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Nonholonomic Motion Planning: Steering Using Sinusoids*

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Abstract

In this paper we investigate methods for steering systems with nonholonomic constraints between arbitrary configurations. Early work by Brockett derives the optimal controls for a set of canonical systems in which the tangent space to the configuration manifold is spanned by the input vector fields and their (first order) Lie brackets. Using Brockett's result as motivation, we derive suboptimal trajectories for systems which are not in canonical form and consider systems in which it takes more than one level of bracketing to achieve controllability. These trajectories use sinusoids at integrally related frequencies to achieve motion at a given bracketing level. We define a class of systems which can be steered using sinusoids (chained systems) and give conditions to convert arbitrary given systems to this form.

Keywords: nonholonomic motion planning, controllability, control Lie algebras

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NONHOLONOMIC MOTION PLANNING: STEERING USING SINUSOIDS

RICHARD M. MURRAY AND S. SHANKAR SASTRY

1. Introduction

Motion planning for robots has a rich history. The traditional difficulty in planning robot trajectories is the avoidance of obstacles. This problem is referred to as the piano mover's problem, in which we attempt to move an object (the piano) through a cluttered environment. This problem is solved by investigating the free configuration space of the piano—all configurations for which the piano does not intersect an obstacle. If the start and goal locations of the piano lie in the same connected component of the free configuration space, the motion planning problem is solvable.

In recent years there has been a great deal of activity in the generation of efficient motion planning algorithms for robots. Most of this work has concentrated on the global problem of determining a path when the obstacle positions are known and dynamic constraints are not considered. This has resulted in a rather complete understanding of the complexity of the computational effort required to plan the trajectories of robots to avoid both fixed and moving obstacles [8, 25, 19]. Other approaches include the use of potential functions for navigating in cluttered environments [22, 21] and compliant motion planning for navigating in the presence of uncertainty [10, 11, 34].

Our interests in motion planning are not along the lines of the aforementioned approaches, but are complementary: they involve motion planning in the presence of nonholonomic or non-integrable constraints. That is, we consider systems in which there are constraints on the velocities of the robots which cannot be integrated to give constraints which are exclusively a function of the configuration variables. These situations arise in a number of different ways and we describe a few of the sources of their origin:

(1) *Mobile robots navigating in a cluttered environment*
The kinematics of the drive mechanisms of robot carts result in constraints on the instantaneous velocities that can be achieved. For instance, a cart with two forward drive wheels and two back wheels cannot move sideways. This was first pointed out by Laumond in the context of motion planning for the Hilare mobile robot [26, 27].

(2) *Multifingered hands manipulating a grasped object*
If an object is twirled through a cyclic motion that returns the object to its

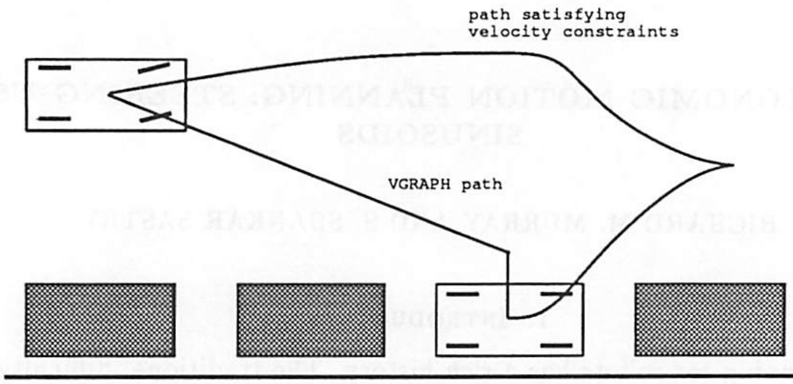

FIGURE 1. Paths generated by conventional path planners may ignore nonholonomic constraints. The straight line path in the figure indicates the path that a conventional path planner might generate. The curved path is one which satisfies the kinematic constraints of the car.

initial position and orientation, and the fingers roll without slipping on the surface of the object, the fingers do not necessarily return to their initial configurations. This feature can be used to plan the regrasp of a poorly grasped object or to choose the nature of this grasp. This application of nonholonomic motion planning was first pointed out by Li [32, 31] (see also [35]).

(3) *Space robotics*
Unanchored robots in space are difficult to control with either thrusters or internal motors since they conserve total angular momentum. This is anonintegrable constraint. The motion of astronauts on space walks is of this ilk, so that planning a strategy to reorient an astronaut is a nonholonomic motion planning problem [45]. Other examples of this effect include gymnasts and springboard divers.

Nonholonomic constraints arise either from the nature of the controls that can be physically applied to the system or from conservation laws which apply to the system. Conventional path planners implicitly assume that arbitrary motion in the configuration space is allowed as long as obstacles are avoided. If a system contains nonholonomic constraints, many of these path planners cannot be directly applied. If we attempt to ignore the constraint, the paths generated by a path planner may not be feasible (see Figure 1). For this reason, it is important to understand how to efficiently compute paths for nonholonomic systems.

To be more specific, we are interested in mechanical systems with linear velocity constraints of the form

$$\omega_i(x)\dot{x} = 0 \qquad i = 1, \cdots, k \tag{1}$$

Here x is the configuration of the system being controlled and $\omega_i(x)$ is a row vector in $\mathbb{R}^n$. These are constraints on the *velocities* of the system. In some cases, the constraints may be explicitly integrable, giving constraints of the form

$$h_i(x) = c_i$$

for some constant c_i. If this is possible, motion of the system is restricted to a level surface of h_i. Such a constraint is said to be *holonomic*. By choosing coordinates for the surface, configuration space methods can be applied. In the instance that there is only one constraint on the velocity of the system, it's integrability may be determined by checking the symmetry of the derivative of $\omega_1(x)$. There is no easy extension of this characterization to the case of multiple constraints.

A constraint is said to be *nonholonomic* if it cannot be written as an algebraic constraint in the configuration space. There are many types of nonholonomic constraints, corresponding to different physical situations.

It will be convenient for us to convert problems with nonholonomic constraints into steering problems for control systems. Consider the problem of constructing a path $x(t) \in \mathbb{R}^n$ between a given x_0 and x_1 subject to k constraints which are linear in $\dot{x}$:

$$\omega_i(x)\dot{x} = 0 \qquad i = 1, \cdots, k$$

We assume the ω_i's are smooth and linearly independent over the ring of smooth functions. Formally, these constraints are exterior differential one-forms on $\mathbb{R}^n$. Specific examples of such systems are given in Section 2.4. Rather than use the machinery of exterior differential systems, we convert the problem to one in control theory. Roughly speaking, we would like to convert the constraint specification from describing the directions in which we can't move to those in which we can. Formally, we choose a basis for the right null space of the constraints, denoted by $g_i(x) \in \mathbb{R}^n$, $i = 1, \cdots, n-k$. The path planning problem can be restated as finding an input function, $u(t) \in \mathbb{R}^m$, such that the control system

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m$$

is driven from x_0 to x_1. It will be shown that if the ω_i's are smooth and linearly independent (over the ring of smooth functions), then the g_i's inherit these properties.

The outline of this paper is as follows: in Section 2, we collect some mathematical preliminaries from the literature on controllability of nonlinear systems and on classification of free Lie algebras. These are drawn from classical references in control theory [4, 17, 18, 36, 40] and Lie algebras [15, 43]. In Section 3, using some outstanding results of Brockett on optimal steering of certain classes of systems as motivation [5], we discuss the use of sinusoidal inputs for steering systems of first order, i.e., systems where controllability is achieved after just one level of Lie brackets of the input vector fields. Section 4 attempts to expand the domain of applicability of these results to more complex systems, where several orders of Lie brackets are needed to obtain the full Lie algebra associated with the input distribution. The style of the paper is self-contained so as to make it accessible to both robotics and control researchers and several examples are sustained through the paper.

A target problem which we set ourselves at the start of this research was that of parking of a car with N trailers. This problem remains unsolved and indeed has generated some fascinating new ideas in the field. It is not a "toy problem" since efforts are underway to automate baggage handling by carts with multiple trailers in airports (not to mention trucks with multiple trailers). However, it is fair to say that the study of nonholonomic motion planning is in it's infancy. There have however been notable contributions by Laumond et al. [29, 30, 26, 20, 28] and by Barraquand and Latombe [2] on motion planning for mobile robots in a cluttered field. While this work represents important initial progress, we feel that less computationally intensive and more insightful approaches are possible by conducting a systematic research program on motion planning of dynamical systems with nonholonomic constraints. We are joined by by several complementary efforts, notably those of Li and co-workers [32, 12] and Sussmann and co-workers [24, 42]. We have also applied the techniques of this paper to steering of space robots using sinusoids in [45].

