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**STABILITY AND DYNAMICS OF DELAY-TYPE  
AND NONLINEAR CELLULAR NEURAL  
NETWORKS**

by

T. Roska, C. W. Wu, M. Balsi, and L. O. Chua

Memorandum No. UCB/ERL M91/110

10 December 1991

# 1 Introduction

Cellular Neural Networks (CNN) represent a new paradigm for nonlinear analog signal processing [1, 2]. Its applications for various practical problems have been demonstrated [3] and the first fully-tested working chip [4] with a 0.3 tera XPS ( $10^{12}$  analog operations per second) capacity reveals its enormous computing power.

The specific CNN functionality is defined by the analog cloning template which is a geometric and analog code of the weights of local interactions of each cell (uniform analog processing unit).

Nonlinear and delay-type CNNs were introduced recently [5]. The delay-type templates proved to be essential in motion related CNN-applications [6]. The analog model of the triade synapse arrangement (TSA) [7] can also be represented by a very simple CNN template [8]. Complex motion related biological models [9, 10] also contain delays. In a more general view of neurobiological models [11] nonlinear and delay-type synapse models seem inevitable.

The introduction of delay in nonlinear CNN dynamics poses the complex question of stability. Following the first proposition in [5], in this paper a set of stability results are presented with additional properties of the dynamics.

The main results of this paper are as follows:

- i) The existence and uniqueness of solutions as well as the conditions of isolated equilibria are stated and proved.
- ii) The range of dynamics is given in terms of cloning template values.
- iii) If  $A$  and  $A^\tau$  are positive templates and the central (self feedback) term of the sum of the feedback and delayed-feedback templates ( $A + A^\tau$ ) is greater than  $1/R_x$ , then the stable outputs are  $\pm 1$ .
- iv) Several stability conditions are presented. The positive cell-linking property [12] and the equivalent sign transformations [13] proved to be important sufficient conditions when applied to  $A$ ,  $A^\tau$  and  $A + A^\tau$ .
- v) Stability conditions have been found for nonlinear and delay-type templates as well.

The key mathematical results used are in [14, 15].

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TITLE PAGE

# Stability and Dynamics of Delay-type and Nonlinear Cellular Neural Networks.

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## Abstract

Delay-type cellular neural networks, introduced recently, proved to be important in different application areas including motion detection. In this paper several stability results and additional properties of the delay-type CNN dynamics are proved. As a typical example, it is proved that if the feedback and delayed feedback template  $A$  and  $A^r$  are positive and the sum  $(A + A^r)$  is cell-linking then the CNN is stable. Moreover, if in addition the central element (self-feedback) of this sum is greater than  $1/R_x$  then all of the stable outputs will be  $\pm 1$ . Results covering nonlinear and delay-type templates are also given.

# 1 Introduction

Cellular Neural Networks (CNN) represent a new paradigm for nonlinear analog signal processing [1, 2]. Its applications for various practical problems have been demonstrated [3] and the first fully-tested working chip [4] with a 0.3 tera XPS ( $10^{12}$  analog operations per second) capacity reveals its enormous computing power.

The specific CNN functionality is defined by the analog cloning template which is a geometric and analog code of the weights of local interactions of each cell (uniform analog processing unit).

Nonlinear and delay-type CNNs were introduced recently [5]. The delay-type templates proved to be essential in motion related CNN-applications [6]. The analog model of the triade synapse arrangement (TSA) [7] can also be represented by a very simple CNN template [8]. Complex motion related biological models [9, 10] also contain delays. In a more general view of neurobiological models [11] nonlinear and delay-type synapse models seem inevitable.

The introduction of delay in nonlinear CNN dynamics poses the complex question of stability. Following the first proposition in [5], in this paper a set of stability results are presented with additional properties of the dynamics.

The main results of this paper are as follows:

- i) The existence and uniqueness of solutions as well as the conditions of isolated equilibria are stated and proved.
- ii) The range of dynamics is given in terms of cloning template values.
- iii) If  $A$  and  $A^T$  are positive templates and the central (self feedback) term of the sum of the feedback and delayed-feedback templates ( $A + A^T$ ) is greater than  $1/R_x$ , then the stable outputs are  $\pm 1$ .
- iv) Several stability conditions are presented. The positive cell-linking property [12] and the equivalent sign transformations [13] proved to be important sufficient conditions when applied to  $A$ ,  $A^T$  and  $A + A^T$ .
- v) Stability conditions have been found for nonlinear and delay-type templates as well.

The key mathematical results used are in [14, 15].

In section 2, the general framework, the existence and uniqueness of solutions, and the determination of the range of dynamics are presented. Section 3 contains the key stability results. In section 4 two examples are presented to illustrate the relevance of the results. In section 5 the results are extended to nonlinear templates.

## 2 General framework and basic properties

An  $M \times N$  CNN is governed by the following set of differential equations [1]:

$$C_x \frac{dv_{x_{ij}}(t)}{dt} = -\frac{1}{R_x} v_{x_{ij}}(t) + \sum_{C_{kl} \in N_r(ij)} A(i, j; k, l) v_{y_{kl}}(t) + \sum_{C_{kl} \in N_r(ij)} B(i, j; k, l) v_{u_{kl}}(t) + I \quad (1)$$

$$N_r(ij) = \{C_{kl} : \max(|k-i|, |l-j|) \leq r\} \quad \text{for some positive integer } r \quad (2)$$

where  $v_{x_{ij}}$ ,  $v_{u_{ij}}$  and  $v_{y_{ij}}$  denote the state voltage, input voltage and output voltage of a cell, respectively. We assume that the input is continuous and has magnitude less than 1. We also assume that the A-template is space-invariant; i.e., for all  $i, j, k, l, m$  and  $n$  such that  $1 \leq i, k, i+m, k+m \leq M$  and  $1 \leq j, l, j+n, l+n \leq N$ ,  $A(i, j; k, l) = A(i+m, j+n; k+m, l+n)$ .  $v_{y_{ij}}(t) = \hat{f}(v_{x_{ij}}(t))$ ,  $\hat{f}(x) \approx \frac{1}{2} [|x+1| - |x-1|]$ .<sup>1</sup>

Symmetric, non-symmetric positive cell-linking and other types of A-templates have been shown to have stable dynamics [1, 12, 13].

