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Adaptive Linearization and Model Reference Control of a Class of MIMO Nonlinear Systems

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Abstract

This paper discusses two schemes for the adaptive control of classes of MIMO nonlinear systems with parametric uncertainty in their dynamics. First, the problem of tracking a reference trajectory is considered and an adaptive version of the input-output decoupling algorithm of [DM87] for general right invertible MIMO systems is proposed. Then on the basis of some results of [DB90a], [DB90b] on asymptotic model matching, a scheme is presented for Model Reference Adaptive Control and a solution is given for input-output linearizable systems. Moreover, the non-adaptive model matching results are extended to yield a solution to the problem of tracking by static state feedback.

1 Introduction

In recent years there has been a great deal of research effort in the adaptive control of nonlinear systems. This research has been primarily focused on SISO systems for which there exist, broadly speaking, three types of approaches: those relying on the existence of *certain matching or structural conditions* for the location of the unknown parameters (see for example [KKM89], [TKMK89] and [KKM91]), the second relying on certain assumptions on *the type of the nonlinearities* in the plant (see for example, [SI89], [NA88], [KTKS91])

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and the third using a Lyapunov function for exhibiting stabilization of the non-adaptive controller([PP89], [PBPJ90]). While both the first and third approaches mentioned above are specifically geared to dealing with polynomial, non globally Lipschitz nonlinearities the structural conditions necessitated in the first approach appear to be restrictive and not easy to verify. Further, the third approach does not appear to have an easy extension to problems of adaptive tracking rather than stabilization. In the second category, there is a complete solution in [SI89] to the problem of adaptive linearization and tracking in SISO systems with bounded states; there is, however, the assumption that the zero dynamics are *exponentially stable* and the nonlinearities are Lipschitz continuous in the domain of applicability of the schemes. It is our goal in this paper to commence a study of model reference adaptive control and tracking control of MIMO nonlinear systems. Thus, our approach is closest in philosophy to that of [SI89]. Specifically, we consider adaptive control of square MIMO nonlinear systems P of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where $x(t) \in X$, an open connected subset of $\mathbf{R}^n$, $u(t) \in \mathbf{R}^m, y(t) \in \mathbf{R}^m$. Further we will assume that f and the columns of g, namely g_i, are analytic vector fields on $\mathbf{R}^n$ and the functions h_i are real analytic functions on $\mathbf{R}^n$.

The first topic that we cover is adaptive input-output linearization of general right-invertible MIMO systems. An adaptive version of the dynamic decoupling algorithm of [DM87] for dynamic input-output decoupling. In the process, we give a precise characterization of the prior information needed to build the adaptive controller.

Next, we investigate the problem of general model reference adaptive control of nonlinear systems. We take advantage of some recent results on (non-adaptive) asymptotic model matching with stability for general MIMO nonlinear systems ([DB90b], [DB90a], [CDB90], [GM89]) to begin this program. First, we specialize these results to non-adaptive tracking using static state feedback for general MIMO nonlinear systems. Then, an adaptive version of this algorithm is given and the prior information needed to implement the scheme and is also discussed.

These results on adaptive control of MIMO systems are general in the sense that they need no specific matching conditions for the parametric dependence of the systems. However, as in [SI89] some form of exponential attractivity of the zero dynamics is required. The exponential minimum phase hypothesis of that paper is weakened here to an hypothesis of exponential attractivity (a precise definition is given in Section 2), which guarantees that the state variables rendered unobservable by the linearization remain bounded.

A brief outline of this paper is as follows: Section 2 contains a review of the results of [SI89] on SISO adaptive linearization. The exponential minimum phase hypothesis of [SI89] is weakened to exponential attractivity of the zero dynamics. Section 3 first contains a review of the scheme of [DM87] on linearization by dynamic extension. Then an adaptive version of this algorithm is presented with a statement of the prior information required for the implementation of the algorithm. In Section 4.1 we review the results of [DB90a], [CDB90] on non adaptive model matching by static state feedback. Section 4.2 contains some extension of these results to the derivation of a non-adaptive, static tracking control law. In section 4.3, adaptive model matching for MIMO nonlinear systems is investigated. The proof of convergence of the scheme needs an extra condition on the plant: namely, that it be input-output linearizable (but not necessarily decouplable) by static state feedback. The algorithm, we conjecture, has a proof of convergence in other more general circumstances as well. Section 5 contains some concluding remarks.

2 A Review of SISO Adaptive Linearization

One method developed in the literature to solve the tracking problem consists first of input-output linearizing the system and then applying a linear tracking control. An *adaptive* version of this two step tracking control law was proposed in [SI89] for SISO systems. We recall some results from this paper to allow for a better understanding of the differences with respect to the MIMO situation illustrated in Section 3.

Consider the system (1) with one input and one output. Let x_0 be an equilibrium point of the undriven system, that is $f(x_0) = 0$, and such that the output is zero at x_0, i.e. $h(x_0) = 0$. We will assume that the system (1) has *strict relative degree* γ at x_0 [Isi89].

One can choose a new set of coordinates $\xi_1 = h(x)$ $,\xi_2 = L_f h(x)$ $,\ldots$ $,\xi_\gamma = L_f^{\gamma-1}h(x)$ and $\eta \in \mathbf{R}^{n-\gamma}$ such that $d\eta_i g \equiv 0$ so as to exhibit the system of (1) in the normal form [Isi89]:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_\gamma &= a(\xi,\eta) + b(\xi,\eta)u \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{2}$$

Here $a(\xi,\eta) = L_f^\gamma h(x), b(\xi,\eta) = L_g L_f^{\gamma-1} h(x)$ and $q(\xi,\eta) = L_f \eta$.

The *zero dynamics* are described by the following dynamical system in $\mathbf{R}^{n-\gamma}$:

$$\dot{\eta} = q(0,\eta) \tag{3}$$

The system is said to be *(exponentially) minimum phase* if the equilibrium point 0 is a (exponentially) stable equilibrium point of (3). It may be verified using a converse argument (see [SI89]) that asymptotic tracking with bounded states can be obtained if the system is *exponentially minimum phase.* The zero dynamics of (3) are said to be **exponentially attractive** to a large enough ball which is $\subset X$, if

$$\eta^T q(0,\eta) \leq -\alpha|\eta|^2 \text{ for } |\eta| \geq R \tag{4}$$

Also, $q(\xi,\eta)$ satisfies the following Lipschitz like continuity condition referred to as **conic continuity** of $q(\xi,\eta)$ in ξ uniformly in η:

$$|q(\xi,\eta) - q(0,\eta)| \leq k|\xi| \tag{5}$$

Proposition 2.1 Globally Bounded Tracking
Assume that the normal form of (2) for the system of (1) valid on X. Further assume that the zero dynamics are exponentially attractive in the sense of (4) and satisfy a conic continuity condition. Then, with the tracking control law

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)}(-L_f^{\gamma} h(x) + y_M^{\gamma} + \alpha_1(y_M^{\gamma-1} - y^{\gamma-1}) + \cdots + \alpha_{\gamma}(y_M - y)) \tag{6}$$

it follows that $y \rightarrow y_M$ with x bounded so long as $y_M, \dot{y}_M, \ldots, y_M^{\gamma-1}$ are bounded.

