HORSESHOES IN THE TWIST AND FLIP MAP

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Horseshoes in the Twist and Flip Map

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Abstract

We derive an analytical relationship between the parameters of a square-wave forced, non-linear, two dimensional ordinary differential equation which determines conditions under which the Poincare map has a horseshoe. This provides an analytical test for chaos for this equation.

In doing this we show that the Poincare map has a closed form expression as a transformation of $\mathbf{R}^2$ of the form FTFT, where F is a flip, ie a 180 degree rotation about the origin and T is a twist centered at $(a, 0)$, for $a > 0$. We show that this derivation is quite general.

We also show how to relate our results to ODE's with continuous periodic forcing (e.g., the sinusoidal-forced Duffing equation).

Finally, we provide a conjecture as to a sufficient condition for chaos in square-wave forced, non-linear ODE's.

1 Introduction

This paper is divided into five sections and an Appendix. This section provides background information, defines the twist and flip map, and describes a general method of reducing the Poincare map of a second order, square wave forced, differential equation to the double twist and flip map. Sec. 2 presents a proof of the horseshoe[1] twist theorem, which is the main result of this paper. Sec. 3 presents some examples of the applications of the horseshoe twist theorem, and Sec. 4 shows how to construct a dissipative transformation from the twist and flip map which is the analogue of the Poincare map of a second order, square-wave forced ODE having a dissipative term. The existence of a strange attractor for this dissipative system is illustrated. Sec. 5 shows the relationship of our results to the sinusoidally forced Duffing equation.

The formulae used throughout this paper are derived in the Appendix.

1.1 Background

The theorem presented in this paper (Sec. 2) was motivated by an analysis of the general Duffing equation with a sinusoidal forcing term which has been studied by many authors, for example see [Guckenheimer & Holmes, 1983]. This equation is:

$$\ddot{x} + b\dot{x} + cx + x^3 = a\cos(t)$$

In an effort to discover the fundamental mechanism for chaos in this equation some simplifications were made which are similar to the analysis in [Tanaka et. al., 1984] where a study of square-wave forced circuits may be found. As a result, the following equation was chosen as the starting point of our analysis:

$$\ddot{x} + b\dot{x} + cx + x^3 = a\ \mathrm{sgn}(\sin(\omega t)) \tag{1}$$

where,

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

[1]See [Guckenheimer & Holmes, 1983] for an explanation of the Smale horseshoe.

Reasonable physical and mathematical intuition suggests that chaos will be no less prevalent in the square-wave forced equation than in the cosine forced equation. However, the square-wave forced equation offers some significant analytical advantages that we now explain.

1.2 Definition of the Twist and Flip Map

In this section and throughout this paper we will use F to denote a flip (a 180 degree rotation about the origin) and T to denote a twist. In this paper[2], a two-dimensional twist is defined as a transformation, T, of $\mathbf{R}^2$ that has two properties. First, it preserves a continuum of closed curves [3] in the sense that each closed curve is invariant with respect to T (i.e. $T(p) \in \Gamma$, for any $p \in \Gamma$, where Γ is any member of a continuum of closed curves) and the curves in the continuum are uniquely parameterized by a non-negative scalar, henceforth called the energy. Second, on each integral curve, the angular displacement (in polar coordinates) with respect to the "center", (which need not be the origin) of the curves is eventually[4] a decreasing or increasing function of the *energy*. We note that we include decreasing in addition to increasing due to the convention that in polar coordinates the angular velocity is negative rather than positive. What is important is that the sign of the angular velocity not change and that the angular velocity is either increasing or decreasing.

In order to fix these ideas we present the following example: Consider

$$\ddot{x} + x^3 = 0$$

This equation has the first integral $\dot{x}^2/2 + x^4/4 = \mathrm{H}$. In the phase plane this equation describes the family of closed curves $y^2/2 + x^4/4 =$

[2]We stress that in the literature the term *twist* has a more restricted meaning than we are presenting here. Our work indicates the need to generalize the present notion of the twist.

[3]Which may be thought of as *integral curves*, much like the integral curves defined by a periodic two-degree of freedom Hamiltonian system.

[4]the term "eventually" used here is for convienence of this paper only. In a forthcomming article this definition will be refined and the term "eventually" will be discarded

H. It may be verified that $\tan(\theta) = y/x^3$ provides an orthogonal family of curves to the closed energy curves. Together these two sets of curves provide a new coordinate system that allows us to compute $\partial\dot{\theta}/\partial H$. Doing this we find that $\partial\dot{\theta}/\partial H = -6\sin^2(\theta)/x^2 < 0$.

[FIGURE 1]

The simplest example of a twist [5] is the transformation in polar coordinates defined by the equation $\mathrm{T}(r,\theta) = (r, \theta + r\tau)$, where r is measured in radians and $\tau \neq 0$. The only *integral curves* which are preserved by this twist are the circles of radius r. This twist, restricted to each such circular integral curve is a function of r which plays a role similar to that of energy. The effect of the twist in this case is that the angle of rotation is a function of the energy. The greater the energy, or radius r, the greater the angle of rotation (positive or negative) of the transformation along a circular energy curve. Fig. 1 shows an example of a simple twist for $\tau = 1, a = 1$ which is not centered at the origin. In general the twist is centered at $(a, 0)$, where $a > 0$. As illustrated in Fig. 1, the simple twist, and twists which are translates of the simple twist, transform straight lines emanating from the center of the twist into spirals.

The twist used in this paper will be a translation of the simple twist and it will always be assumed that $a > 0$. The rectangular coordinate equation for a twist of this type centered at $(a, 0)$ will be given in Sec. 2.

Remark It should be noted that the concept of a twist can be generalized to n-dimensions (e.g. $(r, \theta_1, \theta_2, \ldots, \theta_n) \rightarrow (r, r\tau_1 + \theta_1, r\tau_2 + \theta_2, \ldots, r\tau_n + \theta_n)$), where as before r is measured in radians and $\tau_i \neq 0$ for all i. Doing this would be one means of establishing a general theory for equations of higher dimensions. A second route to generalization would be to factor equations in higher dimensions into lower dimensional components. For example, the sinusoidally forced Duffing equation can be shown to be equivalent to an autonomous equation of degree two in three dimensions. This observation establishes the

[5] When the term *twist* is used in the literature, this example is usually what is meant.

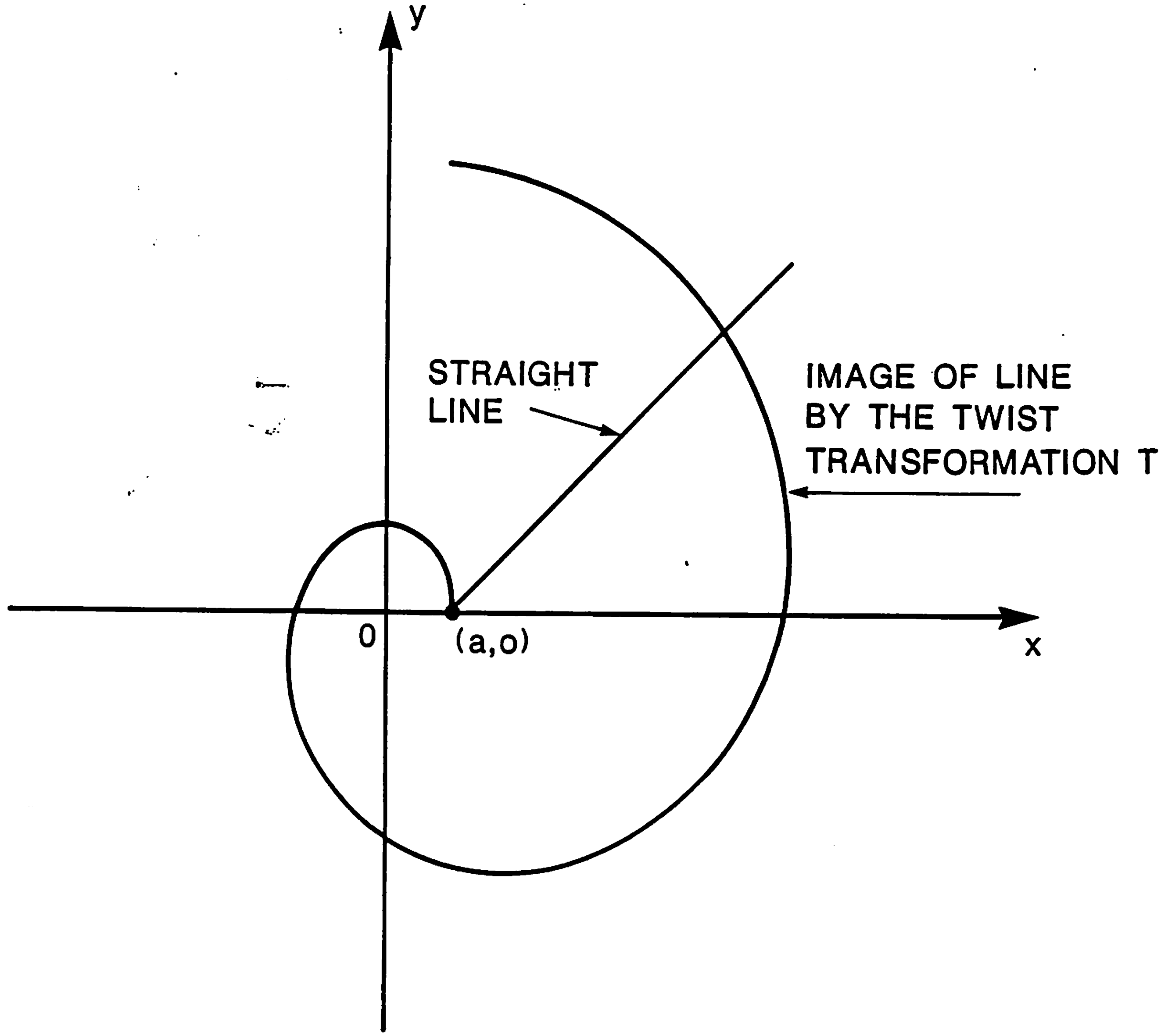

y
STRAIGHT
LINE
IMAGE OF LINE
BY THE TWIST
TRANSFORMATION T
0
(a,o)
x

Fig. 1

needed link between the two-dimensional case and higher dimensional equations. Both of these directions will be treated in a later paper where it will be shown that our results can be extended to equations in higher dimensions.

1.3 Reduction of the Poincare Map of the Square-wave Forced Duffing Equation to a Twist and Flip

In this section we will explain how to reduce the Poincare map [6] of certain second order ODEs to a twist and flip. Although the method is quite general, we use the Duffing equation 1 as a specific example.

The first step in this reduction is to notice that the square-wave forced Duffing equation can be written as a pair of autonomous equations:

$$\ddot{x} + b\dot{x} + cx + x^3 = a, \text{ whenever } \sin(\omega t) > 0 \tag{2}$$

$$\ddot{x} + b\dot{x} + cx + x^3 = -a, \text{ whenever } \sin(\omega t) < 0 \tag{3}$$

To use this pair of equations in place of the square-wave equation we follow the first equation up to time π/ω(half the period of the forcing function), at which time we use the position and velocity at the time $t = \pi/\omega$ as the initial conditions of the second equation. We then follow the second equation for the same time and switch back to the first.

We note that the second equation contains no new information, since its vector field in the $(x, \dot{x})$ phase plane is a 180 degree rotation of the first(substitute $(-x, -\dot{x})$ for $(x, \dot{x})$). From this we conclude that the solution, from any initial state $(x(0), \dot{x}(0))$, of Eq. 1 at the times of $2n\pi/\omega$ can be generated by integrating Eq. 2 over a time interval equal to π/ω, then taking the position and velocity at time $t = \pi/\omega$ and flipping it($(x(\pi/\omega), \dot{x}(\pi/\omega)) \rightarrow (-x(\pi/\omega), -\dot{x}(\pi/\omega))$ and integrating

[6] see [Parker & Chua, 1989]

Eq. 2 again over the same time interval with this flipped initial condition, then flipping the final output again to obtain $(x(2\pi/\omega), \dot{x}(2\pi/\omega))$. Recall that the solution of a periodically forced differential equation after one period of the forcing term is a point of the orbit of the Poincare map. Hence, the sequence of points that make up the orbit of $(x(0), \dot{x}(0))$ under the Poincare map is simply obtained by iterating the above algorithm starting with the point $(x(0), \dot{x}(0))$.

