X}$ such that $(\mathbf{X} \circ \xi)(\mathbf{x})$ is a reciprocal vector field of order (p, q). As in the proof of theorem 1.4, it is necessary and sufficient that

$$\mathbf{X} \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}$$

and

$$\mathbf{X} \begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}$$

to be reciprocal matrices of order $(p, n-p)$ matrices for $(\mathbf{X} \circ \xi)(\mathbf{x})$ to be a reciprocal vector field of order $(p, n-p)$. Thus

$$\mathbf{X} \in \mathrm{Pi}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, p\right) \bigcap \left(\bigcap_{j=1}^{m} \mathrm{Pi}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}, p\right)\right). \qquad \blacksquare$$

Definition 2.5. For vectors $\mathbf{v}, \mathbf{w}$ define the set

$$\mathrm{Pj}(\mathbf{v},\mathbf{w},p) = \bigcup_{k\in\Re}\left\{\mathbf{X} : \begin{bmatrix} v_1 \\ \vdots \\ v_p \\ v_{p+1} \\ \vdots \\ v_n \end{bmatrix} = k\mathbf{X}\begin{bmatrix} w_1 \\ \vdots \\ w_p \\ -w_{p+1} \\ \vdots \\ -w_n \end{bmatrix}\right\}.$$

Theorem 2.6. *Let ξ be a vector field of the form*

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|.$$

There exists matrix $\mathbf{X}$ and reciprocal vector field $\zeta(\mathbf{x})$ of order $(p, n-p)$ such that $\xi(\mathbf{x}) = (\mathbf{X}\circ\zeta)(\mathbf{x})$ if and only if

$$\zeta\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix} + \begin{bmatrix} b'_{11} & \dots & b'_{1n} \\ \vdots & & \vdots \\ b'_{n1} & \dots & b'_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} k_j\begin{bmatrix} \beta_{j\,1} \\ \vdots \\ \beta_{j\,p} \\ -\beta_{j\,p+1} \\ \vdots \\ -\beta_{j\,n} \end{bmatrix}\left|\begin{bmatrix} \beta_{j\,1} \\ \vdots \\ \beta_{j\,p} \\ \beta_{j\,p+1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|,$$

$$\mathbf{X} \in \bigcap_{j=1}^{m}\mathrm{Pj}\left(\begin{bmatrix} \alpha_{j\,1} \\ \vdots \\ \alpha_{j\,p} \\ \alpha_{j\,p+1} \\ \vdots \\ \alpha_{j\,n} \end{bmatrix}, \begin{bmatrix} \beta_{j\,1} \\ \vdots \\ \beta_{j\,p} \\ -\beta_{j\,p+1} \\ \vdots \\ -\beta_{j\,n} \end{bmatrix}, p\right)$$

$$\begin{bmatrix} b'_{11} & \dots & b'_{1n} \\ \vdots & & \vdots \\ b'_{n1} & \dots & b'_{nn} \end{bmatrix}$$

is reciprocal of order $(p, n-p)$ and

$$\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} = \mathbf{X}\begin{bmatrix} b'_{11} & \dots & b'_{1n} \\ \vdots & & \vdots \\ b'_{n1} & \dots & b'_{nn} \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{X}\begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix}.$$

Proof. If $\zeta(\mathbf{x})$ is a reciprocal vector field such that $\xi(\mathbf{x}) = (\mathbf{X}\circ\zeta)(\mathbf{x})$ then ζ is not differentiable along the same points that ξ is not differentiable. As ζ is also a reciprocal vector field then it has the form

$$\zeta\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix} + \begin{bmatrix} b'_{11} & \dots & b'_{1n} \\ \vdots & & \vdots \\ b'_{1n} & \dots & b'_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} k_j\begin{bmatrix} \beta_{j\,1} \\ \vdots \\ \beta_{j\,p} \\ -\beta_{j\,p+1} \\ \vdots \\ -\beta_{j\,n} \end{bmatrix}\left|\begin{bmatrix} \beta_{j\,1} \\ \vdots \\ \beta_{j\,p} \\ \beta_{j\,p+1} \\ \vdots \\ \beta_{j\,n} \end{bmatrix}^t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|$$

with

$$\begin{bmatrix} b'_{11} & \dots & b'_{1n} \\ \vdots & & \vdots \\ b'_{n1} & \dots & b'_{nn} \end{bmatrix}$$

reciprocal of order $(p, n-p)$. From the equalities

$$\xi_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \xi_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 2 \begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ p} \\ \alpha_{j\ p+1} \\ \vdots \\ \alpha_{j\ n} \end{bmatrix} \left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right)$$

$$(\mathbf{X} \circ \zeta)_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (\mathbf{X} \circ \zeta)_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 2k_j\mathbf{X} \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ p} \\ -\beta_{j\ p+1} \\ \vdots \\ -\beta_{j\ n} \end{bmatrix} \left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right)$$

$$\xi_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}(\mathbf{x}) = (\mathbf{X} \circ \zeta)_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_m}(\mathbf{x})$$

$$\xi_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}(\mathbf{x}) = (\mathbf{X} \circ \zeta)_{i_1,\dots,i_{j-1},-1,i_{j+1},\dots,i_m}(\mathbf{x})$$

it follows that

$$\begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ p} \\ \alpha_{j\ p+1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} = k_j\mathbf{X} \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ p} \\ -\beta_{j\ p+1} \\ \vdots \\ -\beta_{j\ n} \end{bmatrix}.$$

Thus

$$\mathbf{X} \in \bigcap_{j=1}^{m} \mathrm{Pj} \left(\begin{bmatrix} \alpha_{j\ 1} \\ \vdots \\ \alpha_{j\ p} \\ \alpha_{j\ p+1} \\ \vdots \\ \alpha_{j\ n} \end{bmatrix}, \begin{bmatrix} \beta_{j\ 1} \\ \vdots \\ \beta_{j\ p} \\ \beta_{j\ p+1} \\ \vdots \\ \beta_{j\ n} \end{bmatrix}, p \right).$$

Now consider the two linear vector fields

$$\xi_{1,\dots,1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left(\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right)$$

and

$$(\mathbf{X}\circ\zeta)_{1,\ldots,1}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\mathbf{X}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}+\mathbf{X}\begin{bmatrix}b'_{11} & \cdots & b'_{1n}\\ \vdots & & \vdots\\ b'_{1n} & \cdots & b'_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}+\sum_{j=1}^{m}k_j\mathbf{X}\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}\left(\begin{bmatrix}\beta_{j1}\\ \vdots\\ \beta_{jn}\end{bmatrix}^t\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}-\gamma_j\right)$$

which agree on the set $A_{1,\ldots,1}$. Equating the derivatives

$$\frac{\partial}{\partial x_i}\xi_{1,\ldots,1}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}b_{1i}\\ \vdots\\ b_{ni}\end{bmatrix}+\sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}\beta_{ji}$$

$$\frac{\partial}{\partial x_i}(\mathbf{X}\circ\zeta)_{1,\ldots,1}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\mathbf{X}\begin{bmatrix}b'_{1i}\\ \vdots\\ b'_{ni}\end{bmatrix}+\sum_{j=1}^{m}k_j\mathbf{X}\begin{bmatrix}\beta_{j\,1}\\ \vdots\\ \beta_{j\,p}\\ -\beta_{j\,p+1}\\ \vdots\\ -\beta_{jn}\end{bmatrix}\beta_{ji}$$

gives

$$\begin{bmatrix}b_{1i}\\ \vdots\\ b_{ni}\end{bmatrix}=\mathbf{X}\begin{bmatrix}b'_{1i}\\ \vdots\\ b'_{ni}\end{bmatrix}$$

for $1\le i\le n$. Thus,

$$\begin{bmatrix}b_{11} & \cdots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \cdots & b_{nn}\end{bmatrix}=\mathbf{X}\begin{bmatrix}b'_{11} & \cdots & b'_{1n}\\ \vdots & & \vdots\\ b'_{1n} & \cdots & b'_{nn}\end{bmatrix}.$$

Equating the constant terms in the linear vector fields requires that

$$\mathrm{d}(\xi)_{1,\ldots,1}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}-\sum_{j=1}^{m}\begin{bmatrix}\alpha_{j1}\\ \vdots\\ \alpha_{jn}\end{bmatrix}\gamma_j$$

$$\mathrm{d}(\mathbf{X}\circ\zeta)_{1,\ldots,1}=\mathbf{X}\begin{bmatrix}\alpha'_1\\ \vdots\\ \alpha'_n\end{bmatrix}-\sum_{j=1}^{m}k_j\mathbf{X}\begin{bmatrix}\beta_{j\,1}\\ \vdots\\ \beta_{j\,p}\\ -\beta_{j\,p+1}\\ \vdots\\ -\beta_{j\,n}\end{bmatrix}\gamma_j$$

are identical from which it follows that

$$\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}=\mathbf{X}\begin{bmatrix}\alpha'_1\\ \vdots\\ \alpha'_n\end{bmatrix}.$$

Conversely, assuming that the stated equalities hold, then it is an easy matter to check that $\xi(\mathbf{x})=(\mathbf{X}\circ\zeta)(\mathbf{x})$ where $\zeta(\mathbf{x})$ is a reciprocal vector field. ∎

§3. Auxiliary results.

The following are some auxiliary results needed to ensure that the alogrithms to be presented can indeed be implemented in a deterministic fashion. Unlike existence proofs where it is sufficient to demonstrate validity of a claim, constructive proofs are much more useful in the design and implementation of functional algorithms.

The first two results deal with properties of polynomials while the rest deal with symmetric matrices.