The preceding result has been critically examined in the literature but it has not been appreciated that the condition of (5) is not a global Lipschitz condition on the function $q(\xi,\eta)$ but only bears some resemblance to one such which, in any event, should read:

$$|q(\xi_1,\eta_1) - q(\xi_2,\eta_2)| \leq k|\xi_1 - \xi_2| + k|\eta_1 - \eta_2|$$

Now, for adaptive tracking, assume that the vector fields f, g in (1) and the function h in (1) are unknown but may be parameterized by finitely many parameters $\theta^* \in \mathbf{R}^l$. The exact nature of the dependence on θ^* is not important in what follows. However, to give some definiteness to the discussion that follows immediately hereafter, we will assume that f, g, h depend linearly by parameters $\theta^* \in \mathbf{R}^l$ in the form

$$\begin{aligned} f(x) &= \textstyle\sum_{i=1}^{l} \theta_i^* f_i(x) \\ g(x) &= \textstyle\sum_{i=1}^{l} \theta_i^* g_i(x) \\ h(x) &= \textstyle\sum_{i=1}^{l} \theta_i^* h_i(x) \end{aligned} \tag{7}$$

where the vector fields f_i, g_i and the functions h_i are known functions of x. In the equation (7) above, it follows that, if some of the θ_i^* are known, they are replaced by their values. Now the linearizing control laws of the previous section are replaced by their estimates depending on the current estimate $\hat{\theta}(t)$ of θ^* in accordance with a heuristic known as the *certainty equivalence principle.* Thus if the "true" system is *known to have relative degree* γ then the control law is given by

$$u = \frac{1}{L_g \widehat{L_f^{\gamma-1}} h(x)}(-L_f^\gamma \widehat{h(x)} + \hat{v}) \tag{8}$$

Here $L_g \widehat{L_f^{\gamma-1}} h(x), L_f^\gamma \widehat{h(x)}$ stand for the estimates of $L_g L_f^{\gamma-1} h(x), L_f^\gamma h(x)$ derived by first expressing these function in terms of the known vector fields f_i, g_j and known functions h_k and multilinear products of the form $\theta_i \ldots \theta_m$ and then replacing the multilinear product by an estimate of the form $\theta_i \widehat{\ldots} \theta_m$. We define the multilinear product as a new parameter and estimate it and $\hat{v}$ stands for the estimate of the tracking control law given by

$$\hat{v} = y_M^\gamma + \alpha_1 (y_M^{\gamma-1} - \widehat{L_f^{\gamma-1}} h) + \cdots + \alpha_\gamma (y_M - \hat{h})$$

Note that the $L_f^i \widehat{h(x)}$ are all multilinear functions of θ. Consequently, if one defines $\Theta \in \mathbf{R}^k$ to be the vector of all multilinear products of the θ_i up to terms of degree γ, it follows that the control law of (8) is affine in Θ. [1] Defining the parameter error in Θ to be $\Phi := \hat{\Theta} - \Theta^*$ and the output error to be $e = y - y_M$ an easy calculation yields that

$$e^\gamma + \alpha_1 e^{\gamma-1} + \cdots + \alpha_\gamma e = \Phi^T W(x, \hat{\Theta}) \tag{9}$$

for some appropriately defined $W(x, \hat{\Theta}) \in \mathbf{R}^k$ (in order, for $W(x, \hat{\Theta})$ to be a smooth function of x, we need $\hat{L}_g L_f^{\gamma-1} h$ to be bounded away from 0. Define the (model) transfer function

$$M(s) = \frac{1}{s^\gamma + \alpha_1 s^{\gamma-1} + \cdots + \alpha_\gamma} \tag{10}$$

and an augmented error e_1 to be

$$e_1 = e + (\hat{\Theta}^T(t) M(s) W(x, \hat{\Theta}) - M(s) \hat{\Theta}^T(t) W(x, \hat{\Theta})) \tag{11}$$

Note that the last two terms above are not equal and refer respectively to each component of W being filtered by $M(s)$ before being multiplied by $\hat{\Theta}(t)$ and filtering $\hat{\Theta}(t)^T W(x, \hat{\Theta})$ by $M(s)$. If $\hat{\Theta}$ were indeed constant, $e_1 = e$. Now combining (9) with (11) yields

$$e_1 = \Phi^T M(s) W(x, \hat{\Theta}) \tag{12}$$

It is convenient to denote the filtered regressor $M(s) W(x, \hat{\Theta})$ by

$$W_1(x, \hat{\Theta}) := M(s) W(x, \hat{\Theta})$$

[1] The linear dependence of the control law on a new parameter vector Θ is key to what follows.

Theorem 2.2 Adaptive Tracking

Consider the system of (1) with the vector fields f, g and the function h parameterized as in (7). Assume that the system can be globally converted into the normal form coordinates of (2). Further assume that the zero dynamics of the system are exponentially attractive inside X in the sense of (4) and satisfy the conic continuity conditions of (5).

Then given a bounded trajectory y_M with first $\gamma - 1$ derivatives all bounded it follows that the control law of (8) with the parameter update law

$$\dot{\hat{\Theta}} = \dot{\Phi} = -\frac{W_1 e_1}{1 + W_1^T W_1} \tag{13}$$

yields bounded tracking, i.e. $y(t) \rightarrow y_M(t)$ with all the states x bounded.

Remarks:

1. The parameter update law is specified for $\hat{\Theta}$. This neglects the multilinear dependence of terms inside the vector. However, this is necessitated by the lack of a systematic theory of nonlinear parameter estimation or identification. In a practical setting, the following heuristic may be used to speed up the parameter convergence: when the multilinear parameter, say $\widehat{\theta_i\theta_j}$ appears to be close to convergence, project it's update law in the direction of $\hat{\theta}_i\hat{\theta}_j$. The heuristic is provably convergent if the estimate $\widehat{\theta_i\theta_j}$ is close enough to $\theta_i^*\theta_j^*$ (!).

2. In order for the regressor W_1 to be bounded it is necessary to confine the parameter estimates in such a range as to keep $\widehat{L_g L_f^{\gamma-1} h}$ to be bounded away from zero. This may be achieved by projecting the parameter error estimates into a region where this is the case.

3. Given the form of the linear error equation there is a large choice available to us for parameter update laws. We choose the *normalized gradient type algorithm* of (13) here for reasons of brevity but we hasten to add that several other normalized algorithms (such as the *normalized least squares* will do as well (see [SB89])).

4. The proof of this theorem is a modification of that in [SI89] using the weaker hypotheses of equations (4, 5).

5. It is useful to note that the **prior information** required for doing adaptive control of SISO systems is knowledge of the relative degree γ of the true plant. The hypotheses of the theorem are additional: that the true plant be minimum phase and that $\widehat{L_g L_f^{\gamma-1} h}$ be bounded away from 0.

3 Adaptive Linearization of General MIMO Systems

As in the case of SISO systems, one could achieve nonadaptive tracking by first input-output linearizing the given system and then applying a linear tracking control law. In this instance, if the plant is not input-output linearizable by static state feedback, then a dynamic compensator is needed and the right-invertibility of the plant ensures the existence of a dynamically linearizing controller [DM87]. Several algorithms have been proposed in the literature for the construction of such a compensator and we now recall the one of [DM87]. We change notation slightly to refer to the process P as Σ_0. The following algorithm starts at $k = 0$ and with state variable $x^e = x$.

Step 1 Let r_i be the relative degree of the i th output of Σ_k, i.e. the largest integer such that

$$L_{g_j} L_f^l h_i(x^e) \equiv 0 \ \ \forall l < r_i - 1 \ \ \forall 1 \leq j \leq m$$

and for all x^e near x_0^e. Define the decoupling matrix $A_k(x^e)$ to have its ij th entry

$$a_{ij}(x) = L_{g_j} L_f^{r_i - 1} h_i(x^e)$$

and denote its normal or generic rank by s_k. If $s_k = m$, stop.

Step 2 If $s_k < m$, assume that the first s_k rows of $A_k(x^e)$ are linearly independent at each point of an open, dense set of X^e (this can always be achieved by a permutation of the components of the output). Apply the regular static state feedback

$$u = \alpha_k(x^e) + \beta_k(x^e)v \tag{14}$$

with α_k, β_k analytic functions of x^e such that the decoupling matrix of Σ_k with the control law of (14) is of the form

$$A_{k1}(x^e) = \begin{bmatrix} I_{s_k \times s_k} & 0 \\ M(x^e) & 0 \end{bmatrix}$$

This may be achieved by choosing α_k, β_k to be solutions of the equations

$$dL_f^{r_i - 1} h(x^e)(f(x^e) + g(x^e)\alpha_k(x^e)) \equiv 0 \ \ \forall 1 \leq i \leq s_k$$

and

$$dL_f^{r_i - 1} h(x^e)(g(x^e)\beta_k(x^e))_j = \delta_{ij} \ \ \forall 1 \leq i \leq s_k \quad 1 \leq j \leq m$$

where $(g(x^e)\beta_k(x^e))_j$ denotes the j th column of the matrix $g(x^e)\beta(x^e)$.

Step 3 There exist q_k columns of $A_1(x^e)$ (without loss of generality the first q_k) with two or more non zero elements. Put an integrator in series with q_k corresponding input channels, i.e. define the dynamic extension of Σ_k composed with (14) as

$$\dot{\zeta}_i = v_i \tag{15}$$

for $i = 1, \ldots, q_k$. Let Σ_{k+1} be the new system obtained by composing Σ_k with (14) and (15), new inputs $v_1, \ldots, v_{q_k}, u_{q_k+1}, \ldots, u_m$ and return to step 1 to resume the procedure with $k \leftarrow k+1$ and the new state variables $x^e \leftarrow \{x^e\} \cup \{\zeta_i\}$.

