SINGULARLY PERTURBED ZERO DYNAMICS OF NONLINEAR SYSTEMS

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SINGULARLY PERTURBED ZERO DYNAMICS OF NONLINEAR SYSTEMS

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Abstract

Stability properties of zero dynamics are among the crucial input-output properties of both linear and nonlinear systems. Unstable, or "non-minimum phase", zero dynamics are a major obstacle to input-output linearization and high gain designs. An analysis of the effects of regular perturbations in system equations on zero dynamics shows that, whenever a perturbation decreases the system's relative degree, it manifests itself as a singular perturbation of zero dynamics. Conditions are given under which the zero dynamics evolve in two timescales characteristic of a standard singular perturbation form that allows a separate analysis of slow and fast parts of the zero dynamics. The slow part is shown to be identical to the zero dynamics of the unperturbed system, while the fast part, represented by the so called *boundary layer system*, describes the effects of perturbations.

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1 Introduction

In the geometric theory of nonlinear feedback systems [1], the notion of a "nonlinear zero" has also emerged as a fundamental concept. It was introduced in [2], where the dynamics of the part which is rendered maximally unobservable after state feedback are considered as a nonlinear analog of the dynamics associated with the zeros of a linear system. Further research [3, 4] led to a precise definition of nonlinear zero dynamics as the internal dynamics arising in a system when its input and its initial states are chosen to constrain the system output to remain zero over some time interval.

In linear systems, many design tasks: tracking, disturbance rejection and noninteracting control; to be performed exactly, require pole-zero cancellation. A technique to solve such tasks in nonlinear systems is input-output linearization, that is the use of state feedback to achieve an exact cancellation of the effects of nonlinearities on system input-output behavior. This technique is a nonlinear analog of the pole-zero cancellation achieved by incorporating into the control law an inverse of the controlled plant. An alternative is to use high-gain feedback and achieve the same cancellation asymptotically, as the gain tends to infinity. A high-gain view of nonlinear zero dynamics was introduced in [6] and related design methods were proposed in [7]. As shown already in [8], systems with high-gain feedback are a class of singularly perturbed systems [9].

The robustness properties of both input-output linearization and high-gain designs crucially depend on what effects modeling imperfections may have on zero dynamics. This problem was begun to be addressed in [10], where it was shown that small regular perturbations in the state equations may, and in most cases will, result in singularly perturbed zero dynamics. The investigation in [10] was motivated by the flight control study for VTOL aircraft in [5], where a small misalignment of the roll jets on the wings caused by the dihedral angle of the wing manifested itself as fast timescale zero dynamics with an unstable equilibrium point, that is a slightly non-minimum phase system. A difficulty with the analysis in [10] was that several scaling steps were to be made to account for the effects of the peaking phenomenon.

We avoid this difficulty here by augmenting the system with pre-scaled chains of integrators which accommodate all the peaking phenomena, leaving the zero dynamics unaltered. This technique also allows for generalizations to multi-input multi-output systems.

2 The Analysis of SISO Systems

Consider a family of SISO nonlinear systems depending on a real-valued parameter ϵ, described by equations of the form:

$$\dot{x} = f(x,\epsilon) + g(x,\epsilon)u \qquad (2.1)$$

$$y = h(x,\epsilon)$$

where $f(x,\epsilon)$, $g(x,\epsilon)$, $h(x,\epsilon)$ are assumed to be smooth functions defined in a neighborhood of $(x^\circ, 0)$ in $\mathbb{R}^n \times \mathbb{R}$. The system corresponding to the value $\epsilon = 0$ of the parameter is called the *unperturbed* system, while any system of the family corresponding to a nonzero value of ϵ is called a *perturbed* system. The state $x = x^\circ$ is an equilibrium point of the unperturbed system, i.e., $f(x^\circ, 0) = 0$ and, without loss of generality, it is assumed that $h(x^\circ, 0) = 0$.

2.1 Two timescales assumption

In what follows, we assume that each system of the family (2.1) has *relative degree* $r(\epsilon)$ at the point x°. Either, $r(\epsilon)$ is constant in a neighborhood of $\epsilon = 0$, or it abruptly changes at $\epsilon = 0$. In the latter case, which is of major interest in the analysis of the zero dynamics, there exist two positive integers, r and d, with $d \neq 0$, such that

$$\begin{array}{ll} r(\epsilon) = r & \text{for all } \epsilon > 0 \text{ in a neighborhood of } \epsilon = 0 \\ r(0) = r + d. & \end{array}$$