We now reduce Eq. 2 to a system of two first order equations in the phase plane by the substitutions $\dot{x} = y$ and $\dot{y} = -ay - bx - x^3 + c$.

Define T as follows: Given a pair of initial conditions (x_0, y_0), integrate the first order system of equations over the time π/ω to get the value $(x(\pi/\omega), y(\pi/\omega))$. This vector is $\mathrm{T}(x_0, y_0)$.

If F is the flip then the Poincare map defined by observing the solution of the square-wave forced equation at the intervals of $2\pi/\omega$ is simply the transformation FTFT. Unfortunately, since the trajectory, $(x(t), y(t))$, after each period of this dissipative ($a > 0$) system is not closed, the map T is not a twist as defined in this paper. Since in this paper we are presenting a theorem that gives necessary conditions for a flip and twist to have horseshoes, we simplify this equation by removing the damping factor $-ay$.[7] In the following subsection and in Sec.4 we indicate how our results can be extended to include a dissipative term. However, a rigorous treatment of this case will appear in a later paper.

The continuum of closed curves around which the twisting takes place is the first integral curve of the ODE. For Eq. 2 this integral curve is given by

$$\frac{\dot{x}^2}{2} + c\frac{x^2}{2} + \frac{x^4}{4} = ax + \mathrm{H} \tag{4}$$

which is parameterized by the energy H.

In phase plane coordinates this one-parameter family of closed curves is given by

$$\frac{y^2}{2} + c\frac{x^2}{2} + \frac{x^4}{4} = ax + \mathrm{H}$$

[7]The analytical advantage of removing this factor is that the resulting transformation T, as defined above is a twist and our theorem provides a result for the case of a simple twist.

It can be easily verified by the coordinate transformation used earlier that the angular velocity, $\dot{\theta}$ (in radians per second) of T is negative along each integral curve, and that the angular velocity is eventually an increasing/decreasing function of the energy H (i.e.$\partial\dot{\theta}/\partial$H < 0 or $\partial\dot{\theta}/\partial$H > 0 sufficiently large H. Observe that, in Equ. 4, $\partial\dot{\theta}/\partial H = -6x\sin^2(\theta)/(x^3 - c)$. Thus, for small H this partial undergoes a change of sign as x goes from zero to infinity, but for large enough x the sign does not change). Hence T is a twist by our definition.

We have exchanged the problem of analyzing the Poincare map for the square-wave forced Duffing equation for the problem of analyzing the double twist and flip map, FTFT, where the components F and T are quite simple and well understood. If FT has fixed points, these points are also fixed for FTFT. Therefore, when fixed points exist for FT, and these fixed points imply the existence of horseshoes for FT, they are also horseshoes for FTFT.

The most important fact about the FT *factorization* of the Poincare map of square-wave forced ODEs such as the Duffing equation is that the the presence of horseshoes for FT implies the presence of chaos for the system described by the ODE.

Further, it appears at this time that square-wave forcing is the extreme case of chaos in ODEs and that cosine forcing has fewer horseshoes than square-wave forcing. This is because the cosine forced ODE has the effect of varying the amplitude a as it moves continuously through its range $[-1, 1]$. We know that for small amplitudes there is a greater chance of an unstable manifold being homoclinic. We have concluded informally that damped, sinusoidal forced ODE have fewer horseshoes still.

In section 5 we will show how to generalize the Poincare map of the square-wave forced ODEs to sinusoidally forced ODEs.

As a point of clarification of our results we note here that we are not presenting a general theory of chaos in Poincare maps at this time. What we are presenting is a special case of chaos in a particular square-wave forced ODE for which the Poincare map is the simple twist and flip. We believe that this example is generic and provides a foundation for a general theory of chaos in second order ODEs. It is possible at this time to provide a more complex example using Duffing's equation and the Jacobi elliptic functions, however, we believe that the simplicity of the twist and flip map is particularly useful in forshadowing the general theory. A careful reading of the proof will show that it is (with a few removable exceptions) entirely independent of the particular example we are using.

The system of first order ODEs for which the simple twist (as defined in Fig. 1) and flip is the Poincare map is

$$\begin{aligned}\dot{x} &= -y\sqrt{(x-a\ \mathrm{sgn}(\sin(\omega t)))^2+y^2}\\ \dot{y} &= (x-a\ \mathrm{sgn}(\sin(\omega t)))\sqrt{(x-a\ \mathrm{sgn}(\sin(\omega t)))^2+y^2}\end{aligned}$$

The Poincare map sampling interval is, therefore, $2\pi/\omega$, and the Poincare map is FTFT, where T defined by sampling the solution of

$$\begin{aligned}\dot{x} &= -y\sqrt{(x-a)^2+y^2}\\ \dot{y} &= (x-a)\sqrt{(x-a)^2+y^2}\end{aligned}$$

at the times π/ω.

We may write out the solution of this autonomous vector ODE explicitly:

$$\begin{aligned}x(t) &= (x_0-a)\cos(r_0t)-y_0\sin(r_0t)+a\\ y(t) &= (x_0-a)\sin(r_0t)+y_0\cos(r_0t)\end{aligned}$$

where $r_0=\sqrt{(x_0-a)^2+y_0^2}$.

Note that although this solution appears to be that of a linear differential equation, the initial conditions occur in a non-linear way. The vector equation for this solution may be written as

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}\cos(r_0t) & -\sin(r_0t)\\ \sin(r_0t) & \cos(r_0t)\end{pmatrix}\begin{pmatrix}x_0-a\\ y_0\end{pmatrix}+\begin{pmatrix}a\\ 0\end{pmatrix}$$

The Poincare map is obtained by evaluating FTFT, where T is evaluated at $t=\pi/\omega$

The relevant energy curves are given by

$$(x-a)^2 + y^2 = r^2 = \mathrm{H}$$

and the partial of the angular velocity with respect to r is exactly 1.

We note that the parameters ω, a occur in the Poincare map and so any analytical condition for these parameters that assures the existence of a horseshoe for the Poincare map will imply the existence of chaos for the solutions of the above system of ODEs.

1.4 An Example of FT with Dissipation

Although the dissipative case will be treated in a later paper, we present here a closed form Poincare map from a dissipative system related to the twist system for which we will prove our theorem.

This system of first order ODEs is:

$$\begin{aligned}
\dot{x} &= -\alpha(x - a\ \mathrm{sgn}(\sin(\omega t))) + \alpha y\sqrt{(x - a\ \mathrm{sgn}(\sin(\omega t)))^2 + y^2} \\
\dot{y} &= -\alpha y - \alpha(x - a\ \mathrm{sgn}(\sin(\omega t)))\sqrt{(x - a\ \mathrm{sgn}(\sin(\omega t)))^2 + y^2}
\end{aligned}$$

The Poincare map sampling interval is, as before $2\pi/\omega$, and the Poincare map is FTFT, where T is defined by sampling the solution of

$$\begin{aligned}
\dot{x} &= -\alpha(x-a) + \alpha y\sqrt{(x-a)^2 + y^2} \\
\dot{y} &= -\alpha y - \alpha(x-a)\sqrt{(x-a)^2 + y^2}
\end{aligned}$$

at the times π/ω.

We may write out the solution of this autonomous vector ODE explicitly:

$$\begin{aligned}
x(t) &= \exp(-\alpha t)((x_0 - a)\cos(r_0(\exp(-\alpha t) - 1)) - y_0 \sin(r_0(\exp(-\alpha t) - 1)) + a \\
y(t) &= \exp(-\alpha t)((x_0 - a)\sin(r_0(\exp(-\alpha t) - 1)) + y_0 \cos(r_0(\exp(-\alpha t) - 1)))
\end{aligned}$$

where we know that $r_0 \exp(-\alpha t) = \sqrt{(x-a)^2 + y^2}$.

2 The Horseshoe Twist Theorem

In this section we prove the horseshoe twist theorem for the simple example of the twist described above . This theorem states the conditions between the parameters a, ω and the initial conditions for which FT is a chaotic transformation (has horseshoes). This section is in four parts. Part one describes the strategy of the proof of the horseshoe twist theorem. Part two contains definitions used in the proof of the horseshoe twist theorem, part three contains some lemmas needed in the proof of the horseshoe twist theorem, and part four is the proof of the horseshoe twist theorem.

2.1 Strategy of the Proof of the Horseshoe Twist Theorem

In order to simplify our discussion of chaos we provide the following definition:

Definition. Assume that we are given a hyperbolic fixed point of a diffeomorphism having non-trivial stable and unstable manifolds (in the two dimensional case this means that one eigenvalue lies inside the unit circle and the other lies outside the unit circle) If the unstable manifold M has a transverse homoclinic point or a point of homoclinic tangency[8] with the stable manifold that produces a horseshoe, then it will be called a *c-manifold* (for chaotic manifold).

To prove the existence of horseshoes, one must prove:

(1) The existence of hyperbolic fixed points for which there are non-trivial stable and unstable manifolds. In the two dimensional case this means that the hyperbolic fixed point must be a saddle point.

(2) The existence of a c-manifold. (In the case of a tangential crossing, it will be sufficient to show that the stable and unstable manifolds are symmetric images of one another about the vertical axis and that the manifold is not homoclinic.)[9]

[8]See [Guckenheimer & Holmes, 1983], theorem 6.6.1

[9]See [Guckenheimer & Holmes, 1983]

The existence of fixed points as well as the fact that all fixed points lie on the vertical axis follows by direct computation (this is lemma 1).

The existence of hyperbolic (saddle) points for the derivative of FT is determined by direct computation also (see lemmas 2 and 3). There we show that the determinant of the derivative of FT is 1 and that for $y \geq 2\omega/\pi$, the trace is strictly greater than 2. These two facts combine to prove that the eigenvalues of D(FT) at these fixed points are of the form λ and $1/\lambda$, where $\lambda > 1$.

We may now use the stable manifold theorem to find a local unstable manifold and from this we obtain a global unstable manifold (there exist a unique continuation of the local unstable manifold since the FT map is analytic). To prove there exist a c-manifold we first note by lemmas 4, 5, and 6 the unstable manifold is the symmetric image of the stable manifold about the vertical axis [10]. Thus if the unstable manifold crosses the vertical axis, then so does the stable manifold, and at the same point. If such manifolds are not homoclinic, we are done.

At this point we have reduced the problem of finding horseshoes to the problem of finding unstable manifolds which cross the vertical axis and are not homoclinic.

We solve these two problems separately. We first show that all unstable manifolds cross the vertical axis, (lemmas 7- 14 and proposition 1). Then we present conditions under which the unstable manifold is not homoclinic, (theorem 1.)

To show that all unstable manifolds cross the vertical axis we need only find a point of an unstable manifold on each side of the vertical axis,(because the unstable manifold is a connected set, it must have crossed the vertical axis). There is one technicality standing in the way of this strategy. The local unstable manifold has two connected components joined at the fixed point. The global unstable manifold will be two connected components joined at the fixed point. It can happen that FT maps one of these connected components onto the other. We will prove for all hyperbolic fixed points of FT on the positive vertical

[10]This *symmetry* of T is, in this case, a result of the fact that the velocity appears only as an even power in the energy (or first integral) Eq. 4 above. This is not true for the dissipative system and this will be addressed in Sec. 4

axis that this cannot happen (it can and does happen for hyperbolic fixed points lying on the negative vertical axis). Lemmas 7, 8, 10 and 11 do this.

Having shown this, it remains to prove that if we start with a point of the local unstable manifold, sufficiently near the fixed point, lying on the RHS of the vertical axis, its iterates under FT cannot all be on the RHS of the vertical axis. This is done by showing that any sequence of iterates of such a point must remain within a simply described bounded region (see lemma 12). Also, since these iterates must have strictly monotonically decreasing energy (see lemma 13), the energy must approach a limit which means that the sequence of iterates must approach a limiting energy curve. Further, if the sequence of energy iterates converges, the sequence of iterates must converge to the vertical axis (see lemma 14). We conclude from this that the limit of the iterates must be either a period-two point or a fixed point. The unstable manifold of a fixed point cannot terminate at a period-two point and we prove in lemma 9 there are no heteroclinic points.[11]. Thus, there must be a point at which the unstable manifold crosses the vertical axis.