2. Mathematical Preliminaries

This section collects a variety of results from differential geometry and nonlinear control theory which will prove useful in studying nonholonomic systems. There are several good references for the material presented here, although no single book is adequate. For basic definitions and concepts in differential geometry, see Boothby [3] or Spivak [39]. A good introduction to nonlinear control theory which includes many of the necessary differential geometric concepts can be found in Isidori [18] or Nijmeijer and van der Schaft [36]. We begin with a brief review of differential geometry for the purpose of fixing notation.

2.1. Differential Geometry. We restrict our attention to a smooth (C^∞) n-dimensional manifold M. Let T_xM denote the tangent space to M at a point $x \in M$. A *vector field* on M is a smooth map which assigns to each point on $x \in M$ a tangent vector $f(x) \in T_xM$. In local coordinates, we represent f as a column vector whose elements depend on x,

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

Alternatively, if $(x_1, \cdots, x_n)$ is a set of local coordinates for M, we write

$$f(x) = f_1(x)\frac{\partial}{\partial x_1} + \cdots + f_n(x)\frac{\partial}{\partial x_n}$$

The symbol $\frac{\partial}{\partial x_i}$ is to be thought of as a basis element for the tangent space with respect to a given set of local coordinates. A vector field is smooth if each $f_i(x)$ is smooth; this can be shown to be independent of choice of coordinates.

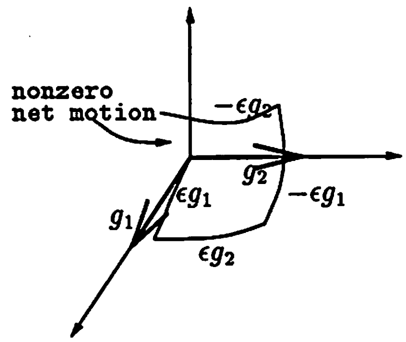

FIGURE 2. A Lie bracket motion

To any vector field we define the *flow* of a vector field to represent the action of integration along a vector field. Specifically, $\phi_t^f : M \to M$ satisfies

$$\frac{d}{dt}\phi_t^f(x) = f(\phi_t^f(x)) \qquad x \in M$$

A vector field is *complete* if its flow is defined for all t. All differentiable vector fields are locally complete. For each given t, ϕ_t^f is a local diffeomorphism of M onto itself and $\phi_t^f \circ \phi_s^f = \phi_{t+s}^f$ for all t, s.

Given two vector fields f and g, we define the *Lie bracket* as

$$[f,g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g$$

We can interpret this quantity using flows. Consider the flow depicted in Figure 2; the net motion satisfies

$$\phi_\epsilon^{-g} \circ \phi_\epsilon^{-f} \circ \phi_\epsilon^{g} \circ \phi_\epsilon^{f}(x_0) = \epsilon^2[f,g](x_0) + o(\epsilon^2). \tag{2}$$

The Lie bracket is the infinitesimal motion that results from flowing around a square defined by two tangent vectors. If $[f,g] = 0$ then f and g commute and it can be shown that the right hand side of equation (2) is identically zero; i.e., we return to the starting point. A *Lie product* is a nested set of Lie brackets, for example,

$$[[f,g],[f,[f,g]]]$$

A *distribution* assigns a subspace of the tangent space to each point in M in a smooth way. A special case is a distribution defined by a set of smooth vector fields, $g_1, \cdots, g_m$. In this case we define the distribution as

$$\Delta = \operatorname{span}\{g_1, \cdots, g_m\}$$

where we take the span over the set of smooth real-valued functions on M. At any point the distribution is a linear subspace of the tangent space

$$\Delta_x = \operatorname{span}\{g_1(x), \cdots, g_m(x)\} \subset T_x M$$

A distribution is *involutive* if it is closed under the Lie bracket:

$$\Delta \text{ involutive} \iff \forall f,g \in \Delta, \quad [f,g] \in \Delta$$

For a finite dimensional distribution it suffices to check that the basis elements are contained in the distribution. The *involutive closure* of a distribution, denoted $\overline{\Delta}$, is the closure of Δ under bracketing. That is, $\overline{\Delta}$ is the smallest distribution containing Δ such that if $f,g \in \overline{\Delta}$ then $[f,g] \in \overline{\Delta}$.

A submanifold $N \subseteq M$ is an *integral manifold* of Δ if $\Delta_x = T_xN$ at every x. That is, the distribution spans the tangent space of the submanifold at every point. A distribution is *integrable* if at every point $x \in M$ there exists a manifold $N \subseteq M$ which is an integral manifold for Δ. Integral manifolds are related to involutive distributions by the following theorem:

Theorem 1 (Frobenius). *A distribution is integrable if and only if it is involutive.*

The following interpretation of Frobenius' theorem is also useful. If Δ is an m-dimensional involutive distribution, then locally there exist $n-m$ functions $h_i \colon M \to \mathbf{R}$ such that integral manifolds of Δ are given by the level surfaces of $h = (h_1, \cdots, h_{n-m})$. These level surfaces form a *foliation* of M. A single level surface is called a *leaf of the foliation.*

Associated with the tangent space T_xM is the dual space T_x^*M, the set of linear functions on T_xM. Just as we defined vector fields on T_xM, on T_x^*M we can define a *one-form*: for each $x \in M$, $\omega(x) \in T_x^*M$. In local coordinates we represent a smooth one form as

$$\omega(x) = \omega_1(x)dx_1 + \cdots + \omega_n(x)dx_n$$

where each ω_i is smooth. The symbols dx_i represent the basis dual to the basis $\frac{\partial}{\partial x_i}$ on T_xM and are defined as

$$dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. A one-form acts on a vector field to give a real-valued function on M,

$$\omega \cdot f = \left(\sum_i \omega_i dx_i\right) \cdot \left(\sum_j f_j \frac{\partial}{\partial x_j}\right) = \sum_i \omega_i f_i$$

A *codistribution* assigns a subspace of T_x^*M to each $x \in M$.

2.2. Nonlinear Controllability. We begin the study of controllability by converting a set of linear velocity constraints into a control system. Consider the problem of constructing a path $x(t) \in M$ between a given x_0 and x_1 subject to the constraints

$$\omega_i(x)\dot{x} = 0 \qquad i = 1, \cdots, k$$

The ω_i's are linear functions on the tangent spaces of M, i.e., one-forms. We assume that the ω_i's are smooth and linearly independent over the ring of smooth functions.

Proposition 2. *There exist vector fields $g_j(x)$, $j = 1, \cdots, n-k$ that annihilate the $\omega_i(x)$'s ($\omega_i(g_j) = 0$) such that the g_i's are smooth and linearly independent over the ring of smooth functions.*

Proof. The ω's form a codistribution of dimension k in $\mathbf{R}^n$. We can choose local coordinates and a basis such that

$$\omega_i = dx_i + \sum_{l=k+1}^{n} \alpha_{il} dx_l \qquad i = 1, \cdots, k$$

where $\alpha_{il} \colon \mathbf{R} \to \mathbf{R}$ is a smooth function. Set

$$g_j = \frac{\partial}{\partial x_{j+k}} + \sum_{l=1}^{k} -\alpha_{l(j+k)} \frac{\partial}{\partial x_l} \qquad j = 1, \cdots, n-k$$

The g_j's are linearly independent. Furthermore they annihilate the constraint since

$$\begin{aligned} \omega_i \cdot g_j &= (dx_i + \textstyle\sum_{l=k+1}^{n} \alpha_{il} dx_l) \cdot (\frac{\partial}{\partial x_{j+k}} + \sum_{l=1}^{k} -\alpha_{l(j+k)} \frac{\partial}{\partial x_l}) \\ &= \alpha_{i(j+k)} - \alpha_{i(j+k)} = 0 \end{aligned}$$

Smoothness of $\{g_j\}$ follows directly from smoothness of α_{ij} and hence $\{\omega_i\}$. □

We now restrict our attention to *drift free* control systems of the form

$$\Sigma: \qquad \dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \qquad \begin{matrix} x \in M \\ u \in U \subset \mathbf{R}^m \end{matrix} \tag{3}$$

In view of the previous proposition, we assume that the g_i's are smooth, linearly independent vector fields on M. We will further assume that their flows are defined for all time (i.e., g_i is *complete*). We wish to determine conditions under which we can steer from $x_0 \in M$ to an arbitrary $x_1 \in M$ by appropriate choice of $u(\cdot)$. We review the formulation of Hermann and Krener [17], specialized to systems having the form of equation (3).