Systems in which this situation occurs are said to have a *singularly perturbed zero dynamics*. The integer r is the relative degree of the perturbed systems, and the integer $r + d$ is the relative degree of the unperturbed system. By definition, the change from $r(\epsilon) = r$ to $r(0) = r + d$ implies that:

$$L_g h(x,\epsilon) = \cdots = L_g L_f^{r-2} h(x,\epsilon) = 0 \qquad \text{for all } x \text{ near } x^\circ \text{ and all } \epsilon \qquad (2.2a)$$

$$L_g L_f^{r-1} h(x^\circ, \epsilon) \neq 0 \qquad \text{for all } \epsilon > 0 \tag{2.2b}$$

$$L_g L_f^{r-1} h(x, 0) = \cdots = L_g L_f^{r+d-2} h(x, 0) = 0 \qquad \text{for all } x \text{ near } x^\circ \tag{2.2c}$$

$$L_g L_f^{r+d-1} h(x^\circ, 0) \neq 0. \tag{2.2d}$$

The following developments are based on an assumption about the ϵ-family of systems (2.1). As we shall see later on, this assumption allows the singularly perturbed zero dynamics to possess only two timescales. First of all we assume that the rates with which the terms in (2.2c) tend to zero as $\epsilon \to 0$ are determined by

$$L_g L_f^{r-1} h(x, \epsilon) = \epsilon^m \alpha_0(x, \epsilon) \tag{2.3}$$

where $\alpha_0(x, \epsilon)$ is a smooth function, nonzero at $(x^\circ, 0)$ and m is an integer. Without loss of generality, m can be set equal to d. Otherwise ϵ can be replaced by $\epsilon^{d/m}$. Then, our *two timescales assumption* is that, for all $0 \leq k \leq d$,

$$L_g L_f^{r-1+k} h(x, \epsilon) = \epsilon^{d-k} \alpha_k(x, \epsilon) \tag{2.4}$$

where each $\alpha_k(x, \epsilon)$ is a smooth function in a neighborhood of $(x^\circ, 0)$. Note that for $k = d$ in (2.4), the condition (2.2d) implies $\alpha_d(x^\circ, 0) \neq 0$, while, for $k = 0$ and $m = d$, assumption (2.4) with $\alpha_0(x^\circ, 0) \neq 0$ incorporates (2.3).

2.2 Augmented form of perturbed zero dynamics

We proceed now to identify the zero dynamics of the perturbed system and to show how they are related to the zero dynamics of the unperturbed system. In doing so, we take advantage of the fact that the zero dynamics of a system are not altered by addition of integrators to the input channel (see [1], page 389). Because of this property, we can identify the zero dynamics of any system of the family (2.1) with those of the following family of *augmented* systems:

$$\begin{aligned} \dot{x} &= f(x, \epsilon) + g(x, \epsilon) z_1 \\ \epsilon \dot{z}_1 &= z_2 \\ \epsilon \dot{z}_2 &= z_3 \\ &\cdots \\ \epsilon \dot{z}_d &= v \\ y &= h(x, \epsilon). \end{aligned} \tag{2.5}$$

In order to identify the zero dynamics of (2.5), we set to zero the output y and its derivatives, until we arrive at a relation that can be solved for some $v = v(x,z,\epsilon)$. Then, we set $v = v(x,z,\epsilon)$ in (2.5) and consider the restriction of the systems thus obtained on the submanifold (of the state space) on which the output and its derivatives vanish. We first obtain

$$\begin{gathered} y = h(x,\epsilon) \\ y^{(1)} = L_f h(x,\epsilon) \\ \cdots \\ y^{(r-1)} = L_f^{r-1} h(x,\epsilon) \\ y^{(r)} = L_f^r h(x,\epsilon) + \alpha_0(x,\epsilon)\epsilon^d z_1 \end{gathered}$$

Because of the special structure of (2.5), the $(r+1)$th time derivative of y has the form

$$\begin{aligned} y^{(r+1)} = {} & L_f^{r+1} h(x,\epsilon) + \alpha_1(x,\epsilon)\epsilon^{d-1} z_1 + \alpha_0(x,\epsilon)\epsilon^{d-1} z_2 \\ & +(L_f \alpha_0(x,\epsilon))\epsilon^d z_1 + (L_g \alpha_0(x,\epsilon))\epsilon^d z_1^2. \end{aligned}$$

In a similar way, we obtain:

$$\begin{aligned} y^{(r+k)} &= L_f^{r+k} h(x,\epsilon) + \alpha_k(x,\epsilon)\epsilon^{d-k} z_1 + \cdots + \alpha_0(x,\epsilon)\epsilon^{d-k} z_{k+1} \\ & \quad +\epsilon^{d-k+1} P_k(x,z_1,\cdots,z_k,\epsilon) \end{aligned}$$

where $P_k(x,z_1,\cdots,z_k,\epsilon)$ is a polynomial in $z_1,\cdots,z_k$, whose coefficients are smooth functions of x and ϵ. The last two iterations give:

$$\begin{aligned} y^{(r+d-1)} &= L_f^{r+d-1} h(x,\epsilon) + \alpha_{d-1}(x,\epsilon)\epsilon z_1 + \cdots + \alpha_0(x,\epsilon)\epsilon z_d \\ & \quad +\epsilon^2 P_{d-1}(x,z_1,\cdots,z_{d-1},\epsilon) \\ y^{(r+d)} &= L_f^{r+d} h(x,\epsilon) + \alpha_d(x,\epsilon) z_1 + \cdots + \alpha_1(x,\epsilon) z_d + \alpha_0(x,\epsilon) v \\ & \quad +\epsilon P_d(x,z_1,\cdots,z_d,\epsilon). \end{aligned}$$