Definition 3.1. Let $\alpha \in (\mathrm{IN} \cup \{0\})^k$, then

$$|\alpha| = \sum_{i=1}^{k} \alpha_i$$

and

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_k^{\alpha_k}.$$

Proposition [3] 3.2. *Let*

$$f(x_1) = \sum_{i=0}^{r} c_i x_1^i$$

be a polynomial in the variable x_1 of degree r with $c_r \neq 0$. There exists $y_1 \neq 0$ such that $f(y_1) \neq 0$.

Proof. If $0 = r$ then let $y_1 = 1$. In this case, $f(y_1) = c_0 \neq 0$. Assume that $1 \leq r$, then

$$f(x_1) = c_r x_1^r + \sum_{i=0}^{r-1} c_i x_1^i.$$

It may be assumed that $c_r > 0$, otherwise consider $-f(x_1)$ instead of $f(x_1)$.

Let

$$M = \max\{|c_i| : i = 0, \dots, r-1\}$$
$$y_1 = \max\{1, \frac{Mr+1}{c_r}\}.$$

Then

$$\begin{aligned} f(y_1) &= c_r y_1^r + \sum_{i=0}^{r-1} c_i y_1^i \\ &\geq c_r y_1^r - M \sum_{i=0}^{r-1} y_1^i \\ &\geq c_r y_1^r - M r y_1^{r-1} \\ &= y_1^{r-1}(c_r y_1 - Mr) \\ &> 0. \end{aligned}$$

∎

Proposition [3] 3.3. *Let*

$$f(x_1,\ldots,x_k)=\sum_{i=0}^{r}\left(\sum_{\substack{|\alpha|=r\\ \alpha\in(\mathbb{N}\cup\{0\})^k}} c_\alpha \mathbf{x}^\alpha\right)$$

be a polynomial in k variables of degree r with $c_\alpha \neq 0$ for some $|\alpha| = r$. There exists $y_1,\ldots,y_k \neq 0$ such that $f(y_1,\ldots,y_k) \neq 0$.

Proof. If $0 = r$ then let $y_1 = \ldots = y_k = 1$. In this case $f(1,\ldots,1) = c_{(0,\ldots,0)} \neq 0$. Assume that $1 \leq r$, then

$$f(x_1,\ldots,x_k)=\sum_{|\alpha|=r} c_\alpha \mathbf{x}^\alpha+\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i} c_\alpha \mathbf{x}^\alpha\right).$$

Consider the nontrivial homogeneous polynomial given by

$$g(x_1,\ldots,x_k)=\sum_{|\alpha|=r} c_\alpha \mathbf{x}^\alpha.$$

Define new variables $y_1,\ldots,y_k$ by $y_i = x_1^{(r+1)^{i-1}}$. Then $g(y_1,\ldots,y_k) = h(x_1)$ is a polynomial in x_1 of order at most $r(r+1)^{k-1}$. By proposition 3.2 there is a value $y_1 \neq 0$ such that $h(y_1) \neq 0$. Then the values $y_1,\ldots,y_1^{(r+1)^{k-1}} \neq 0$ satisfy $g(y_1,\ldots,y_1^{(r+1)^{k-1}}) = K \neq 0$. It may be assumed that $K > 0$, otherwise consider $-f(x_1,\ldots,x_k)$ instead of $f(x_1,\ldots,x_k)$.

Let

$$M=\max\{|c_\alpha| : |\alpha| = 0,\ldots,r-1\}$$

$$\epsilon=\max\{|y_i| : i = 1,\ldots,k\}$$

and choose

$$\lambda=\max\left\{1,\frac{1}{\epsilon},\frac{1}{K}\left(M\frac{(r+k-2)!}{(k-1)!}r\epsilon^{r-1}+1\right)\right\}.$$

Then

$$\begin{aligned}
f(\lambda y_1,\ldots,\lambda y_k) &= \sum_{|\alpha|=r} c_\alpha(\lambda\mathbf{y})^\alpha+\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i} c_\alpha(\lambda\mathbf{y})^\alpha\right)\\
&=\lambda^r\sum_{|\alpha|=r} c_\alpha \mathbf{y}^\alpha+\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i} c_\alpha\lambda^i\mathbf{y}^\alpha\right)\\
&\geq \lambda^r K - M\sum_{i=0}^{r-1}\left(\sum_{|\alpha|=i}\lambda^i\mathbf{y}^\alpha\right)\\
&\geq \lambda^r K - M\sum_{i=0}^{r-1}\binom{i+k-1}{k-1}\lambda^i\epsilon^i\\
&\geq \lambda^r K - M\frac{(r+k-2)!}{(k-1)!}r\lambda^{r-1}\epsilon^{r-1}\\
&=\lambda^{r-1}\left(\lambda K - M\frac{(r+k-2)!}{(k-1)!}r\epsilon^{r-1}\right)\\
&> 0.
\end{aligned}$$

∎

Proposition [3] 3.4. *Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$ be a basis for a linear manifolds of matrices. An element*

$$\sum_{i=0}^{q} x_i \mathbf{w}_i$$

is symmetric if and only if

$$\sum_{i=0}^{q} x_i(\mathbf{w}_i - \mathbf{w}_i^t) = 0.$$

Proposition [3] 3.5. *Let the given symmetric matrix be*

$$\mathbf{Y}_n = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix}.$$

Define the symmetric submatrices

$$\mathbf{Y}_n = \begin{bmatrix} y_{11} & \cdots & y_{1i} \\ \vdots & & \vdots \\ y_{i1} & \cdots & y_{ii} \end{bmatrix}$$

for $1 \le i \le n$. The matrix $\mathbf{Y}_n$ is postive definite if and only if $\det(\mathbf{Y}_i) > 0$ *for $1 \le i \le n$.*

§4. The pseudo-reciprocal vector field $\xi = \mathbf{X} \circ \zeta$ with X an invertible matrix.

If ξ is a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is an invertible matrix then $\mathbf{X}^{-1} \circ \xi$ is a reciprocal vector field. Thus, a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ invertible, has an invertible matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. Conversely, if there does not exist an invertible matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ an invertible matrix and ζ a reciprocal vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form.

Let the vector field ξ be given by

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|,$$

then an algorithm to determine the existence of invertible matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a reciprocal vector field is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \ldots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{Pi}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s \right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathrm{Pi}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for $\mathrm{span}(S) \cap \mathrm{span}(T)$.

Step 3.3: Let $S = R$.

Step 4: Form the matrix

$$\mathbf{Y}(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

and let $f(x_1, \ldots, x_{q_m})$ be the polynomial given by

$$f(x_1, \ldots, x_{q_m}) = \det \mathbf{Y}(x_1, \ldots, x_{q_m}).$$

Step 5: Determine if $f(x_1, \ldots, x_{q_m})$ is identically the zero function. If it is then go to step 6 else choose values for $x_1, \ldots, x_{q_m}$ such that $f(x_1, \ldots, x_{q_m}) \neq 0$ and go to step 7.

Step 6: In this case, all matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are non-invertible. If $s < n$ then let $s = s + 1$ and go to step 2, otherwise there do not exist invertible matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of any order. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is invertible and ζ is a reciprocal vector field.

Step 7: In this case, there exists a set of values $x_1, \ldots, x_{q_m}$ such that the matrix

$$\mathbf{Y}(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is invertible and $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$. Thus ξ can be written in the form $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ with $\mathbf{Y}^{-1}$ invertible and $\mathbf{Y} \circ \xi$ a reciprocal vector field.

Example 4.1. (Figure 2.) This example will demonstrate a case where the desired decomposition does not exist. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 0 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|.$$

Step 1: Let $s = 1$.

Step 2: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x_{12} \\ 0 & x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(1,1)$. Thus $x_{12}=0$ and a basis for S is given by the vectors

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}.$$

Step 3: Since $m=1$ then steps 3.1 to 3.3 need only be used once.

Step 3.1: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix}x_{11} & x_{12}\\x_{21} & x_{22}\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}=\begin{bmatrix}x_{12} & 0\\x_{22} & 0\end{bmatrix}$$

is a reciprocal matrix of order $(1,1)$. Thus $x_{22}=0$, a basis for the T is given by the vectors

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}.$$

Step 3.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0\\0 & 1 & 0 & 0 & 0 & 1\\0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}$$

which is the span of the vectors

$$\left\{\begin{bmatrix}-1\\0\\0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\\0\\0\\1\end{bmatrix}\right\}.$$

Thus, R is given by the span of the vectors

$$\left\{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\-1\\0\end{bmatrix}\right\}.$$

Step 3.3: Let S be the span of the vectors

$$\left\{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\-1\\0\end{bmatrix}\right\}.$$

Step 4: The matrix $\mathbf{Y}(x_1,x_2)$ is given by

$$\begin{aligned}\mathbf{Y}(x_1,x_2)&=x_1\begin{bmatrix}-1 & 0\\0 & 0\end{bmatrix}+x_2\begin{bmatrix}0 & 0\\-1 & 0\end{bmatrix}\\&=\begin{bmatrix}-x_1 & 0\\-x_2 & 0\end{bmatrix}\end{aligned}$$

and the function $f(x_1, x_2)$ is given by $f(x_1, x_2) = \det(\mathbf{Y}(x_1, x_2)) = 0$.

Step 5: It is clear that $f(x_1, x_2)$ is identically the zero function.

Step 6: It can be concluded that ξ may not be decomposed as the composition of an invertible matrix $\mathbf{X}$ and a reciprocal vector field ζ of oreder $(1,1)$. As $s < 2$ then let $s = s + 1 = 2$.

Step 2: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x_{12} \\ 0 & x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(2,0)$. Thus $x_{12} = 0$ and a basis for S is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 3: Since $m = 1$ then steps 3.1 to 3.3 need only be used once.