It remains to find a non-homoclinic manifold. Theorem 1 states sufficient conditions under which a manifold is a c-manifold. It is not the most general theorem possible, but it cover all but a small set of cases.

A more general result which we do not prove, which is believed to be true is as follows: If $(0,y)$ is a hyperbolic fixed point, with $r = \sqrt{a^2+y^2}$, and $r_0 = \mathrm{Int}(r/2\omega)$, where $\mathrm{Int}(.)$ denotes the integer function, then we require that the following two circles intersect:

$$\begin{aligned}(x+a)^2+y^2 &= r^2\\(x-a)^2+y^2 &= r_0^2\end{aligned}$$

From this intersection condition follows a variety of algebraic formulae.

Although the theorem we prove is not the strongest possible its proof is very intuitive, simple, and geometric. Further it illustrates all

[11]see [Guckenheimer & Holmes, 1983]

the properties of chaos observed for square-wave forced ODE's such as the Duffing equation.

2.2 Definitions

T is the simple twist centered at the point $(a, 0)$ with $a > 0$. We define T by $\mathrm{T}(r, \theta) = (r, \pi r/\omega + \theta)$, where the polar coordinates (r, θ) is measured about the point $(a, 0)$ and r is in radians. In rectangular coordinates the equation for T is as follows:

Let $\mathbf{z}$ denote a vector in $\mathbf{R}^2$ and let $\mathbf{a}$ denote the vector $(a, 0)$ then

$$\mathrm{T}(\mathbf{z}) \quad = \quad \mathbf{A}(\pi r/\omega)(\mathbf{z} - \mathbf{a}) \quad + \quad \mathbf{a}$$

where

$$\mathbf{A}(\pi r/\omega) = \begin{pmatrix} \cos(\pi r/\omega) & -\sin(\pi r/\omega) \\ \sin(\pi r/\omega) & \cos(\pi r/\omega) \end{pmatrix}$$

and $r = \|\mathbf{z} - \mathbf{a}\|$.

For any given vector $\mathbf{z}$ define $\rho(\mathbf{z}) = \|\mathbf{z} - \mathbf{a}\|$

F will denote a flip (180 degree rotation) about the origin. The equation for F is

$$\mathrm{F}(\mathbf{z}) = -\mathbf{I}\mathbf{z} = -\mathbf{z}$$

where $\mathbf{I}$ denotes the identity matrix.

Let Φ denote the twist and flip map, i.e.,

$$\Phi = \mathrm{FT}$$

Define the reflection operator about the horizontal axis by the matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the reflection operator about the vertical axis by the matrix

$$\mathbf{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that the flip is given by the equation $\mathrm{F} = \mathbf{PR} = \mathbf{RP}$

We define $\mathbf{C}_r$ to be the circle of radius r centered at $(a, 0)$ and define $\mathbf{D}_r$ to be the circle of radius r centered at $(-a, 0)$.

Using these definitions we define $\mathbf{G}_r$ to be the intersection of $\mathbf{D}_r$ and the right half plane $\{(x, y)|x > 0\}$ and let $\mathbf{H}_r$ denote the intersection of $\mathbf{C}_r$ and the left half plane. Figure 2 illustrates these two regions which will be important in our proof of the horseshoe twist theorem.

We will use r_0 to denote $2\omega\mathrm{Int}(r/2\omega)$

We will use D(FT) to denote the derivative of FT.

The derivation of the relevant formulae for FT and its derivative are given in the Appendix.

Throughout all proofs, we will use polar and rectangular coordinates, depending on which provides the simplest expression within a given proof. For simplicity, we use the abbreviations RHS and LHS to denote the right-hand side and left-hand side, respectively.

[FIGURE 2]

2.3 Lemmas

Lemma 1 FT *has an infinite number of fixed points. Moreover, they all lie on the vertical axis and approach infinity in both directions.*

Proof: At a fixed point, $\mathbf{z}$, we have the equation: $\mathrm{F}(\mathbf{z}) = \mathrm{T}(\mathbf{z})$. Consequently the following equation holds:

$$\mathbf{A}(\pi r/\omega)(\mathbf{z} - \mathbf{a}) = -(\mathbf{z} + \mathbf{a}).$$

From this matrix equation follows three scalar equations (see Appendix A.2):

$$\|\mathbf{z} - \mathbf{a}\| = \|\mathbf{z} + \mathbf{a}\|$$

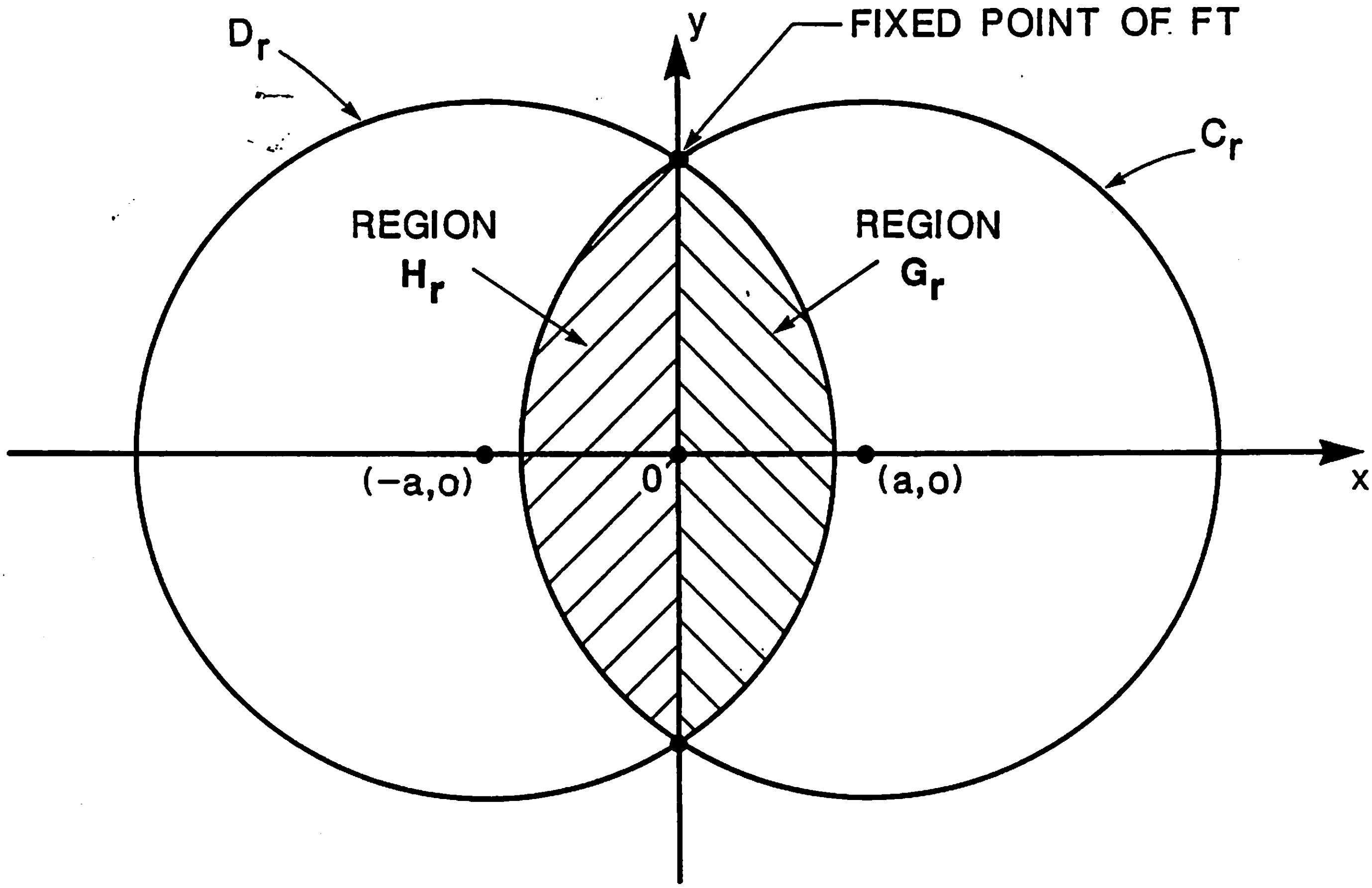


Fig. 2

which shows that all fixed points lie on the vertical axis. Next we have

$$\cos(\pi r/\omega) = 1 - 2(y/r)^2, \text{ and } \sin(\pi r/\omega) = 2ay/r^2$$

which shows where the fixed points lie exactly(see appendix A.2 for derivation).

The existence of an infinite number of fixed points for FT may also be deduced from the fact that $\rho(\mathbf{z})$, the energy, is continuous and strictly increasing as a function of y and a. ■

Lemma 2 *$Det(\mathrm{D}\Phi) = 1$ at each fixed point.*

Proof: See Appendix A.4.■

Lemma 3 *The Trace of $D\Phi$ at a fixed point is given by*

$$tr = 2(1 + \frac{ay\pi}{r\omega}) - 4(\frac{a}{r})^2$$

and $tr > 2$ for $y \geq 2\omega/\pi$. Consequently, all fixed points for which $y \geq 2\omega/\pi$ along the positive y-axis are hyperbolic.

Proof: See Appendix A.4.■

Remark: The most general condition for a fixed point to be hyperbolic is

$$r^2 - a^2 > 4(a/r)^2(\omega/\pi)^2$$

which is obtained by setting $tr > 2$ and simplifying.

Lemma 4 $\mathbf{PTPT} = \mathbf{I}$. *Thus,* $\mathbf{PT} = (\mathbf{PT})^{-1}$

Proof: The above condition is equivalent to $\mathbf{PAPA} = \mathbf{I}$ which is clearly true by a direct computation.■

Lemma 5 $\mathbf{R}\Phi = \Phi^{-1}\mathbf{R}$.

Proof: $\mathbf{R}(\mathrm{FT}) = \mathbf{PT} = (\mathbf{PT})^{-1}$, (by lemma 4) $= (\mathbf{RFT})^{-1} = (\mathrm{FT})^{-1}\mathbf{R}$.■

Remark: lemma 5 states that Φ is topologically conjugate[12] to its inverse.

[12] see [Guckenheimer & Holmes, 1983]

Lemma 6 *Given a hyperbolic fixed point of Φ, the stable manifold is the reflection of the unstable manifold about the y-axis.*

Proof: Follows from lemma 5.■

Lemma 7 *Let $(0, y)$ be a fixed point on the positive y-axis. The slope of the expanding eigenvector of the unstable manifold is strictly less than $-a/y$ for this point, consequently there exist an arc of the unstable manifold pointing into the circle $\mathbf{D}_r$.*

Proof: See Fig. 3. In Appendix A.5 it is shown that

$$\text{slope} = -\frac{a}{y}\sqrt{\frac{(\mathrm{tr}/2)+1}{(\mathrm{tr}/2)-1}} \quad .$$

■

[FIGURE 3]

Lemma 8 *For any hyperbolic fixed point where $y \geq 0$, there exist an arc of the unstable manifold, call it α, starting at the fixed point and lying entirely inside the region $\mathbf{G}_r$ We may choose α to be a closed arc having, therefore, a definite end point.*

Proof: Lemma's 1, 2, and 3 prove the existence of hyperbolic fixed points. Lemma 7 implies that an arc of any unstable manifold must lie in the stated region. It follows from the Stable Manifold Theorem[Parker & Chua, 1989] that there is a sufficiently small arc α which is contained in $\Phi(\alpha)$ and lying entirely in $\mathbf{G}_r$.■

Lemma 9 *(1)If for any point $\mathbf{z}$, on the vertical axis, $\mathrm{T}(\mathbf{z})$ is also on the vertical axis, then $\mathbf{z}$ is either a fixed point of T or a fixed point of* FT.

(2) There are no heteroclinic points for Φ.