Recall that, by assumption, $\alpha_0(x^\circ,0) \neq 0$. Thus, in a neighborhood of $x = x^\circ$, $z_1 = \cdots = z_d = 0$, $\epsilon = 0$, the equation $y^{(r+d)} = 0$ can be solved for v, as a *smooth* function of $x, z_1, \cdots, z_k, \epsilon$. This function is a feedback law of the form:

$$v(x,z,\epsilon) = \sum_{i=1}^{d} a_i(x,\epsilon) z_i + b(x,\epsilon) + \epsilon Q_d(x,z,\epsilon) \tag{2.6}$$

where $z = (z_1,\cdots,z_k)$ and $a_1(x^\circ,0) = -(\alpha_d(x^\circ,0)/\alpha_0(x^\circ,0)) \neq 0$.

Equating to zero y and its first $r+d-1$ derivatives provides a set of $r+d$ constraints delineating the subset of the state space on which the zero dynamics of (2.5) are defined. These constraints are conveniently expressed in the form:

$$0 = H(x,\epsilon) + \epsilon R(x,z,\epsilon) \tag{2.7}$$

where:

$$H(x,\epsilon) = \text{ col } (h(x,\epsilon), L_f h(x,\epsilon), \cdots, L_f^{r+d-1} h(x,\epsilon)) \tag{2.8}$$

and $R(x,z,\epsilon)$ is an $(r+d)$-dimensional vector of smooth functions satisfying $R(x,0,\epsilon) = 0$. Observe now that, at $\epsilon = 0$, the right-hand-side of (2.7) reduces to $H(x,0)$ and that, at $x = x^\circ$, the differentials of the $r+d$ entries of $H(x,0)$ are linearly independent because the system (2.1) has relative degree exactly equal to $r+d$ at $(x,\epsilon) = (x^\circ, 0)$ (see [1], page 148). As a consequence, at $(x,z,\epsilon) = (x^\circ,0,0)$, the Jacobian matrix of (2.7) has rank $r+d$. It follows that, for each ϵ near 0, the set of points (x,z) which satisfy (2.7) near $(x^\circ,0)$ is a smooth $(n-r-d)$-dimensional submanifold. This submanifold, denoted Z^*, is rendered invariant by the feedback law (2.6) and the restriction of the flow of the closed-loop system (2.5)-(2.6) to Z^* describes the zero dynamics of (2.5), i.e., the zero dynamics of the family (2.1).

2.3 Singular perturbation analysis

In order to represent the zero dynamics explicitly, we note that since the differentials of the entries of (2.8) are linearly independent at x° for each small ϵ, the functions:

$$\xi_i(x,\epsilon) = L_f^{i-1} h(x,\epsilon) \qquad 1 \le i \le r+d$$

can be used as a (partial) set of new local coordinates for (2.1). The partial transformation thus defined is smooth in x and ϵ. If $r+d$ is strictly less than n, then the choice of new coordinates must be completed by a set of additional smooth functions $\eta_i(x,\epsilon)$, $1 \le i \le n-r-d$. Since, for all (x,ϵ) near $(x^\circ,0)$, we have:

$$(\text{span } \{g(x,\epsilon)\}) \cap (\text{span } \{dh(x,\epsilon), \cdots, dL_f^{r+d-1} h(x,\epsilon)\})^\perp = \{0\}$$

we can always choose $\eta(x,\epsilon)$ (see [1], page 150) in such a way that $L_g \eta(x,\epsilon) = 0$ for all (x,ϵ) near $(x^\circ,0)$. Without loss of generality, it is possible to choose $\eta(x,\epsilon)$ so that $\eta(x^\circ,0) = 0$. In the new coordinates: $\xi = (\xi_1, \cdots, \xi_{r+d})$ $\eta = (\eta_1, \cdots, \eta_{n-r-d})$ the system (2.1) assumes the form:

$$\dot{\xi} = c(\xi,\eta,\epsilon) + d(\xi,\eta,\epsilon)u$$

$$\dot{\eta} = q(\xi, \eta, \epsilon).$$

Recall that, by definition, the system: $\dot{\eta} = q(0, \eta, 0)$ represents the *zero dynamics of the unperturbed system* (2.1).