□

At each step of the previous algorithm, the following dynamic compensator is applied

$$\begin{aligned} \dot{\zeta}_k &= v' \\ u &= \alpha_k(x^e) + \beta_k(x^e) \begin{bmatrix} \zeta_k \\ v'' \end{bmatrix} \end{aligned}$$

where $v' = \text{col}(v_1, \ldots, v_{q_k}), v'' = \text{col}(v_{q_k+1}, \ldots, v_m), \zeta_k = \text{col}(\zeta_1, \ldots, \zeta_{q_k})$. In the preceding algorithm, the use of α_k to cancel part of the drift in the k th step of the algorithm is not strictly essential and its non usage may help simplify the adaptive version of the algorithm. Moreover, in order to simplify the adaptive version of the algorithm, at each step k, an integrator can be added to each of the first s_k inputs (rather than q_k). This could lead to a dynamic decoupling compensator which is not necessarily of minimal dimension.

If the original system is right invertible, then the procedure converges in a finite number of steps to a system, denoted Σ^e, having vector relative degree $(r_1^e, \ldots, r_m^e)$. Let (f^e, g^e, h^e) be the triple characterizing Σ^e, $x^e = (x, \zeta)$ its state, u^e its input and y^e its output. Construct a local change of coordinates $\phi(x) = (\xi, \eta)$ with $\xi = \text{col}(\xi_i)$ by setting

$$\begin{aligned} \xi_i = &\ \text{col } (h_i^e(x^e), L_{f^e} h_i^e(x^e), \ldots, L_{f^e}^{r_i^e - 1} h_i^e(x^e)) \\ := &\ \text{col } (\xi_1^i, \xi_2^i, \ldots, \xi_{r_i^e}^i) \end{aligned}$$

and using some complementary coordinates η. Then, Σ^e takes the standard form ([Isi89], pg. 240):

$$\begin{aligned} \dot{\eta} &= q(\xi, \eta) + p(\xi, \eta) u^e \\ \dot{\xi}_1^i &= \xi_2^i \\ &\vdots \\ \dot{\xi}_{r_i^e - 1}^i &= \xi_{r_i^e}^i \\ \dot{\xi}_{r_i^e}^i &= b_i^e(\xi, \eta) + \textstyle\sum_{j=1}^m a_{ij}^e(\xi, \eta) u_j^e \\ y_i^e &= \xi_1^i \end{aligned} \tag{16}$$

for $i = 1, \ldots, m$ and

$$a_{ij}^e(\xi, \eta) = L_{g_j^e} L_{f^e}^{r_i^e - 1} h_i^e(\phi^{-1}(\xi, \eta))$$

for $1 \leq i, j \leq m$ and

$$b_i^e(\xi, \eta) = L_{f^e}^{r_i^e} h_i^e(\phi^{-1}(\xi, \eta))$$

for $1 \leq i \leq m$. At this point, asymptotic tracking may be obtained by first applying to (16) the decoupling and linearizing control law and then the standard linear tracking control law.

For the adaptive version of this scheme, we consider the case where the dynamics of the plant depend on unknown parameters as

$$
\begin{aligned}
\dot{x} &= f(x,\theta^*) + g(x,\theta^*)u \\
y &= h(x,\theta^*)
\end{aligned}
\tag{17}
$$

We assume that Σ is right invertible for the true value of the parameter θ^*. In this instance, the form of the equation (16) is replaced by one of the form:

$$
\begin{aligned}
\dot{\eta} &= q(\xi,\eta,\theta^*) + p(\xi,\eta,\theta^*)u^e \\
\dot{\xi}_1^i &= \xi_2^i \\
&\vdots \\
\dot{\xi}_{r_i^e-1}^i &= \xi_{r_i^e}^i \\
\dot{\xi}_{r_i^e}^i &= b_i^e(\xi,\eta,\theta^*) + \textstyle\sum_{j=1}^m a_{ij}^e(\xi,\eta,\theta^*)u_j^e \\
y_i^e &= \xi_1^i
\end{aligned}
\tag{18}
$$

Now the nonadaptive tracking law is of the form

$$
u^e = (A^e)^{-1}(-b^e + \begin{bmatrix} y_{M1}^{r_1^e} + \alpha_{11}(y_{M1}^{r_1^e-1} - \xi_{r_1^e-1}^1) + \cdots + \alpha_{1r_1^e}(y_{M1} - \xi_1^1) \\ \cdots \\ y_{Mm}^{r_m^e} + \alpha_{m1}(y_{Mm}^{r_m^e-1} - \xi_{r_m^e-1}^m) + \cdots + \alpha_{mr_m^e}(y_{Mm} - \xi_1^m) \end{bmatrix}) \tag{19}
$$

where the polynomials $s^{r_i^e} + \alpha_{i1}s^{r_i^e-1} + \cdots + \alpha_{ir_i^e}$ are all Hurwitz.

Prior Information required for adaptive control

The variables A^e, b^e, ξ_i^j are all functions of the unknown parameter θ^*. Also, the matrices α_k, β_k defined at the k th step of the dynamic decoupling algorithm are functions of θ^*. To estimate these one needs the knowledge of the relative degrees r_i of the system Σ_k at every step in the procedure above. In particular we need to know the vector relative degree of the system with dynamic extension Σ^e at the last step namely $(r_1^e, \ldots, r_m^e)$ is known. [2]

Moreover, we need to assume that the integers s_k representing the rank of the decoupling matrices A_k representing the number of integrators to be added at each step in the dynamic decoupling algorithm described above are known and independent of θ. Also, it will need to be assumed that the s_k columns which contribute to the rank of A_k are also known to be independent of θ.

From this prior information it is possible to compute α_k, β_k as a function of θ. As in the SISO case, we will assume that it is possible to choose a new parameterization $\Theta \in \mathbb{R}^l$ such that all of the variables $(A^e)^{-1}, (A^e)^{-1}b^e, \xi_i^j, \alpha_k, \beta_k$ depend linearly on Θ.

[2]This is the counterpart to the assumption frequently made in the context of linear adaptive control of MIMO adaptive systems that the Hermite form of the plant is known (see, for example, Section 6.3 of [SB89]).

The adaptive version of the dynamic state feedback control law follows by using certainty equivalence in estimating α_k, β_k. As a consequence, the normal form equation (18) takes the form

$$
\begin{aligned}
\dot{\eta} &= q(\xi,\eta,\theta^*) + p(\xi,\eta,\theta^*)u^e \\
\dot{\xi}_1^i &= \xi_2^i + w_1^i(x^e,\hat{\Theta})\Phi \\
&\vdots \\
\dot{\xi}_{r_i^e-1}^i &= \xi_{r_i^e}^i + w_{r_i^e-1}^i(x^e,\hat{\Theta})\Phi \\
\dot{\xi}_{r_i^e}^i &= y_{Mi}^{r_i^e} + \alpha_{i1}(y_{Mi}^{r_i^e-1} - \xi_{r_i^e-1}^i) + \cdots + \alpha_{ir_i^e}(y_{Mi} - \xi_1^i) + w_{r_i^e}^i(x^e,\hat{\Theta})\Phi \\
y_i^e &= \xi_1^i
\end{aligned}
\tag{20}
$$

with $\Phi = \Theta - \Theta^*$ denoting the parameter error. Note the presence of regressors at each step of equation (20). This is caused by the fact that the α_k, β_k are no longer exact when $\Theta \neq \Theta^*$. Thus the error equations for the tracking errors $e_i = y_i - y_{Mi}$ are given by

$$e_i = M_1^i(s)w_1^i(x^e,\hat{\Theta})\Phi + \cdots + M_{r_i^e-1}^i(s)w_{r_i^e-1}^i(x^e,\hat{\Theta})\Phi + M_{r_i^e}^i(s)w_{r_i^e}^i(x^e,\hat{\Theta})\Phi \tag{21}$$

where

$$
\begin{aligned}
M_1^i(s) &= \frac{s^{r_i^e-1}+\alpha_{i1}s^{r_i^e-2}+\cdots+\alpha_{i(r_i^e-1)}}{s^{r_i^e}+\alpha_{i1}s^{r_i^e-1}+\cdots+\alpha_{ir_i^e}} \\
M_2^i(s) &= \frac{s^{r_i^e-2}+\alpha_{i1}s^{r_i^e-3}+\cdots+\alpha_{i(r_i^e-2)}}{s^{r_i^e}+\alpha_{i1}s^{r_i^e-1}+\cdots+\alpha_{ir_i^e}} \\
&\vdots \\
M_{r_i^e}^i &= \frac{1}{s^{r_i^e}+\alpha_{i1}s^{r_i^e-1}+\cdots+\alpha_{ir_i^e}}
\end{aligned}
\tag{22}
$$