An  $M \times N$  delay-type CNN is described by the state equations [5]:

$$C_x \frac{dv_{x_{ij}}(t)}{dt} = -\frac{1}{R_x} v_{x_{ij}}(t) + \sum_{k,l} A(i, j; k, l) v_{y_{kl}}(t) + \sum_{k,l} A^\tau(i, j; k, l) v_{y_{kl}}(t - \tau_{i,j;k,l}^A) + \sum_{k,l} B(i, j; k, l) v_{u_{kl}}(t) + \sum_{k,l} B^\tau(i, j; k, l) v_{u_{kl}}(t - \tau_{i,j;k,l}^B) + I \quad (3)$$

and an  $M \times N$  delay-type CNN with nonlinear templates is described by the state equations

---

<sup>1</sup>We use the symbol  $\approx$  since in the following proofs, we will assume  $\hat{f}(x_i)$  to be a smooth ( $C^1$ ), strictly increasing and odd approximation of the piecewise linear function, bounded between -1 and 1.

$$\begin{aligned}
C_x \frac{dv_{x_{ij}}(t)}{dt} &= -\frac{1}{R_x} v_{x_{ij}}(t) + \sum_{k,l} \hat{A}(i,j;k,l)(v_{y_{kl}}(t), v_{y_{ij}}(t)) \\
&+ \sum_{k,l} A^\tau(i,j;k,l)v_{y_{kl}}(t - \tau_{ij;k,l}^A) + \sum_{k,l} \hat{B}(i,j;k,l)(v_{u_{kl}}(t), v_{u_{ij}}(t)) \\
&+ \sum_{k,l} B^\tau(i,j;k,l)v_{u_{kl}}(t - \tau_{ij;k,l}^B) + I
\end{aligned} \tag{4}$$

where  $\tau_{ij;kl}^A, \tau_{ij;kl}^B \geq 0$  for all  $i, j, k, l$  and generally  $\tau_{ij;kl}^A, \tau_{ij;kl}^B$  are space-invariant as well. Without loss of generality, we will assume that  $C_x = 1, R_x = 1$ .

We relabel the state variables  $v_{x_{ij}}$  into a vector  $\tilde{x}$  of size  $n = MN$ . Similarly, the input and output variables  $v_{u_{ij}}$  and  $v_{y_{ij}}$  are relabeled into  $\tilde{u}$  and  $\tilde{y}$  using the same labeling order. The invertible ordering will be called  $\sigma$ , i.e.  $\tilde{x}_{\sigma(i,j)} = v_{x_{ij}}$ . Furthermore,  $\tau_{ij;kl}^A$  is ordered into a matrix  $\tilde{\tau}^A$  such that  $\tilde{\tau}_{\sigma(i,j),\sigma(k,l)}^A = \tau_{ij;k,l}^A$ . The same ordering is used on  $\tau_{ij;kl}^B, A, A^\tau$  and  $\hat{A}$  to obtain  $\tilde{\tau}^B, \tilde{A}, \tilde{A}^\tau$  and  $\tilde{A}_n$  respectively. Without loss of generality, we assume  $\tilde{\tau}_{ij}^A = 0$  if  $\tilde{A}_{ij}^\tau = 0$ .

Define

$$\tilde{\tau}_j = \max_i \tilde{\tau}_{ij}^A \tag{5}$$

$\tilde{\tau}_j$  is the maximum delay of all the interactions from cell  $j$  onto other cells.

$$C_{\tilde{\tau}} = \prod_{i=1}^n C([- \tilde{\tau}_i, 0], \mathbb{R}) \tag{6}$$

where  $C(H, \mathbb{R})$  is the set of continuous functions into  $\mathbb{R}$  defined on the interval  $H$  and  $C_{\tilde{\tau}}$  can be thought of as continuous functions defined into  $\mathbb{R}^n$  where the interval of definition is different on each component. We also define  $C_{\tilde{\tau}}^+$  as the subset of  $C_{\tilde{\tau}}$  mapping into  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$ . Throughout this paper, we will use the  $\infty$ -norm for functions in  $C_{\tilde{\tau}}$ .

The initial conditions for the delay-type CNN is given by:

$$v_{x_{ij}}(t) = v_{0_{ij}}(t) \quad t \in [-\tilde{\tau}_{\sigma(i,j)}, 0] \tag{7}$$

We will assume that  $v_{0_{ij}}(t)$  is a continuous function ( $v_0(t) \in C_{\tilde{\tau}}$ ).