Step 3.1: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x_{12} & 0 \\ x_{22} & 0 \end{bmatrix}$$

is a reciprocal matrix of order $(2,0)$. Thus $x_{22} = 0$, a basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 3.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Step 3.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Step 4: The matrix $\mathbf{Y}(x_1, x_2)$ is given by

$$\mathbf{Y}(x_1,x_2) = x_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -x_1 & 0 \\ -x_2 & 0 \end{bmatrix}$$

and the function $f(x_1, x_2)$ is given by $f(x_1, x_2) = \det(\mathbf{Y}(x_1, x_2)) = 0$.

Step 5: It is clear that $f(x_1, x_2)$ is identically the zero function.

Step 6: It can be concluded that ξ may not be decomposed as the composition of an invertible matrix $\mathbf{X}$ and a reciprocal vector field ζ of order $(2, 0)$. As $s = 2$, the algorithm terminates without a successful decomposition.

EXAMPLE 4.2. (Figure 3.) This example will demonstrate a case where a desired decomposition exists. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 1 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|.$$

Step 1: Let $s = 1$.

Step 2: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3x_{11} + x_{12} & 3x_{11} + 2x_{12} \\ 3x_{21} + x_{22} & 3x_{21} + 2x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(1, 1)$. Thus, a basis for S is given by the vectors

$$\left\{ \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 3: Since $m = 1$ then steps 3.1 to 3.3 need only be used once.

Step 3.1: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -x_{11} - x_{12} & -x_{11} - x_{12} \\ -x_{21} - x_{22} & -x_{21} - x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(1, 1)$. Thus, a basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 3.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} -\frac{2}{3} & -1 & -\frac{1}{3} & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 3.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 4: The matrix $\mathbf{Y}(x_1, x_2)$ is given by

$$\begin{aligned} \mathbf{Y}(x_1, x_2) &= x_1 \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - x_2 & -2x_1 \\ x_2 & x_1 \end{bmatrix} \end{aligned}$$

and the function $f(x_1, x_2)$ is given by $f(x_1, x_2) = \det(\mathbf{Y}(x_1, x_2)) = x_1^2 + x_1 x_2$.

Step 5: It is clear that $f(x_1, x_2)$ is not identically the zero function, the values of $x_1 = 1, x_2 = 1$ satisfies $f(x_1, x_2) \neq 0$.

Step 7: It can be concluded that ξ may be decomposed as the composition of an invertible matrix $\mathbf{X}$ and a reciprocal vector field ζ as

$$\begin{aligned} \xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \circ \xi \right) \\ &= \frac{1}{2} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \circ \left(\begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 1 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right| \right). \end{aligned}$$

§5. The pseudo-reciprocal vector field $\xi = \mathbf{X} \circ \zeta$ with X an invertible symmetric matrix.

If ξ is a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is an invertible symmetric matrix then $\mathbf{X}^{-1} \circ \xi$ is a reciprocal vector field. Thus, a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ invertible symmetric, has an invertible symmetric matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. Conversely, if there does not exist an invertible symmetric matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ an invertible symmetric matrix and ζ a reciprocal vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form. However, if there does not exist an invertible symmetric matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field there may still exist invertible matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a reciprocal vector field.

Let the vector field ξ be given by

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|,$$

then an algorithm to determine the existence of invertible symmetric matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ reciprocal vector fields is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{Pi}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}, s \right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \dots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \dots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathrm{Pi}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \dots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \dots & \alpha_{jn}\beta_{jn} \end{bmatrix}, s \right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \dots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \dots, \mathbf{w}_{q_i}^i\}$ form a basis for $\mathrm{span}(S) \cap \mathrm{span}(T)$.

Step 3.3: Let $S = R$.

Step 4: From the equation

$$\sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m - (\mathbf{w}_i^m)^t) = 0$$

determine a set of independent variables $x_1, \dots, x_k$ and dependent variables $x_{k+1}, \dots, x_{q_m}$. Form the matrix

$$\mathbf{Y}(x_1, \dots, x_k) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

and let $f(x_1,\ldots,x_k)$ be the polynomial given by

$$f(x_1,\ldots,x_k) = \det\mathbf{Y}(x_1,\ldots,x_k).$$

Step 5: Determine if $f(x_1,\ldots,x_k)$ is identically the zero function. If it is then go to step 6 else choose values for $x_1,\ldots,x_k$ such that $f(x_1,\ldots,x_k) \neq 0$ and go to step 7.

Step 6: In this case, all symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y}\circ\xi$ is a reciprocal vector field of order $(s, n-s)$ are non-invertible. If $s < n$ athen let $s = s+1$ and go to step 2, otherwise there do not exist invertible symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y}\circ\xi$ is a reciprocal vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X}\circ\zeta$ where $\mathbf{X}$ is invertible symmetric and ζ is a reciprocal vector field.

Step 7: In this case, there exists a set of values $x_1,\ldots,x_k$ such that the matrix

$$\mathbf{Y}(x_1,\ldots,x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is invertible symmetric and $\mathbf{Y}\circ\xi$ is a reciprocal vector field of oreder $(s, n-s)$. Thus ξ can be written in the form $\xi = \mathbf{Y}^{-1}\circ(\mathbf{Y}\circ\xi)$ with $\mathbf{Y}^{-1}$ invertible symmetric and $\mathbf{Y}\circ\xi$ a reciprocal vector field.

EXAMPLE 5.1. (Figure 4.) This example will demonstrate a case where a desired decomposition exists. Let the vector field be given by

$$\xi\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 & 5 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix}\left|\begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1\right|.$$

Step 1: Let $s = 1$.

Step 2: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\begin{bmatrix} 8 & 5 \\ 7 & 7 \end{bmatrix} = \begin{bmatrix} 8x_{11}+7x_{12} & 5x_{11}+7x_{12} \\ 8x_{21}+7x_{22} & 5x_{21}+7x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(1,1)$. Thus, a basis for S is given by the vectors

$$\left\{\begin{bmatrix} -\frac{7}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{8}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Step 3: Since $m = 1$ then steps 3.1 to 3.3 need only be used once.

Step 3.1: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\begin{bmatrix} 5 & 10 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 5x_{11}+4x_{12} & 10x_{11}+8x_{12} \\ 5x_{21}+4x_{22} & 10x_{21}+8x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(1,1)$. Thus, a basis for the T is given by the vectors

$$\left\{\begin{bmatrix} -\frac{4}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Step 3.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} -\frac{7}{5} & -\frac{8}{5} & -\frac{7}{5} & -\frac{4}{5} & -\frac{1}{2} & -\frac{2}{5} \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \\ \frac{5}{3} \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -\frac{11}{6} \\ 1 \\ 0 \\ \frac{11}{6} \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} \frac{14}{15} \\ -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{29}{30} \\ -\frac{11}{6} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 3.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} \frac{14}{15} \\ -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{29}{30} \\ -\frac{11}{6} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 4: The equation

$$x_1 \left(\begin{bmatrix} \frac{14}{15} & -\frac{5}{3} \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{14}{15} & -\frac{5}{3} \\ 0 & 1 \end{bmatrix}^t \right) + x_2 \left(\begin{bmatrix} \frac{29}{30} & -\frac{11}{6} \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{29}{30} & -\frac{11}{6} \\ 1 & 0 \end{bmatrix}^t \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

determines that

$$x_1 \begin{bmatrix} 0 & -\frac{5}{3} \\ \frac{5}{3} & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & -\frac{17}{6} \\ \frac{17}{6} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from which $x_2 = -\frac{10}{17}x_1$. Thus,

$$\begin{aligned} \mathbf{Y}(x_1) &= x_1 \begin{bmatrix} \frac{14}{15} & -\frac{5}{3} \\ 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} \frac{29}{30} & -\frac{11}{6} \\ 1 & 0 \end{bmatrix} \\ &= x_1 \begin{bmatrix} \frac{31}{85} & -\frac{10}{17} \\ -\frac{10}{17} & 1 \end{bmatrix} \end{aligned}$$

and $f(x_1) = \det(\mathbf{Y}(x_1)) = \frac{27}{1445}x_1^2$.

Step 5: It is clear that $f(x_1)$ is not identically the zero function. The value of $x_1 = 85$ satisfies $f(x_1) \neq 0$.

Step 7: It can be concluded that ξ may be decomposed as the composition of an invertible symmetric matrix $\mathbf{X}$ and a reciprocal vector field ζ as

$$\begin{aligned} \xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 31 & -50 \\ -50 & 85 \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} 31 & -50 \\ -50 & 85 \end{bmatrix} \circ \xi \right) \\ &= \frac{1}{135} \begin{bmatrix} 85 & 50 \\ 50 & 31 \end{bmatrix} \circ \left(\begin{bmatrix} 31 \\ -50 \end{bmatrix} + \begin{bmatrix} -102 & -195 \\ 195 & 345 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -45 \\ 90 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 \right| \right). \end{aligned}$$

§6. The pseudo-reciprocal vector field $\xi = \mathbf{X} \circ \zeta$, X a symmetric positive definite matrix. If ξ is a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is a symmetric positive definite matrix then $\mathbf{X}^{-1} \circ \xi$ is a reciprocal vector field. Thus, a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ symmetric positive definite, has a symmetric positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. Conversely, if there does not exist a symmetric positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ a symmetric positive definite matrix and ζ a reciprocal vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form. However, if there does not exist a symmetric positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field there may still exist invertible symmetric matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a reciprocal vector field.