Note that all the transfer functions M_j^i are proper, stable transfer functions. The first point of business is to define an augmented error to simplify the form of the error equation. To this end, we define

$$e_{1i} = e_i + (M_1^i(s)w_1^i)\hat{\Theta}(t) - M_1^i(s)(w_1^i\hat{\Theta}(t)) + \cdots + (M_{r_i^e}^i(s)w_{r_i^e}^i)\hat{\Theta}(t) - M_{r_i^e}^i(s)(w_{r_i^e}^i\hat{\Theta}(t)) \tag{23}$$

It is easy to see that the augmented error is of the form

$$e_{1i} = W_i(x^e,\hat{\Theta})\Phi \tag{24}$$

where $W_i(x^e,\hat{\Theta}) = M_1^i(s)w_1^i(x^e,\hat{\Theta}) + \cdots + M_{r_i^e}^i(s)w_{r_i^e}^i(x^e,\hat{\Theta})$ is a filtered regressor. Repeating this procedure for each of the outputs yields an augmented error equation of the form

$$e_1 = W_1(x^e,\hat{\Theta})\Phi$$

with $e_1 \in \mathbf{R}^m$ so that the same update law as before, namely,

$$\dot{\hat{\Theta}} = \dot{\Phi} = -\frac{W_1 e_1}{1 + tr(W_1^T W_1)} \tag{25}$$

may be used. There are some differences in the construction of this augmented error from the SISO case. The first difference is in the fact that the parameter error may show up in several places in the normal form of the extended system and not just in the equations for $\dot{\xi}^i_{r^e_i}$. This is necessitated by the fact that in an adaptive version of the procedure described above, α_k, β_k need to be estimated. The second difference is in the construction of the augmented error, even for the i th output, different regressors are filtered by different transfer functions. Now, however under the same hypothesis as in Theorem 2.2, the same theorem holds. There is however one difference in the proof from the SISO case, namely that the zero dynamics are indeed driven by the input u^e in the MIMO case. As a consequence, as in the case of the proof of Theorem 3.3 we need to insist that the initial conditions of the states x^e, the initial parameter error $\Phi(0)$ and the tracking output y_M and their appropriate derivatives are small enough so as to guarantee the conclusions of the theorem.

Example

Consider the following example of a two input two output system with one unknown parameter modified slightly from [DBGM89]:

$$\begin{aligned} \dot{x}_1 &= x_3 u_1 + \theta^* x_3 x_1 u_2 + x_4 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u_1 + \theta^* x_1 u_2 \\ \dot{x}_4 &= u_2 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned} \tag{26}$$

We first give the non-adaptive control law. Following the algorithm above we differentiate the outputs y_1 once and y_2 twice to obtain

$$\begin{bmatrix} \dot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 & \theta^* x_3 x_1 \\ 1 & \theta^* x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with $A_0(x)$ having rank 1. Choosing $\beta_0(x)$ to be

$$\begin{bmatrix} 1 & -\theta^* x_1 \\ 0 & 1 \end{bmatrix}$$

and $\dot{\zeta}_1 = v_1$ yields for $x^e = \{x\} \cup \{\zeta_1\}$ the equations

$$\begin{aligned} \dot{x}_1 &= x_3 \zeta_1 + x_4 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \zeta_1 \\ \dot{x}_4 &= v_2 \\ \dot{\zeta}_1 &= v_1 \end{aligned}$$

As far as the normal form variables are concerned we now have $\xi_1^1 = x_1, \xi_1^2 = x_2, \xi_2^2 = x_3, \xi_2^1 = x_4 + x_3\zeta_1, \xi_3^2 = \zeta_1$. Differentiating the outputs again we get

$$\begin{bmatrix} y_1^{(2)} \\ y_2^{(3)} \end{bmatrix} = \begin{bmatrix} \zeta_1^2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

and now, we see that the control law

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ -\zeta_1^2 + w_1 - x_3 w_2 \end{bmatrix}$$

yields a decoupled and linearized system $y_1^{(2)} = w_1, y_2^{(3)} = w_2$.

For the adaptive version of this algorithm, we use the prior information that the ranks of A_0, A_1 are 1,2 respectively. We also assume that the integrator is placed on the first channel after the first step. There is only one unknown parameter θ and it enters the control law linearly through $\beta_0(x)$ alone. We choose the control input at the first step to be

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -\hat{\theta} x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ v_2 \end{bmatrix}$$

Using the integrator on channel 1 we get the state equation obtained from using $\hat{\beta}_0$ rather than β_0

$$\begin{aligned} \dot{x}_1 &= x_3\zeta_1 + x_4 - \phi x_3 x_1 v_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \zeta_1 - \phi x_1 v_2 \\ \dot{x}_4 &= v_2 \\ \dot{\zeta}_1 &= v_1 \end{aligned}$$

Here $\phi = \hat{\theta} - \theta^*$. Defining ξ_i^j as before and noting that the choice of control law at Step 1 of the algorithm above does not need knowledge of the parameter θ yields the following perturbed normal form

$$\begin{aligned} \dot{\xi}_1^1 &= \xi_2^1 - \phi x_3 x_1 v_2 \\ \dot{\xi}_2^1 &= w_1 - \phi x_1 \zeta_1 v_2 \\ \dot{\xi}_1^2 &= \xi_2^2 \\ \dot{\xi}_2^2 &= \xi_3^2 - \phi x_1 v_2 \\ \dot{\xi}_3^2 &= w_2 \end{aligned}$$

This establishes the nature of the perturbation in the normal form discussed above.

4 Model Reference Control of MIMO Systems

In this section we first review the results of [DB90a], [CDB90] on stable nonadaptive model matching using static state feedback. We use these results then to propose a solution for determining a *static* state feedback for achieving asymptotic tracking. Finally, we give an adaptive version of both the model reference and tracking schemes.

4.1 Nonadaptive Asymptotic Model Matching: a Review

Consider the plant P to be a square, nonlinear plant of the form (1). Define $G(x) := \text{span}\{g_1(x), \dots, g_m(x)\}$ (over the ring of analytic functions) and assume that the dimension of the distribution G is m for all $x \in X$. Consider also a model M of the form

$$\begin{aligned} \dot{z} &= f_M(z) + g_{M1}(x)v_1 + \cdots + g_{Mm}(x)v_m \\ y_{M1} &= h_{M1}(z) \\ &\vdots \\ y_{Mm} &= h_{Mm}(z) \end{aligned} \tag{27}$$

Here $z \in X_M$, an open subset of $\mathbf{R}^{n_M}$. It will cause no confusion to refer to the collection of vector fields $[g_{M1} \cdots g_{Mm}]$ as g_M. We will need to assume that f_M, g_M are analytic vector fields and that h_M is an analytic function. The notation $y_M(t)$ is used to mean the output of the model starting from state z_0 at 0 if there is no need to highlight the dependence on the initial state.

An extended system Σ^E is associated with the plant and model as follows:

$$\begin{aligned} \dot{x^E} &= f^E(x^E) + \hat{g}(x^E)u + \hat{p}(x^E)v \\ y^E &= h^E(x^E) \end{aligned} \tag{28}$$

with state $(x^E)^T := (x^T, z^T) \in X \times X_M$, inputs u, v and

$$f^E(x^E) = \begin{bmatrix} f(x) \\ f_M(z) \end{bmatrix}, \hat{g}(x^E) = \begin{bmatrix} g(x) \\ 0 \end{bmatrix}, \hat{p}(x^E) = \begin{bmatrix} 0 \\ g_M(z) \end{bmatrix}$$

$$h^E(x^E) = h(x) - h_M(z)$$

Further define

$$g^E(x^E) = \begin{bmatrix} \hat{g}(x^E) & \hat{p}(x^E) \end{bmatrix}$$

Also, define the dynamical system with state x^E, input u and output y^E described by the triple $(f^E, \hat{g}, h^E)$ to be $\hat{\Sigma}$. Now, consider a point $x_0^E = (x_0, z_0)$ which is an equilibrium point of f^E and also produces zero output for the system $\hat{\Sigma}$, i.e.

$$f^E(x_0^E) = 0 \ , \ \ h^E(x_0^E) = 0$$

Now, assume that

Assumption A1 : (Regularity of $\hat{\Sigma}$)

x_0^E is a regular point for the zero dynamics algorithm applied to $\hat{\Sigma}$ (regular in the sense of [Isi89], page 302).