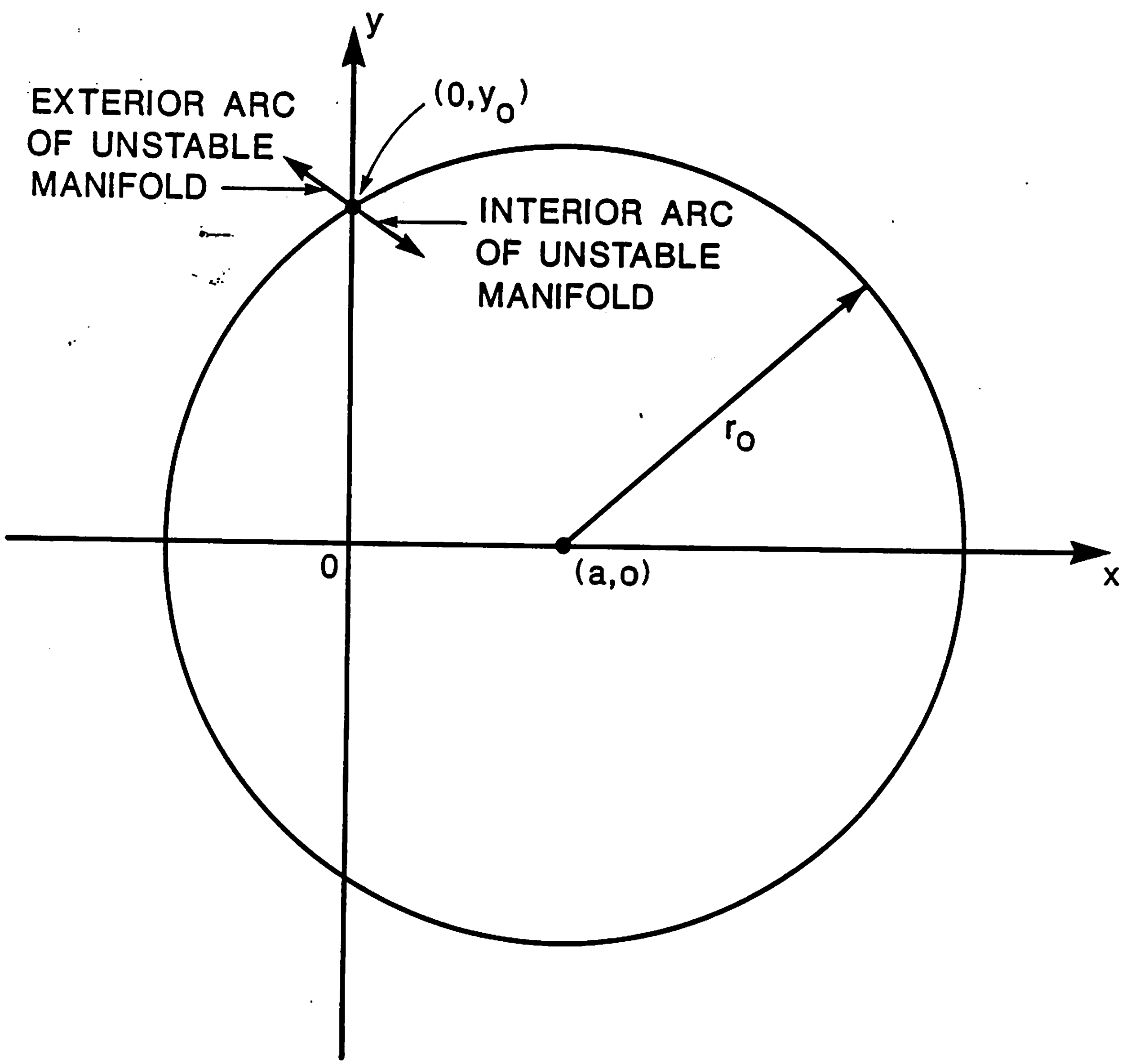

y
EXTERIOR ARC
OF UNSTABLE
MANIFOLD
$(0,y_o)$
INTERIOR ARC
OF UNSTABLE
MANIFOLD
r_o
0
(a,o)
x

Fig. 3

Proof:(1) Note that $\|\mathbf{z}-\mathbf{a}\| = \|\mathrm{T}(\mathbf{z})-\mathbf{a}\|$, and so $\mathrm{T}(\mathbf{z}) = \pm\mathbf{z}$.

(2) See Fig. 4. If two hyperbolic fixed points are separated by a period-two point we are done, since the heteroclinic manifold *would* have to cross a circle of fixed points of T to reach another fixed point. In doing this, it must cross the vertical axis creating either a c-manifold or a homoclinic manifold. Thus, assume there are two consecutive hyperbolic fixed points having no period-two point between them.

The straight line between these two points is a curve of Lipschitz constant 1 (see lemma 5 and the proof of the stable manifold theorem to see that this line has Lipschitz constant 1 because it bisects the angle between the stable and unstable manifolds of D(T). Since this is true for every hyperbolic fixed point, the specific fixed point is irrelevant) for each unstable manifold and by lemma 7 and the stable manifold theorem must be mapped, by Φ to the RHS of the vertical axis for the top point and to the LHS for the bottom point. Thus, there must be a point, call it $\mathbf{z}_1$, mapped to a point on the vertical axis, call it $\mathbf{z}_2$, by Φ. That is, $\Phi(\mathbf{z}_1) = \mathbf{z}_2$ where $\mathbf{z}_1, \mathbf{z}_2$ lie on the same energy curve of T. We have therefore $\mathrm{T}(\mathbf{z}_1) = -\mathbf{z}_2$. Now $\mathbf{z}_2$ cannot lie on the negative vertical axis, otherwise it is fixed by Φ contrary to assumption. We conclude that $\mathbf{z}_1 = -\mathbf{z}_2$ and thus $\mathbf{z}_1$ must be a period-two point of Φ.■

Remark: There can be two consecutive fixed points but one of them will be elliptic.

[FIGURE 4]

Lemma 10 *Let* $(0, y)$ *be the rectangular coordinates of a hyperbolic fixed point. Let* $y_1 = \sqrt{r_0^2 - a^2}$. *Then the vertical line between* $(0, y)$ *and* $(0, y_1)$ *is mapped into the region* $\mathbf{G}_r$ *by* $\mathbf{\Phi}$.

Proof: See Fig. 4 again. We know by the stable manifold theorem that a small arc of this line near the fixed point is contracted toward the unstable manifold, thus into region $\mathbf{G}_r$. Further, by (1) of lemma 9 if a point on the vertical axis is mapped onto the vertical axis by T it must be either a fixed point or a period-two point. But the period-two points separate the fixed points for all fixed points that lie above the

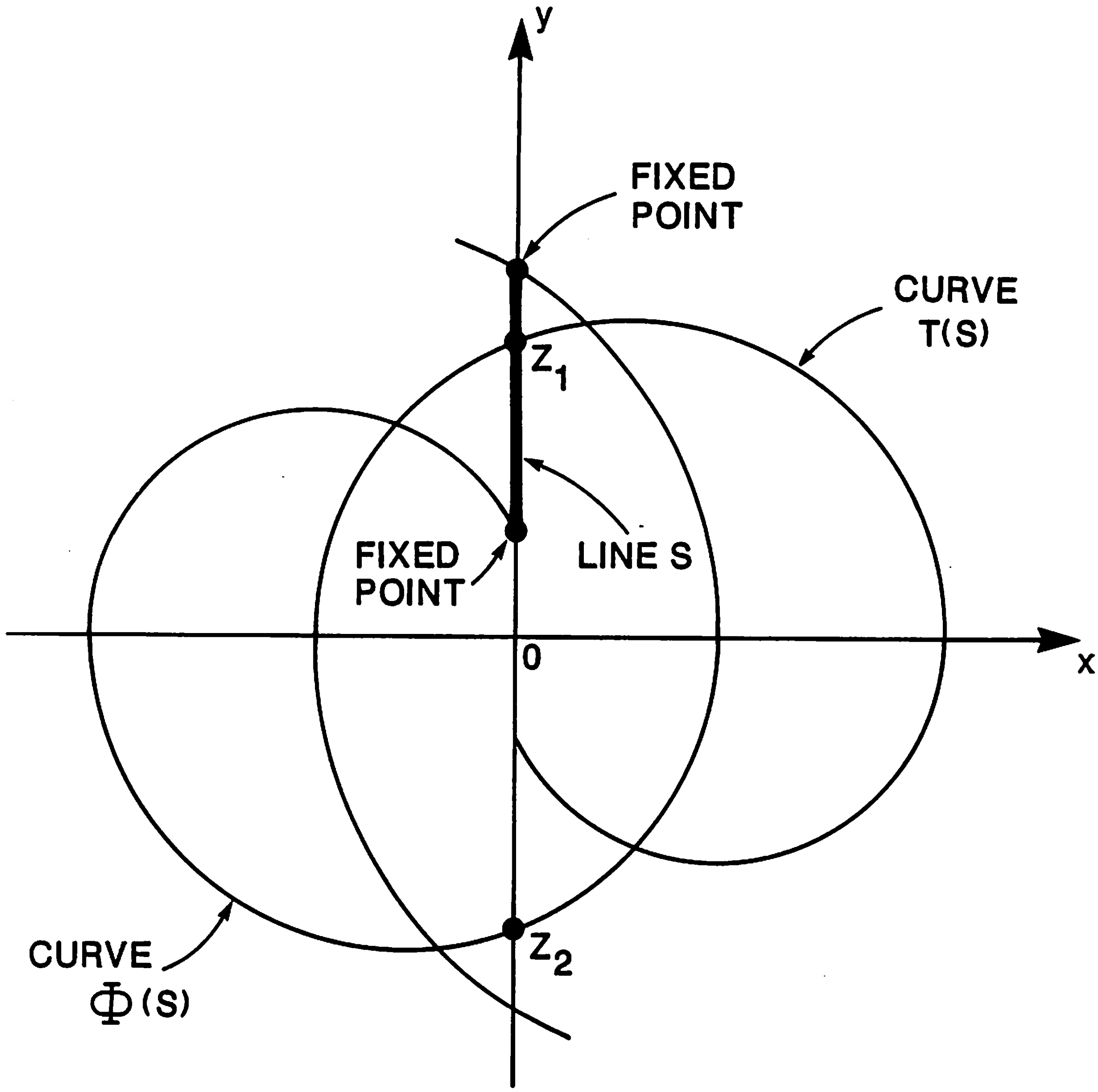


Fig. 4

first period-two point. Hence this line segment cannot return to the vertical axis before y_1.■

Lemma 11 *For all hyperbolic fixed points on the positive vertical axis, the component of the unstable manifold of a hyperbolic fixed point that points into the circle* $\mathbf{C}_r$ *is mapped onto itself by* Φ.

Proof: We will refer to the component pointing into the interior of the circle centered at $(a, 0)$ and passing through the fixed point as the interior component and the component that points outside of this circle as the exterior component.

By lemma 8 there must be a small arc of the interior component of the unstable manifold, lying in $\mathbf{G}_r$, and lying above the circle $\|\mathbf{z}-\mathbf{a}\| = \sqrt{(y-\delta)^2 + a^2}$, for some δ. By lemma 10, δ can be chosen so that this is mapped into the region $\mathbf{H}_r$ by T. This arc must be mapped into the region $\mathbf{G}_r$ by F and thus the interior component of the unstable manifold is not mapped onto the exterior component of the unstable manifold for $y \geq 2\omega/\pi$.■

Remark: Using the stable manifold theorem we can prove that for all hyperbolic fixed points on the negative vertical axis the two components of the unstable manifold are interchanged (fliped) at each iteration. For FT maps in which the partial of the angular velocity with respect to H is negative, the situation is reversed: The hyperbolic fixed points on the positive y-axis are fliped.

Lemma 12 *(Boundary) Let S denote the line segment of lemma 10.*

(1) An arc of the unstable manifold lies in the region bounded by three curves. The first curve is S, the second is the right hand boundary of $\mathbf{G}_r$ *(we will call this boundary* B_r*), the third is the curve* $\mathrm{T}^{-1}(S)$*.(Note that* $\mathrm{T}^{-1}(S)$ *is a continuous curve from S to* B_r*, since the lower end point of S is fixed by T).*

(2) If the unstable manifold crosses the curve $\mathrm{T}^{-1}(S)$*, it must also cross the vertical axis.*

(3) Let A be an arc of the unstable manifold of lemma 8 which lies in $\mathbf{G}_r$ *but does not intersect* $\mathbf{D}_r$*. Then* $\mathrm{FT}(A)$ *cannot intersect the circle* $\mathbf{D}_r$*.*

Proof: See Fig. 5. (1) follows from lemma 10.

(2)The second result follows from applying FT to $\mathrm{T}^{-1}(S)$, given that the unstable manifold crosses this curve.