Theorem 6.1. *Let ξ be a vector field of the form*

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right| .$$

There exists a symmetric positive definite matrix $\mathbf{X}$ such that $(\mathbf{X} \circ \xi)(\mathbf{x})$ is a reciprocal vector field of order $(s, n-s)$ if and only if

$$\mathbf{X} \in \mathrm{Pi}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}, s \right) \bigcap \left(\bigcap_{j=1}^{m} \mathrm{Pi}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \dots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \dots & \alpha_{jn}\beta_{jn} \end{bmatrix}, s \right) \right),$$

$\mathbf{X}$ *is symmetric and for $i = 1, \dots, n$,*

$$0 < \det \begin{bmatrix} x_{11} & \dots & x_{1i} \\ \vdots & & \vdots \\ x_{i1} & \dots & x_{ii} \end{bmatrix} .$$

PROOF. Immediate from theorem 2.4 and proposition[3] 3.5. ∎

Let the vector field ξ be given by

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right| ,$$

then an algorithm to determine the existence of symmetric positive definite matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ reciprocal vector fields is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0, \dots, \mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{Pi}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}, s \right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i, \ldots, \mathbf{v}_{p_i}^i\}$ form a basis for

$$\text{Pi}\left(\begin{bmatrix} \alpha_{j1}\beta_{j1} & \ldots & \alpha_{j1}\beta_{jn} \\ \vdots & & \vdots \\ \alpha_{jn}\beta_{j1} & \ldots & \alpha_{jn}\beta_{jn} \end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i, \ldots, \mathbf{w}_{q_i}^i\}$ form a basis for span$(S) \cap$ span(T).

Step 3.3: Let $S = R$.

Step 4: From the equation

$$\sum_{i=1}^{q_m} x_i(\mathbf{w}_i^m - (\mathbf{w}_i^m)^t) = 0$$

determine a set of independent variables $x_1, \ldots, x_k$ and dependent variables $x_{k+1}, \ldots, x_{q_m}$. Form the matrix

$$\mathbf{Y}_n(x_1, \ldots, x_k) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m.$$

Define the matrices

$$\mathbf{Y}_i(x_1, \ldots, x_k) = \begin{bmatrix} \mathbf{Y}_n(x_1, \ldots, x_k)_{11} & \ldots & \mathbf{Y}_n(x_1, \ldots, x_k)_{1i} \\ \vdots & & \vdots \\ \mathbf{Y}_n(x_1, \ldots, x_k)_{i1} & \ldots & \mathbf{Y}_n(x_1, \ldots, x_k)_{ii} \end{bmatrix}$$

and let $f_i(x_1, \ldots, x_k)$ be the polynomial given by

$$f_i(x_1, \ldots, x_k) = \det \mathbf{Y}_i(x_1, \ldots, x_k)$$

for $1 \leq i \leq n$.

Step 5: Determine if there exist values $x_1, \ldots, x_k$ such that the following set of inequalities hold simultaneously,

$$\begin{aligned} f_1(x_1, \ldots, x_k) &> 0 \\ &\vdots \\ f_n(x_1, \ldots, x_k) &> 0. \end{aligned}$$

If such values do not exist then go to step 6 else go to step 7.

Step 6: In this case, all symmetric matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are either non-invertible or invertible and not positive definite. If $s < n$ then let $s = s + 1$ and go to step 2, otherwise there do not exist symmetric positive definite matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is symmetric positive definite and ζ is a reciprocal vector field.

Step 7: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\mathbf{Y}_n(x_1, \ldots, x_{q_m}) = \sum_{i=1}^{q_m} x_i \mathbf{w}_i^m$$

is symmetric positive definite and $\mathbf{Y}_n \circ \xi$ is a reciprocal vector field of order $(s, n-s)$. Thus ξ can be written in the form $\xi = \mathbf{Y}_n^{-1} \circ (\mathbf{Y}_n \circ \xi)$ with $\mathbf{Y}_n^{-1}$ symmetric positive definite and $\mathbf{Y}_n \circ \xi$ a reciprocal vector field.

EXAMPLE 6.2. (Figure 5.) This example will demonstrate a case where a desired decomposition exists. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 24 & 36 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ -16 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 1 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \right|.$$

Step 1: Let $s = 1$.

Step 2: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 24 & 36 \end{bmatrix} = \begin{bmatrix} 5x_{11} + 24x_{12} & 7x_{11} + 36x_{12} \\ 5x_{21} + 24x_{22} & 7x_{21} + 36x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(1,1)$. Thus, a basis for S is given by the vectors

$$\left\{ \begin{bmatrix} -\frac{36}{7} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{7} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{24}{7} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 3: Since $m = 1$ then steps 3.1 to 3.3 need only be used once.

Step 3.1: By lemma 2.3, it is required to solve for $\mathbf{X}$ where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} -3 & -3 \\ -16 & -16 \end{bmatrix} = \begin{bmatrix} -3x_{11} - 16x_{12} & -3x_{11} - 16x_{12} \\ -3x_{21} - 16x_{22} & -3x_{21} - 16x_{22} \end{bmatrix}$$

is a reciprocal matrix of order $(1,1)$. Thus, a basis for the T is given by the vectors

$$\left\{ \begin{bmatrix} -\frac{16}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{16}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step 3.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} -\frac{36}{7} & -\frac{5}{7} & -\frac{36}{7} & -\frac{16}{3} & -1 & -\frac{16}{3} \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

which is the span of the vectors

$$\left\{ \begin{bmatrix} -10 \\ 0 \\ 1 \\ 10 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ \frac{3}{2} \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vectors

$$\left\{ \begin{bmatrix} 48 \\ -10 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 3.3: Let S be the span of the vectors

$$\left\{ \begin{bmatrix} 48 \\ -10 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Step 4: The equation

$$x_1 \left(\begin{bmatrix} 48 & -10 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 48 & -10 \\ 0 & 1 \end{bmatrix}^t \right) + x_2 \left(\begin{bmatrix} 7 & -\frac{3}{2} \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 7 & -\frac{3}{2} \\ 1 & 0 \end{bmatrix}^t \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

determines that

$$x_1 \begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from which $x_2 = -4x_1$. Thus,

$$\begin{aligned} \mathbf{Y}_2(x_1) &= x_1 \begin{bmatrix} 48 & -10 \\ 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 7 & -\frac{3}{2} \\ 1 & 0 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 20 & -4 \\ -4 & 1 \end{bmatrix} \\ \mathbf{Y}_1(x_1) &= [20x_1] \end{aligned}$$

and

$$f_2(x_1) = \det(\mathbf{Y}_2(x_1)) = 4x_1^2$$

$$f_1(x_1) = \det(\mathbf{Y}_1(x_1)) = 20x_1.$$

Step 5: It is clear that the inequalities

$$4x_1^2 > 0$$

$$20x_1 > 0$$

can be satisfied simultaneously by $x_1 = \frac{1}{4}$.

Step 7: It can be concluded that ξ may be decomposed as the composition of a symmetric positive definite matrix $\mathbf{X}$ and a reciprocal vector field ζ as

$$\begin{aligned} \xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 5 & -1 \\ -1 & \frac{1}{4} \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} 5 & -1 \\ -1 & \frac{1}{4} \end{bmatrix} \circ \xi \right) \\ &= \begin{bmatrix} 1 & 4 \\ 4 & 20 \end{bmatrix} \circ \left(\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left| \begin{bmatrix} 1 \\ 1 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \right| \right). \end{aligned}$$

§7. The pseudo-reciprocal vector field $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ a diagonal positive definite matrix.

If ξ is a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is a diagonal positive definite matrix then $\mathbf{X}^{-1} \circ \xi$ is a reciprocal vector field. Thus, a pseudo-reciprocal vector field of the form $\xi = \mathbf{X} \circ \zeta$, $\mathbf{X}$ diagonal positive definite, has a diagonal positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. Conversely, if there does not exist a diagonal positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field then ξ cannot be decomposed as $\xi = \mathbf{X} \circ \zeta$ with $\mathbf{X}$ a diagonal positive definite matrix and ζ a reciprocal vector field. It is immediate that if such a matrix $\mathbf{Y}$ exists then $\xi = \mathbf{Y}^{-1} \circ (\mathbf{Y} \circ \xi)$ is a valid decomposition of the desired form. However, if there does not exist a diagonal positive definite matrix $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field there may still exist symmetric positive definite matrices $\mathbf{Y}$ with $\mathbf{Y} \circ \xi$ a reciprocal vector field.