Actually, the assumption A1 is a sufficient condition in order to apply the zero dynamics algorithm to the system $\hat{\Sigma}$ around x_0^E. Let $\hat{M}_k$ denote the submanifold defined at step k of the algorithm and $\hat{M}^*$ denote the zero dynamics manifold obtained at the conclusion of the algorithm; there further exists a unique smooth control $u_0 : \hat{M}^* \longrightarrow \mathbf{R}^m$ so as to make $\hat{M}^*$ invariant, i.e. $f^E(x^E) + \hat{g}(x^E)u_0$ is tangent to $\hat{M}^*$. The vector field $f^E(x^E) + \hat{g}(x^E)u_0$ restricted to $\hat{M}^*$ is referred to as the zero dynamics of $\hat{\Sigma}$. It can also be shown that $\hat{M}^*$ can be expressed in a neighborhood of x_0^E as

$$\hat{M}^* = \{x^E \in X \times X_M : \hat{H}^*(x^E) = 0\}$$

The following theorem uses the procedure of the zero dynamics algorithm to solve the model matching problem as follows [DB90a], [CDB90]:

Theorem 4.1 Stable Model Matching
Consider the system of (28) and assume that there exists an x_0^E such that

1. *A1 holds,*
2. *$\hat{\Sigma}$ is minimum phase at x_0^E, and*
3. *span $\{\hat{p}(x^E)\} \subset T_{x^E}\hat{M}_k +$ span $\{\hat{g}(x^E)\}$ in a neighborhood of x_0^E in $\hat{M}_k$ for all k.*

Then, there exist neighborhoods U of x_0 and U_M of z_0, an integer ν, a compensator Q defined by

$$\begin{aligned} \dot{\chi} &= a(\chi, x) + b(\chi, x)v \\ u &= c(\chi, x) + d(\chi, x)v \end{aligned} \tag{29}$$

for appropriately defined analytic a, b, c, d and $\chi \in \mathbf{R}^\nu$, a function $F : U \times U_M \longrightarrow \mathbf{R}^\nu$ and a constant $L \in \mathbf{R}_+$ such that
a) If $v(t) \equiv 0$ then the point $(x_0, \chi_0 := F(x_0, z_0))$ is an asymptotically stable equilibrium point of the closed loop $P \circ Q$, i.e. of the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)c(x, \chi) \\ \dot{\chi} &= a(\chi, x) \end{aligned}$$

b) If $|v(t)| < L$ for all $t \geq 0$ then

$$\lim_{t \to \infty} y^{P \circ Q}(x, F(x, z), t) - y_M(z, t) = 0$$

for all $(x, z) \in U \times U_M$.

Remarks:

1. In view of the propositions of [Isi89], Appendix B.2 the fulfillment of (a) above guarantees that given $\epsilon > 0$ there exist δ, K such that if $|(x(0), \chi(0))| < \delta$ and $|v(.)| < K$, then $|(x(t), \chi(t))| < \epsilon$ for all $t \geq 0$.

2. The proof of the preceding theorem in the previously mentioned references is constructive and the compensator may be shown to be of the form

$$\begin{aligned} \dot{\chi} = & \; f_M(\chi) + g_M(\chi)v \\ u = & \; u(\chi, x, v) \end{aligned} \tag{30}$$

initialized at $\chi_0 = z_0$, i.e. $\chi_0 = F(x_0, z_0) = z_0$. As a consequence we have that $\chi(t) \equiv z(t)$ and one may define the control law in terms of x^E alone rather than x, χ, z as

$$u(x^E, v) := u^*(x^E, v) + M^{-1}(x^E)K\hat{H}^*(x^E) \tag{31}$$

where $M(x^E) \in \mathbf{R}^{m\times m} := d\hat{H}^*(x^E)\hat{g}(x^E)$ and $u^*(x^E, v) := u_0(x^E) + u_1(x^E)v$ is the unique solution u of the equation

$$d\hat{H}^*(x^E)(f^E(x^E) + \hat{g}(x^E)u + \hat{p}(x^E)v) = 0$$

so that

$$u_0(x^E) = M^{-1}(x^E)d\hat{H}^*(x^E)f^E(x^E)$$

and

$$u_1(x^E) = M^{-1}(x^E)d\hat{H}^*(x^E)\hat{p}(x^E)$$

Further $K \in \mathbf{R}^{m\times m}$ is chosen to stabilize part of the system dynamics as specified below.

Let x_0 (respectively, x_0^E) be an equilibrium of P (respectively, $\hat{\Sigma}$) such that $h(x_0) = 0$ (respectively, $h^E(x_0^E) = 0$). Then it is shown in [DB90a] that because of the structure of $\hat{\Sigma}$, the following two assumptions are equivalent

Assumption A1' : (Strong Regularity of $\hat{\Sigma}$)

$\hat{\Sigma}$ is right-invertible and $(x_0^E, y^E \equiv 0)$ is a locally strongly regular pair for $\hat{\Sigma}$ (strongly regular in the sense of [DBG90]).

Assumption A2 : (Strong Regularity of P)

P is right-invertible and $(x_0, y \equiv 0)$ is a locally strongly regular pair for P.

Clearly A2 $\Rightarrow$ A1 as well. Weaker hypotheses than A2 are also sufficient for our purposes, for example the regularity hypothesis of [Isi89], pg. 302 where in (ii) the constancy of rank is to be assumed in a neighborhood of x_0 in X. For simplicity, we will use A2 in what follows. If in Theorem 3.1, the hypothesis A1 is replaced by A2, then one can construct a local change of coordinates $(\xi, \eta, z') = \Psi(x, z)$ with $z' = z - z_0$, $\Psi(x_0, z_0) = 0$ and $\xi = \hat{H}^*(x, z)$ such that the plant with the controller of equation (30) has the form (see [DB90a]):

$$\begin{aligned} \dot{\xi} = & \; A\xi + q_1(\xi, \eta, z') + p_1(\xi, \eta, z')v \\ \dot{z}' = & \; f_M(z' + z_0) + g_M(z' + z_0)v \\ \dot{\eta} = & \; \psi(\xi, \eta, z') + \phi(\xi, \eta, z')v \end{aligned} \tag{32}$$

where the matrix A is rendered Hurwitz by appropriate choice of K in (31). The states ξ contain in particular the output errors as some of their entries. Also the functions q_1 and p_1 satisfy some extra conditions, namely

$$q_1(0, \eta, z') \equiv 0 \quad \frac{\partial q_1}{\partial \xi}(0,0,0) = 0$$

and

$$p_1(0, \eta, z') \equiv 0$$

Further, the dynamical system

$$\begin{aligned} \dot{z}' &= f_M(z' + z_0) \\ \dot{\eta} &= \psi(0, \eta, z') \end{aligned} \tag{33}$$

represents the zero dynamics of $\hat{\Sigma}$ and the system

$$\dot{\eta} = \psi(0, \eta, 0) \tag{34}$$

represents the zero dynamics of P. The zero dynamics manifold of $\hat{\Sigma}$ is now given by

$$\hat{M}^* = \{(\xi, \eta, z') \mid \xi = 0\}$$

The form (33) of the zero dynamics of the system $\hat{\Sigma}$ shows that it is minimum phase if the zero dynamics of P and the undriven model dynamics are asymptotically stable.