A system Σ is controllable if for any $x_0, x_1 \in M$ there exists a $T > 0$ and $u \colon [0,T] \to U$ such that Σ satisfies $x(0) = x_0$ and $x(T) = x_1$. A system is *small-time locally controllable* at x_0 if we can reach nearby points in arbitrarily small amounts of time and stay near to x_0 at all times. Given an open set $V \subseteq M$, define

$$\begin{aligned} \mathcal{R}^V(x_0, T) = \{&x \in M : \exists\, u \colon [0,T] \to U \text{ that steers } \Sigma \text{ from } x(0) = x_0 \text{ to} \\ &x(T) = x_1 \text{ and satisfies } x(t) \in V \text{ for } 0 \le t \le T\} \end{aligned}$$

$\mathcal{R}^V(x_0, T)$ is the set of states which are reachable from x_0 in time T that also remain in V. We also define

$$\mathcal{R}^V(x_0, \le T) = \bigcup_{0 < \tau \le T} \mathcal{R}^V(x_0, \tau)$$

A system is small-time locally controllable (*locally controllable* for brevity) if $\mathcal{R}^V(x_0, \le T)$ contains a neighborhood of x_0 for all neighborhoods V of x_0 and $T > 0$. Define $\Delta = \{g_1, \cdots, g_m\}$ and let $\overline{\Delta}$ be the involutive closure of Δ. We wish

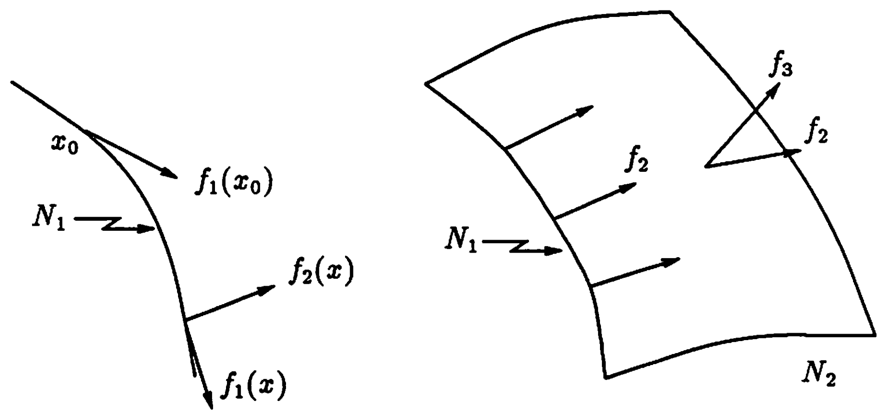

FIGURE 3. Proof of local accessibility. At each step we can find a vector field which is not in N_k.

to establish the following implications for a given system Σ, in a neighborhood of a point:

$$\overline{\Delta}_x = T_xM \;\Rightarrow\; \operatorname{int}\mathcal{R}^V(x_0, \leq T) \neq \{\} \iff \Sigma \text{ is locally controllable}$$

This result is referred to as *Chow's Theorem* [9] and asserts that the drift-free system Σ is controllable if the involutive closure of the vector fields spans TM.

Theorem 3 (Controllability Rank Condition).
If $\overline{\Delta}_x = T_xM$ for all x is some neighborhood of x_0, then for any $T > 0$ and neighborhood V of x_0, $\operatorname{int}\mathcal{R}^V(x_0, \leq T)$ is nonempty.

Proof. The proof is by recursion. Choose $f_1 \in \Delta$. For $\epsilon_1 > 0$ sufficiently small,

$$N_1 = \{\phi^{f_1}_{t_1}(x_0) : 0 < t_1 < \epsilon_1\}$$

is a manifold of dimension 1 which contains points arbitrarily close to x_0; without loss of generality, we can take $N_1 \subset V$. Assume $N_k \subset V$ is a k-dimensional manifold. If $k < n$, there exists $x \in N_k$ and $f_{k+1} \in \Delta$ such that $f_{k+1} \notin T_xN_k$. If this were not so then $\Delta_x \in T_xN_k$ for any x in some open set $W \subset N_k$, which would imply $\overline{\Delta}|_W \subset TN_k$. This cannot be true since $\dim \overline{\Delta}_x = n > \dim N_k$. For ϵ_{k+1} sufficiently small

$$N_{k+1} = \{\phi^{f_{k+1}}_{t_{k+1}} \circ \cdots \circ \phi^{f_1}_{t_1}(x_0) : 0 < t_i < \epsilon_i,\, i = 1, \cdots, k+1\}$$

is a $k+1$ dimensional manifold. Since ϵ can be made arbitrarily small, we can assume $N_{k+1} \subset V$.

If $k = n$, $N_k \subset V$ is an n-dimensional manifold and by construction $N_k \subset \mathcal{R}^V(x_0, \leq \epsilon_1 + \cdots + \epsilon_n)$. Hence $\mathcal{R}^V(x_0, \epsilon)$ contains an open set. By restricting each $\epsilon_i \leq T/n$, we can find such an open set for any $T > 0$. This proof is illustrated in Figure 3. $\square$

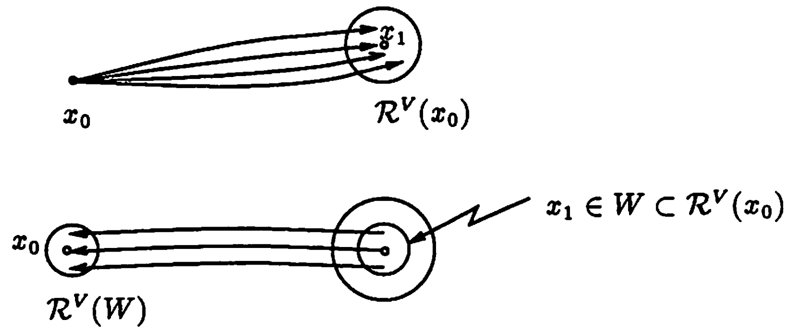

FIGURE 4. Proof of local controllability. To show $\mathcal{R}^V(x_0)$ contains a neighborhood of the origin, we move to any point x_1 and map a neighborhood of x_1 to a neighborhood of x_0 by reversing our original path.

Theorem 4 (Local Controllability).
$\operatorname{int} \mathcal{R}^V(x_0, \leq T)$ is nonempty for all neighborhoods V of x_0 and $T > 0$ $\iff$ Σ is locally controllable at x_0.

Proof. The sufficiency follows from the definition of locally controllable. To prove necessity, we need to show that $\mathcal{R}^V(x_0, \leq T)$ contains a neighborhood of x_0. Choose a piecewise constant $u\colon [0, T/2] \to U$ such that u steers x_0 to some $x_1 \in \mathcal{R}^V(x_0, \leq T/2)$ and $x(t) \in V$. Let ϕ_t^u be the flow corresponding to this input (as given in the proof of the previous theorem). Since Σ is symmetric, we can flow backwards from x_1 to x_0 using $u'(t) = -u(T/2 - t)$, $t \in [0, T/2]$. The flow corresponding to u' is $(\phi_t^u)^{-1}$. By continuity of the flow, there exists $W \subset \mathcal{R}^V(x_0, T/2)$ such that $x_1 \in W$ and $(\phi_t^u)^{-1}(W) \subset V$ for all t. Furthermore, $(\phi_{T/2}^u)^{-1}(W)$ is a neighborhood of x_0. It follows that $\mathcal{R}^V(x_0, \leq T)$ contains a neighborhood of x_0 since we can concatenate the inputs which steer x_0 to $x_1 \in W$ with u' to obtain an open set containing x_0. This is illustrated in Figure 4. □

2.3. Classification of Lie Algebras. We now develop some concepts which allow us to classify nonholonomic systems. A more complete treatment can be found in the work of Vershik [13, 44]. Basic facts concerning Lie algebras are taken from Varadarajan [43]. Let $\Delta = \operatorname{span}\{g_1, \cdots, g_m\}$ be the distribution associated with the control system (3). Define $G_1 = \Delta$ and

$$G_i = G_{i-1} + [G_1, G_{i-1}]$$

where

$$[G_1, G_{i-1}] = \operatorname{span}\{[g, h] : g \in G_1, h \in G_{i-1}\}$$

The set of all G's defines the *filtration* associated with a distribution. Each G_i is defined to be spanned by the input vector fields plus the vector fields formed by taking up to $i-1$ Lie brackets. The Jacobi identity implies $[G_i, G_j] \subset [G_1, G_{i+j-1}] \subset G_{i+j}$.