Consider first the delay-type CNN with linear template elements. The state equations (3) assumes the form of a system of functional differential equations (FDE):



$$\begin{aligned}\dot{\tilde{x}} &= F(t, \tilde{x}_t) = -\tilde{x}(t) + \tilde{A}(f_1(\tilde{x}(t))) + \tilde{A}^\tau f_2(\tilde{x}(t - \tau)) \\ &\quad + \tilde{B}(\tilde{u}(t)) + \tilde{B}^\tau(\tilde{u}(t - \tau)) + \tilde{I}\end{aligned}\tag{8}$$

where  $f_1(\tilde{x}(t))$  is defined as:

$$(f_1(\tilde{x}))_i = \hat{f}(\tilde{x}_i)\tag{9}$$

and  $\tilde{A}^\tau f_2(\tilde{x}(t - \tau))$  is defined as:

$$\tilde{A}^\tau f_2(\tilde{x}(t - \tau))_i = \sum_{k=1}^n \tilde{A}_{ik}^\tau \cdot \hat{f}(\tilde{x}_k(t - \tilde{\tau}_{ik}^A))\tag{10}$$

$\tilde{B}^\tau \tilde{u}(t - \tau)$  is defined as:

$$\tilde{B}^\tau \tilde{u}(t - \tau)_i = \sum_{k=1}^n \tilde{B}_{ik}^\tau \cdot \tilde{u}_k(t - \tilde{\tau}_{ik}^B)\tag{11}$$

and  $\tilde{x}_t \in C_{\tilde{\tau}}$  is defined as

$$\tilde{x}_t(\theta)_i = \tilde{x}(t + \theta)_i \quad \theta \in [-\tilde{\tau}_i, 0]\tag{12}$$

**Proposition 2.1** *Given the initial condition*

$$\tilde{x}_0(t) = \phi(t), \quad \phi(t) \in C_{\tilde{\tau}}\tag{13}$$

*then the delay-type CNN has a unique continuous solution for  $t \in [0, \infty)$ .*

**Proof:** we need to show that (8) has a unique solution. First we show that  $F(t, \tilde{x}_t)$  is globally Lipschitzian, i.e.

$$|F(t, \psi) - F(t, \phi)| \leq L |\psi - \phi| \quad \text{for all } \psi, \phi \in C_{\tilde{\tau}} \text{ and all } t\tag{14}$$

for some constant  $L$ . If we define  $L_c = \left( \sum_{i,j} \sum_{k,l} |A^\tau(i, j; k, l)| + \sum_{i,j} \sum_{k,l} |A(i, j; k, l)| \right) + 1$ , then  $L_c$  qualifies as our Lipschitz constant.

Since the input is continuous,  $F(t, \psi)$  is continuous with respect to  $t$  for all  $\psi$ . The conclusion then follows from [16, page 308–309]. ■

**Proposition 2.2** *If the initial condition are bounded by  $K$ , then all states  $v_{x_{i,j}}$  of a delay-type cellular neural network are bounded for all time in absolute value by the sum:*

$$v_{max} = K + |I| + \max_{i,j} \left[ \sum_{k,l} (|A^T(i,j;k,l)| + |A(i,j;k,l)| + |B^T(i,j;k,l)| + |B(i,j;k,l)|) \right] \quad (15)$$

and the  $\omega$ -limit points of  $v_{x_{i,j}}(t)$  are bounded in absolute value by  $v_{max} - K$ .

Proof: it is sufficient to follow the proof of theorem 1 of [1] to see that also in this case it is possible to recast the equations of the network in the same form of eq. (4a) of [1]:

$$\frac{dv_{x_{ij}}}{dt}(t) = -v_{x_{ij}} + f_{ij}(t) + g_{ij}(t) \quad (16)$$

where  $f_{ij}$  depends only on  $v_{y_{kl}}(t)$  and  $v_{y_{kl}}(t - \tau_{i,j;k,l}^A)$  and  $g_{ij}$  on the inputs and bias current, and for both it is possible to compute an upper bound in the same way as in [1]. ■

### 3 Stability results for linear and delay-type templates

To be able to use the results on cooperative systems, we will assume in the following that the input is constant. This give us an autonomous system  $\dot{x} = F(x_t)$ .

$F(x_t)$  is differentiable in the sense that

$$F(\phi) = F(\psi) + dF(\psi)(\phi - \psi) + o(\|\phi - \psi\|) \quad (17)$$

such that for each  $\psi$ ,  $dF(\psi)$  is a bounded linear operator from  $C_{\bar{\tau}}$  to  $\mathbb{R}^n$ . ( $dF(\psi)$  is called the Fréchet derivative.)  $L(\psi) = dF(\psi)$  has the following representation:

$$L_i(\psi)(\phi) = \sum_j \int_{-\bar{\tau}_{ij}}^0 \phi_j(\Theta) d\theta \eta_{ij}(\Theta, \psi) \quad 1 \leq i \leq n \quad (18)$$

**Definition 3.1 (from [15])** *A system of FDE  $\dot{x} = F(x_t)$  is cooperative in  $C_{\bar{\tau}}$  if for every  $\psi \in C_{\bar{\tau}}$  and for every  $\phi \in C_{\bar{\tau}}^+$  with  $\phi_i(0) = 0$ ,  $dF(\psi)(\phi) \geq 0$ .*

A system of FDE  $\dot{x} = F(x_t)$  is cooperative and irreducible in  $C_{\bar{\tau}}$  if

(i) it is cooperative;

(ii) for every  $\psi \in C_{\bar{\tau}}$  the matrix  $P(0)$  defined by

$$P(0) = \text{col}(dF(\hat{e}_1), dF(\hat{e}_2), \dots, dF(\hat{e}_n)) = (\eta_{ij}(0, \psi)) \quad (19)$$

is irreducible; and

(iii) for every  $j$  for which  $\bar{\tau}_j > 0$ , there exists  $i$  such that for all  $\psi \in C_{\bar{\tau}}$  and all small  $\epsilon > 0$ ,

$$\eta_{ij}(-\bar{\tau}_j + \epsilon, \psi) > 0 \quad (20)$$

We define the saturation region as the region in  $\mathbb{R}^n$  such that all components have magnitude greater than 1. We also define the following region in  $\mathbb{R}^n$ :

$$\text{region 1} = \{x \in \mathbb{R}^n : \hat{f}'(x_i) \approx 1 \text{ for some } i \text{ and if } \hat{f}'(x_j) \not\approx 1 \text{ then } \hat{f}'(x_j) \approx 0\} \quad (21)$$

By choosing an  $\hat{f}$  closely approximating the piecewise linear function, region 1 can be made arbitrarily close to the complement of the saturation region. And the  $\approx$  sign in the above definition can be made arbitrarily close to being  $=$ . We will assume that  $\hat{f}$  is chosen such that the approximations are close enough for our purposes.