4.2 Nonadaptive Tracking by Static State Feedback

The decomposition (32) can be used to extend the proof of Theorem 4.1 to cover the case where, instead of assuming the asymptotic stability of the zero dynamics of $\hat{\Sigma}$, one assumes that the variables z' are bounded by a sufficiently small constant and that the zero dynamics of P is asymptotically stable. This can then be usefully applied to solve trajectory tracking as a special case of the model matching problem in which the desired trajectory y_M is generated by a model consisting of chains of integrators driven by the appropriate derivatives of the y_{Mi}. More precisely, define μ_i to be the essential order of the i th output of the plant y_i as defined in [GM89]. Then, define the model to be matched to have state $z = \text{col } (z_i, i = 1, \ldots, m)$ with dynamics

$$\begin{aligned} \dot{z}_{i1} &= z_{i2} \\ \dot{z}_{i2} &= z_{i3} \\ &\cdots \\ \dot{z}_{i\mu_i} &= v_i \\ y_{Mi} &= z_{i1} \end{aligned} \tag{35}$$

We need to verify that this model corresponding to $y_{Mi}^{(\mu_i)} = v_i$ satisfies the hypothesis 3 of Theorem 3.1. Indeed, applying the zero dynamics algorithm to Σ^E corresponds to the application of the inversion algorithm to P with $y = y_M$. If one chooses μ_i to be the largest derivative of y_i which appears in the inversion algorithm, it follows that the largest derivative of y_{Mi} which appears in the zero dynamics algorithm is $y_{Mi}^{(\mu_i)}$. Thus, it follows from the choice of the model of (35) above that $d\hat{H}_k\hat{p} \equiv 0$ for $k < k^*$. At k^* the satisfaction of the third hypothesis of Theorem 3.1 is guaranteed by the fact that at this step (by Assumption A2) the matrix $d\hat{H}^*\hat{g}$ is invertible. Consequently, the choice of the model guarantees that

$$\text{span}\{\hat{p}(x^E)\} \subset T_{x^E}\hat{M}_k + \text{span}\{\hat{g}(x^E)\}$$

for all $k \leq k^*$. An interesting by-product of this calculation is that in this case the form of the equations (32) is somewhat simplified in that the term p_1 no longer exists since terms in v occur only at $k = k^*$ in the equations for $\dot{\xi}$ and are, in any event, cancelled by the choice of the control law to yield:

$$\begin{aligned} \dot{\xi} &= A\xi + q_1(\xi, \eta, z') \\ \dot{z}' &= f_M(z' + z_0) + g_M(z' + z_0)v \\ \dot{\eta} &= \psi(\xi, \eta, z') + \phi(\xi, \eta, z')v \end{aligned} \tag{36}$$

For the next theorem, define $\mu = \max_i \mu_i$.

Theorem 4.2 MIMO Asymptotic Tracking

Assume that Assumption A2 above holds and that P is minimum phase at x_0. Then, there exist constants δ_1, δ_2 and a compensator Q of the form

$$u = c(x, y_M, \dot{y}_M, \ldots, y_M^{(\mu-1)}) + d(x, y_M, \dot{y}_M, \ldots, y_M^{(\mu-1)})v \tag{37}$$

such that

1. *If $y_M(t) \equiv 0$ the closed loop system $P \circ Q$ is asymptotically stable with equilibrium point x_0.*

2. *When $\sup_{t>0}(|y_M(t)|, \ldots, |y_M^{\mu-1}(t)|) < \delta_1$, and $|x_0| < \delta_2$ then*

$$\lim_{t\to\infty} y^{P\circ Q}(t) = y_M(t)$$

Proof: It is an extension of the proof of Theorem 3.1. Define the compensator Q by the formula of (30) above with the states of the model replaced by the appropriate derivatives of y_M. Then, by the discussion above, there exist a set of new local coordinates such that the controlled plant is described by equations of the form of (36) with the added linearity of the model. Since the plant is minimum phase at x_0, it follows from the Lemma of page 442 [Isi89] that the equilibrium $(\xi, \eta) = 0$ of the system

$$\begin{aligned} \dot{\xi} &= A\xi + q_1(\xi, \eta, 0) \\ \dot{\eta} &= \psi(\xi, \eta, 0) \end{aligned} \tag{38}$$

is asymptotically stable. Further, from the lemma on page 444 of [Isi89], it follows that given $\epsilon > 0$, $\exists \delta_1(\epsilon), \delta_2(\epsilon)$ and $V(\epsilon)$ such that for the system of (36)

$$|\xi(0)|, |\eta(0)| < \delta_1, \quad |z'(.)| < \delta_2, |v(.)| < V \Rightarrow |\xi(t)|, |\eta(t)| < \epsilon$$

Now, following the proof of Lemma 3.1 of [CDB90] we consider the system

$$\dot{\xi} = A\xi + q_1(\xi, \eta, z') \tag{39}$$

The matrix A is Hurwitz and by the assumptions on q_1 and the bounds on $|\xi(t)|, |\eta(t)|$ it follows that

$$|q_1(\xi, \eta, z')| \leq k_1(\epsilon)|\xi|$$

for some continuous function $k_1(\epsilon)$ with $k_1(0) = 0$ (roughly speaking, q_1 is higher order in ξ and both ξ, η are of order ϵ). Now, using the Gronwall inequality on (39) it follows that

$$|\xi(t)| \leq |\xi(0)| e^{-[\beta - \alpha k_1(\epsilon)]t}$$

where β is a bound on the rate of convergence of the equation

$$\dot{\xi} = A\xi$$

and α is a constant. k_1, V are as described above. From the properties of k_1 it follows that for ϵ small enough and V small enough

$$\beta - \alpha k_1(\epsilon) > 0$$

so that

$$\lim_{t \to \infty} \xi(t) = 0$$

This completes the proof since the output errors $y^{P \circ Q} - y_M$ are among the state variables ξ.
□

Example

To illustrate the implications of the assumptions A1' and A2, we consider a simple, conceptual example of a three-input three-output system taken from [Isi89], pp. 304-307. In principle, the model for tracking needs to be chosen by determining the essential orders of the outputs. In point of fact, we will determine the essential orders during the course of verifying the hypothesis A2. We assume that the model is of the form (35) with the integers μ_i to be determined. Thus, we start with the extended system outputs $y_i^E = y_i - y_{Mi} = y_i - z_{i1}$ for $i = 1,2,3$. $\hat{M}_0$ is now defined to be $\{x^E \in X \times X_M : y^E \equiv 0\}$. By the regularity hypothesis A1', it is a manifold of codimension 3. For the next step in the algorithm, differentiate these outputs to yield

$$\begin{array}{rll} \dot{y}_1^E = & L_f h_1 - z_{12} + L_g h_1 u & = 0 \\ \dot{y}_2^E = & L_f h_2 - z_{22} + L_g h_2 u & = 0 \\ \dot{y}_3^E = & L_f h_3 - z_{32} + L_g h_3 u & = 0 \end{array}$$

Similar to the example in [Isi89], it is assumed that $L_g h_3(x) \equiv 0$ and that $L_g h$ has rank 1 around x_0 so that $L_g h_2(x) = -\gamma(x) L_g h_1(x)$ for some analytic function γ. Thus, we define $\hat{M}_1 = \{x^E \in \hat{M}_0 : \gamma(x)(L_f h_1 - z_{12}) + (L_f h_2 - z_{22}) := \phi_2(x) - \gamma(x) z_{12} - z_{22} = \hat{\phi}_2(x,z) = 0; L_f h_3 - z_{32} := \phi_3(x) - z_{32} = \hat{\phi}_3(x,z) = 0\}$. Now differentiating $\hat{\phi}_2, \hat{\phi}_3$ we obtain

$$\begin{array}{rll} \dot{\hat{\phi}}_2 = & L_f \phi_2 - L_f \gamma z_{12} - \gamma z_{13} - z_{23} & +(L_g \phi_2 - L_g \gamma z_{12}) u \\ \dot{\hat{\phi}}_3 = & L_f \phi_3 - z_{33} & + L_g \phi_3 u \end{array}$$