A filtration is *regular* in a neighborhood U of x_0 if

$$\text{rank } G_i(x) = \text{rank } G_i(x_0) \qquad \forall x \in U$$

We say a system is regular if the corresponding filtration is regular. If a filtration is regular, then at each step of its construction, G_i either gains dimension or the construction terminates. If rank G_{i+1} = rank G_i then G_i is involutive and hence $G_{i+j} = G_i$ for all $j \geq 0$. Clearly rank $G_i \leq n$ and hence if a filtration is regular, then there exists an integer $p < n$ such that $G_i = G_{p+1}$ for all $i \geq p+1$. We refer to p as the *degree of nonholonomy* of the distribution.

For a regular system, Chow's theorem is particularly easy to prove.

Theorem 5 (Chow's Theorem for Regular Systems).
For a regular system, a path exists between two arbitrary points in an open set $U \subset M$ if and only if $G_p(x) = T_xM \approx \mathbb{R}^n$ for all $x \in U$.

Proof. (Necessity) Suppose that $G_p(x) = \mathbb{R}^k$ for $k < n$. Since the system is regular, G_p is involutive and Frobenius' theorem implies that G_p is integrable. Let N be the integrable submanifold for G_p, of dimension k. Since all trajectories of the system are confined to N, any $x \in M$ such that $x \notin N$ is not reachable.

(Sufficiency) Given two points $x_0, x_1 \in M$, we can connect the points with a line $x_0 + t(x_1 - x_0)$. At each point on this line, the reachable states form an open set (since the system is controllable). Since the line has finite length, it is a compact set (in the relative topology) and hence we can pick a finite subcover of reachable sets. Since a path exists between each intersecting pair of this finite subcover, a path between x_0 and x_1 exists by concatenating path segments. □

A system (or distribution) satisfying the conditions of the theorem is said to be *maximally nonholonomic*. This version of Chow's theorem is considerably weaker than our previous version, which holds for non-regular systems. If a regular system is not maximally nonholonomic, then by Frobenius' theorem we can restrict ourselves to a manifold on which the system is maximally nonholonomic.

It is also useful to record the dimension of each G_i. For a regular system, we define the *growth vector* $r \in \mathbb{Z}^{p+1}$ as

$$r_i = \text{rank } G_i$$

We define the *relative growth vector* $\sigma \in \mathbb{Z}^{p+1}$ as $\sigma_i = r_i - r_{i-1}$ and $r_0 := 0$. The growth vector for a system is a convenient way to represent information about the associated control Lie algebra. For a distribution with finite rank, the growth vector is bounded from above at each step. To properly determine this bound, we must determine the rank of G_i taking into account skew-symmetry and the Jacobi identity. A careful calculation [37] gives

$$\overline{\sigma}_i = \frac{1}{i}\left((\overline{\sigma}_1)^i - \sum_{j|i,j<i} j\overline{\sigma}_j\right) \qquad i > 1 \tag{4}$$

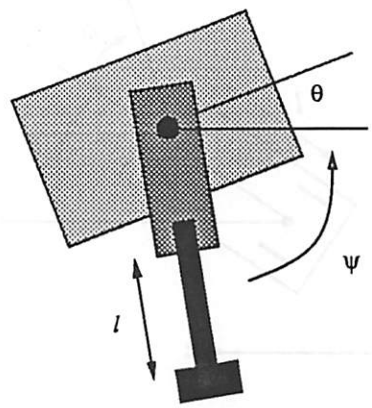

FIGURE 5. A simple hopping robot. The robot consists of a leg which can both rotate and extend. The configuration of the mechanism is given by the angle of the body and the angle and length (extension) of the leg.

where $\overline{\sigma}_i$ is the maximum relative growth at the i^{th} stage and $j|i$ means all integers j such that j divides i. If $\sigma_i = \overline{\sigma}_i$ for all i, we say Δ has *maximum growth.*

2.4. Examples of Nonholonomic Systems. To illustrate the classification of nonholonomic systems, we present several detailed examples. These examples are used in later sections as a basis for testing planning algorithms.

Example 1 (Hopping robot). As our first example, we consider a hopping robot as shown in Figure 5. This robot consists of a body with an actuated leg that can rotate and extend. We are interested in studying the robot when it is in free flight. The "constraint" on the system is conservation of angular momentum.

The (ψ, l, θ) be the body angle, leg extension and leg angle of the robot. For simplicity, we take the body mass to be 1 and concentrate the mass of the leg, m_l, at the foot. The upper leg length is also taken to be 1, with l representing the extension of the leg past this point. Since we control the leg angle and extension directly, we choose them as our inputs. The angular momentum of the robot is given by

$$\dot{\theta} + m_l(l+1)^2(\dot{\theta} + \dot{\psi}) = 0$$

Thus our equations become

$$\begin{aligned} \dot{\psi} &= u_1 \\ \dot{l} &= u_2 \\ \dot{\theta} &= -\frac{m_l(l+1)^2}{1+m_l(l+1)^2} u_1 \end{aligned}$$

In vector field form we have

$$\begin{aligned} g_1 &= \frac{\partial}{\partial \psi} - \frac{m_l(l+1)^2}{1+m_l(l+1)^2}\frac{\partial}{\partial \theta} \\ g_2 &= \frac{\partial}{\partial l} \\ g_3 = [g_1, g_2] &= \frac{2m_l(l+1)}{(1+m_l(l+1)^2)^2}\frac{\partial}{\partial \theta} \end{aligned}$$

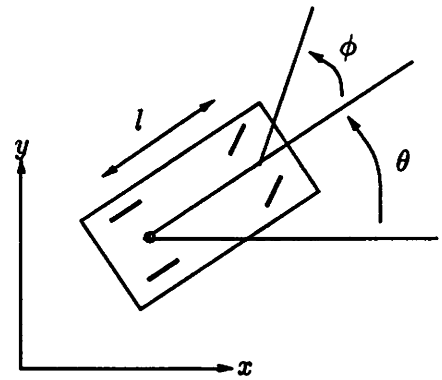

FIGURE 6. Kinematic model of an automobile. The configuration of the car is determined by the Cartesian location of the back wheels, the angle the car makes with the horizontal and the steering wheel angle relative to the car body. The two inputs are the velocity of the rear wheels and the steering velocity.

In a neighborhood of $l = 0$, $\{g_1, g_2, g_3\}$ is full rank and hence the hopping robot has degree of nonholonomy 3 with growth vector $(2, 3)$.

Example 2 (Kinematic car). Figure 6 shows an automobile with front and rear tires. The rear tires are aligned with the car while the front tires are allowed to spin about the vertical axes. To simplify the derivation, we model the front and rear pairs of wheels as single wheels at the midpoints of the axles. The constraints on the system arise by allowing the wheels to roll and spin, but not slip.

Let (x, y, ϕ, θ) denote the configuration of the car, parameterized by the location of the rear wheel(s), the angle of the car body with respect to the horizontal (θ), and the steering angle with respect to the car body (ϕ). The constraints for the front and rear wheels are formed by writing the sideways velocity of the wheels:

$$\begin{aligned}\tfrac{d}{dt}(x + l\cos\theta)\cdot\sin(\theta+\phi) - \tfrac{d}{dt}(y + l\sin\theta)\cdot\cos(\theta+\phi) &= 0\\ \dot{x}\sin\theta - \dot{y}\cos\theta &= 0\end{aligned}$$

Written as one forms we have

$$\begin{aligned}\omega_1 &= \sin(\theta+\phi)dx - \cos(\theta+\phi)dy - l\cos\phi d\theta\\ \omega_2 &= \sin\theta dx - \cos\theta dy\end{aligned}$$

Converting this to a control system gives

$$\begin{aligned}\dot{x} &= \cos\theta\, u_1\\ \dot{y} &= \sin\theta\, u_1\\ \dot{\phi} &= u_2\\ \dot{\theta} &= \tfrac{1}{l}\tan\phi\, u_1\end{aligned}$$

For this choice of vector fields, u_1 corresponds to the forward velocity of the rear wheels of the car and u_2 corresponds to the velocity of the steering wheel.