**Proposition 3.1** *All equilibria of a CNN in the saturation region are isolated. If the corresponding FDE is cooperative, then all equilibria in the saturation region are stable. If the center element of  $A + A^\tau$  is greater than 1, and the corresponding FDE is cooperative, then all equilibria in region 1 are unstable.*

Proof: an equilibrium point  $e$  in the saturation region means that  $e_i \notin (-1, 1)$  for all  $i$ . Therefore  $Df_1(e) \approx 0$ , so  $DF(e) \approx -E$ , where  $E$  is the identity matrix, is nonsingular. This implies that  $e$  is isolated. At the equilibrium point  $e$ ,

$$P(0)_{ij} = \int_{-\bar{\tau}_j}^0 e^{0\theta} d\eta_{ij}(\theta) = \bar{A}_{ij}^{\tau}(\hat{f}'(e_j)) + \bar{A}_{ij}(\hat{f}'(e_j)) - \delta_{ij} \quad (22)$$

For  $\mathbf{e}$  in the saturation region,  $\hat{f}'(\mathbf{e}_i) \approx 0$ , so  $P(0) \approx -E$ . So all the eigenvalues of  $P(0)$  have negative real parts. If the FDE is also cooperative, then  $\mathbf{e}$  is stable by corollary 3.2 in [15].

To show that  $\mathbf{e}$  in region 1 is unstable, we use a technique similar to that used in [17]. By reordering the states such that the states in the saturation region come first,  $P(0)$  is close to a matrix of the following form:

$$P(0) \approx \begin{pmatrix} -1 & X & X & \cdots \\ 0 & -1 & X & \cdots \\ \vdots & 0 & \ddots & X \\ & \vdots & & P' \end{pmatrix} \quad (23)$$

where  $P'$  is the submatrix corresponding to the cells not in the saturation region. The eigenvalues of the matrix in (23) consist of  $-1$ 's and the eigenvalues of  $P'$ . Because  $\bar{A}_{ii} + \bar{A}_{ii}^\tau > 1$ , the diagonal elements of  $P'$  are positive and therefore  $P'$  has a positive trace which implies that  $P'$  has some eigenvalues with positive real parts. Then  $P(0)$  has eigenvalues with positive real parts, and  $\mathbf{e}$  is unstable (again by corollary 3.2 in [15]).  $\blacksquare$

**Proposition 3.2** *A CNN with linear A-templates and delay-type A-templates such that*

- (i) *A has only nonnegative elements in the off-center locations;*
- (ii)  *$A^\tau$  has only nonnegative elements;*
- (iii)  *$A + A^\tau$  is a cell-linking template;*

*is cooperative and irreducible in  $C_\tau$ .*

**Proof:** condition (i) implies that  $\bar{A}$  is an off-diagonally nonnegative matrix. Condition (ii) implies that  $\bar{A}^\tau$  is a nonnegative matrix. Condition (iii) implies that  $\bar{A} + \bar{A}^\tau$  is an irreducible matrix.

Let  $L(\psi, \phi) = dF(\psi)(\phi)$ . Then

$$\begin{aligned} L_i(\psi, \phi) &= \sum_j \bar{A}_{ij}^\tau \hat{f}'(\psi_j(-\tau_{ij})) \cdot \phi_j(-\tau_{ij}) + \sum_j \bar{A}_{ij} \hat{f}'(\psi_j(0)) \cdot \phi_j(0) - \phi_i(0) \\ &= \sum_j \int_{-\tau_{ij}}^0 \phi_j(\Theta) d\eta_{ij}(\Theta) \end{aligned} \quad (24)$$

therefore

$$\eta_{ij}(\Theta) = \tilde{A}_{ij}^{\tau}(\hat{f}'(\psi_j(-\tilde{\tau}_{ij})))u(\Theta + \tilde{\tau}_{ij}) + [\tilde{A}_{ij}(\hat{f}'(\psi_j(0))) - \delta_{ij}]u(\Theta) \quad (25)$$

Where  $u(t)$  is the unit step function. Given conditions (i) and (ii), the cooperative condition is satisfied.

$P(0) = \eta_{ij}(0)$  is irreducible if and only if  $\tilde{A} + \tilde{A}^{\tau}$  is irreducible. For each  $j$  for which  $\tilde{\tau}_j > 0$ , let  $i$  be such that  $\tilde{\tau}_j = \tilde{\tau}_{ij} > 0$ . Because  $\tilde{\tau}_{ij} > 0$ ,  $\tilde{A}_{ij}^{\tau} \neq 0$ . Then for all sufficiently small positive  $\epsilon$ ,

$$\eta_{ij}(-\tilde{\tau}_j + \epsilon) > 0 \quad (26)$$

■

**Proposition 3.3** *If a CNN with linear A-templates and delay-type A-templates satisfies:*

- (i) *A has only nonnegative elements in the off-center locations;*
- (ii)  *$A^{\tau}$  has only nonnegative elements;*
- (iii)  *$A + A^{\tau}$  is a cell-linking template, and*
- (iv) *all the equilibria are isolated,*

*then the union of the basins of attraction of all stable equilibrium points will be a dense open set in  $C_{\tilde{\tau}}$ . This means that the limit cycles or strange attractors, if they exist, cannot be stable.*