Similar to the example in [Isi89], it is assumed that the matrix

$$\begin{bmatrix} L_g h_1 \\ L_g \phi_2 \\ L_g \phi_3 \end{bmatrix}$$

has rank 2 around x_0. Now Assumption A1' implies that the matrix

$$\begin{bmatrix} L_g h_1 \\ L_g \phi_2 - (L_g \gamma) z_{12} \\ L_g \phi_3 \end{bmatrix}$$

has rank 2 around $x_0, z_0 = 0$. As a consequence there exist functions $\hat{\delta}_1(x,z), \hat{\delta}_2(x,z)$ such that $\hat{\delta}_1 L_g h_1 + \hat{\delta}_2 [L_g \phi_2 - (L_g \gamma) z_{12}] + L_g \phi_3 = 0$. Thus, we have that $\hat{M}_2 = \{x^E \in \hat{M}_1 : \hat{\delta}_1 [L_f h_1 - z_{12}] + \hat{\delta}_2 [L_f \phi_2 - L_f \gamma z_{12} - \gamma z_{13} - z_{23}] + L_f \phi_3 - z_{33} := \hat{\psi}_3(x^E) = 0\}$. This implies in particular also that $L_g \gamma$ is in the row span of $L_g h_1, L_g \phi_2$. Further it is easy to see that $\hat{\delta}_1(x,0) = \delta_1(x), \hat{\delta}_2(x,z) = \delta_2(x)$ where the functions δ_1, δ_2 are as in the example of [Isi89]. Further, define $\bar{\psi}_3$ as

$$\bar{\psi}_3(x^E) = \hat{\delta}_1 L_f h_1 + \hat{\delta}_2 L_f \phi_2 + L_f \phi_3$$

so that $\bar{\psi}_3(x,0) = \psi_3(x)$ and $\hat{\psi}_3 = \bar{\psi}_3 - *z$ for some suitably defined matrix $*$. Now, differentiating $\hat{\psi}_3$ we get the coefficient of u to be of the form of $L_g \bar{\psi}_3 + ** \, z$ for some suitable matrix $**$. Like in the example of [Isi89], we assume that the matrix

$$\begin{bmatrix} L_g h_1 \\ L_g \phi_2 \\ L_g \psi_3 \end{bmatrix}$$

has constant rank 3 around x_0. Now, Assumption A2 implies that the following matrix multiplying u

$$\begin{bmatrix} L_g h_1 \\ L_g \phi_2 - L_g \gamma z_{12} \\ L_g \bar{\psi}_3(x,z) + {}^{**}z \end{bmatrix}$$

also has rank 3 around $(x_0, 0)$. Therefore the algorithm terminates at this point, $\hat{M}^* = \hat{M}_2$ and can be locally described as the zero set of the functions $h_i(x) - z_{i1}, \hat{\phi}_2(x,z), \hat{\phi}_3(x,z), \hat{\psi}_3(x,z)$. An easy calculation shows that at the end of the algorithm we are left with the derivatives of z_{13}, z_{23}, z_{33}. Thus, a suitable choice of model for the purpose of tracking is $\dot{z}_{i3} = v_i$ for $i = 1, \ldots, 3$. This is consistent with the independent calculation of the essential orders of the plant to be $3, 3, 3$.

4.3 Model Reference Adaptive Control for a Class of MIMO Systems

In the previous section, it was shown that, under the hypotheses of Theorem 4.1, the output of the plant P controlled by (30) (with u of the form of (31)) asymptotically tracks the output of the model M. In this section, we will present an adaptive version of that Theorem for P belonging to the class of systems input-output linearizable by state feedback. ([Isi89], p. 267). We consider systems of the form of equation (1) with the added feature that the dynamics of the plant depend on certain unknown parameters $\theta^* \in \mathbb{R}^l$, i.e.

$$\begin{aligned} \dot{x} = &\ f(x,\theta^*) + g(x,\theta^*)u \\ y = &\ h(x,\theta^*) \end{aligned} \tag{40}$$

The assumption A2 of the previous section is assumed to hold for the true value of the plant parameter. Carrying forward the dependence on θ through the derivation of the compensator (30) will yield the manifold $\hat{H}^*(x^E, \theta)$ and the control law of (31), namely

$$u(x^E, v, \theta^*) := u^*(x^E, v, \theta^*) + M^{-1}(x^E, \theta^*) K \hat{H}^*(x^E, \theta^*) \tag{41}$$

Hypotheses Needed for Adaptive Control

The assumption that we can indeed determine $\hat{H}^*(x^E, \theta^*)$ as a function of θ^* contains within it some assumptions about the structure of the plant. Indeed, one needs that at every step in the zero dynamics algorithm modified as described above for stable, model matching, the manifold $\hat{M}_k$, described as the zero set of the function $\hat{H}_k(x^E, \theta)$, satisfies the condition that

$$d\hat{H}_k(x^E, \theta)\hat{g}(x^E, \theta)$$

has a left null space of constant dimension as a function of θ since the new function $\hat{H}_{k+1}$ is defined to be $\hat{H}_k$ along with

$$\gamma_k(x^E,\theta)d\hat{H}_k(x^E,\theta)f^E(x^E,\theta)$$

where $\gamma_k(x^E,\theta)$ is a basis for the left null space of $d\hat{H}_k(x^E,\theta)\hat{g}(x^E,\theta)$. Note that the model is assumed to be known and independent of θ.

Adaptive Control Law

By looking at the form of the control law of (41), we see that if one assumes that

$$u^*(x^E,v,\theta^*), M(x^E,\theta^*), \hat{H}^*(x^E,\theta^*)$$

can be reparameterized to depend linearly on some new parameter Θ, then the control law can be linearly parameterized as

$$u(x^E,v,\theta^*) = \bar{u}(x^E,v) + W_1(x^E,v,\Theta^*)\Theta^* \tag{42}$$

for an appropriately defined matrix $W_1(x^E,v,\Theta^*) \in \mathbf{R}^{m\times k}$ and parameter vector $\Theta^* \in \mathbf{R}^k$. Actually both $\bar{u}$ and W_1 are affine in v. As a consequence of (42), the adaptive model matching control law is given by

$$u(x^E,v,\hat{\theta}(t)) := \bar{u}(x^E,v) + W_1(x^E,v,\hat{\Theta})\hat{\Theta}(t) \tag{43}$$

Denoting the parameter error $\Phi(t) = \hat{\Theta}(t) - \Theta^* \in \mathbf{R}^k$ the following modification of (36) will be obtained

$$\begin{aligned} \dot{\xi} =& \; A\xi + q_1(\xi,\eta,z') + p_1(\xi,\eta,z')v + W_2(\xi,\eta,z',v,\hat{\Theta})\Phi \\ \dot{z}' =& \; f_M(z'+z_0) + g_M(z'+z_0)v \\ \dot{\eta} =& \; \psi(\xi,\eta,z') + \phi(\xi,\eta,z')v + W_3(\xi,\eta,z',v,\hat{\Theta})\Phi \end{aligned} \tag{44}$$

The dependence of the matrices W_2, W_3 on the data, and for that matter on W_1 above, is involved. The equations (44) are affine in Φ as a consequence of the linear parameterization of the control law by the unknown parameter Θ. Note that when $\Phi = 0$ the equations (44) reduce to (36).

We are not as yet able to give a stability proof for a parameter update law derived on the basis of a composite Lyapunov function involving the system of equation (44) and an equation for $\dot{\Phi}$. There appear to be two reasons for this difficulty:

1. The nonadaptive proof as presented in Theorem 3.2 is not a one step proof based on the use of a Lyapunov function.

2. Though the terms q_1 and p_1 in the differential equation for $\dot{\xi}$ satisfy the conditions

$$q_1(0,\eta,z') \equiv 0 \quad \frac{\partial q_1}{\partial \xi}(0,0,0) \equiv 0 \quad p_1(0,\eta,z') \equiv 0$$

they are sufficiently complicated so as to not allow for a simple construction of a Lyapunov function for the ξ, η system.

However, there at least two special cases for which an adaptive scheme can be derived:

1. $W_2 \equiv 0$. This case corresponds to the situation when the parameter variation is a disturbance which is rejected by the non-adaptive control law. In this instance, no parameter update is necessary. This case will not be treated in what follows for its obvious simplicity. It is however, important to guarantee that the η variables remain bounded since they are driven by the parameter error Φ, if boundedness of all the state variables is an issue.

2. $q_1 \equiv 0$ and $p_1 \equiv 0$.
$q_1 \equiv 0$ in the case that the plant P is input-output linearizable by regular static state feedback. To see this, the following two facts are useful:

   Fact 1 P is input-output linearizable by static state feedback if and only if $\hat{\Sigma}$ is.

   Fact 2 $q_1 \equiv 0$ if and only if $\hat{\Sigma}$ is linearizable by static state feedback.
Indeed, from studying the application of the zero dynamics algorithm to $\hat{\Sigma}$ as in [DB90a] it may be shown that $q_1 \equiv 0$ is equivalent to requiring that the matrices whose rows span the orthogonal complement to $L_{\hat{g}}\hat{H}_k(x^E)$ at each step k do not depend on x^E. In other words, $q_1 \equiv 0$ if and only if

$$\text{rank}_{\mathcal{K}} L_{\hat{g}}\hat{H}_k(x^E) = \text{Rrank} L_{\hat{g}}\hat{H}_k(x^E) \tag{45}$$

   where the left hand side is the rank over the field $\mathcal{K}$ of meromorphic functions of x^E and Rrank denotes the dimension of the real vector space spanned by the rows of $L_{\hat{g}}\hat{H}_k(x^E)$. But (45) is equivalent to the input-output linearizability of $\hat{\Sigma}$ and this proves Fact 2.