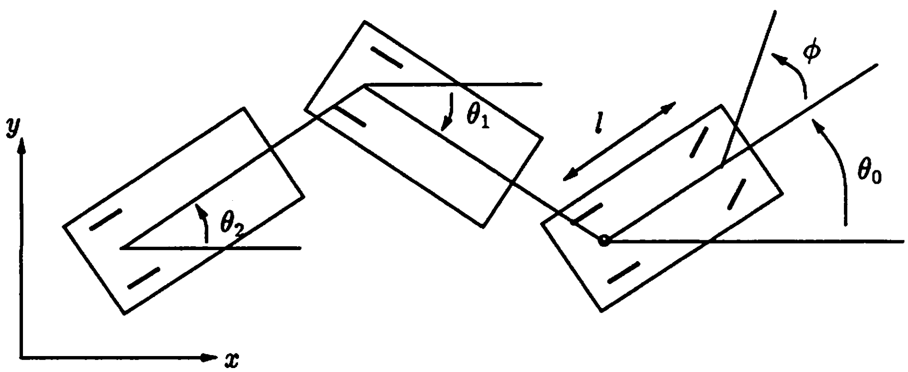

FIGURE 7. Kinematic car with trailers. The trailer configuration is described the the angle the trailer makes with the horizontal, ψ. The rear wheels of the trailer are fixed and constrained to move along the line in which they point or rotate about their center. The inputs to the system are the inputs to the tow car: the driving velocity (of the front wheels) and the steering velocity. This system is an example of a fourth degree system; higher degree systems can be generated by adding extra trailers.

To calculate the growth vector, we build the filtration

$$\begin{aligned}
g_1 &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + \frac{1}{l}\tan\phi\frac{\partial}{\partial\theta} \\
g_2 &= \frac{\partial}{\partial\phi} \\
g_3 = [g_1, g_2] &= \frac{-1}{l\cos^2\phi}\frac{\partial}{\partial\theta} \\
g_4 = [g_1, g_3] &= \frac{-\sin\theta}{l\cos^2\phi}\frac{\partial}{\partial x} + \frac{\cos\theta}{l\cos^2\phi}\frac{\partial}{\partial y} \\
g_5 = [g_2, g_3] &= \frac{-2\tan\phi}{l\cos^2\phi}\frac{\partial}{\partial\theta}
\end{aligned}$$

$\{g_1, g_2, g_3, g_4\}$ are linearly independent when $\phi \neq \pm\pi$. Thus the system has degree of nonholonomy 2 with growth vector $r = (2,3,4)$ and relative growth vector $\sigma = (2,1,1)$. The system is regular away from $\phi = \pm\pi$, where g_1 is undefined.

Example 3 (Car with N trailers). Figure 7 shows a car with N trailers attached. We attach the hitch of each trailer to the center of the rear axle of the previous trailer. The wheels of the individual trailers are aligned with the body of the trailer. The constraints are again based on allowing the wheels only to roll and spin, but not slip. The dimension of the state space is $4 + N$ with 2 controls.

We parameterize the configuration by the states of the automobile plus the angles of each of the trailers with respect to the horizontal. For consistency we will write θ_0 for the angle of the car. Calculation of the constraints becomes tedious since we have to write the velocity of the wheels of each trailer, which depend on all previous trailers. Instead, we choose to use the same inputs as the automobile and calculate the effect on the trailer angles.

At each trailer, we can write the hitch velocity as the sum of two components: the velocity in the direction the trailer is pointing and its perpendicular. The perpendicular component causes the trailer to spin. Letting v_{i-1} be the forward velocity of the previous trailer, we have

$$\begin{aligned}\dot\theta_i &= \tfrac{1}{d_i}\sin(\theta_{i-1}-\theta_i)v_{i-1}\\ v_i &= \cos(\theta_{i-1}-\theta_i)v_{i-1}\end{aligned}$$

Aggregating these equations gives

$$\begin{aligned}\dot x &= \cos\theta_0\, u_1\\ \dot y &= \sin\theta_0\, u_1\\ \dot\phi &= u_2\\ \dot\theta_0 &= \tfrac{1}{l}\tan\phi\, u_1\\ \dot\theta_i &= \tfrac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_{j-1}-\theta_j)\right)\sin(\theta_{i-1}-\theta_i)\, u_1\end{aligned} \tag{5}$$

The filtration corresponding to the N trailer problem is very complex. For small values of N, controllability can be verified directly. For the general case, a very detailed and well-organized calculation by Laumond [28] shows that the system is controllable with degree of nonholonomy $N+2$ and relative growth vector $\sigma = (2,1,\cdots,1)$.[1]

2.5. Philip Hall Bases for Lie Algebras. We will be interested in the sequel in constructing nonholonomic systems which are canonical in the sense that they allow for the maximal growth of the filtration associated with a given set of vector fields $\Delta = \text{span}\{g_1,\ldots,g_m\}$.

To construct such systems with a given number of inputs and degree of nonholonomy, it is necessary to introduce some additional machinery. In constructing canonical nonholonomic systems we must observe the fundamental restrictions imposed by the Lie bracket: skew-symmetry and the Jacobi identity. Our search for a set of vector fields which have a given degree of nonholonomy is equivalent to searching for a basis for a free, finitely generated, finite-dimensional Lie algebra. One basis set for such a distribution is a Philip Hall basis [15, 37]:

Given a set of generators $\{X_1,\cdots,X_m\}$, we define the length of a Lie product recursively as

$$\begin{aligned}l(X_i) &= 1 \qquad\qquad i=1,\cdots,m\\ l([A,B]) &= l(A)+l(B)\end{aligned}$$

where A and B are themselves Lie products. A Lie algebra is *nilpotent* if there exists an integer k such that all Lie products of length greater than k are zero. k is called the degree of nilpotency. A nilpotent Lie algebra is finite-dimensional. A *P. Hall basis* is an ordered set of Lie products $H = \{B_i\}$ satisfying

(PH1) $X_i \in H$, $i = 1,\cdots,m$

[1] Laumond uses a slightly different system, obtained by ignoring ϕ and choosing u_1 and $u_1\tan\phi$ as inputs. Since setting $u_1 = 0$ allows us to steer ϕ independently, controllability for the system given here follows from Laumond's result.

(PH2) If $l(B_i) < l(B_j)$ then $B_i < B_j$

(PH3) $[B_i, B_j] \in H$ if and only if

(a) $B_i, B_j \in H$ and $B_i < B_j$ *and*

(b) either $B_j = X_k$ for some k or

$B_j = [B_l, B_r]$ with $B_l, B_r \in H$ and $B_l \leq B_i$

The proof that a P. Hall basis is a basis for the free Lie algebra generated by $\{X_1, \cdots, X_m\}$ can be found in [15, 37]. The construction above is a clever way of keeping track of the conditions imposed by the skew symmetry and the Jacobi identity.

A P. Hall basis with degree of nilpotency k can be constructed from a set of generators using the definition. The simplest approach is to construct all possible Lie products with length less than k and use the definition to eliminate elements which fail to satisfy one of the properties. In practice, the basis can be built in such a way that only (PH3) need be checked.

Example 4. A basis for the nilpotent Lie algebra of degree 3 generated by $\{X, Y, Z\}$ is

$$\begin{array}{llll} X & Y & Z & \\ [X,Y] & [X,Z] & [Y,Z] & \\ [X,[X,Y]] & [X,[X,Z]] & [Y,[X,Y]] & [Y,[X,Z]] \\ [Y,[Y,Z]] & [Z,[X,Y]] & [Z,[X,Z]] & [Z,[Y,Z]] \end{array}$$

Note that $[X,[Y,Z]]$ does not appear since

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

and two of the three terms are already present.