**Proof:** from proposition 3.2, conditions (i), (ii) and (iii) imply cooperativeness and irreducibility in  $C_{\tilde{\tau}}$ . Because of the piecewise-linear structure of the CNN, isolated equilibria implies that they are finite.

$$|F(\phi)| \leq \left( 1 + \sum_i \sum_j (|\tilde{A}_{ij}^{\tau}| + |\tilde{A}_{ij}|) \right) |\phi| + \sum_i \sum_j (|\tilde{B}_{ij}^{\tau}| + |\tilde{B}_{ij}|) |u| + |I| \quad (27)$$

Therefore  $F$  maps bounded subsets of  $C_{\tilde{\tau}}$  into bounded subsets of  $\mathbb{R}^n$ . Since  $x(t)$  is continuous,  $C_{\tilde{\tau}}$  is positively invariant. From proposition 2.2  $\omega(\phi)$ , the  $\omega$ -limit set through  $\phi$ , is bounded<sup>2</sup> for all  $\phi$  in  $C_{\tilde{\tau}}$  and the map  $\phi \rightarrow x_t(\phi)$  maps bounded sets into bounded sets for all  $t \geq 0$ .

---

<sup>2</sup>Note that the bound is independent of  $\phi$ .

By theorem 4.5 in [15], the union of the basins of attraction of all stable equilibrium points is a dense open set in  $C_{\bar{r}}$ . ■

Remark 1: The sign transformations in [13] can be applied in proposition 3.3. However, the *same* transformation type should be used to both  $A$  and  $A^T$ .

## 4 Examples

### Example 1

The feedback and delayed-feedback templates below are not necessarily stable by themselves, since they are not cell-linking:

0	0	0		0	-1.9	0
0	2.1	0		0	0.75	0
1.3	0	0		0	0	1.2
$A$			$A^T$			

However, combining them in a single template we obtain

$A + A^T =$	0	-1.9	0
	0	2.85	0
	1.3	0	1.2

The resulting template  $A + A^T$  is cell linking, and  $A$  and  $A^T$  are both positive after applying transformation 3 in [13]. It also satisfies the other conditions in proposition 3.3, so the CNN is stable.

### Example 2

The delay-type template given below detects all moving objects which have speed in the vicinity of 1 pixel per delay-time [6]. The delay used is uniform and the input is continuous. Three consecutive

input snapshots are shown in figure 1, 2 and 3. The object of the left hand side (object A) has the specified speed. Two consecutive output snapshots, corresponding to the second and the third input snapshot, are shown in figure 4 and 5. Object B on the right hand side (having a higher speed) will disappear on the output.

0	0	0
0	1	0
0	0	0

0	0	0
0	6	0
0	0	0

 $I = -2$ 

*A*
*B*

0.68	0.68	0.68
0.68	0.68	0.68
0.68	0.68	0.68

0	0	0
0	0	0
0	0	0

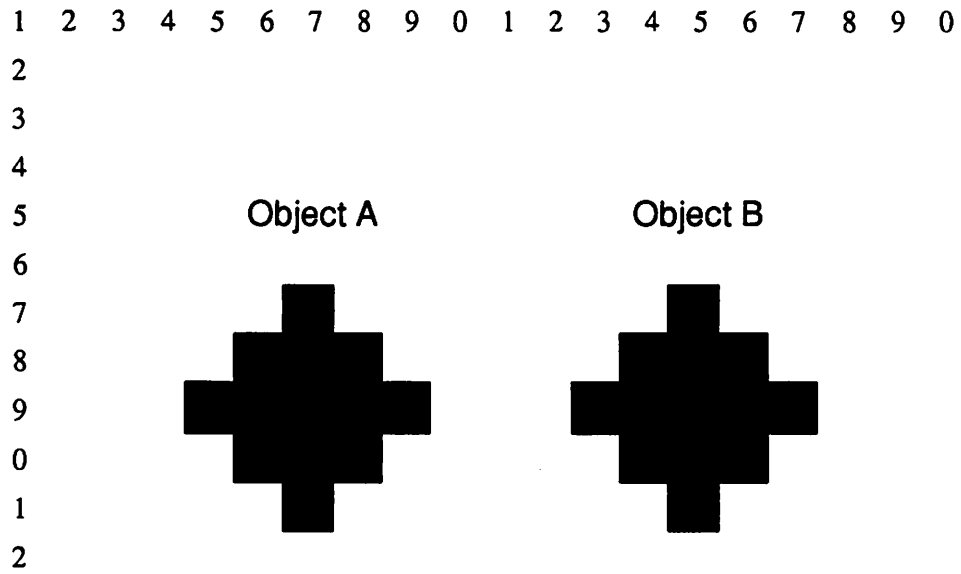
 $\tau \geq 10R_x C_x$ 

*A<sup>τ</sup>*
*B<sup>τ</sup>*

Although the input is not constant, we can still apply the stability criteria if the input is stepwise constant or slowly varying compared to the time constant of the system. So noticing that  $A$  and  $A^\tau$  are positive and

$$A + A^\tau = \begin{array}{|c|c|c|} \hline 0.68 & 0.68 & 0.68 \\ \hline 0.68 & 1.68 & 0.68 \\ \hline 0.68 & 0.68 & 0.68 \\ \hline \end{array}$$

is cell-linking, we can deduce that the CNN is stable. Furthermore, since the center element of  $A + A^\tau$  is greater than 1, the stable equilibria must lie in the saturation region.