(3) We refer to Fig. 5 again. Assume the contrary and let $\mathbf{z}$ be a point of the unstable manifold on the circle $\mathbf{D}_r$. Then $\Phi^{-1}(\mathbf{z})$ must be in $\mathbf{G}_r - \mathbf{D}_r$, by assumption. But $\mathrm{F}(\mathbf{D}_r) = \mathbf{C}_r$ and $\mathrm{T}^{-1}(\mathbf{C}_r) = \mathbf{C}_r$ and $\mathbf{C}_r \cap \mathbf{G}_r = \emptyset$.■

[FIGURE 5]

Lemma 13 *If* $\mathbf{z}_0$ *lies on the* RHS *of the y-axis and* $\mathbf{z}_{n+1} = \Phi(\mathbf{z}_n)$ *is a sequence of iterates of* Φ *that all lie on the* RHS *of the y-axis, then* $\rho(\Phi(\mathbf{z}_n))$ *is a strictly decreasing sequence of positive numbers.*

Proof: See Fig. 6. The lemma follows from the fact that if $\mathbf{z}$ is on the LHS of the vertical axis, then $\rho(\mathbf{R}(\mathbf{z})) < \rho(\mathbf{z})$. This follows from the relation:

$$\rho(\mathbf{z}) = \rho(\mathrm{T}(\mathbf{z})) = \rho(\mathbf{P}\mathrm{T}(\mathbf{z})) = \rho(\mathbf{R}\Phi(\mathbf{z}))$$

Thus, if for any $\mathbf{z}$ on the LHS of the y-axis ($x < 0$), we have $\rho(\mathbf{R}(\mathbf{z})) < \rho(\mathbf{z})$, we are done. But this is true by the law of cosines.

Another way to see this is to observe that for two consecutive iterates $r_{n+1}^2 - r_n^2 = 4ax_n$, so that the energy increases or decreases depending on whether x is positive or negative. ■

[FIGURE 6]

Lemma 14 *If for some point* $\mathbf{z}, \rho(\Phi^n(\mathbf{z})) \to r_0$*, then* $\Phi^n(\mathbf{z})$ *converges to a fixed point or it oscillates between a pair of period-two points.*

Proof: Let $\rho(\Phi^n(\mathbf{z})) = r_n$. Given two successive iterates of $\mathbf{z}$ we have $|r_{n+1}^2 - r_n^2| = 4a|x_n|$, where x_n is the horizontal rectangular coordinate of $\Phi^n(\mathbf{z})$. If $r_n \to r_0$ then $x_n \to 0$ and $\mathbf{z}_n$ must converge to the vertical axis. Therefore $\Phi^n(\mathbf{z})$ must converge to a point which is the intersection of the vertical axis and the circle $\|\mathbf{z} - \mathbf{a}\| = r_0$. This is either a fixed point or a pair of period-two points.■

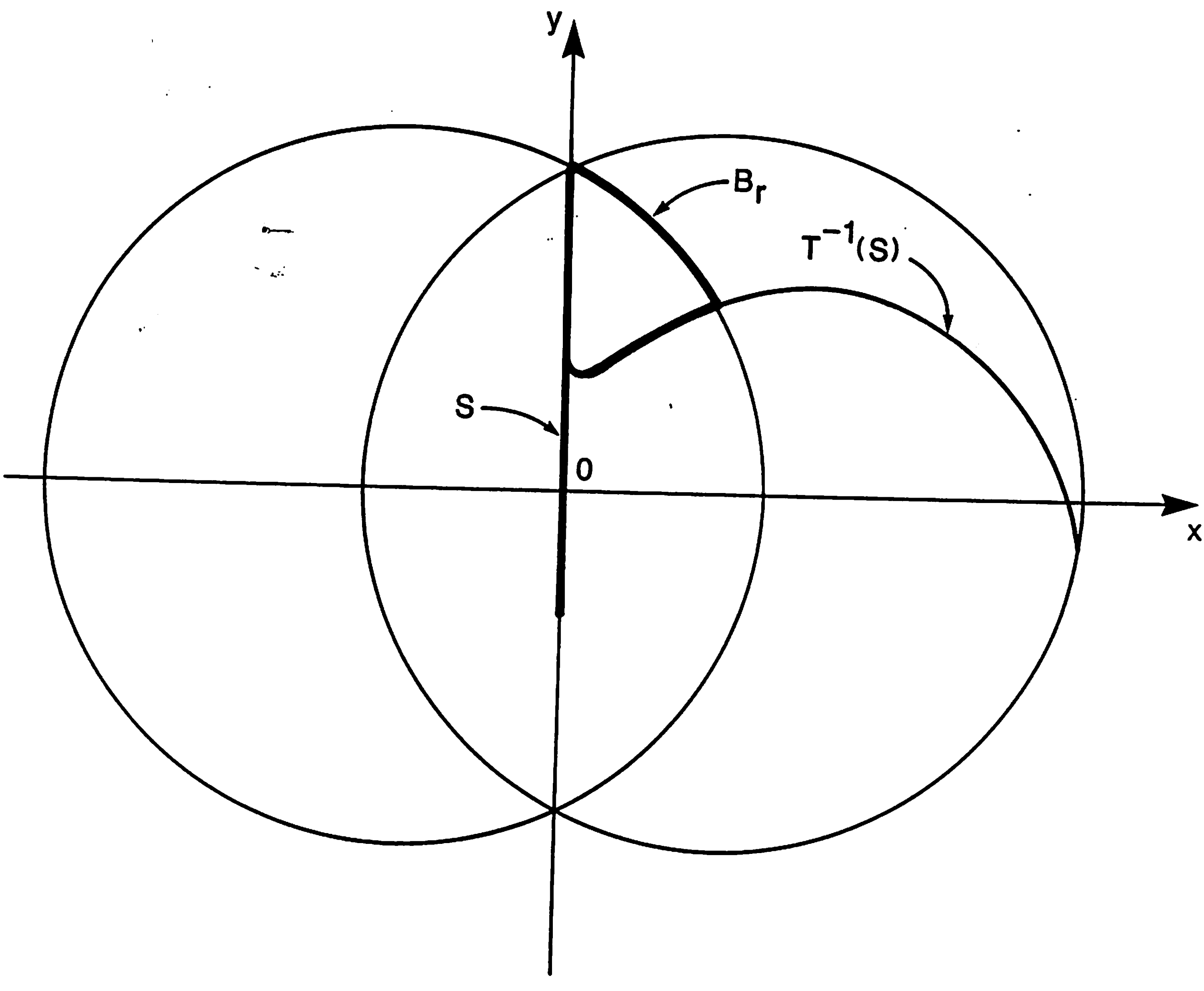


Fig. 5

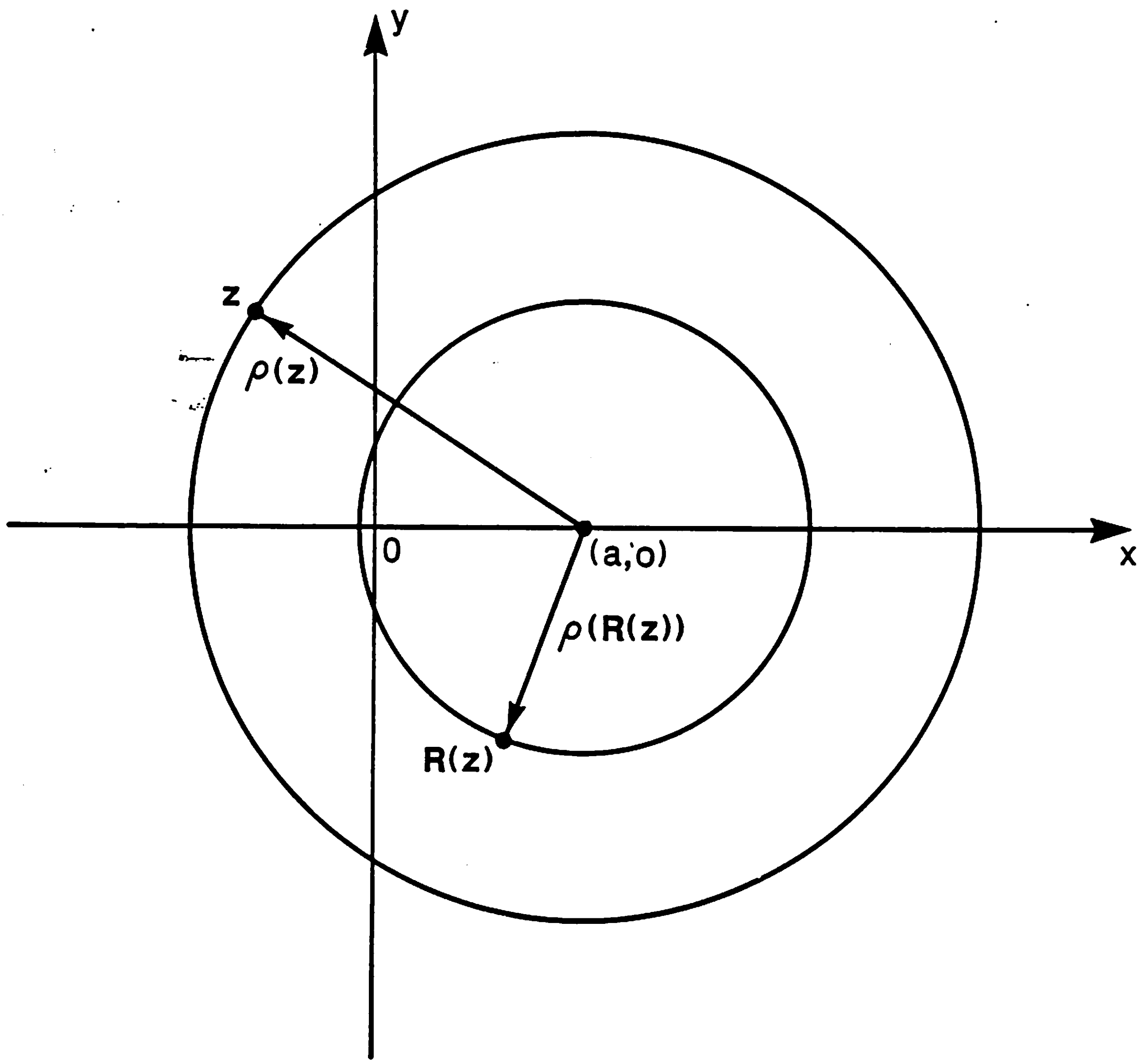


Fig. 6

2.4 Statement and Proof of the Horseshoe Twist Theorem

Proposition 1 *Every unstable manifold crosses the vertical axis.*

Proof:This proposition is equivalent to the requirement that some iterate of the arc of lemma 8 must cross the y-axis. Assume that it does not. Then the iterates of the end point of this arc define an infinite sequence of points of the unstable manifold for which r_n is a strictly decreasing sequence. This sequence is bounded by the curves of lemmas 12, any one of which if crossed implies a crossing of the vertical axis. Since the set bounded by these curves is compact, the sequence of iterates must either cross a boundary or have an accumulation point, which must be a limit of the sequence r_n, which is a fixed point or a period-two point. But there are no heteroclinic points for Φ by lemma 9.■

Theorem 1 *Let* $(0, y)$ *be a hyperbolic fixed point and let* M *be the unstable manifold, and assume that* $r_0 > a$. *Define* $\bar{\mathrm{K}}, q$ *as in appendix A.6. If* $0 \leq q\mathrm{K} \leq .5(1 + \sqrt{1 - \mathrm{K}})$ *and* $r_0^2 \geq (c - a)^2$, *then* M *is a c-manifold for* FT.

Proof:

We need only show that the manifold at $(0, y)$ is not homoclinic. Assume M is a homoclinic manifold. It must cross the circle C_{r_0}. Thus it contains a fixed point, **z**, of T. It must also contain F(**z**) and by symmetry of the homoclinic manifold it must contain the reflection of these points about the vertical axis. Thus we may assume that **z** is in the first quadrant of the plane. The flip of this point must be in the third quadrant. Now $\mathrm{T}^{-1}(\mathrm{F}(\mathbf{z}))$ is also on the manifold and must be in the first quadrant closer to the fixed point since a homoclinic manifold cannot cross the vertical in more than one point. Similarly, T(**z**) must be in the fourth quadrant. But because the manifold is homoclinic, it must lie entirely in the region $\mathbf{G}_r \cup \mathbf{H}_r$. We now refer to Fig. 7.