In new coordinates, the augmented system (2.5) with feedback control (2.6) assumes the form:

$$\begin{aligned} \dot{\xi} &= c(\xi, \eta, \epsilon) + d(\xi, \eta, \epsilon) z_1 \\ \dot{\eta} &= q(\xi, \eta, \epsilon) \\ \epsilon \dot{z}_1 &= z_2 \\ \epsilon \dot{z}_2 &= z_3 \\ &\cdots \\ \epsilon \dot{z}_d &= \textstyle\sum_{i=1}^{d} \hat{a}_i(\xi, \eta, \epsilon) z_i + \hat{b}(\xi, \eta, \epsilon) + \epsilon \hat{Q}_d(\xi, \eta, z, \epsilon). \end{aligned} \tag{2.9}$$

Note that, in the new coordinates (ξ, η), the constraint (2.7) identifying the zero dynamics manifold becomes:

$$0 = \xi + \epsilon \hat{R}(\xi, \eta, z, \epsilon).$$

Solving this relation for ξ yields:

$$\xi = \epsilon F(\eta, z, \epsilon) \tag{2.10}$$

where $F(\eta, z, \epsilon)$ is a smooth function. We see, then, that in the (ξ, η, z) coordinates, the zero dynamics manifold Z^* of the perturbed system (2.5) can be expressed as the graph of a smooth mapping; in particular, on this manifold the ξ coordinates are of order ϵ.

Now, to make the zero dynamics of (2.5) explicit, it suffices to substitute (2.10) into (2.9) and obtain:

$$\begin{aligned} \dot{\eta} &= q(\epsilon F(\eta, z, \epsilon), \eta, \epsilon) \\ \epsilon \dot{z}_1 &= z_2 \\ \epsilon \dot{z}_2 &= z_3 \\ &\cdots \\ \epsilon \dot{z}_d &= \textstyle\sum_{i=1}^{d} \hat{a}_i(\epsilon F(\eta, z, \epsilon), \eta, \epsilon) z_i + \hat{b}(\epsilon F(\eta, z, \epsilon), \eta, \epsilon) \\ &\quad + \epsilon \hat{Q}_d(\epsilon F(\eta, z, \epsilon), \eta, z, \epsilon). \end{aligned} \tag{2.11}$$

These equations describe *the zero dynamics of the perturbed systems* for each small nonzero ϵ. They appear in a singularly perturbed form:

$$\begin{aligned} \dot{\eta} &= f(\eta, z, \epsilon) \\ \epsilon \dot{z} &= g(\eta, z, \epsilon) \end{aligned} \tag{2.12}$$

The Jacobian matrix of the fast subsystem is

$$\left(\frac{\partial g(\eta, z, 0)}{\partial z}\right) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \hat{a}_1(0,\eta,0) & \cdot & \cdot & \cdots & \hat{a}_d(0,\eta,0) \end{bmatrix} \tag{2.13}$$

It is nonsingular for all η near 0, since $\hat{a}_1(0,0,0)$ is nonzero (recall that $\xi(x^\circ,0) = 0$, $\eta(x^\circ,0) = 0$, and that $\hat{a}_i(\xi,\eta,\epsilon)$ derives from $a_i(x,\epsilon)$ with x expressed as a function of (ξ,η,ϵ)).

From the above analysis we see that, under the assumptions (2.2), (2.3) and (2.4), the zero dynamics of the family of regularly perturbed systems (2.5) exhibit the behavior of a singularly perturbed system in the standard two timescales form, (see [9], page 9). Its slow or "reduced" system, defined by setting $\epsilon = 0$ and eliminating the variable z, is exactly the system

$$\dot{\eta} = q(0,\eta,0) \tag{2.14}$$

which describes the zero dynamics of the unperturbed system. Its fast or "boundary layer" system, obtained by rescaling time as $\tau = t/\epsilon$ and then setting $\epsilon = 0$, has the form of a family of *linear* systems parameterized by a *constant* vector η, namely:

$$\frac{dz}{d\tau} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \hat{a}_1(0,\eta,0) & \cdot & \cdot & \cdots & \hat{a}_d(0,\eta,0) \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ \hat{b}(0,\eta,0) \end{bmatrix}. \tag{2.15}$$

Note that, if at least one eigenvlaue of the Jacobian matrix of (2.15) is in the right half plane, the boundary layer system is unstable and so are the members of the family of the perturbed system. If this is the case, the system is said to be *slightly nonminimum phase* (see [10]).

2.4 Discussion

Comparing the boundary layer system (2.15) with its counterpart in [10], the linearity of (2.15) may seem surprising. We will now show that the linear form of (2.15) is due to the scaling of z and, in particular, to the fact that in the augmented system (2.5) the variable

z_1 plays the role of the control input into the original system. A property of the standard singular perturbation form (2.12) is the absence of *peaking*, which means that, as a function of ϵ, the solution η, z of (2.12) will remain bounded as $\epsilon \to 0$, whenever its initial condition is bounded and independent of ϵ, (see [9], chapter 1). In view of (2.14) this implies that no peaking in the control input u is allowed. Such an "automatic scaling" was not present in [10].