Object A

Object B

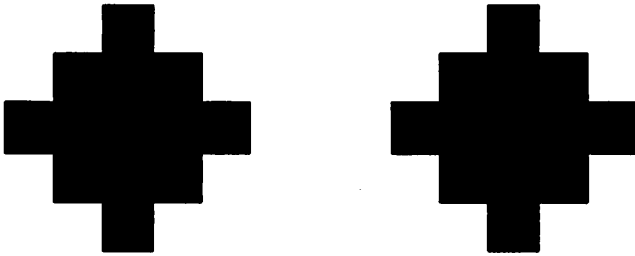


Figure 1: First input snapshot

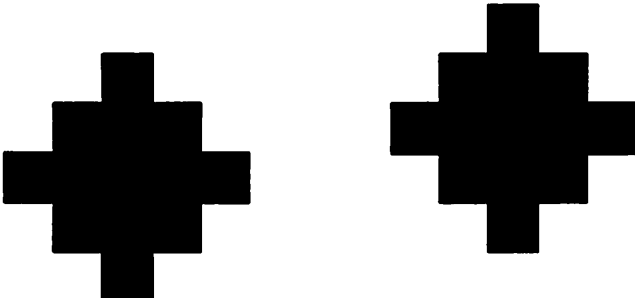
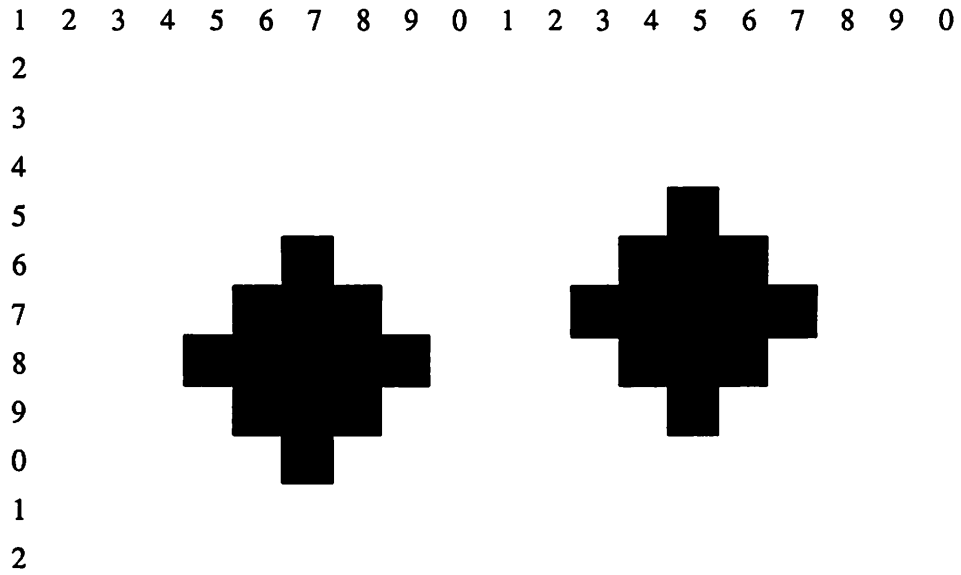