It is sufficient to show that under these conditions that it is not possible that both $\mathrm{T}(-\mathbf{z})$ and $\mathrm{T}^{-1}(-\mathbf{z})$ lie on the RHS of the vertical axis.The remainder of this proof is so uneventful and tedious that it is, for the most part, relegated to the appendix.

We use the definitions of appendix A.6 where $\mathbf{z} = \mathbf{z}_0$. We note from plane geometry that the cords from $\mathbf{z}_1$ to $-\mathbf{z}_0$ and from $\mathbf{z}_2$ to $-\mathbf{z}_0$ (as defined in appendix A.6) are representative of the angles they subtend. It is sufficient to show that the length of the cord from $\mathbf{z}_2$ to $-\mathbf{z}_0$ is always greater than that from $\mathbf{z}_1$ to $-\mathbf{z}_0$. This is done in appendix A.6. ■

[FIGURE 7]

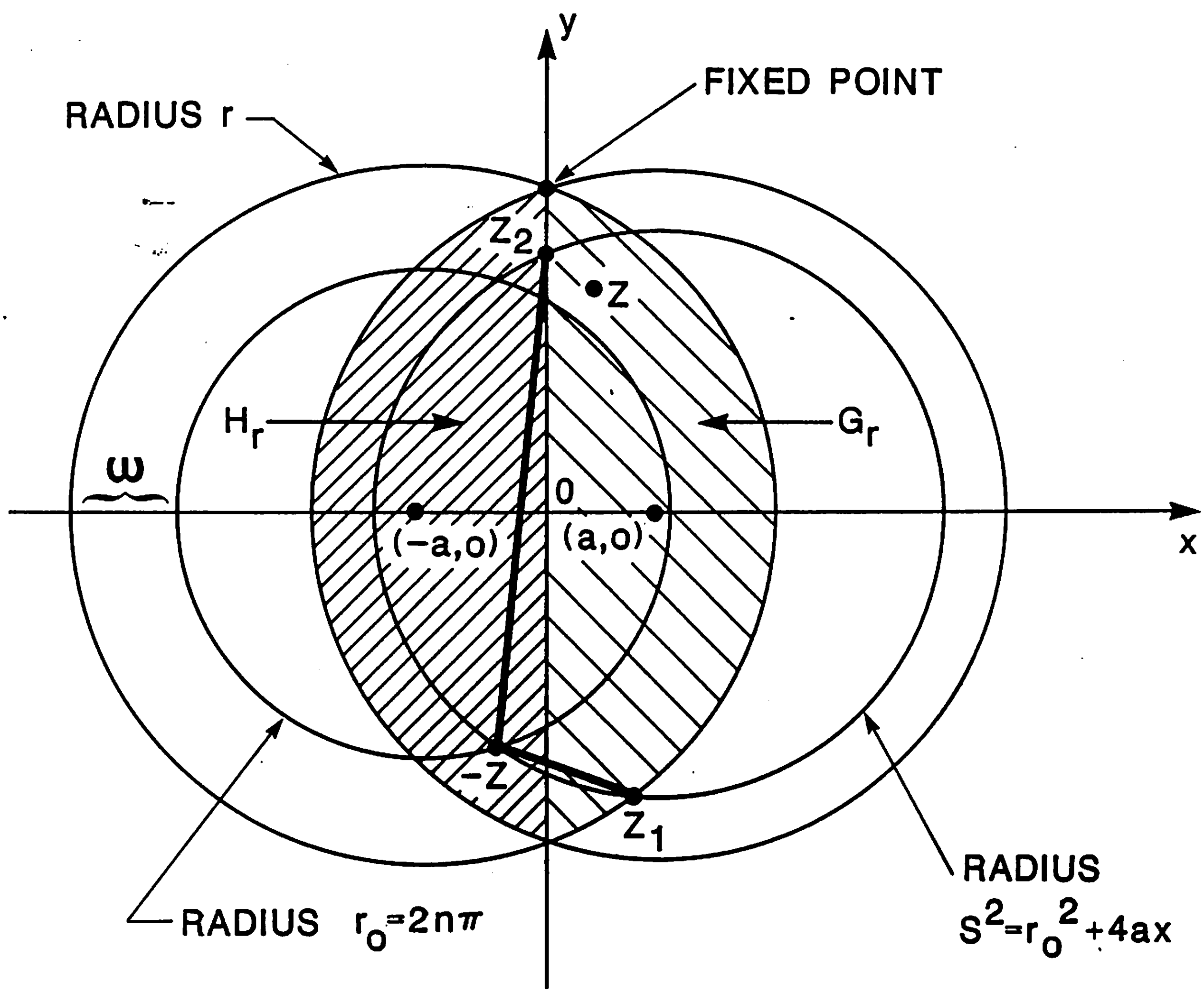
Proposition 2 *Assume the definitions of theorem 1.*
(1) If $s \geq 0.13$ and $q\mathrm{K} > .5(1 + \sqrt{1 - \mathrm{K}})$, then M *is a c-manifold.*
(2) Also, if $a < r_0 - 0.52$, and $q < \bar{\mathrm{K}}$, then M *is a c-manifold.*

Proof: (1) See appendix A.6. The proof of (2) is a direct computation using the definitions of appendix A.6.■

We present three conjectures that will be more fully discussed in a later paper. The setting is quite general. Consider the following ODE where $p(x)$ is any polynomial in x whose highest power is odd, greater than 1, and whose coefficient is positive:

$$\ddot{x} + p(x) = a\mathrm{sgn}(\sin(\omega t))$$

For this equation we may factor the Poincare map as FT, where T is obtained from the autonomous equation $\ddot{x} + p(x) = a$ by evaluating this equation at the times $t = \pi/\omega$. A first integral of the autonomous equation is of the form $\dot{x}^2/2 + q(x) = ax + \mathrm{H}$. Where the highest power of $q(x)$ is even. Note that the first integral defines a set of closed energy curves and that T is a twist according to our definition. The center of the twist depends on the polynomial $p(x)$. Assume there is a hyperbolic fixed point (which will be on the vertical axis) and that H defines the energy surface passing through this fixed point. Call this closed curve $\mathbf{C}_\mathrm{H}$ (It is the analogue of the curve $\mathbf{C}_r$ for the simple twist) and let $\mathbf{D}_\mathrm{H}$ be the curve which is the analogue to $\mathbf{D}_r$. Let H_0 define any energy curve, $\mathrm{C}_{\mathrm{H}_0}$, of fixed points of T where $\mathrm{H}_0 < \mathrm{H}$. Let M be the unstable manifold passing through the designated fixed point. We have the following conjecture:



y
FIXED POINT
RADIUS r
Z2
Z
Hr
Gr
ω
0
(-a,o)
(a,o)
x
-Z
Z1
RADIUS
S2=ro2+4ax
RADIUS ro=2nπ

Fig. 7

Conjecture 1 *The manifold* M *of* FT *is a c-manifold if* M *contains a fixed point of* T.

Conjecture 2 *A sufficient condition for the above described manifold to be a c-manifold is that the two curves,* $\mathbf{D}_{\mathrm{H}}$ *and* $\mathbf{C}_{\mathrm{H}_0}$ *intersect.*

Returning to the simple twist we have the following special case:

Conjecture 3 *A sufficient condition for a manifold of* Φ *to be a c-manifold is that the two circles,* $(x-a)^2+y^2 = r_0^2$ *and* $(x+a)^2+y^2 = r^2$ *intersect.*

There are various formulae possible based on this conjecture: $r \leq r_0 + 2a$, $r_0 \geq |c-a|$.

It is a simple matter to write down ODEs for which these conjectures apply. Among the easiest are the ODEs for which the hyper-elliptic functions are solutions. The following example demonstrates the ease in doing this.

Let $x = \lambda \cos(f(\lambda)t+\theta)$ and let $y = \lambda \sin(f(\lambda)t+\theta)$. These functions solve the following first order system of ODEs:

$$\begin{aligned} \dot{x} &= -f(\sqrt{x^2+y^2})y \\ \dot{y} &= f(\sqrt{x^2+y^2})x \end{aligned}$$

It is simple to add the necessary translates and square-wave forcing terms to produce equations whose Poincare maps are of the form FT.

The same approach may repeated with the Jacobi elliptic cosine, cn(t): For example let $x = \lambda \mathrm{cn}(\lambda^2 t + \theta)$, differentiate twice, and use the identity $\ddot{\mathrm{cn}}(t) = -\mathrm{cn}^3(t)$, then use the energy equation to eliminate λ to get an ODE free of constants.

2.5 Summary of Main Result

We now summarize the main result of this paper. We begin by restating the definition of the simple twist:

T is the simple twist centered at the point $(a, 0)$ with $a > 0$. We define T by $\mathrm{T}(r,\theta) = (r, \pi r/\omega + \theta)$, where the polar coordinates (r, θ)

is measured about the point $(a, 0)$ and r is in radians. In rectangular coordinates the equation for T is as follows:

Let $\mathbf{z}$ denote a vector in $\mathbf{R}^2$ and let $\mathbf{a}$ denote the vector $(a, 0)$ then

$$\mathrm{T}(\mathbf{z}) \quad = \quad \mathbf{A}(\pi r/\omega)(\mathbf{z}-\mathbf{a}) \quad + \quad \mathbf{a}$$

where

$$\mathbf{A}(\pi r/\omega) = \begin{pmatrix} \cos(\pi r/\omega) & -\sin(\pi r/\omega) \\ \\ \sin(\pi r/\omega) & \cos(\pi r/\omega) \end{pmatrix}$$

and $r = \|\mathbf{z}-\mathbf{a}\|$.

Recall that the twist and flip map is FT, where T is defined above and F is a 180 degree rotation. FT is the Poincare map for a specific non-linear ODE presented earlier.

Horseshoe Twist Theorem. *Let* FT *be the twist and flip map, where* T *is given above. Let* $(0, y)$ *be a hyperbolic fixed point for* FT, *and* $r = \sqrt{a^2 + y^2}$. *Let*

$$\begin{array}{ll} \mathrm{N} = \mathrm{Int}(r/2\omega) & r_0 = 2\omega\mathrm{N} \\ \rho = (r - r_0)/(\omega) & \mathrm{K} = (4\mathrm{N}\rho + \rho^2)/(16\mathrm{N}^2) \\ q\mathrm{K} = (a/r_0)^2 & c = a/q \end{array}$$

If $r_0^2 \geq (c-a)^2$ *and* $q\mathrm{K} \leq (1+\sqrt{1-\mathrm{K}})/2$, *then* FT *has a horseshoe.*

2.6 Two Examples of the Use the Horseshoe Twist Theorem

We illustrate use of the horseshoe twist with the following two examples. Since the computation of the trace of D(FT) at these fixed points is a routine application of our formulas, it is ommited.

Example 1: The amplitude of the forcing function, $a = 3$; the frequency of the forcing, $\omega = \pi$

A fixed point for the Poincare map FT is found at $(0, 8.19045)$. At this point

$$
\begin{aligned}
r &= 8.722584 \\
\mathrm{N} &= 1 \\
r_0 &= 2\pi \\
\rho &= 0.7764845 \\
q &= 0.983471 \\
\mathrm{K} &= 0.2318041 \\
\bar{\mathrm{K}} &= 4.313987 \\
c &= 3.050421 \\
(c-a) &= 0.050421 \\
(1+\sqrt{1-\mathrm{K}})/2 &= 0.9382339 \\
q\mathrm{K} &= 0.2279727
\end{aligned}
$$

We see that $r_0 > |c-a|$ and that $q\mathrm{K} < .5(1+\sqrt{1-\mathrm{K}})$ so that the unstable manifold at this fixed point is a c-manifold, i.e., FT has a horseshoe.

Example 2: The amplitude of the forcing function, $a = 2$; the frequency of the forcing, $\omega = \pi$

A fixed point for the Poincare map FT is found at approximately $(0, 8.75)$. At this point

$$
\begin{aligned}
r &= 8.975661 \\
\mathrm{N} &= 1 \\
r_0 &= 2\pi \\
\rho &= 0.8570417 \\
q &= 0.3894453 \\
\mathrm{K} &= 0.2601679 \\
\bar{\mathrm{K}} &= 3.843671 \\
c &= 5.135509 \\
(c-a) &= 2.135509 \\
(1+\sqrt{1-\mathrm{K}})/2 &= 0.9300675 \\
q\mathrm{K} &= 0.1013212
\end{aligned}
$$

Again we see that $r_0 > |c-a|$ and that $q\mathrm{K} < .5(1+\sqrt{1-\mathrm{K}})$ so that the unstable manifold at this fixed point is a c-manifold as well and so FT has a horseshoe.