Let us clarify this point a bit further. A detailed inspection of the constraint (2.7) which identifies the zero dynamics manifold Z^* of the unperturbed system shows that the vector $R(x, z, \epsilon)$ can be rewritten as:

$$R(x,z,\epsilon) = \begin{bmatrix} 0 \\ S(x,z,\epsilon) \end{bmatrix}$$

with:

$$S(x,z,\epsilon) = A(x,\epsilon)z + \epsilon Q(x,z,\epsilon)$$

where:

$$A(x,\epsilon) = \begin{bmatrix} \alpha_0(x,\epsilon)\epsilon^{d-1} & 0 & \cdots & 0 \\ * & \alpha_0(x,\epsilon)\epsilon^{d-2} & \cdots & 0 \\ . & . & \cdots & . \\ * & * & \cdots & \alpha_0(x,\epsilon) \end{bmatrix}$$

and $Q(x, z, \epsilon)$ is a function which vanishes at $z = 0$ together with its first order derivative with respect to z. Therefore, choosing ξ' and ξ'' to denote:

$$\begin{aligned} \xi' &= \text{col}\,(\xi_1, \cdots, \xi_r) \\ \xi'' &= \text{col}\,(\xi_{r+1}, \cdots, \xi_{r+d}) \end{aligned}$$

the constraint (2.7), expressed in the (ξ, η, z) coordinates, has a more specific form:

$$\begin{aligned} 0 &= \xi' \\ 0 &= \xi'' + \epsilon \hat{A}(\xi', \xi'', \eta, \epsilon)z + \epsilon^2 \hat{Q}(\xi', \xi'', \eta, z, \epsilon) \end{aligned} \tag{2.16}$$

where $\hat{A}(0, 0, 0, \epsilon)$ is nonsingular for small nonzero ϵ. In the above analysis, we have solved (2.16) for ξ' and ξ'' in order to express Z^* as a graph of a mapping. To proceed in the reverse direction and solve for z, would bring us back to the form used in [10]. However, it is apparent from (2.16) that, for z to remain bounded as $\epsilon \to 0$, the magnitudes of ξ'' and η would have to be restricted to an appropriate order of ϵ.

Let us conclude this section with the remark that if the two timescales assumption (2.4) fails to hold, i.e., if any of the functions $\alpha_i(x,\epsilon)$, $1 \le i \le d-1$ has a singularity at $\epsilon = 0$, most of the previous analysis is still valid, and in particular the expression (2.11) for the zero dynamics of the perturbed system. However, the equations in question are not any more in the standard two timescales form.

3 The Analysis of MIMO Systems

Because of the large variety of situations that may occur in the case of MIMO systems, a general treatment would be extremely involved. It is, therefore, more useful to focus on some elementary, but also typical, situations, hereafter discussed in the case of two-input two-output systems. To start with, we need to distinguish between the following two situations in which the zero dynamics are singularly perturbed:

(i) *Each system* in the ϵ-family *has* a (vector) relative degree (see [1], page 235) at a given point x°, but, as a function of ϵ, its value is not constant in a neighborhood of $\epsilon = 0$;

(ii) *Some systems* of the ϵ-family *do not* have a (vector) relative degree at a given point x°; the relative degree may be well defined for nonzero ϵ, but not for $\epsilon = 0$.

Case (i)

In the first case, we consider a system of the form

$$\begin{aligned} \dot{x} &= f(x,\epsilon) + g_1(x,\epsilon)u_1 + g_2(x,\epsilon)u_2 \\ y_1 &= h_1(x,\epsilon) \\ y_2 &= h_2(x,\epsilon) \end{aligned} \tag{3.1}$$

and suppose that the system has vector relative degree $\{r_1(\epsilon), r_2(\epsilon)\}$ for each ϵ in a neighborhood of $\epsilon = 0$, with $r_1(\epsilon)$, $r_2(\epsilon)$ satisfying

$$\begin{aligned} r_1(\epsilon) &= s \qquad \text{for all } \epsilon \\ r_2(\epsilon) &= r \qquad \text{for all } \epsilon > 0 \\ r_2(0) &= r + d. \end{aligned}$$

By definition, this assumption implies that in the matrix

$$\begin{bmatrix} L_{g_1}L_f^i h_1(x,\epsilon) & L_{g_2}L_f^i h_1(x,\epsilon) \\ L_{g_1}L_f^j h_2(x,\epsilon) & L_{g_2}L_f^j h_2(x,\epsilon) \end{bmatrix} \tag{3.2}$$

- the first row will be identically zero for x near x° and ϵ near 0 if $i < s-1$, nonzero at $(x^\circ, 0)$ if $i = s-1$;

- the second row will be identically zero for x near x° and ϵ near 0 if $j < r-1$, zero at $(x^\circ, 0)$ but not at (x°, ϵ) with $\epsilon \neq 0$ if $r-1 \leq j \leq r+d-2$, nonzero at $(x^\circ, 0)$ if $j = r+d-1$

- the matrix itself will be nonsingular at (x°, ϵ) with $\epsilon \neq 0$ if $i = s-1$ and $j = r-1$ and also at $(x^\circ, 0)$ if $i = s-1$ and $j = r+d-1$.