Definition 7.1. Define the set

$$\mathbf{E}\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}, \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}, p\right) = \left\{\begin{bmatrix}d_1\\ \vdots\\ d_p\\ d_{p+1}\\ \vdots\\ d_n\end{bmatrix} : \exists\lambda \in \Re \ni \begin{bmatrix}d_1\alpha_1\\ \vdots\\ d_p\alpha_p\\ d_{p+1}\alpha_{p+1}\\ \vdots\\ d_n\alpha_n\end{bmatrix} = \lambda \begin{bmatrix}\beta_1\\ \vdots\\ \beta_p\\ -\beta_{p+1}\\ \vdots\\ -\beta_n\end{bmatrix}\right\}.$$

Lemma 7.2. *There exists vectors such that*

$$\mathbf{E}\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}, \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}, p\right) = \left\{\sum_{i=1}^{k} t_i \begin{bmatrix}c_{i1}\\ \vdots\\ c_{in}\end{bmatrix} : t_i \in \Re\right\}.$$

PROOF. It is required to solve the equations

$$\begin{aligned} d_1\alpha_1 &= \lambda\beta_1\\ &\vdots\\ d_p\alpha_p &= \lambda\beta_p\\ d_{p+1}\alpha_{p+1} &= -\lambda\beta_{p+1}\\ &\vdots\\ d_n\alpha_n &= -\lambda\beta_n. \end{aligned}$$

If there exist $\alpha_i = 0$ with $\beta_i \neq 0$ then $0 = \lambda\beta_i$ from which $\lambda = 0$ and the above equations reduce to

$$\begin{aligned} d_1\alpha_1 &= 0\\ &\vdots\\ d_n\alpha_n &= 0. \end{aligned}$$

Let $\mathbf{e}_i$ denote the i-th coordinate vector. If $\alpha_{i_1}, \ldots, \alpha_{i_j} \neq 0$ and $\alpha_{i_{j+1}}, \ldots, \alpha_{i_n} = 0$ then

$$\mathrm{E}\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}, \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}, p\right) = \left\{\sum_{k=j+1}^{n} t_{i_k}\mathbf{e}_{i_k} : t_{i_k} \in \Re\right\}.$$

If it happens that whenever $\alpha_i = 0$ that $\beta_i = 0$ then consider $\alpha_{i_1}, \ldots, \alpha_{i_j} \neq 0, \alpha_{i_{j+1}}, \ldots, \alpha_{i_n} = 0$. The equations reduce to

$$\begin{aligned} d_{i_1}\alpha_{i_1} &= \lambda\beta_{i_1} \\ &\vdots \\ d_{i_l}\alpha_{i_l} &= \lambda\beta_{i_l} \\ d_{i_{l+1}}\alpha_{i_{l+1}} &= -\lambda\beta_{i_{l+1}} \\ &\vdots \\ d_{i_j}\alpha_{i_j} &= -\lambda\beta_{i_j} \end{aligned}$$

for $i_1 < \ldots i_l \leq p < p+1 \leq i_{l+1} < \ldots, i_n$. Thus

$$\mathrm{E}\left(\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}, \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}, s\right) = \left\{\lambda\left(\sum_{k=1}^{l}\frac{\beta_{i_k}}{\alpha_{i_k}}\mathbf{e}_{i_k} + \sum_{k=l+1}^{j}\frac{-\beta_{i_k}}{\alpha_{i_k}}\mathbf{e}_{i_k}\right) + \sum_{k=j+1}^{n} t_{i_k}\mathbf{e}_{i_k} : \lambda, t_{i_k} \in \Re\right\}. \quad \blacksquare$$

Definition 7.3. Define the set

$$\mathrm{F}\left(\begin{bmatrix}b_{11} & \ldots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \ldots & b_{nn}\end{bmatrix}, p\right) = \left\{\begin{bmatrix}d_1\\ \vdots\\ d_n\end{bmatrix} : \begin{bmatrix}d_1b_{11} & \ldots & d_1b_{1n}\\ \vdots & & \vdots\\ d_nb_{n1} & \ldots & d_nb_{nn}\end{bmatrix} \text{ reciprocal of order } (p, n-p)\right\}.$$

Lemma 7.4. *There exists vectors such that*

$$\mathrm{F}\left(\begin{bmatrix}b_{11} & \ldots & b_{1n}\\ \vdots & & \vdots\\ b_{n1} & \ldots & b_{nn}\end{bmatrix}, p\right) = \left\{\sum_{i=1}^{k} t_i \begin{bmatrix}d_{i1}\\ \vdots\\ d_{in}\end{bmatrix} : t_i \in \Re\right\}.$$

PROOF. It is required to solve for a matrix

$$\begin{bmatrix}d_1b_{11} & \ldots & d_1b_{1n}\\ \vdots & & \vdots\\ d_nb_{n1} & \ldots & d_nb_{nn}\end{bmatrix}$$

satisfying the equalities

$$d_ib_{ij} = d_jb_{ji}$$

for $1 \leq i, j \leq p$,

$$d_ib_{ij} = -d_jb_{ji}$$

for $1 \le i \le p$, $p+1 \le j \le n$,

$$d_i b_{ij} = -d_j b_{ji}$$

for $p+1 \le i \le n$, $1 \le j \le p$, and

$$d_i b_{ij} = d_j b_{ji}$$

for $p+1 \le i, j \le n$.

The above equations can be rewritten as

$$\begin{bmatrix} c_{1\,1} & \dots & c_{1\,n} \\ \vdots & & \vdots \\ c_{n\times n\,1} & \dots & c_{n\times n\,n} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By linear algebra theory there are vectors such that

$$\mathrm{F}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}, p \right) = \left\{ \sum_{i=1}^{k} t_i \begin{bmatrix} d_{i1} \\ \vdots \\ d_{in} \end{bmatrix} : t_i \in \Re \right\}. \qquad \blacksquare$$

Definition 7.5. Given $d_1, \dots, d_n \in \Re$ then

$$\Lambda(d_1, \dots, d_n) = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}.$$

Theorem 7.6. *Let ξ be a vector field of the form*

$$\xi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} \left| \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j \right|.$$

There exists a diagonal positive definite matrix $\Lambda(d_1, \dots, d_n)$ such that $(\Lambda(d_1, \dots, d_n) \circ \xi)(\mathbf{x})$ is a reciprocal vector field of order $(s, n-s)$ if and only if

$$\Lambda(d_1, \dots, d_n) \in \mathrm{F}\left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}, s \right) \bigcap \left(\bigcap_{j=1}^{m} \mathrm{E}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}, \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}, s \right) \right),$$

and $0 < d_i$ for $i = 1, \dots, n$.

Proof. Assume that there exists a matrix $\Lambda(d_1, \dots, d_n)$ such that $(\Lambda(d_1, \dots, d_n) \circ \xi)(\mathbf{x})$ is a reciprocal vector field of order $(s, n-s)$. As in the proof of theorem 1.4, it is necessary and sufficient that

$$\begin{bmatrix} d_1 b_{11} & \dots & d_1 b_{1n} \\ \vdots & & \vdots \\ d_n b_{n1} & \dots & d_n b_{nn} \end{bmatrix}$$

be reciprocal of order $(s, n-s)$ and

$$\begin{bmatrix} d_1\alpha_{j1} \\ \vdots \\ d_n\alpha_{jn} \end{bmatrix} = \lambda_j \begin{bmatrix} \beta_{j1} \\ \vdots \beta_{js} \\ \beta_{js+1} \\ \vdots \\ \beta_{jn} \end{bmatrix}$$

for $(\Lambda(d_1,\ldots,d_n)\circ\xi)(\mathbf{x})$ to be a reciprocal vector field of order $(s, n-s)$. Thus, by lemmas 7.2 and 7.4,

$$\Lambda(d_1,\ldots,d_n) \in \mathrm{F}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s\right) \bigcap \left(\bigcap_{j=1}^{m} \mathrm{E}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}, \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}, s\right)\right).$$

The condition that $0 < d_i$ is necessary and sufficient for $\Lambda(d_1,\ldots,d_n)$ to be a diagonal positive definite matrix. ∎

Let the vector field ξ be given by

$$\xi\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \sum_{j=1}^{m}\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}\left|\begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \gamma_j\right|,$$

then an algorithm to determine the existence of diagonal positive definite matrices $\mathbf{Y}$ with $\mathbf{Y}\circ\xi$ reciprocal vector fields is given by the following sequence of steps:

Step 1: Let $s = 1$.

Step 2: Let $S = \{\mathbf{w}_1^0,\ldots,\mathbf{w}_{q_0}^0\}$ where the vectors $\{\mathbf{w}_1^0,\ldots,\mathbf{w}_{q_0}^0\}$ form a basis for

$$\mathrm{F}\left(\begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}, s\right).$$

Step 3: For i=1 to m repeat the steps 3.1 through to 3.3.

Step 3.1: Let $T = \{\mathbf{v}_1^i,\ldots,\mathbf{v}_{p_i}^i\}$ where the vectors $\{\mathbf{v}_1^i,\ldots,\mathbf{v}_{p_i}^i\}$ form a basis for

$$\mathrm{E}\left(\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix}, \begin{bmatrix} \beta_{j1} \\ \vdots \\ \beta_{jn} \end{bmatrix}, s\right).$$

Step 3.2: Let $R = \{\mathbf{w}_1^i,\ldots,\mathbf{w}_{q_i}^i\}$ where the vectors $\{\mathbf{w}_1^i,\ldots,\mathbf{w}_{q_i}^i\}$ form a basis for $\mathrm{span}(S)\cap \mathrm{span}(T)$.

Step 3.3: Let $S = R$.

Step 4: Determine if there exist values $x_1,\ldots,x_{q_m}$ such that the following set of inequalities hold simultaneously,

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &> 0 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &> 0. \end{aligned}$$

If such values do not exist then go to step 5 else go to step 6.

Step 5: In this case, all diagonal matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field of order $(s, n-s)$ are either non-invertible or invertible and not positive definite. If $s < n$ then let $s = s+1$ and go to step 2, otherwise there do not exist diagonal positive definite matrices $\mathbf{Y}$ such that $\mathbf{Y} \circ \xi$ is a reciprocal vector field. The vector field ξ cannot be written in the form $\xi = \mathbf{X} \circ \zeta$ where $\mathbf{X}$ is diagonal positive definite and ζ is a reciprocal vector field.

Step 6: In this case, there exists a set of values $x_1, \ldots, x_k$ such that the matrix

$$\Lambda(y_1, \ldots, y_n)$$

with

$$y_j = \sum_{i=1}^{q_m} x_i (\mathbf{w}_i^m)_j$$

is diagonal positive definite and $\Lambda(y_1, \ldots, y_n) \circ \xi$ is a reciprocal vector field of order $(s, n-s)$. Thus ξ can be written in the form $\xi = \Lambda(y_1, \ldots, y_n)^{-1} \circ (\Lambda(y_1, \ldots, y_n) \circ \xi)$ with $\Lambda(y_1, \ldots, y_n)^{-1}$ diagonal positive definite and $\Lambda(y_1, \ldots, y_n) \circ \xi$ a reciprocal vector field.