3 Some Illustrations Using the Twist and Flip Map

In this section we will demonstrate that, apart from its value in understanding the structure of the Poincare map of certain non-linear square-wave forced ODEs, the twist and flip map can be particualrly useful in illustrating some fundamental features of non-linear dynamics and chaos. Its advantage over other algorithms is its simplicity, intuitive appeal, ease of computation (it provides about a 100:1 reduction in computer time in producing these illustrations as compared with a conventional numerical technique for solving an ODE).[13]

All figures are generated on an IBM PC using quick basic. The fixed point is computed by trial and error and does not have to be very accurate due to the compressing and stretching action of FT near the fixed point.

The unstable manifold is produced by iteration of the map FT on 5000 points of a small line segment (length of about .1) having the same slope as the unstable manifold for the hyperbolic fixed point. (see our formula for the slope of the unstable manifold at the fixed point) The value of a has been chosen arbitrarily to be 1 and $\omega = \pi$. The fixed point is at $(0, 1.95)$. The slope of the unstable manifold at this point is about -1.176.

Elliptic regions[14] are generated by 2000 iterations of FT on an initial point near the elliptic fixed point.

[FIGURES 8- 11]

The number of iterations varies for each figure in order to present the most informative illustration.

Fig. 8 is an unstable manifold generated with 10 iterations of FT.

Fig. 9 illustrates homoclinic tangles for the fixed point. Here, 10 iterations were used.

Fig. 10 shows that both elliptic and hyperbolic regions coexist and that the hyperbolic region encircles the ellitic region as a result ofthe

[13]Compare these examples with those found in [Arnold & Avez,1968] and [Ozorio de Almeida, 1988].

[14][Arnold & Avez, 1968]

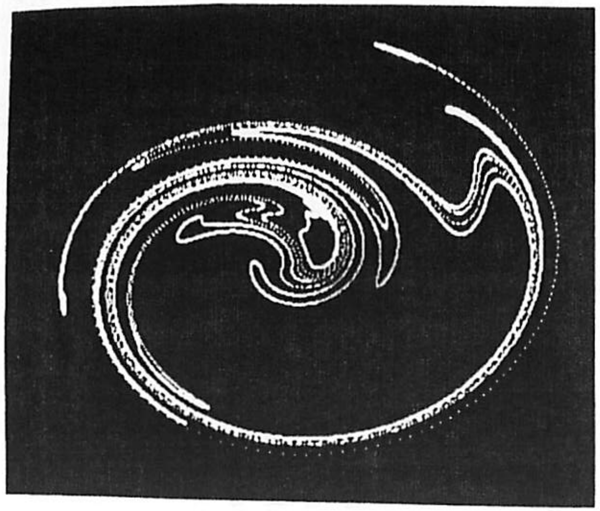


Fig. 8

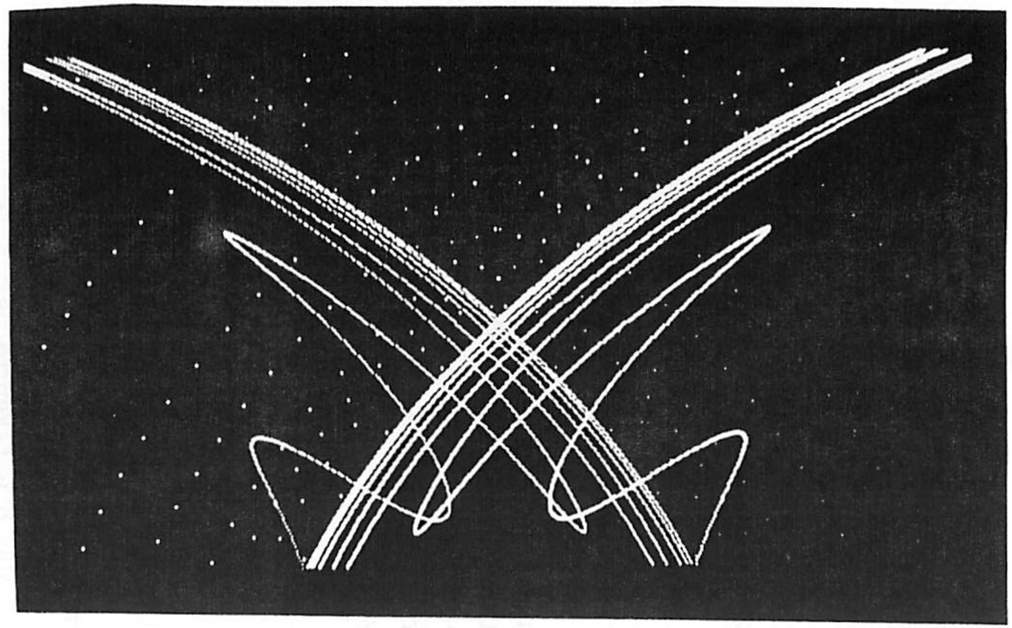


Fig. 9

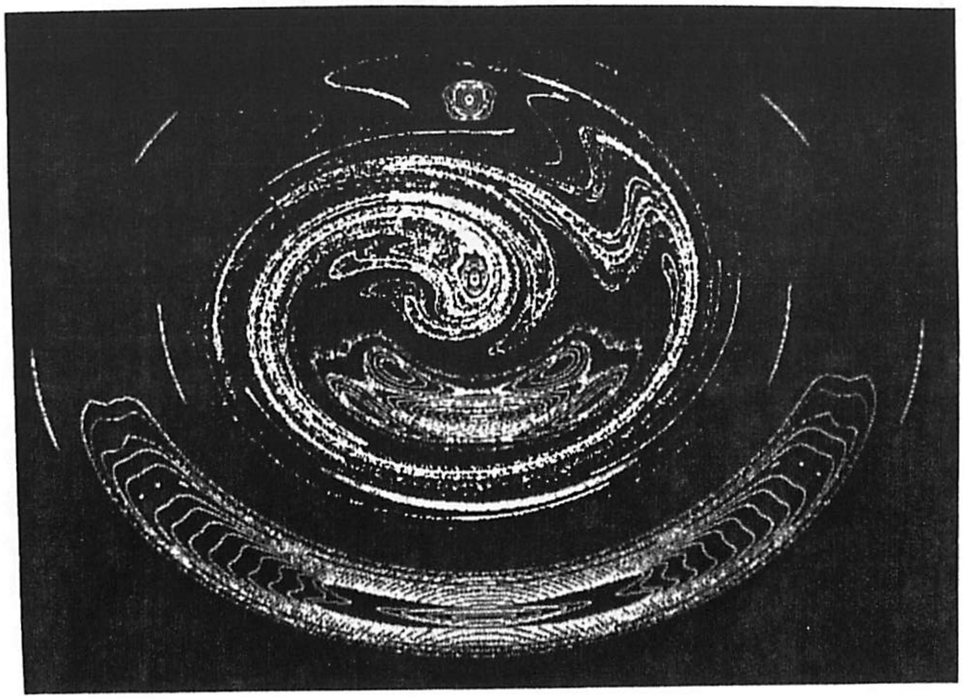


Fig. 10

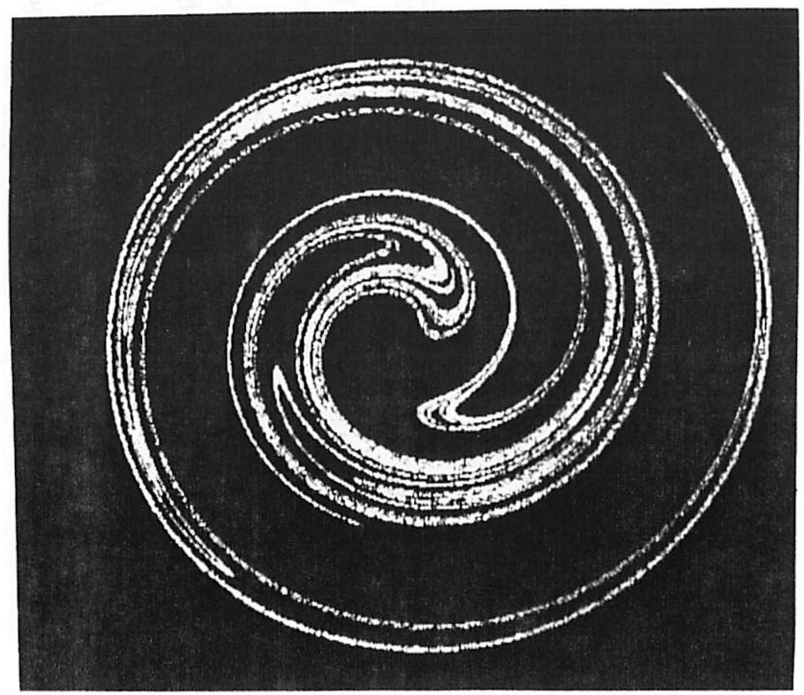


Fig. 11

unstable manifold "wrapping" around the elliptic regions. We used 16 iterations of FT.

4 Twist with Dissipation

As an alternative to the equation of subsection 1.4, we may put dissipation into the transformation FT directly without knowing the ODE for which FT is the Poincare map. The damped FT map is very useful in generating strange attractors. The only modification needed is for T. The damped twist in this case is defined by

$$\mathrm{T}(\mathbf{z}) \quad = \quad \exp(-\alpha r)\mathbf{A}(\pi r/\omega)(\mathbf{z}-\mathbf{a}) \quad + \quad \mathbf{a}$$

We define the matrix $\mathbf{A}(\pi r/\omega)$ as before:

$$\mathbf{A}(\pi r/\omega) = \begin{pmatrix} \cos(\pi r/\omega) & -\sin(\pi r/\omega) \\ \\ \sin(\pi r/\omega) & \cos(\pi r/\omega) \end{pmatrix}$$

Also as before, we have $r = \|\mathbf{z}-\mathbf{a}\|$. The scalar α is taken to be positive, but small. We may recompute the fixed point equations as before and find

$$\mathbf{A}(\pi r/\omega)(\mathbf{z}-\mathbf{a}) = -\exp(\alpha r)(\mathbf{z}+\mathbf{a}).$$

As before, this matrix equation leads to three scalar equations:

$$\|\mathbf{z}-\mathbf{a}\| = \exp(\alpha r)\|\mathbf{z}+\mathbf{a}\|$$

$$\cos(\pi r/\omega) = \exp(\alpha r)(1-2(y/r)^2), \text{ and } \sin(\pi r/\omega) = \exp(\alpha r)2ay/r^2$$

These equations show that no matter how small α is, there exist only a finite number of fixed points, (unlike the undamped case) and that all fixed points lie on the LHS of the vertical axis. Further, for relatively small damping factors such as .05 there is only one fixed point. In short, it appears that damping drastically reduces the domain of initial conditions that can lead to chaos.

In Fig. 11 we show a strange attractor generated by the damped twist. We use 100000 iterations with $\alpha = 0.1$ and $\lambda = 3$ and $a = 1$. The initial conditions are $(0, 1)$.

It is possible to obtain a strange attractor for the equation of Sec.1.4. Take $\omega = \pi$, $a = 30$, $\alpha = .05$. However, a general treatment of these Poincaré maps will be presented in a later paper.

5 Factorization of the Duffing Equation with Sinusoidal Forcing

We illustrate our theory here with the undamped Duffing equation with the forcing term $a\sin(t)$. The modifications needed to obtain a general theory including damping are obvious.

Consider the equation

$$\ddot{y} + y^3 = a\sin(t)$$

We obtain a uniform approximation of $\sin(t)$ by a simple function:

$$\sin(t) \cong \sum_{i=1}^{2n} a_i \chi_{[\pi i/n, \pi(i+1)/n]}(t)$$

Where $\chi_{[a,b]}$ is the characteristic function for the interval $[a, b]$:

$$\chi_{[a,b]}(t) = 1, \text{ for } t \in [a, b]$$

and 0 otherwise.