In order to avoid multiple time scaling in the zero dynamics of (3.1), we extend the *two time scale assumption* of (2.4) by assuming that, for all $0 \leq k \leq d$

$$\begin{bmatrix} L_{g_1} L_f^{s-1} h_1(x,\epsilon) & L_{g_2} L_f^{s-1} h_1(x,\epsilon) \\ L_{g_1} L_f^{r-1+k} h_2(x,\epsilon) & L_{g_2} L_f^{r-1+k} h_2(x,\epsilon) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^{d-k} \end{bmatrix} A_k(x,\epsilon) \tag{3.3}$$

where $A_k(x,\epsilon)$ is a matrix of *smooth* functions, and both $A_0(x^\circ, 0)$ and $A_d(x^\circ, 0)$ are *nonsingular.* This assumption is in accord with the above described properties of the the matrix (3.2).

Since the zero dynamics of a system of the form (3.1) are not affected whenever $g_1(x,\epsilon)$ and $g_2(x,\epsilon)$ are changed into new vector fields $\tilde{g}_1(x,\epsilon)$ and $\tilde{g}_2(x,\epsilon)$ by a change of coordinates in the input space, we suppose, without loss of generality, that in (3.3)

$$A_0(x,\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.4}$$

Since the first row of the matrix (3.3) is by definition the same for each k, we will have then

$$A_k(x,\epsilon) = \begin{bmatrix} 1 & 0 \\ \gamma_k(x,\epsilon) & \alpha_k(x,\epsilon) \end{bmatrix} \quad \text{and} \quad \alpha_d(x^\circ, 0) \neq 0. \tag{3.5}$$

As in the SISO case, the zero dynamics of the MIMO family (3.1) will be analyzed using the *augmented* system

$$\begin{aligned} \dot{x} &= f(x,\epsilon) + g_1(x,\epsilon)v_1 + g_2(x,\epsilon)z_1 \\ \epsilon\dot{z}_1 &= z_2 \\ \epsilon\dot{z}_2 &= z_3 \\ &\cdots \\ \epsilon\dot{z}_d &= v_2 \end{aligned} \tag{3.6}$$

obtained by adding an $\frac{1}{\epsilon}$-scaled chain of integrators to the second input channel. The calculations of the zero dynamics for the augmented system (3.6) are straightforward. Setting $y_1 = y_2 = 0$ yields

$$\begin{aligned} 0 &= h_1(x,\epsilon) = \cdots = L_f^{s-1}h_1(x,\epsilon) \\ 0 &= h_2(x,\epsilon) = \cdots = L_f^{r-1}h_2(x,\epsilon) \end{aligned}$$

and in view of (3.3) and (3.4)

$$0 = L_f^s h_1(x,\epsilon) + v_1$$

$$0 = L_f^r h_2(x,\epsilon) + \epsilon^d z_1.$$

The first equation yields $v_1(x,\epsilon) = -L_f^s h_1(x,\epsilon)$. The second equation is, in fact, an additional constraint among the state variables of (3.6), that must be further differentiated with respect to time. Taking its first derivative, one obtains

$$\begin{aligned} 0 &= L_f^{r+1}h_2(x,\epsilon) + L_{g_1}L_f^r h_2(x,\epsilon)v_1(x,\epsilon) + L_{g_2}L_f^r h_2(x,\epsilon)z_1 + \epsilon^{d-1}z_2 \\ &= L_f^{r+1}h_2(x,\epsilon) + \epsilon^{d-1}r_1(x,\epsilon)v_1(x,\epsilon) + \alpha_1(x,\epsilon)\epsilon^{d-1}z_1 + \epsilon^{d-1}z_2 \end{aligned}$$

and, after $d-2$ more differentiations,

$$\begin{aligned} 0 &= L_f^{r+d-1}h_2(x,\epsilon) + \alpha_{d-1}(x,\epsilon)\epsilon z_1 + \cdots + \alpha_1(x,\epsilon)\epsilon z_{d-1} + \epsilon z_d \\ &\quad + \epsilon^2 P_{d-1}(x,\epsilon,z_1,\cdots,z_{d-2}) + \epsilon Q_{d-1}(x,\epsilon) \\ 0 &= L_f^{r+d}h_2(x,\epsilon) + \alpha_d + \alpha_d(x,\epsilon)z_1 + \cdots + \alpha_1(x,\epsilon)z_d + v_2 + \\ &\quad \epsilon P_d(x,\epsilon,z_1,\cdots,z_{d-1}) + Q_d(x,\epsilon) \end{aligned}$$

where the $P_k(x,\epsilon,z_1,\cdots,z_{k-1})$'s are polynomials in $z_1,\cdots,z_{k-1}$ whose coefficients are smooth functions of x and ϵ, and the $Q_k(x,\epsilon)$'s are smooth functions of x and ϵ.