Note that if there exists a solution $y_1, \ldots, y_{q_m}$ to

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &= \epsilon_1 > 0 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &= \epsilon_n > 0 \end{aligned}$$

then there exists a solution $y'_1, \ldots, y'_{q_m}$ to

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &\geq 1 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &\geq 1 \end{aligned}$$

by scaling the original values $y_1, \ldots, y_{q_m}$ with a sufficiently large constant. Decompose the variables $x_1, \ldots, x_{q_m}$ as $x_i = x_i^1 - x_i^2$ for $i = 1, \ldots, q_m$. It then follows that $y_1^1 = y'_1, \ldots, y_{q_m}^1 = y'_{q_m}, y_1^2 = 0, \ldots, y_{q_m}^2 = 0$ is an optimal solution to the linear programming problem of

$$\text{minimise} \sum_{i=1}^{n} v_i$$

subject to

$$\begin{aligned} x_1^1(\mathbf{w}_1^m)_1 - x_1^2(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}^1(\mathbf{w}_{q_m}^m)_1 - x_{q_m}^2(\mathbf{w}_{q_m}^m)_1 + v_1 - u_1 &= 1 \\ &\vdots \\ x_1^1(\mathbf{w}_1^m)_n - x_1^2(\mathbf{w}_1^m)_n + \ldots + x_{q_m}^1(\mathbf{w}_{q_m}^m)_n - x_{q_m}^2(\mathbf{w}_{q_m}^m)_1 + v_n - u_n &= 1 \\ x_1^1, x_1^2, \ldots, x_{q_m}^1, x_{q_m}^2 &\geq 0 \\ u_1, \ldots, u_n, v_1, \ldots, v_n &\geq 0. \end{aligned}$$

Conversely, given an optimal solution to the above linear programming problem, if $0 \leq u_i - v_i$ for $i = 1, \ldots, n$ then $x_i = x_i^1 - x_i^2$ is a solution to the original problem

$$\begin{aligned} x_1(\mathbf{w}_1^m)_1 + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_1 &> 0 \\ &\vdots \\ x_1(\mathbf{w}_1^m)_n + \ldots + x_{q_m}(\mathbf{w}_{q_m}^m)_n &> 0. \end{aligned}$$

EXAMPLE 7.1. (Figure 6.) This example will demonstrate a case where a vector field ξ can be decomposed as $\xi = \mathbf{X} \circ \zeta$ where the matrix $\mathbf{X}$ is diagonal positive definite. Let the vector field be given by

$$\xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} \left| \begin{bmatrix} -3 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 5 \right|.$$

Step 1: Let $s = 1$.

Step 2: A basis for S is given by the vectors

$$\left\{ \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right\}.$$

Step 3: Since $m = 1$ then steps 3.1 to 3.3 need only be used once.

Step 3.1: By lemma 7.2 a basis for T is given by the vector

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} \right\}.$$

Step 3.2: By lemma[3] 3.12, it is needed to find the kernel of the matrix given by

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

which is the span of the vector

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Thus, R is given by the span of the vector

$$\left\{ \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right\}.$$

Step 3.3: Let S be the span of the vector

$$\left\{ \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right\}.$$

Step 4: The equations

$$\begin{aligned} x_1 &> 0 \\ \frac{2}{3} x_1 &> 0 \end{aligned}$$

can be satisfied simultaneously with $x_1 = 1/2$.

Step 6: It can be concluded that ξ may be decomposed as the composition of a diagonal positive definite matrix A(7, 6) and a reciprocal vector field ζ as

$$\begin{aligned} \xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}^{-1} \circ \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \circ \xi \right) \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \circ \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} \left| \begin{bmatrix} -3 \\ 2 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 5 \right| \right). \end{aligned}$$

References.

[1] Chua L.O. and Deng A., "Canonical piecewise-linear modeling." *IEEE Transactions on Circuits and Systems.*, vol.33, pp.511-525, May 1986.

[2] Chua L.O. and Deng A., "Canonical piecewise-linear representations." *IEEE Transactions on Circuits and Systems.*, vol.35, pp.101-111, January 1988.

[3] Lum R., and Chua L.O., "Invariance properties of continuous piecewise-linear vector fields." Electronics Research Laboratory No. UCB/ERL M90/38 3 May 1990.

[4] Parker T.S. and Chua L.O., "Practical numerical algorithms for chaotic systems." Springer-Verlag, New York, 1989.

Figure captions.

Figure 1. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}+\begin{bmatrix}3&4\\-1&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}6\\-2\end{bmatrix}\left|\begin{bmatrix}3\\1\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix}-1\right|+\begin{bmatrix}-1\\2\end{bmatrix}\left|\begin{bmatrix}1\\2\end{bmatrix}^t\begin{bmatrix}x\\y\end{bmatrix}-1\right|.$$

Figure 2. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\left|\begin{bmatrix}1\\0\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}\right|.$$

Figure 3. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}3&3\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}-1\\-1\end{bmatrix}\left|\begin{bmatrix}1\\1\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}\right|.$$

Figure 4. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}8&5\\7&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}5\\4\end{bmatrix}\left|\begin{bmatrix}1\\2\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}-1\right|.$$

Figure 5. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5&7\\24&36\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}-3\\-16\end{bmatrix}\left|\begin{bmatrix}1\\1\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}+1\right|.$$

Figure 6. This is the phase portrait corresponding to the vector field given by

$$\xi\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}4&-2\\3&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}-6\\6\end{bmatrix}\left|\begin{bmatrix}-3\\2\end{bmatrix}^t\begin{bmatrix}x_1\\x_2\end{bmatrix}+5\right|.$$