Example 5. A larger example, which we will use in the sequel, is a basis for a Lie algebra of degree 5 with 2 generators:

$$\begin{array}{rl} B_1 - B_2: & X \quad Y \\ B_3: & [X,Y] \\ B_4 - B_5: & [X,[X,Y]] \quad [Y,[X,Y]] \\ B_6 - B_8: & [X,[X,[X,Y]]] \quad [Y,[X,[X,Y]]] \quad [Y,[Y,[X,Y]]] \\ B_9 - B_{14}: & [X,[X,[X,[X,Y]]]] \quad [Y,[X,[X,[X,Y]]]] \quad [Y,[Y,[X,[X,Y]]]] \\ & [Y,[Y,[Y,[X,Y]]]] \quad [[X,Y],[X,[X,Y]]] \quad [[X,Y],[Y,[X,Y]]] \end{array}$$

Note that B_{13} and B_{14} have the form $[B_3, B_4]$ and $[B_3, B_5]$, requiring careful checking of the condition (PH3).

3. Steering Controllable Systems Using Sinusoids

In this section, we investigate methods for steering systems with nonholonomic constraints between arbitrary configurations. Early work by Brockett derives the optimal controls for a set of canonical systems in which the tangent space to the configuration manifold is spanned by the input vector fields and their (first order) Lie brackets. Using Brockett's result as motivation, we derive suboptimal trajectories for systems which are not in canonical form and consider systems in which it takes more than one level of bracketing to achieve controllability. These trajectories use sinusoids at integrally related frequencies to achieve motion at a given bracketing level. Examples and simulation results are presented.

We consider systems of the form

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \qquad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m \tag{6}$$

with $\{g_i\}$ a set of smooth, linearly independent vector fields in some neighborhood of the origin. We also assume that the system is regular (as defined in Section 2.3) and hence has a well-defined degree of nonholonomy and growth vector.

3.1. First Degree Systems. Control systems in which the first level of brackets together with the input vector fields span the tangent space at each configuration arise in many areas. In classical mechanics, systems with growth vector $r = (n-1, n)$ are called contact structures [1]. A version of the Darboux theorem asserts that for these systems the corresponding constraint can be written as

$$dx_3 = x_2 dx_1$$

(using the notation of exterior differential forms). In $\mathbf{R}^3$ and using control system form, this becomes

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \end{aligned} \tag{7}$$

Brockett considered a more general version of this system [5]; we review his results here. Consider a control system as in equation (6) that is maximally nonholonomic with growth vector $(m, n) = (m, \frac{m(m+1)}{2})$. We would like to find an input $u(t)$ on the interval 0 to 1 which steers the system between an arbitrary initial and final configuration and minimizes

$$\int_0^1 |u|^2 dt$$

This problem is related to finding the geodesics associated with a singular Riemannian metric (Carnot-Caratheodory metric). To solve the problem, Brockett considers a class of systems which have a special canonical form. An equivalent form, which is more useful for our purposes, is

$$\begin{aligned} \dot{x}_i &= u_i \qquad i = 1, \cdots, m \\ \dot{x}_{ij} &= x_i u_j \quad i < j \end{aligned} \tag{8}$$

We see that if $m = 2$, this is exactly the contact system (7). It can be shown that the input vector fields and their pairwise brackets span $\mathbf{R}^n$ and hence the system is controllable with degree of nonholonomy equal to 1.

To find the optimal input between two points, we construct the Lagrangian

$$L(x, \dot{x}) = \sum_{i=1}^{m} \dot{x}_i^2 + \sum_{i,j} \lambda_{ij}(\dot{x}_{ij} - x_i \dot{x}_i) \tag{9}$$

Here we have used the fact that $u_i = \dot{x}_i$. The λ_{ij}'s are the Lagrangian multipliers associated with the constraint imposed by the control system. Substituting equation (9) into the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

it can be shown that the input must satisfy

$$u = e^{\Lambda t} u_0$$

where Λ is a constant skew-symmetric matrix. Thus the inputs are sinusoids at various frequencies. Unfortunately, even for very simple problems, determining Λ and u_0 given an initial and final configuration is very difficult.

A great deal of simplification occurs if we consider moving between configurations where $x_i(1) = x_i(0)$. In this instance the eigenvalues of Λ must be multiples of 2π and Brockett showed that the optimal inputs are sinusoids at integrally related frequencies, namely $2\pi, 2\cdot 2\pi, \cdots, \frac{m}{2}\cdot 2\pi$. This simplifies the problem tremendously and for many examples reduces the search to that of finding u_0. We use this result to propose the following algorithm for steering systems of this type:

Algorithm 1. (Steering first-order canonical systems).

1. Steer the x_i's to their desired values using any input and ignoring the evolution of the x_{ij}'s.
2. Using sinusoids at integrally related frequencies, find u_0 such that the input steers the x_{ij}'s to their desired values. By the choice of input, the x_i's are unchanged.

The resulting trajectories are suboptimal but easily computable and have several nice properties which we will explore.

Example 6. We consider as an example a kinematic hopping robot, as shown in Figure 5. This example has been studied by Li, Montgomery and Raibert [33] using holonomy methods. We wish to reorient the body of robot while in midair and bring the leg rotation and extension to a desired final value. The kinematic equations of the robot (in center of mass coordinates) can be written as

$$\begin{aligned} \dot{\psi} &= u_1 \\ \dot{l} &= u_2 \\ \dot{\theta} &= -\tfrac{m_l(l+1)^2}{1+m_l(l+1)^2} u_1 \end{aligned}$$

where we have used units such that the mass of the body is 1 and the length of the leg at zero extension is also 1. The last equation is a consequence of conservation of angular momentum. Expanding the equation using a Taylor series about $l = 0$

$$\dot{\theta} = -\frac{m_l}{1+m_l}\dot{\psi} - \frac{2m_l}{(1+m_l)^2} l u_1 + o(l) u_1$$

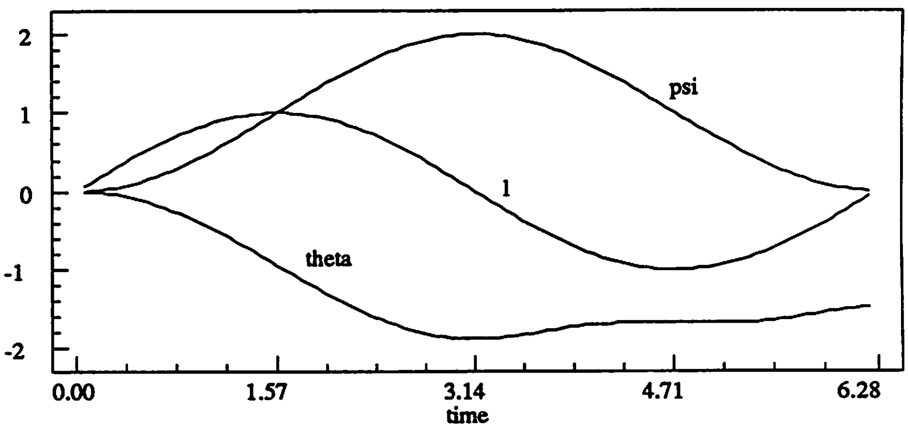

FIGURE 8. Nonholonomic motion for a hopping robot. Using sinusoidal inputs, the leg angle and extension return to their starting values but the body angle goes a net rotation.

This suggests a change of coordinates, $\alpha = \theta + \frac{m_l}{1+m_l}\psi$ to put the equations in the form

$$\begin{aligned}\dot{\psi} &= u_1 \\ \dot{l} &= u_2 \\ \dot{\alpha} &= \tfrac{2m_l}{(1+m_l)^2} l u_1 + o(l)u_1 = f(l)u_1\end{aligned}$$

This equation has the same form locally as the canonical system in equation (8).