Note that these constraints have the same structure of their SISO counterparts. The last one can be solved for a smooth feedback control $v_2(x,z,\epsilon)$. The first $s+r+d$ constraints identify, for each value of ϵ, the zero dynamics submanifold Z^* of the system. Substitution of $v_1(x,\epsilon)$ and $v_2(x,z,\epsilon)$ into (3.6) yields a vector field and its restriction to Z^* defines the zero dynamics vector field. A structure analogous to (2.9) in the SISO case can be obtained and the same conclusions follow. In fact, the zero dynamics of the perturbed system are in the standard form of a two-timescale singularly perturbed system, because $\alpha_d(x^\circ,0) \neq 0$.

Case (ii)

We now proceed to the analysis of the second case and suppose that in the matrix (3.2)

- the first row is identically zero for each (x,ϵ) near $(x^\circ,0)$ for $i<s-1$, and nonzero at $(x^\circ,0)$ if $i=s-1$,

- the second row is identically zero for each (x,ϵ) for $j<r-1$, and nonzero at $(x^\circ,0)$ if $j=r-1$,

- the matrix
$$\begin{bmatrix} L_{g_1}L_f^{s-1}h_1(x^\circ,\epsilon) & L_{g_2}L_f^{s-1}h_1(x^\circ,\epsilon) \\ L_{g_1}L_f^{r-1}h_2(x^\circ,\epsilon) & L_{g_2}L_f^{r-1}h_2(x^\circ,\epsilon) \end{bmatrix} \tag{3.7}$$
is nonsingular for all $\epsilon\neq 0$, but it is singular at $\epsilon=0$.

  This means that the system (3.1) has vector relative degree $\{r_1(\epsilon),r_2(\epsilon)\}=\{s,r\}$ at each $\epsilon\neq 0$, but its relative degree can not be defined at $\epsilon=0$ because the matrix (3.7) is singular.

Without loss of generality, possibly after a change of $g_1(x,\epsilon)$ and $g_2(x,\epsilon)$, we can assume that
$$L_{g_1}L_f^{s-1}h_1(x^\circ,0)\neq 0$$
$$L_{g_2}L_f^{s-1}h_1(x^\circ,0)=L_{g_2}L_f^{r-1}h_2(x^\circ,0)=0.$$

Also in this case we need to postulate suitable hypothesis in order to avoid multiple time scaling in the zero dynamics of (3.1) but, before doing this, we need first to identify the zero dynamics of the unperturbed system. We observe that the constraints $y_1=0$ and $y_2=0$ imply, as always,
$$\begin{array}{rcl} 0 &=& h_1(x,0)=\cdots=L_f^{s-1}h_1(x,0) \\ 0 &=& h_2(x,0)=\cdots=L_f^{r-1}h_2(x,0). \end{array}$$

Moreover
$$0=L_f^s h_1(x,0)+L_{g_1}L_f^{s-1}h_1(x,0)u_1 \tag{3.8a}$$
$$0=L_f^r h_2(x,0)+L_{g_1}L_f^{r-1}h_2(x,0)u_1. \tag{3.8b}$$

We solve (3.8a) for $u_1=\gamma(x,0)$, where
$$\gamma(x,\epsilon)=-\frac{L_f^s h_1(x,\epsilon)}{L_{g_1}L_f^{s-1}h_1(x,\epsilon)}.$$

Substituting $u_1 = \gamma(x,0)$ into (3.8b) we obtain

$$0 = L_f^r h_2(x,0) + L_{g_1} L_f^{r-1} h_2(x,0)\gamma(x,0)$$

and, introducing the notation

$$\lambda(x,\epsilon) = L_f^{r-1} h_2(x,\epsilon), \qquad \tilde{f}(x,\epsilon) = f(x,\epsilon) + g_1(x,\epsilon)\gamma(x,\epsilon)$$

we express this constraint in the form

$$0 = L_{\tilde{f}}\lambda(x,0).$$

Differentiating this once more with respect to time provides

$$0 = L_f L_{\tilde{f}}\lambda(x,0) + L_{g_1} L_{\tilde{f}}\lambda(x,0)\gamma(x,0) + L_{g_2} L_{\tilde{f}}\lambda(x,0)u_2$$

which can be solved for u_2, if $L_{g_2} L_{\tilde{f}}\lambda(x^\circ,0) \neq 0$. If not, then the latter generates a new constraint, namely

$$0 = L_{\tilde{f}}^2\lambda(x,0)$$

. Suppose now that some integer d exists such that for all $0 \leq k < d-1$,

$$L_{g_2} L_{\tilde{f}}^{k+1}\lambda(x,0) = 0$$

for all x near x°, while

$$L_{g_2} L_{\tilde{f}}^{d}\lambda(x^\circ,0) \neq 0.$$

Suppose also that the differentials of

$$h_1(x,0),\cdots,L_f^{s-1}h_1(x,0),h_2(x,0),\cdots,L_f^{r-1}h_2(x,0),L_{\tilde{f}}\lambda(x,0),\cdots,L_{\tilde{f}}^{d}\lambda(x,0),$$

are linearly independent at $x = x^\circ$.