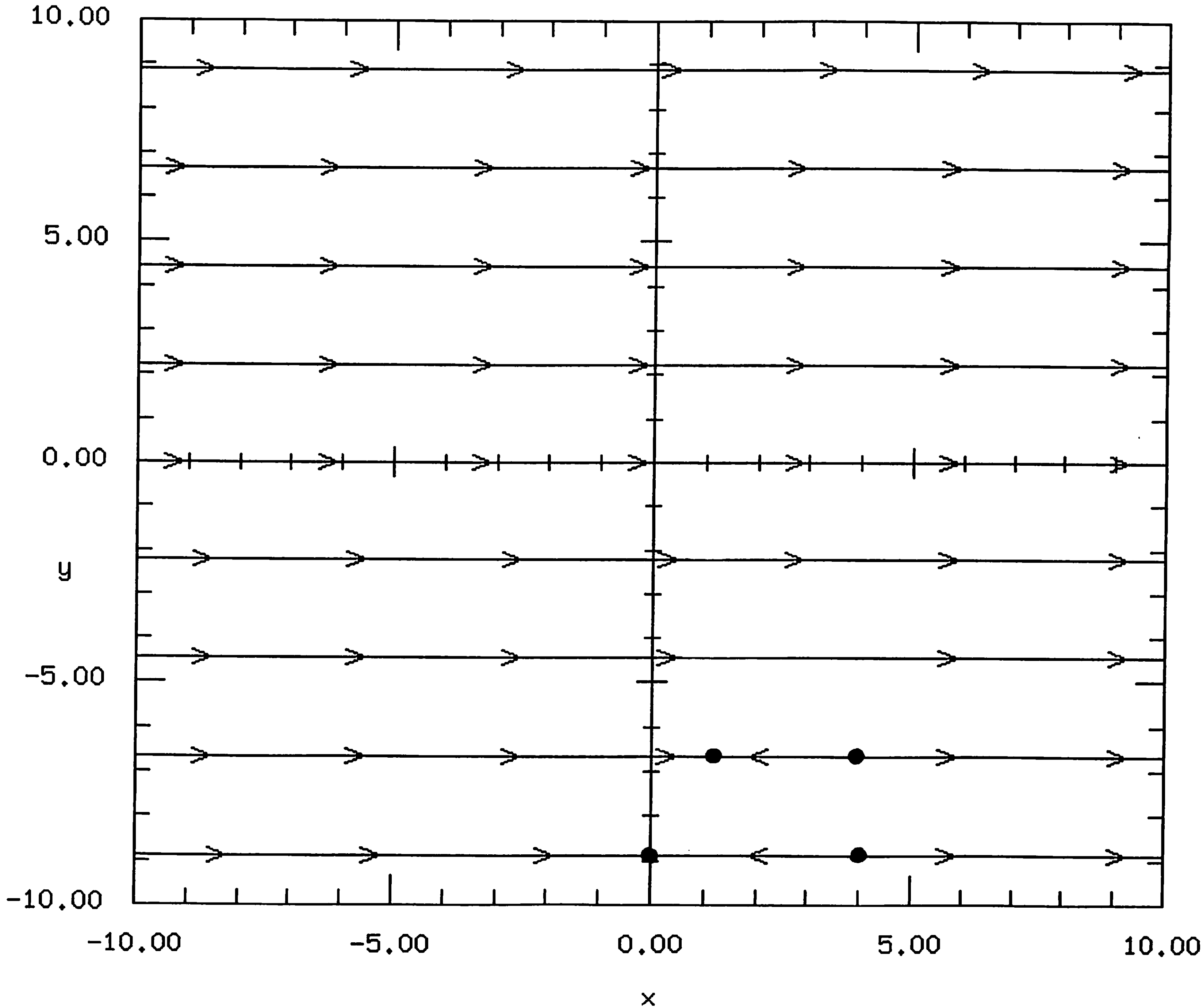

Figure 1

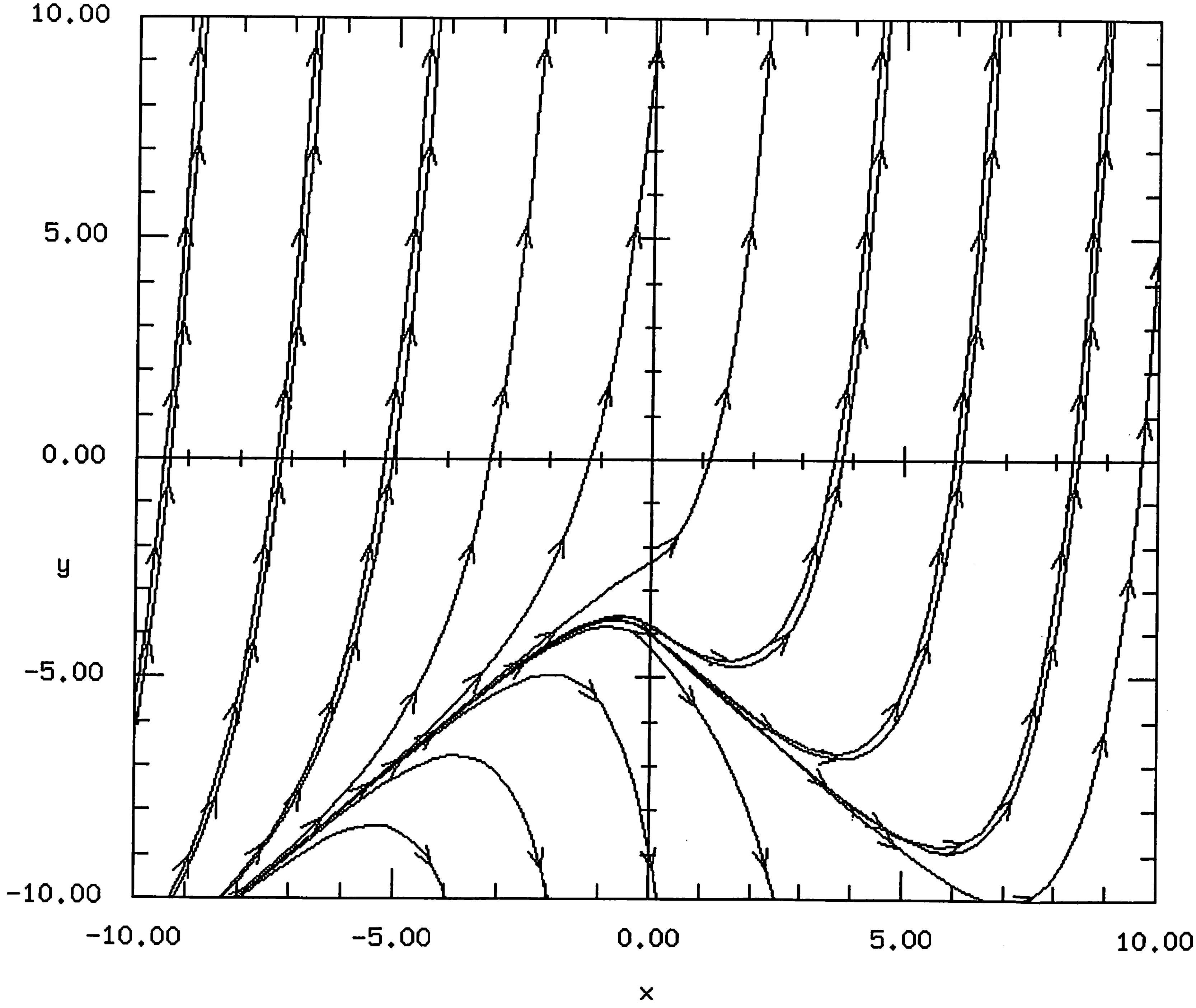

Figure 2

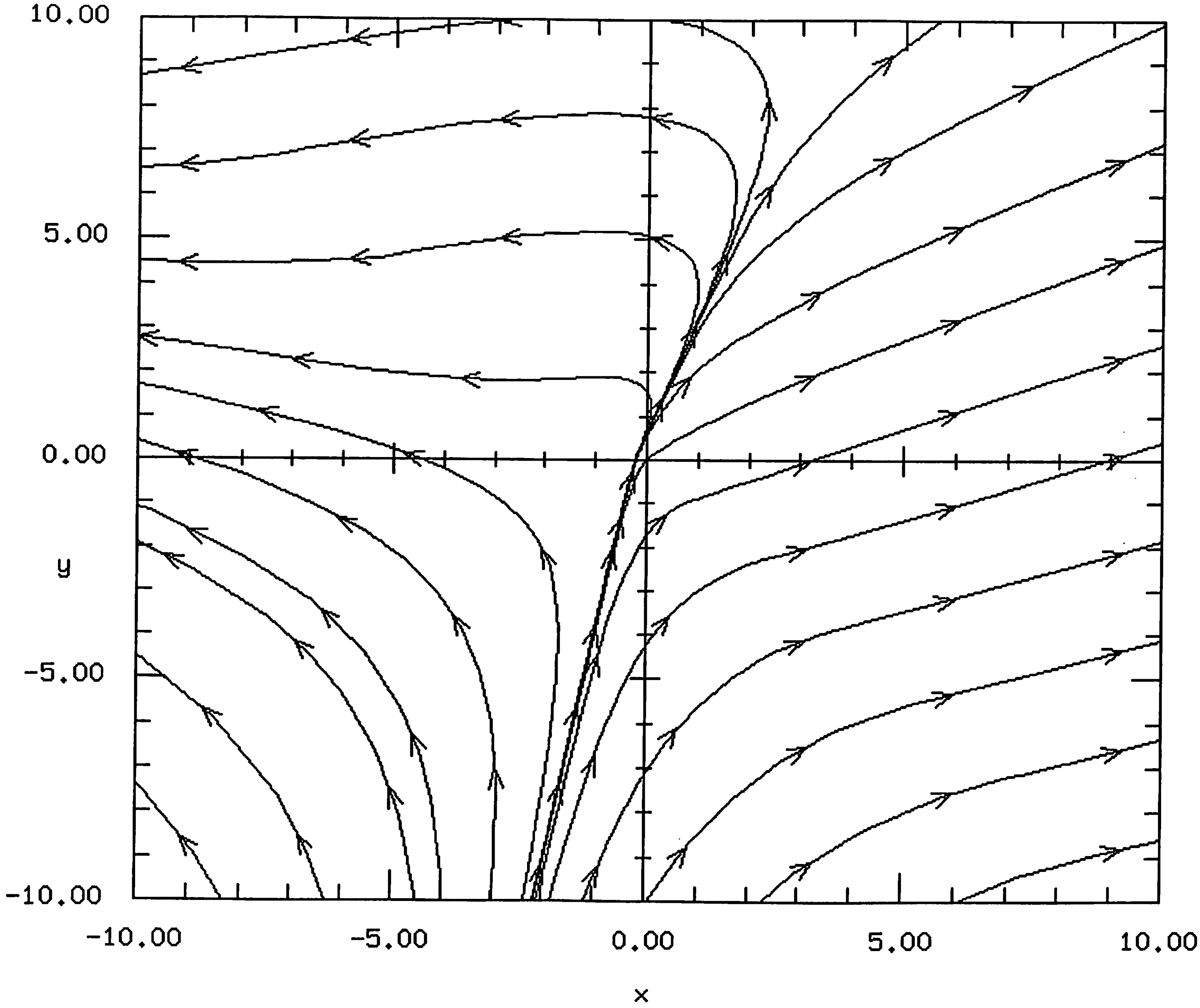

Figure 3