Using this as justification, we attempt to use our proposed algorithm to steer the full nonlinear system. Since we control the ψ and l states directly, we first steer them to their desired values. Then using sinusoids in the ψ and l inputs,

$$\begin{aligned}u_1 &= a_1 \sin \omega t \\ u_2 &= a_2 \cos \omega t\end{aligned}$$

we steer θ to its desired value. By construction, this last motion does not affect the final values of ψ and l. To include the effect of nonlinearity in the first vector field, harmonic analysis can be used. Since l is periodic, we expand f using its Fourier series,

$$f(\frac{a_2}{\omega} \sin \omega t) = \beta_1 \sin \omega t + \beta_2 \sin 2\omega t + \cdots$$

Integrating $\dot{\alpha}$ over one period, only the first term in the expansion contributes to the net motion

$$\begin{aligned}\alpha(\tfrac{2\pi}{\omega}) &= \alpha(0) + \int_0^{\frac{2\pi}{\omega}} \left(\frac{a_1\beta_1}{\omega} \sin^2 \omega t + \frac{a_1\beta_2}{\omega} \sin \omega t \sin 2\omega t + \cdots \right) dt \\ &= \alpha(0) + \tfrac{\pi a_1 \beta_1}{\omega^2}\end{aligned}$$

Figure 8 shows the trajectory for the last motion segment; ψ and l return to their initial values but α (and hence θ) experiences a net change. To compute the required input amplitudes, we plot β_1 as a function of a_2 and choose a_2 such that $\frac{a_1\beta_1}{\pi\omega} = \theta_1 - \theta_0$. Using this procedure, we can (locally) steer between any two configurations.

3.2. Second Degree Systems. We next consider systems in which the first level of bracketing is not enough to span $\mathbf{R}^n$. We begin by trying to extend the previous canonical form to the next higher level of bracketing. Consider a system which can be expressed as

$$\begin{aligned} \dot{x}_i &= u_i && i = 1, \cdots, m \\ \dot{x}_{ij} &= x_i u_j && i < j \\ \dot{x}_{ijk} &= x_{ij} u_k && \text{(mod Jacobi identity)} \end{aligned} \tag{10}$$

Because Jacobi's identity imposes relations between certain brackets, not all x_{ijk} combinations are possible. This is analogous to limiting the x_{ij}'s according to

$$[g_i, [g_j, g_k]] + [g_j, [g_k, g_i]] + [g_k, [g_i, g_j] = 0.$$

Using the calculation in equation (4) shows that this system has relative growth vector $(m, \frac{m(m-1)}{2}, \frac{(m+1)m(m-1)}{3})$. Constructing the Lagrangian (with the same integral cost function) and substituting into the Euler-Lagrange equations does not result in a constant set of Lagrange multipliers. As a consequence, we cannot solve the optimal control problem in closed form.

We can however extend and apply our previous algorithm as follows:

Algorithm 2. (Steering second order canonical systems).

1. Steer the x_i's to their desired values. This causes drift in all other states.
2. Steer the x_{ij}'s to their desired values using integrally related sinusoidal inputs. If the i^{th} input has frequency ω_i then x_{ij} will have frequency components at $\omega_i \pm \omega_j$. By choosing inputs such that we get frequency components at zero, we can generate motion in the desired states.
3. Use sinusoidal inputs a second time to move all previously steered states in a closed loop and generate motion only in the x_{ijk} directions. This requires careful choice of the input frequencies so that $\omega_i \pm \omega_j \neq 0$ but $\omega_i \pm \omega_j \pm \omega_k$ has zero frequency components.

Example 7. To illustrate the algorithm, we consider the motion of a front wheel drive car as shown in Figure 6. The kinematics of this mechanism where derived in the last chapter and can be written as

$$\begin{aligned} \dot{x} &= \cos\theta\, u_1 \\ \dot{y} &= \sin\theta\, u_1 \\ \dot{\phi} &= u_2 \\ \dot{\theta} &= \tfrac{1}{l}\tan\phi\, u_1 \end{aligned} \tag{11}$$

In this form, u_1 does not control any state directly. We use a change of coordinates and a change of input to put the equations in the form

$$\begin{aligned} \dot{x} &= v_1 && v_1 = \cos\theta u_1 \\ \dot{\phi} &= v_2 && v_2 = u_2 \\ \dot{\alpha} &= \tan\phi\, v_1 && \alpha = \sin\theta \\ \dot{y} &= \tfrac{\alpha}{\sqrt{1-\alpha^2}}\, v_1 \end{aligned}$$

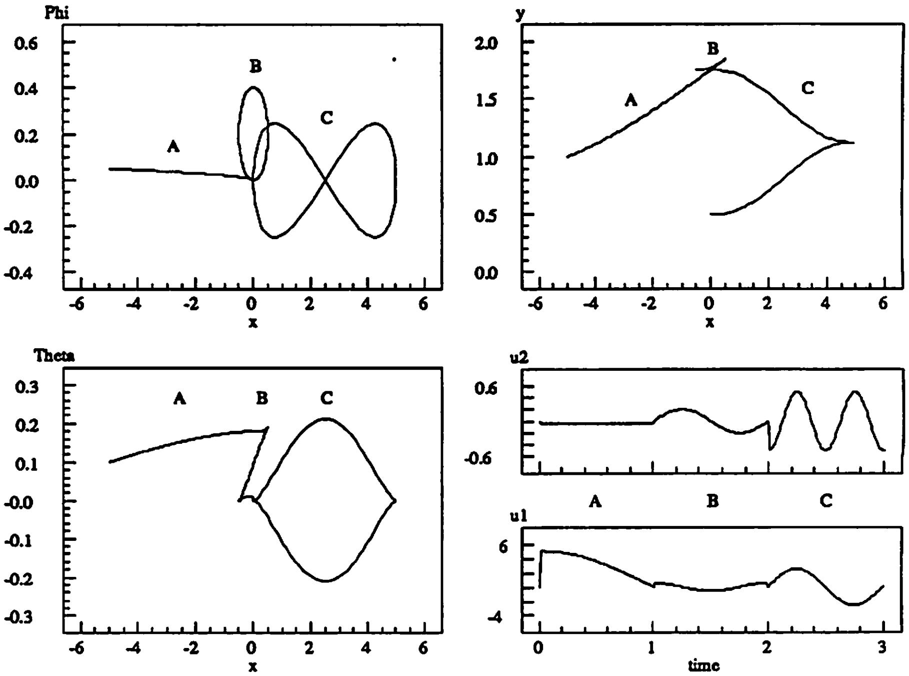

FIGURE 9. Sample trajectories for a car. The trajectory shown is a three stage path which moves the unicycle from $x = -5$, $y = 1$, $\theta = 0.05$, $\phi = 1$) to (0, 0.5, 0, 0). The first three figures show the states versus x; the bottom right figures show the inputs as functions of time.

As before, the linear portion of the nonlinearities matches the canonical system and we can include the effects of the nonlinearities using Fourier series techniques.

An example of the algorithm applied to the car is given in Figure 9. The first portion of the path, labeled A, drives the x and ϕ states to their desired values using a constant input. The second portion, labeled B, uses a periodic input to drive θ while bringing the other two states back to their desired values. The last step brings y to its desired value and returns the other three states to their correct values. The Lissajous figures that are obtained from the phase portraits of the different variables are quite instructive. Consider the portion of the curve labeled C. The upper left plot contains the Lissajous figure for x, ϕ (two loops); the lower left plot is the corresponding figure for x, θ (one loop) and the open curve in x,y shows the increment in the y variable. The very powerful implication here is that the Lie bracket directions correspond to rectification of harmonic periodic motions of the driving vector fields and the harmonic relations are determined by the degree of the Lie bracket corresponding to the desired direction of motion. This point has also been made rather elegantly by Brockett [6] in the context of the rectification of mechanical motion.

4. Chained Systems

We now study more general examples of nonholonomic systems and investigate the use of sinusoids for steering such systems. As in the previous section, we try to generate canonical classes of higher order systems, i.e., systems where more than one level of Lie brackets is needed to span the tangent space to the configuration space. We show that in full generality it is difficult to use sinusoids to steer such systems. This leads us to specialize to a smaller class of higher order systems, which we refer to as chained systems, which can be steered using sinusoids. We give sufficient conditions under which systems can be transformed into a chained canonical form and show the procedure applied to several illustrative examples.

4.1. Maximum Growth Canonical Systems. Using a P. Hall basis, it is possible to construct vector fields which have maximum growth; at each level of bracketing the dimension of the filtration grows by the maximum possible amount. More specifically, we wish construct a set of vector fields $\{X_i\}$ such that when the vector fields are substituted into the expressions for the P. Hall basis elements, the resulting set of vector fields is linearly independent. The method of construction used here is due to Grayson and Grossmann [14]; similar results can be found in the work of Sussmann [41]. We present only the 2-input case for simplicity.