   For $p_1 \equiv 0$ the model M has to fulfill a structural condition, which is explained in what follows: suppose that x_0^E is regular for the controlled invariant distribution algorithm [Isi89], p.237, applied to $\hat{\Sigma}$. Then, one can define $\hat{\Delta}^*$ as being the maximal $(f^E, \hat{g})$ invariant distribution contained in ker dh^E. A sufficient condition for the solvability of the model matching problem is

$$\text{span } (\hat{p}) \subset \hat{\Delta}^* + \text{span } (\hat{g}) \tag{46}$$

(46) can also be expressed as an equality between the structures at infinity of the plant and the extended system Σ^E. If (46) holds, then $d\hat{H}_k\hat{p} \equiv 0$ at each step k and $p_1 \equiv 0$ in the (44). Also, condition (46) implies hypothesis 3 of Theorem 4.1. Further, if M is linear and P is input-output linearizable by static state feedback (46) is a necessary and sufficient condition for the solution of the model matching problem [DBI86].

Thus, to summarize these discussions, we see that $p_1 \equiv 0$ is a consequence of (46), which, in the case of P being input-output linearizable and M linear is the necessary and sufficient condition for the solvability of the model matching problem. Further, the input-output linearizability of P guarantees that $p_1 \equiv 0$.

Now, if p_1, q_1 are both identically zero, the form of the equations (44) is

$$\begin{aligned} \dot{\xi} &= A\xi + W_2(\xi,\eta,z',v,\hat{\Theta})\Phi \\ \dot{z}' &= f_M(z'+z_0) + g_M(z'+z_0)v \\ \dot{\eta} &= \psi(\xi,\eta,z') + \phi(\xi,\eta,z')v + W_3(\xi,\eta,z',v,\hat{\Theta})\Phi \end{aligned} \tag{47}$$

Now, choose $P > 0$ to be the positive definite solution to the equation $A^TP+PA = -I$. Define $e = P\xi$. Since the ξ variables are precisely the variables $\hat{H}(x^E,\Theta^*)$ it follows that e is not available for measurement. However, we have assumed that the variable $\hat{H}(x^E,\theta^*)$ depends linearly on the parameters. As a consequence, we may estimate e by

$$\hat{e} = P\hat{H}(x^E,\hat{\Theta})$$

and we have in addition that $\hat{e} = e + W_4(x^E)\Phi$. Loosely speaking

$$e = P(sI-A)^{-1}(W_2(\xi,\eta,z',v,\hat{\Theta})\Phi)$$

where the hybrid notation refers to the convolution between the respective time domain functions. We also need to define the augmented error

$$e_1 = \hat{e} - P(sI-A)^{-1}(W_2(\xi,\eta,z',v,\hat{\Theta})\hat{\Theta}) + (P(sI-A)^{-1}W_2(\xi,\eta,z',v,\hat{\Theta}))\hat{\Theta}$$

An easy calculation yields that

$$e_1 = \{P(sI-A)^{-1}W_2(\xi,\eta,z',v,\hat{\Theta})\}\Phi + W_4(\xi,\eta,z')\Phi \tag{48}$$

Defining a new regressor

$$\tilde{W}_2 := \{P(sI-A)^{-1}W_2\} + W_4$$

so that

$$e_1 = \tilde{W}_2\Phi$$

From this form of the regressor the following theorem is obtained using the same techniques as in the proof of Theorem 2.2

Theorem 4.3 Model Reference Adaptive Control for Input-Output Linearizable Plants

Consider the system of (40) and the model of (35). Assume that A2 above holds and x_0^E *is a regular point for the controlled invariant distribution applied to* $\hat{\Sigma}$. *Suppose that the zero dynamics of* P *are exponentially attractive inside* X *and that* P *is input-output linearizable by static state feedback. Further, assume that the model is such that (46) holds for the true plant in a neighborhood of* x_0^E. *Then, under reparameterization with the control law of (43) the system* $\hat{\Sigma}$ *can be expressed as in equation (47) with* A *Hurwitz. Assume that the vector fields* $\psi(\xi,\eta,z'), \phi(\xi,\eta,z'), W_3(\xi,\eta,z',v,\hat{\Theta})$ *are Lipschitz continuous in their variables on* $X \times X_M$ *and* $v \in \mathbf{R}^m$. *Further, assume that* W_2, W_4 *have bounded derivatives with respect to* $\xi, \eta, z', \hat{\Theta}$.

Consider the update law

$$\dot{\Phi} = -\frac{\tilde{W}_2 e_1}{1 + \; trace(\tilde{W}_2^T \tilde{W}_2)} \tag{49}$$

with the augmented error defined as in equation (48).

Under this update law, the compensator Q *defined as in (30) initialized at the same initial state as the model, with input* u *as in (43), is such that*

$$\lim_{t\to\infty} y^{P\circ Q}(x,z,t) - y_M(z,t) = 0$$

for all (x,z) *in a neighborhood of* x_0, z_0 *provided that*

$$\begin{array}{rl} |v(t)| < L & \forall t \geq 0 \\ |z(t)| < \delta_1 & \forall t \geq 0 \\ |x_0|, |\Phi(0)| < \delta_2 & \end{array}$$

for suitable constants δ_1, δ_2, L.

Remarks

- A sufficient condition for $|z(t)| < \delta_1$ is that the model be asymptotically stable and $|z_0|, |v(t)|$ are sufficiently small.
- The condition (46) has to be assumed to be satisfied for the true plant and model. This is a *hypothesis* in addition to the prior information utilized earlier.

As in the non-adaptive case, this theorem can be usefully applied to the problem of adaptive tracking by static state feedback. The condition (46) can always be satisfied by an appropriate choice of the integers μ_i; in particular, by choosing μ_i to be the essential order of the i th output. The preceding theorem specializes as follows:

Theorem 4.4 Adaptive tracking for input-output linearizable plants
Consider the system of (40) and the model of (35). Assume that A2 above holds. Suppose that the zero dynamics of P are exponentially attractive inside X and that P is input-output linearizable by static state feedback. Then, under reparameterization with the control law of (43) the system $\hat{\Sigma}$ *with the model of (35) can be expressed as in equation (47) with A Hurwitz. Assume that the vector fields* $\psi(\xi, \eta, z'), \phi(\xi, \eta, z'), W_3(\xi, \eta, z', v, \hat{\Theta})$ *are Lipschitz continuous in their variables on* $X \times X_M$ *and* $v \in \mathbf{R}^m$*. Further, assume that* W_2, W_4 *have bounded derivatives with respect to* $\xi, \eta, z', \hat{\Theta}$*.*
Consider the update law

$$\dot{\Phi} = -\frac{\tilde{W}_2 e_1}{1 + \mathit{trace}(\tilde{W}_2^T \tilde{W}_2)} \tag{50}$$

with the augmented error defined as in equation (48). Under this update law, the control law of (43) yields asymptotic tracking, with bounded states, provided that

$$\sup_{t \geq 0}(|y_M(t)|, \ldots, |y_M^{mu-1}(t)|) < \delta_1 \;\; \mathit{and} |x_0|, |\Phi(0)| < \delta_2$$

for suitable constants δ_1, δ_2*.*

Remark: The knowledge of the integers μ_i is *prior information* needed for adaptive tracking, in addition to the prior information listed above for the zero dynamics algorithm.

5 Conclusions

This paper has investigated two schemes for the adaptive control of MIMO nonlinear systems:

- *Adaptive Linearization by Dynamic Compensation.* Here the underlying non-adaptive algorithm used dynamic extension repeatedly to make the augmented system have vector relative degree. At this point, it was possible to linearize (and decouple) the system.

- *Model Reference Adaptive Control by Static State Feedback* Here the underlying non-adaptive control law used static state feedback of the states of the plant and the model. The model reference scheme also had a specialization to a tracking controller, by a suitable choice of the model.

The proof of convergence of the adaptive linearization by dynamic compensation is complete. The proof of convergence in the case of model reference adaptive control by static state feedback needed the assumption that the plant was input-output linearizable by static state feedback. Thus, our results in the context of adaptive static state feedback model matching represent a first step towards a general theory of Model Reference Adaptive Control for MIMO nonlinear systems. Of course, in a more complete theory, we will also need a theory of adaptive observers to dispense with the necessity of using state feedback (for some recent work on this problem, which we have not discussed in the current paper see [Mar90], [KKM90]).

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