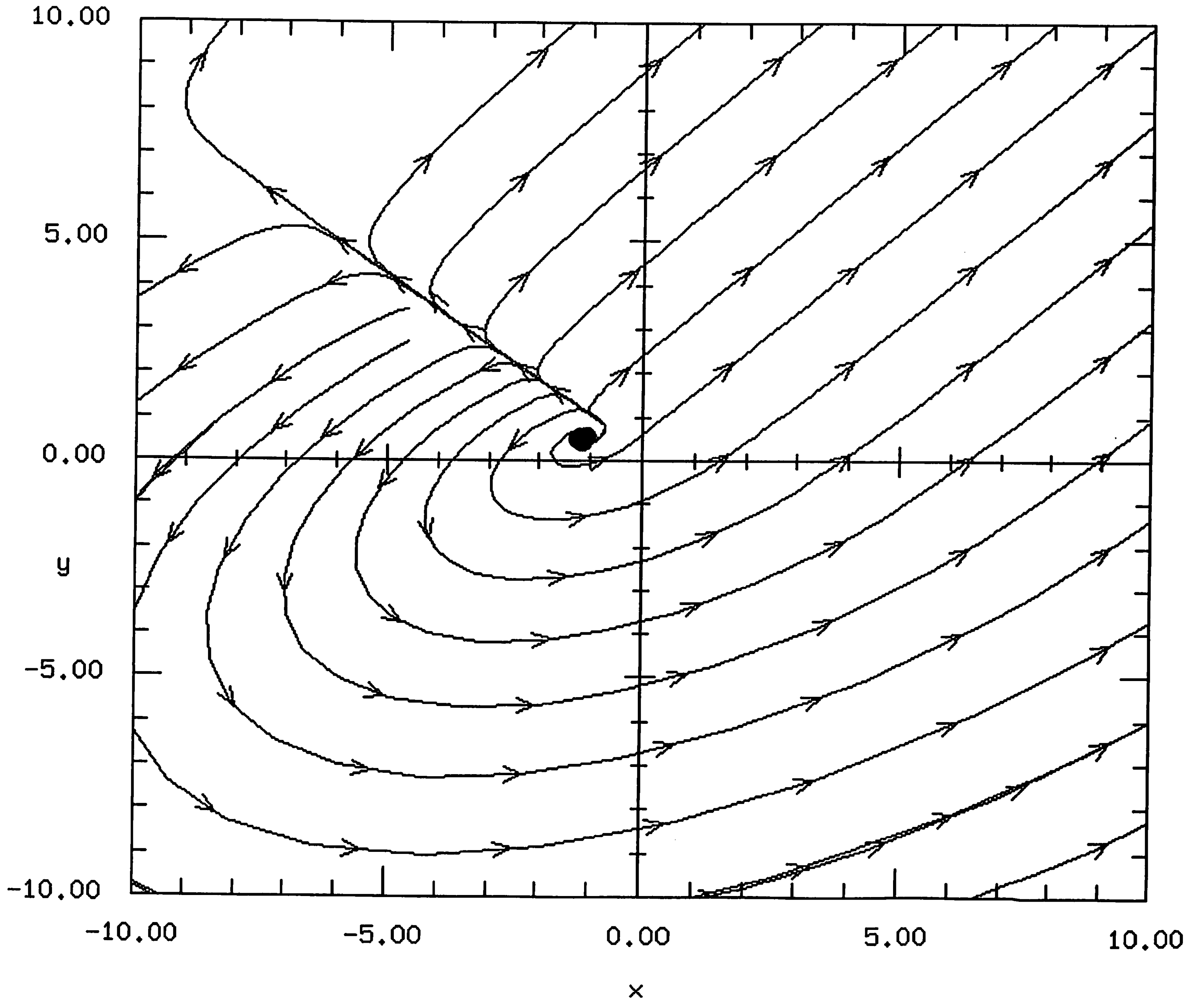

Figure 4

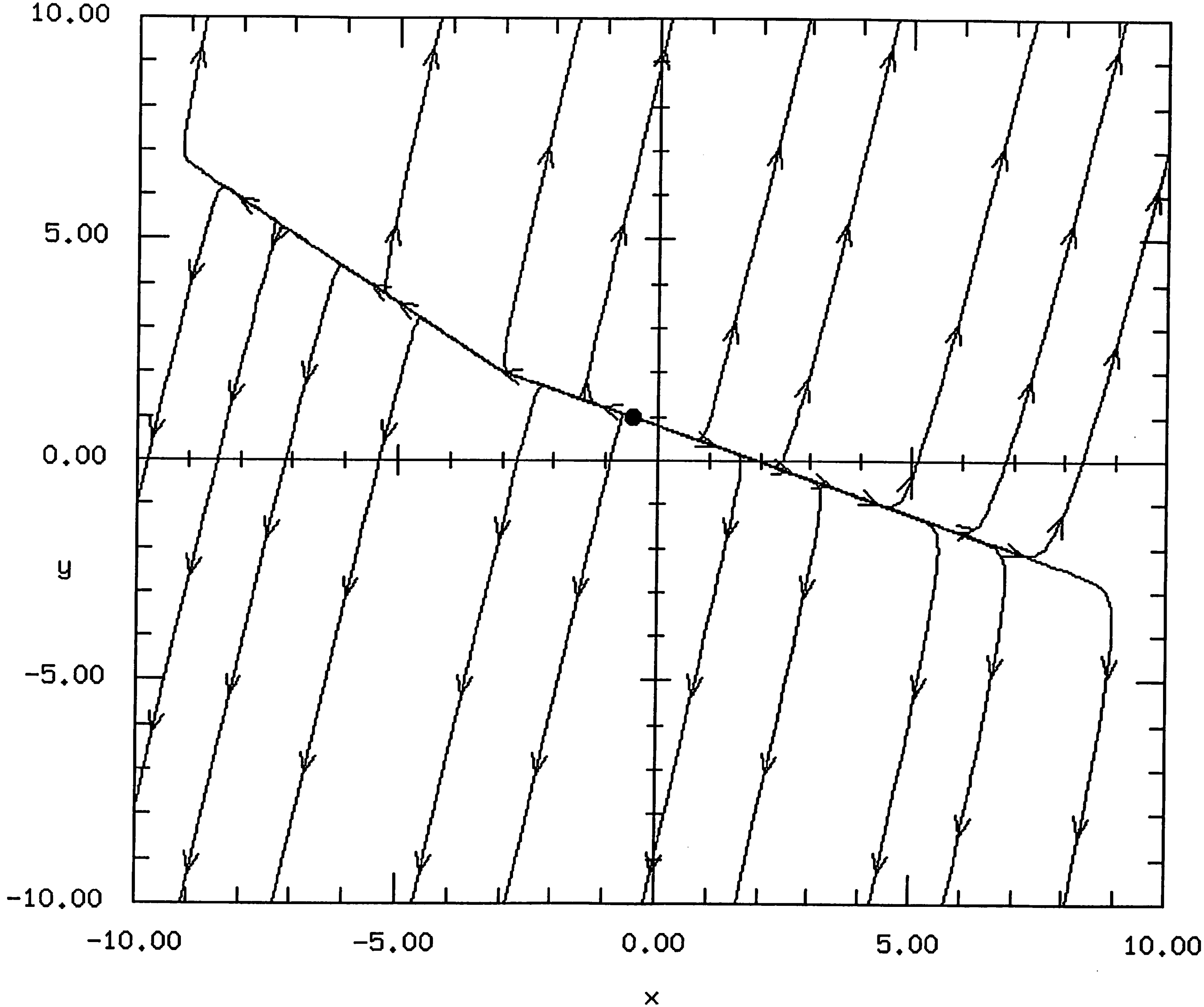

Figure 5

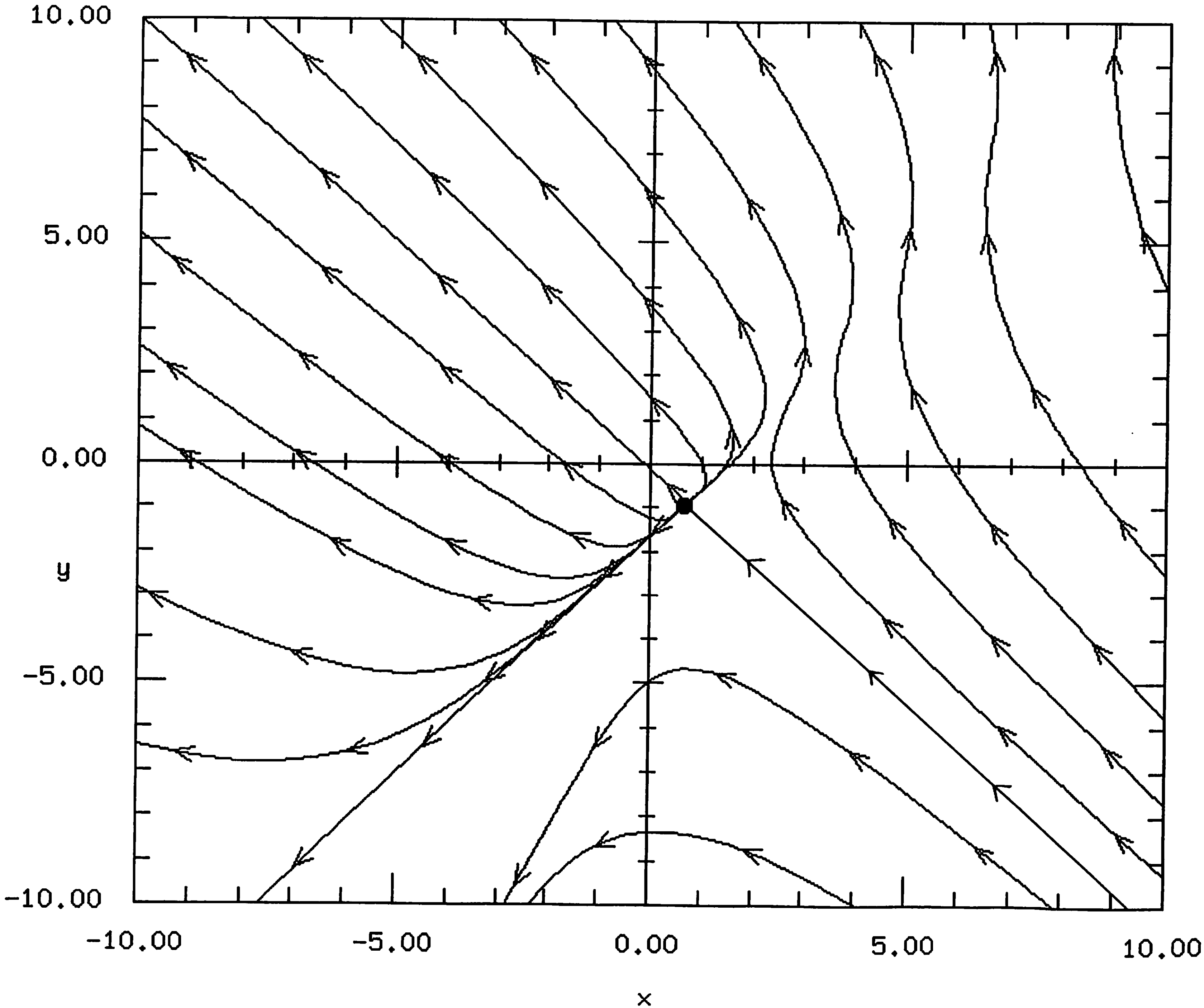

figure 6