Figure 2: Second input snapshot



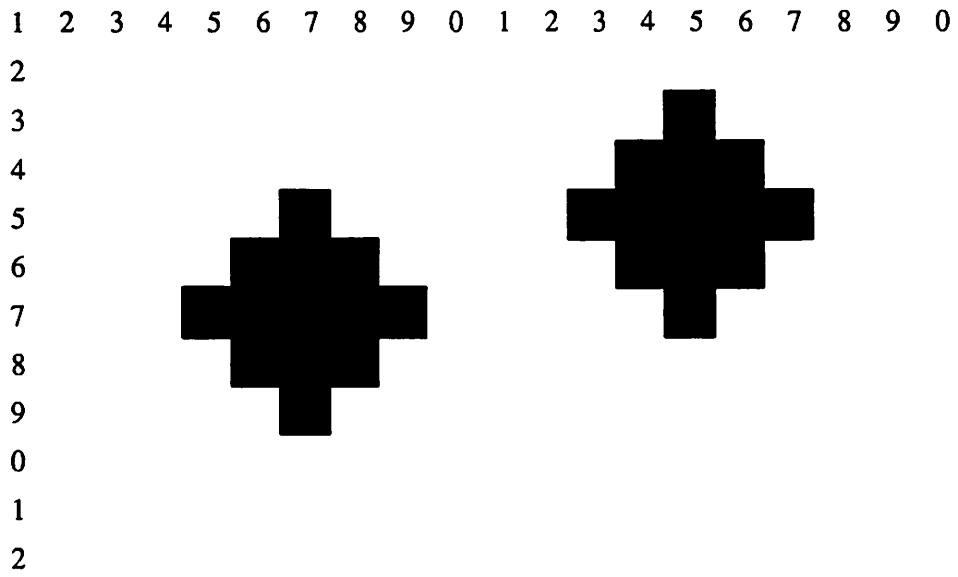


Figure 3: Third input snapshot

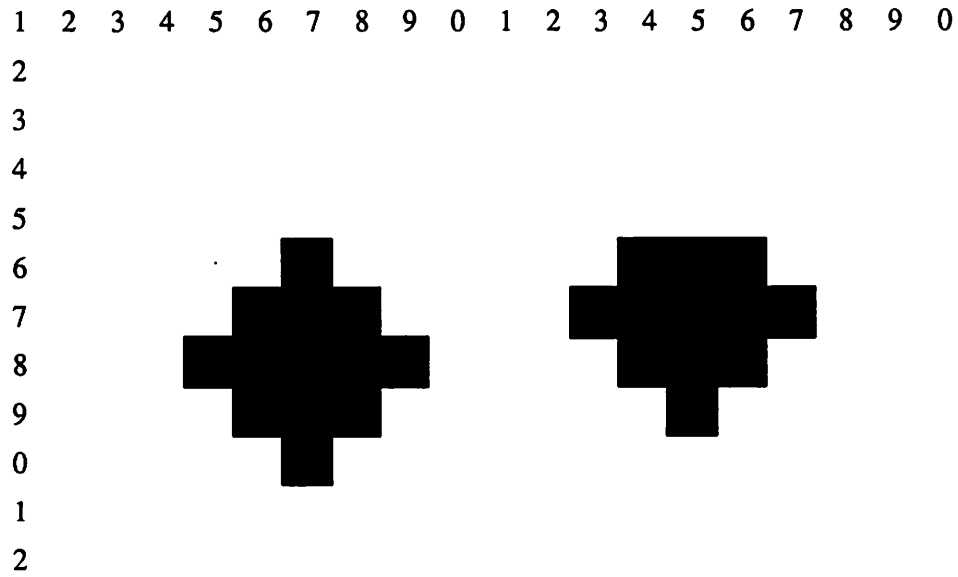


Figure 4: Output snapshot

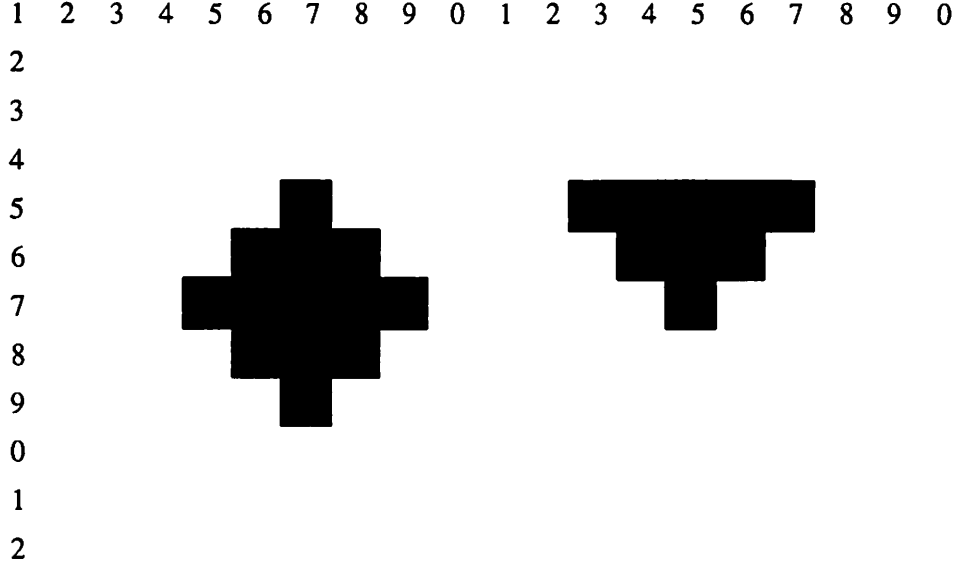


Figure 5: Output snapshot

## 5 Stability results for nonlinear and delay-type templates

After relabeling, the state equations in (4) assume the following form:

$$\begin{aligned} \dot{\tilde{x}} &= F(t, \tilde{x}_t) = -\tilde{x}(t) + \tilde{A}_{nl}(f_1(\tilde{x}(t))) + \tilde{A}^\tau f_2(\tilde{x}(t - \tau)) \\ &\quad + \tilde{B}_{nl}(\tilde{u}(t)) + \tilde{B}^\tau(\tilde{u}(t - \tau)) + \tilde{I} \end{aligned} \quad (28)$$

where  $\tilde{A}^\tau f_2(\tilde{x}(t - \tau))$  is defined as:

$$\tilde{A}^\tau f_2(\tilde{x}(t - \tau))_i = \sum_{k=1}^n (\tilde{A}^\tau)_{ik} \cdot \hat{f}(\tilde{x}_k(t - \tilde{\tau}_{ik}^A)) \quad (29)$$

$\tilde{B}^\tau \tilde{u}(t - \tau)$  is defined as:

$$\tilde{B}^\tau \tilde{u}(t - \tau)_i = \sum_{k=1}^n (\tilde{B}^\tau)_{ik} \cdot \tilde{u}_k(t - \tilde{\tau}_{ik}^B) \quad (30)$$

We will assume that  $\hat{A}$  and  $\hat{B}$  are continuous,

$$\hat{A}_{max}(i, j; k, l) = \sup_{(x,y) \in [-1,1]^2} \hat{A}(i, j; k, l)(x, y) < \infty \quad (31)$$

$$\hat{B}_{max}(i, j; k, l) = \sup_{(x,y) \in [-1,1]^2} \hat{B}(i, j; k, l)(x, y) < \infty \quad (32)$$

**Proposition 5.1** *Given the initial condition:*

$$\tilde{x}_0(t) = \phi(t), \quad \phi(t) \in C_{\tilde{\tau}} \quad (33)$$

*then the nonlinear delay-type CNN has a unique continuous solution for  $t \in [0, \infty)$ .*

Proof:  $F(t, x_t)$  is globally Lipschitzian with Lipschitz constant

$$L_c = \left( \sum_{i,j} \sum_{k,l} |A^\tau(i, j; k, l)| + \sum_{i,j} \sum_{k,l} |\hat{A}_{max}(i, j; k, l)| \right) + 1 \quad (34)$$

Again the conclusion follows from [16]. ■

**Proposition 5.2** *If the initial condition is bounded by  $K$ , then all states  $v_{x_i}$  of a nonlinear delay-type cellular neural network are bounded for all time in absolute value by the sum:*

$$v_{max} = K + |I| + \max_{i,j} \left( \sum_{k,l} (|A^\tau(i, j; k, l)| + |\hat{A}_{max}(i, j; k, l)| + |B^\tau(i, j; k, l)| + |\hat{B}_{max}(i, j; k, l)|) \right) \quad (35)$$

*and the  $\omega$ -limit points of  $v_{x_{ij}}(t)$  are bounded in absolute value by  $v_{max} - K$ .*

Proof: the proof is essentially the same as in proposition 2.2 ■

When the input is constant, we have the following stability results.