Then, the smooth $(n-r-s-d)$ dimensional submanifold defined by the constraints

$$\begin{array}{ll} L_f^{i-1}h_1(x,0) = 0 & 1 \leq i \leq s \\ L_f^{i-1}h_2(x,0) = 0 & 1 \leq i \leq r \\ L_{\tilde{f}}^{i}\lambda(x,0) = 0 & 1 \leq i \leq d \end{array}$$

is the zero dynamics manifold Z^* of (3.1) for $\epsilon = 0$.

We are now ready to investigate how the zero dynamics of (3.1) depend on ϵ. First, we make the following *two timescale assumption* that:

$$\begin{bmatrix} L_{g_1}L_f^{s-1}h_1(x,\epsilon) & L_{g_2}L_f^{s-1}h_1(x,\epsilon) \\ L_{g_1}L_f^{r-1}h_2(x,\epsilon) & L_{g_2}L_f^{r-1}h_2(x,\epsilon) \end{bmatrix} = A_0(x,\epsilon)\begin{bmatrix} 1 & 0 \\ 0 & \epsilon^d \end{bmatrix} \tag{3.9}$$

with $A_0(x,\epsilon)$ a matrix of smooth functions which is nonsingular at $(x^\circ, 0)$, and that

$$L_{g_2}L_{\tilde{f}}^k\lambda(x,\epsilon) = \alpha_k(x,\epsilon)\epsilon^{d-k}$$

where $\alpha_k(x,\epsilon)$, $k = 1, \cdots, d$ are smooth functions with $\alpha_d(x^\circ, 0) \neq 0$.

As before, we introduce an augmented system having exactly the same form as (3.6) and calculate its zero dynamics. We obtain now

$$0 = L_f^s h_1(x,\epsilon) + a_{11}^{(0)}(x,\epsilon)v_1 + a_{12}^{(0)}(x,\epsilon)\epsilon^d z_1 \tag{3.10a}$$

$$0 = L_f^r h_2(x,\epsilon) + a_{21}^{(0)}(x,\epsilon)v_1 + a_{22}^{(0)}(x,\epsilon)\epsilon^d z_1 \tag{3.10b}$$

where $a_{ij}^{(0)}$ denotes the (i,j)-element of the matrix A_0. Again, (3.10a) can be solved for v_1, yielding

$$v_1(x,\epsilon) = \gamma(x,\epsilon) + \delta(x,\epsilon)\epsilon^d z_1$$

with $\gamma(x,\epsilon)$ defined as before and

$$\delta(x,\epsilon) = -\frac{a_{12}^{(0)}(x,\epsilon)}{a_{11}^{(0)}(x,\epsilon)}.$$

Substituting $v_1(x,\epsilon)$ into (3.10b) yields a constraint of the form

$$0 = L_{\tilde{f}}\lambda(x,\epsilon) + \alpha_0(x,\epsilon)\epsilon^d z_1 \tag{3.11}$$

with $\tilde{f}(x,\epsilon)$ defined as before, and

$$\alpha_0(x,\epsilon) = a_{22}^{(0)}(x,\epsilon) - \frac{a_{21}^{(0)}(x,\epsilon)}{a_{11}^{(0)}(x,\epsilon)}a_{21}^{(0)}(x,\epsilon) = \frac{\det A_0(x,\epsilon)}{a_{11}^{(0)}(x,\epsilon)}.$$

Note that $\alpha_0(x^\circ, 0) \neq 0$.

From this point on, the analysis proceeds identically as in the case of a SISO system (in fact, the constraint (3.11) is identical to one of the constraints found in the preceding section). Again one is able to find for the zero dynamics of the perturbed system the standard two timescales form.

4 Conclusion

Whenever a regular perturbation of system equations causes an increase of the relative degree, it manifests itself as a singular perturbation of zero dynamics. Conditions (2.4), (3.3) and (3.11) define classes of perturbations which lead to a timescale separation of zero dynamics into a slow part and a fast part of the standard singular perturbation form (2.12). The slow part described by the *reduced system* (2.14), is identical to the zero dynamics of the unperturbed system. The fast part, described by the *boundary layer system* (2.15) on the fast time scale $\tau = \frac{t}{\epsilon}$. Several remarkable properties of the boundary layer system provide a deeper insight into the effects of small perturbations on zero dynamics.

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