We consider the n differential equations

$$\ddot{y} + y^3 = a_i \operatorname{sgn}(\sin(t)), t \in [\pi i/n, \pi(i+1)/n]$$

We use the output of the i^{th} equation as the input of the $(i+1)^{st}$ with the output of the n^{th} being used as the initial condition of the first equation. As before we need only half of these equations. If we define T_i as the transformation which maps a point of the plane to the solution of the i^{th} equation, then the solution of the simple forced equation is factored as $F \circ T_1 \circ T_2 \circ \cdots \circ T_n \circ F \circ T_n \circ T_{n-1} \circ \cdots \circ T_1$. The square-wave factorization is a special case of this where $n = 1$.

In order to simplify notation we need a symbol to represent the composition of the T_i. In particular, we need to indicate the limit of this composition as $n \to \infty$ since this limit is the solution of the original ODE at the period of the forcing function, the Poincare map. Also, the ODE over any time interval may be factored into a finite composition such as this. If we take the limit of this composition, it is the solution of the ODE. This composition represents for the non-linear ODE an analogue of an integration process, where composition replaces convolution and addition. In the case of linear ODE's this process reduces to the convolution because the general solution is a sum of the homogeneous solution and the particular solution.

We see that the solution is the composition of infinitessimal twists. Note that most of the lemmas proved for the twist carry over to this particular composition of infinitessimal twists.

For lack of a better representation we will define the limit of such a process by the formula:

$$\Pi T dt = \lim_{n \to \infty} \Pi_{k=1}^{n} \circ T_k$$

Acknowledgements

We thank Dr.James Ellenbogen(MITRE Corporation) for the many useful conversations, suggestions, and computer programs he developed that were instrumental in carrying out this research.

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Appendix

A

A.1 Definitions

Define the matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that as a function of u, the rotation matrix $\mathbf{A}$ satisfies the first order ODE,

$$\mathbf{A}'(u) = \mathbf{B}\mathbf{A}(u)$$

We will use the abbreviations, $r_x = \partial r/\partial x$ and $r_y = \partial r/\partial y$

A.2 Fixed Points of FT

At a fixed point $\mathrm{FT}(\mathbf{z}) = \mathbf{z}$, so we have

$$\mathrm{T}(\mathbf{z}) = -\mathbf{z}.$$

Substituting for the definition of T we get an equation for $\mathbf{A}(\pi r/\omega)$:

$$\mathbf{A}(\pi r/\omega)(\mathbf{z} - \mathbf{a}) = -(\mathbf{z} + \mathbf{a}) \tag{5}$$

From which we conclude

$$\|\mathbf{z} - \mathbf{a}\| = \|\mathbf{z} + \mathbf{a}\|$$

$$\cos(\pi r/\omega) = 1 - 2(y/r)^2 = (a/r)^2 - (y/r)^2 = 2(a/r)^2 - 1$$

and,

$$\sin(\pi r/\omega) = 2(ay/r^2)$$

From the first equation we conclude that $x = 0$. The last two equations are obtained by multiplying out Eq. 5 and solving for cosine and sine explicitly.

A.3 Derivative of FT

In this subsection we compute the derivative of FT, i.e., the Jacobian matrix of FT with respect to (x, y). Let $\mu = \pi/\omega$. Note that

$$\mathrm{D(F)} = -\mathbf{I}$$

and

$$\mathrm{D(FT)} = -\mathbf{I}\left(\frac{\partial \mathbf{A}(\mu r)(\mathbf{z}-\mathbf{a})}{\partial x}, \frac{\partial \mathbf{A}(\mu r)(\mathbf{z}-\mathbf{a})}{\partial y}\right)$$

This is equal to

$$-\mu(r_x\mathbf{A}'(\mu r)(\mathbf{z}-\mathbf{a}), r_y\mathbf{A}'(\mu r)(\mathbf{z}-\mathbf{a})) - \mathbf{A}(\mu r)$$

Since $\mathbf{A}'(u) = \mathbf{BA}(u)$, we have

$$\mathrm{D(FT)} = -\mu(r_x\mathbf{BA}(\mu r)(\mathbf{z}-\mathbf{a}), r_y\mathbf{BA}(\mu r)(\mathbf{z}-\mathbf{a})) - \mathbf{A}(\mu r)$$

This determinant may be evaluated using the following formula for the computation of the determinant of the sum of two matrices.

Let $A = (A_1, A_2)$ and let $B = (B_1, B_2)$, where A_i, B_i are 2 dimensional column vectors. Then

$$det(A + B) = det(A) + det(B) + det(A_1, B_2) + det(B_1, A_2)$$

A.4 Derivative of FT at a Fixed Point

Using the fixed point equations in the equation for D(FT) we have, at a fixed point:

$$\mathrm{D(FT)} = \mu(r_x\mathbf{B}(\mathbf{z}+\mathbf{a}), r_y\mathbf{B}(\mathbf{z}+\mathbf{a})) - \mathbf{A}(\mu r)$$

Therefore at a fixed point, the derivative of FT is given by the matrix

$$\mathrm{D(FT)} = \begin{pmatrix} \mu a y/r - \cos(\mu r) & \sin(\mu r) - \mu y^2/r \\ -(\mu a^2/r + \sin(\mu r)) & \mu a y/r - \cos(\mu r) \end{pmatrix}$$

Using these formulae we compute the trace of D(FT) is equal to

$$\text{trace } (\mathrm{D(FT)}) = 2(1 + \mu a y/r) - 4(a/r)^2$$

For $y \geq 2/\mu$, trace is > 2 since

$$2(1 + \mu a \frac{y}{r}) - 4(\frac{a}{r})^2 \geq$$

$$2(1 + 2\frac{a}{r}) - 4(\frac{a}{r})^2 > 2.$$

Using these same identities, we compute the determinant of D(FT):

$$\det(\mathrm{D(FT)} = 1 - \mu(2ay/r\cos(\mu r) + r\sin(\mu r)((a/r)^2 - (y/r)^2))$$

$$= 1 - \mu(r\sin(\mu r)\cos(\mu r) + r\cos(\mu r)\sin(\mu r)) = 1.$$

A more elaborate computation shows that the determinant of the D(FT) is 1 everywhere, but we have not used this fact in our proof.

A.5 Eigenvectors and Eigenvalues of D(FT)

The largest eigenvalue of D(FT) is given by the formula:

$$\lambda = \mathrm{tr}/2 + \sqrt{(\mathrm{tr}/2)^2 - 1}$$

where tr is the trace of D(FT). The slope of the corresponding real eigenvector is given by

$$\text{slope} = (\lambda - \text{tr}/2)/(\sin(\mu r) - \mu y^2/r)$$

and this is equal to

$$\sqrt{(\text{tr}/2)^2 - 1}/(\sin(\mu r) - \mu y^2/r)$$

By appendix A.2 $\sin(\mu r) = 2ay/r^2$ so that

$$\sin(\mu r) - y^2/r = 2ay/r^2 - \mu y^2/r = -(y/a)((\mu y a/r) - 2(a/r)^2)$$

we conclude from this that the slope is given by

$$\text{slope} = -\frac{a}{y}\sqrt{\frac{(\text{tr}/2) + 1}{(\text{tr}/2) - 1}}$$

A.6 Proof of Homoclinic Condition

We define $\mathbf{z}_1$ as the intersection of the curves $\mathbf{C}_s$ and $\mathbf{D}_r$ below the origin, and $\mathbf{z}_2$ as the intersection of the curve $\mathbf{C}_s$ and the positive y-axis. The condition of theorem 1 can be analyzed by considering the difference in the two cords given by

$$\|\mathbf{z}_2 - \mathbf{z}_0\|^2 - \|\mathbf{z}_1 - \mathbf{z}_0\|^2)/2 = (x_0^2 + x_0(a - c) + y_0(|y_1| + y_2) - ac)$$

Where $\mathbf{z}_0 = (x_0, y_0)$, $\mathbf{z}_1 = (x_1, y_1)$, and $\mathbf{z}_2 = (x_2, y_2)$.

We observe that the intersection in the first quadrant of the D_r and the curve $\mathbf{C}_{r_0}$ is given by $c = (r^2 - r_0^2)/4a$. Also, for $r_0 = r - (r \bmod(2\omega))$ we may write $r = r_0 + \rho\omega$, if $r_0 > a$, where $0 < \rho < 1$. These two facts motivate the following definitions:

$$\text{K} = \frac{4n\rho + \rho^2}{16n^2}$$

$$a^2 = q r_0^2 \text{K}$$

$0 \leq q \leq \bar{\text{K}}$, where $\bar{\text{K}} = 1/\text{K}$ This gives us variables which are independent of frequency.

From these two definitions we may conclude that $qc = a, ac = a^2/q$. To guarantee intersections we must have $r_0^2-(c-a)^2 \geq 0$. Given this we parameterize x in terms of c by the formula $x = sc$, where $0 \leq s \leq 1$. Then

$y_0^2 = \alpha(q\bar{\mathrm{K}} - (s-q)^2)$,

$y_1^2 = \alpha(q\bar{\mathrm{K}} - (s-q)^2 + 2(s+q) - 1))$, where $\alpha = \mathrm{K}r_0^2/q$

In these new parameters, and since $y_2 \geq y_0$, half the difference in the two cords is greater than or equal to

$$\alpha(s-1)(s+q) + y_0^2 + y_0|y_1| \tag{6}$$

First consider the estimate of Eq. 6 given by $\alpha((3q-1)s + q(\bar{\mathrm{K}} - q - 1))$. If $q \geq 1/3$ and $q \leq \bar{\mathrm{K}} - 1$, we are done. If $q < 1/3$, then we can combine terms to get this estimate to be $q\bar{\mathrm{K}} - (q-1)^2$, which is positive since we assume that an intersection actually exist. Thus, for all $0 \leq q \leq \bar{\mathrm{K}} - 1$, the result follows.

We now consider $q > \bar{\mathrm{K}} - 1$. In this case we have $|y_1| \geq y_0$, since $s + q \geq 0.5$ (The smallest value of $\bar{\mathrm{K}}$ is 3.2).

We add to our estimate the term y_0^2 to get the estimate

$$\alpha((5q-1)s - s^2 + q(2\bar{\mathrm{K}} - 1 - 2q))$$

Since $q \geq \bar{\mathrm{K}} - 1$, we know that $(5q-1)s - s^2 \geq 0$. Thus, if $q \leq \bar{\mathrm{K}} - .5$, we are done.

The final improvement in this estimate comes from requiring that $y_0(y_0 + |y_1|) \geq ac$ or $y_0|y_1| \geq ac - y_0^2$. Squaring both sides and simplifying we get the condition $4ac \geq c^2 + (ac/y_0)^2$. Substituting our definitions and simplifying we get $4 \geq 1/q + (\mathrm{K}/(1-q\mathrm{K}))$ which is equivalent to the quadratic equation for q:

$$4\mathrm{K}q^2 - 4q + 1 \leq 0$$

From this we conclude $q \leq \bar{\mathrm{K}}(1 + \sqrt{1-\mathrm{K}})/2$.

This is the best estimate for chaos we have proven so far.

We can show by example that this is the best we can do without considering the value of s.

If $s > 0.13$, and $q\mathrm{K} > .5(1+\sqrt{1-\mathrm{K}})$, then $x^2+(a-c)x-ac+2y^2 \geq 0$. This follows from the equation

$$\sqrt{((2q-s)s + 2(s+q) - 1)(2q-s)s} \geq \bar{\mathrm{K}} + s(1 - 3\bar{\mathrm{K}})$$

Where we take $\bar{\mathrm{K}} \geq q \geq .9\bar{\mathrm{K}}$ It is possible to show by example that this is as far as we can go with this estimate (take $\rho = 0.99, n = 3, s = 0.1, q = \bar{\mathrm{K}} - 0.000001$).

We may force a better lower bound on s by requiring that s be large enough that both T and T^{-1} rotate the point $(-x_0, -y_0)$ into the RHS of the vertical axis. But we have not done this yet.

To get the complete result which would prove conjecture 3 must use information about the transformation T. In particular, we must satisfy the two equations: $\cos(r) = 2(a/r)^2 - 1$, and $\sin(r) = 2ay/r^2$, as well as the condition that gaurantees the trace is greater than 2.

These equations should eliminate cases where our lower bound on s fail.

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