**Proposition 5.3** *If the corresponding FDE of a nonlinear and delay-type CNN is cooperative, then all equilibria in the saturation region are stable. If in addition  $(D\tilde{A}_{nl})_{ii} + (\tilde{A}^\tau)_{ii} > 1$  for all  $i$  and for all  $x \in [-1, 1]^n$ , then all equilibria in region 1 is unstable.*

Proof: for  $\mathbf{e}$ , an equilibrium point in region 1,

$$P(0)_{ij} = \int_{-\tilde{\tau}_j}^0 e^{0\theta} d\eta_{ij}(\theta) = (\tilde{A}^\tau)_{ij}(\hat{f}'(\mathbf{e}_j)) + (D\tilde{A}_{nl})_{ij}(f_1(\mathbf{e}))(\hat{f}'(\mathbf{e}_j)) - \delta_{ij} \quad (36)$$

The rest of the proof is similar to that of proposition 3.1. ■

**Proposition 5.4** *A CNN with nonlinear A-templates and delay-type A-templates such that*

- (i)  $\inf_{x \in [-1, 1]^n} D\tilde{A}_{nl}(x)$  is an off-diagonally nonnegative matrix;<sup>3</sup>
- (ii)  $\tilde{A}^\tau$  is a nonnegative matrix,
- (iii)  $D\tilde{A}_{nl}(x) + \tilde{A}^\tau$  is an irreducible matrix for all  $x \in [-1, 1]^n$ ;

is cooperative and irreducible in  $C_{\tilde{\tau}}$ .

**Proof:** let  $L(\psi, \phi) = dF(\psi)(\phi)$ . Then

$$\begin{aligned}
L_i(\psi, \phi) &= \sum_j (\tilde{A}^\tau)_{ij} \hat{f}'(\psi_j(-\tilde{\tau}_{ij})) \cdot \phi_j(-\tilde{\tau}_{ij}) \\
&\quad + \sum_j \left[ D(\tilde{A}_{nl})_{ij}(f_1(\psi(0))) \right] \cdot \hat{f}'(\psi_j(0)) \cdot \phi_j(0) - \phi_i(0) \\
&= \sum_j \int_{-\tilde{\tau}_{ij}}^0 \phi_j(\Theta) d_\theta \eta_{ij}(\Theta, \psi)
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\eta_{ij}(\Theta, \psi) &= (\tilde{A}^\tau)_{ij} (\hat{f}'(\psi_j(-\tilde{\tau}_{ij})) u(\Theta + \tilde{\tau}_{ij})) \\
&\quad + \left[ (D\tilde{A}_{nl})_{ij}(f_1(\psi(0))) \hat{f}'(\psi_j(0)) - \delta_{ij} \right] u(\Theta)
\end{aligned} \tag{38}$$

Given that  $D\tilde{A}_{nl}$  is off-diagonally nonnegative, and  $\tilde{A}^\tau$  is nonnegative, the cooperative condition is satisfied.  $\eta_{ij}(0, \psi)$  is irreducible if and only if  $D\tilde{A}_{nl}(f_1(\psi(0))) + \tilde{A}^\tau$  is irreducible. For each  $j$  for which  $\tilde{\tau}_j > 0$ , let  $i$  be such that  $\tilde{\tau}_j = \tilde{\tau}_{ij} > 0$ . Since  $(\tilde{A}^\tau)_{ij} \neq 0$ , then for all sufficiently small positive  $\epsilon$ ,

$$\eta_{ij}(-\tau_j + \epsilon) > 0 \tag{39}$$

■

**Proposition 5.5** *If a CNN with nonlinear A-templates and delay-type A-templates satisfies:*

- (i)  $\inf_{x \in [-1, 1]^n} D\tilde{A}_{nl}(x)$  is an off-diagonally nonnegative matrix;
- (ii)  $\tilde{A}^\tau$  is a nonnegative matrix,

---

<sup>3</sup>The infimum is taken componentwise.

(iii)  $D\tilde{A}_{nl}(x) + \tilde{A}^\tau$  is an irreducible matrix for all  $x \in [-1, 1]^n$ ;

(iv) the set of equilibria is finite,

then the union of the basins of attraction of all stable equilibrium points will be a dense open set in  $C_{\tilde{\tau}}$ .

Proof: from proposition 5.4, conditions (i), (ii) and (iii) imply cooperativeness and irreducibility in  $C_{\tilde{\tau}}$ .

$$|F(\phi)| \leq \left( 1 + \sum_{i,j} \sum_{k,l} (|A^\tau(i,j,k,l)| + |\hat{A}_{max}(i,j;k,l)|) \right) |\phi| + \sum_{i,j} \sum_{k,l} (|B(i,j;k,l)| + |\hat{B}_{max}(i,j;k,l)|) |u| + |I| \quad (40)$$

Therefore  $F$  maps bounded subsets of  $C_{\tilde{\tau}}$  into bounded subsets of  $R^n$ . Since  $x(t)$  is continuous,  $C_{\tilde{\tau}}$  is positively invariant. From proposition 5.2  $\omega(\phi)$  is bounded for all  $\phi$  in  $C_{\tilde{\tau}}$  and the map  $\phi \rightarrow x_t(\phi)$  maps bounded sets into bounded sets for all  $t \geq 0$ .

By theorem 4.5 in [15], the union of the basins of attraction of all stable equilibrium points is a dense open set in  $C_{\tilde{\tau}}$ . ■

## 6 Conclusions

Many motion related phenomena can be represented and/or modeled by delay-type CNNs. Simple and useful stability conditions have been presented in this paper including the case when the templates are nonlinear and delay-type.

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