An important property of a P. Hall Basis is that each basis element has a unique representation as a set of nested Lie products

$$B_i = [B_{i_1}, [B_{i_2}, \cdots [B_{i_l}, X_j] \cdots]] \tag{12}$$

Given a P. Hall basis element $B = [B_i, B_j]$, we convert it into this form by recursively expanding B_j. We associate with each such basis element a vector $\alpha_i \in \mathbb{Z}^n$ which indicates the number of times each basis element occurs in the expansion (12). Thus $\alpha_i(k)$ is the number of times B_k appears in the expansion for B_i. From the properties of a P. Hall basis, it is clear that $\alpha_i(k) = 0$ if $k \geq i$.

Given a P. Hall basis $H = \{B_1, \cdots, B_n\}$ we construct a vector field on $\mathbb{R}^n$ using coordinates $x \in \mathbb{R}^n$. Assume $B_i = X_i$ for $i = 1, \cdots, m$. Given α_i associated with B_i, $i > m$, we define

$$\begin{aligned} x^{\alpha_i} &= \prod_j x_j^{\alpha_i(j)} \\ \alpha_i! &= \prod_j \alpha_i(j)! \end{aligned}$$

Theorem 6 (Maximal growth 2 input systems [14]).
Fix $k \geq 1$ and let n be the rank of the free, nilpotent Lie algebra of order k with 2 generators. Then

$$X_1 = \frac{\partial}{\partial x_1} \qquad X_2 = \frac{\partial}{\partial x_2} + \sum_{i=3}^{n} \frac{x^{\alpha_i}}{\alpha_i!} \frac{\partial}{\partial x_i}$$

generate a free, nilpotent Lie algebra (of vector fields) of order k at the origin.

The vector fields generated by this theorem are extensions of the canonical forms we have seen for degree of nonholonomy 1 and 2. The degree of nonholonomy for these vector fields is identical to the order of nilpotency. One way to interpret and gain insight into this formula is to note that a Lie product

$$[B_{i_1}, [B_{i_2}, \cdots [B_{i_k}, X_2]]]$$

corresponds to a vector field obtained by taking the derivative of the components of X_2 with respect to $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$. The coefficients of X_2 are chosen such that taking this derivative leaves 1 in the $\frac{\partial}{\partial x_i}$ term.

Example 8. Consider the two input example given previously, but with order of nilpotency 4 instead of 5. The system generated by Theorem 6 is

$$\begin{array}{rclcl}
\dot{x}_1 & = & u_1 & \quad & X \\
\dot{x}_2 & = & u_2 & & Y \\
\dot{x}_3 & = & x_1 u_2 & & [X,Y] \\
\dot{x}_4 & = & \frac{1}{2}x_1^2 u_2 & & [X,[X,Y]] \\
\dot{x}_5 & = & x_1 x_2 u_2 & & [Y,[X,Y]] \\
\dot{x}_6 & = & \frac{1}{6}x_1^3 u_2 & & [X,[X,[X,Y]]] \\
\dot{x}_7 & = & \frac{1}{2}x_1^2 x_2 u_2 & & [Y,[X,[X,Y]]] \\
\dot{x}_8 & = & \frac{1}{2}x_1 x_2^2 u_2 & & [Y,[Y,[X,Y]]]
\end{array}$$

We can now ask ourselves if it is possible to steer these canonical systems using sinusoids. Although the form of the system is different from that we used in Section 3.2, the same approach can be used to steer x_1 through x_5. That is, sinusoids at the same frequency and proper phase give motion in x_3 and sinusoids at frequency 1 and 2 give motion in x_4 and x_5 (switching the input frequency switches between x_4 and x_5). This can be verified by direct calculation.

Steering in the $x_6 - x_8$ directions is more difficult. Consider the effect of using two simple sinusoids as inputs, $u_1 = a\cos\omega_1 t$ and $u_2 = b\sin\omega_2 t$. In order to prevent motion in lower level brackets, we must have $\omega_1 \neq \pm\omega_2$, $\omega_1 \neq \pm 2\omega_2$, $\omega_2 \neq \pm 2\omega_1$. Assuming these relationships hold, we get the following frequency components in the derivatives of the dynamic system:

$$\begin{array}{ll}
x_6: & \omega_1 \pm \omega_2 \quad 3\omega_1 \pm \omega_2 \\
x_7: & \omega_1 \quad 2\omega_1 \quad 2\omega_2 \quad 2\omega_1 \pm 2\omega_2 \\
x_8: & \omega_2 \quad \omega_1 \pm \omega_2 \quad \omega_1 \pm 3\omega_2
\end{array}$$

By choosing frequencies such that the derivative has a term at frequency 0, we get motion in that coordinate. Thus $\omega_2 = 3\omega_1$ gives motion in x_6 (only) and $\omega_1 = 3\omega_2$ gives motion in x_8 (only).

Based on these calculations, it would appear that choosing $2\omega_1 = 2\omega_2$ would give motion in x_7. This is in fact the case, but we *also* get motion in the x_3 direction. It is not possible to get motion *only* in the x_7 direction using simple sinusoids. A direct calculation verifies that adjusting the phasing of the inputs does not resolve this dilemma. It may still be possible to steer the system using combinations of sinusoids at different frequencies for each input or using more complicated periodic functions (such as elliptic functions, see [7]).

Rather than explore the use of more complicated inputs for steering nonholonomic systems, we consider instead a simpler class of systems. The justification for changing the class of systems is simple—most of the systems encountered as examples do not have the complicated structure of our canonical example. Thus there may be a simpler class of systems which is both steerable using simple sinusoids and representative of systems in which we are interested. This is the topic of the next subsection.

4.2. Chained Systems. Consider a two input system of the following form:

$$
\begin{array}{rclcrcl}
\dot{x}_0 & = & u_1 & \qquad & \dot{y}_0 & = & u_2 \\
\dot{x}_1 & = & y_0 u_1 & & (y_1 & := & -x_1) \\
\dot{x}_2 & = & x_1 u_1 & & \dot{y}_2 & = & y_1 u_2 \\
\dot{x}_3 & = & x_2 u_1 & & \dot{y}_3 & = & y_2 u_2 \\
 & \vdots & & & & \vdots & \\
\dot{x}_{n_x} & = & x_{n_x-1} u_1 & & \dot{y}_{n_y} & = & y_{n_y-1} u_2
\end{array}
\tag{13}
$$

or more compactly

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = X u_1 + Y u_2 \qquad
\begin{array}{rcl}
X & = & \frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} x_{i-1} \frac{\partial}{\partial x_i} \\
Y & = & \frac{\partial}{\partial y_0} + \sum_{j=2}^{n} y_{j-1} \frac{\partial}{\partial y_j}
\end{array}
$$

where $y_1 := -x_1$ to account for skew-symmetry of the Lie bracket. We refer to this system as a *two-chain system*. The first item is to check the controllability of these systems. To this end, denote iterated Lie products as $\mathrm{ad}_X^k Y$:

$$
\mathrm{ad}_X Y = [X, Y] \qquad \mathrm{ad}_X^k Y = [X, \mathrm{ad}_X^{k-1} Y] = [X, [X, \cdots, [X, Y] \cdots]]
$$

Lemma 7 (Lie bracket calculations).
For the vector fields in equation (13)

$$
\begin{array}{rcl}
\mathrm{ad}_X^k Y & = & (-1)^k \frac{\partial}{\partial x_k} \\
\mathrm{ad}_Y^k X & = & (-1)^k \frac{\partial}{\partial y_k}
\end{array}
\qquad k > 1
$$

Proof. By induction. Since the first level of brackets is irregular, we begin by expanding $[X, Y]$ and $[X, [X, Y]]$.

$$
\begin{array}{rcl}
[X, Y] & = & \left(\frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial x_1} + \sum x_{i-1} \frac{\partial}{\partial x_i}\right) \left(\frac{\partial}{\partial y_0} + \sum y_{j-1} \frac{\partial}{\partial y_j}\right) - \\
 & & \left(\frac{\partial}{\partial y_0} + \sum y_{j-1} \frac{\partial}{\partial y_j}\right) \left(\frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial x_1} + \sum x_{i-1} \frac{\partial}{\partial x_i}\right) \\
 & = & 0 - \frac{\partial}{\partial x_1} \\
[X, [X, Y]] & = & X(-\frac{\partial}{\partial x_1}) + \frac{\partial}{\partial x_1}(X) = 0 + \frac{\partial}{\partial x_2}
\end{array}
$$

Now assume that $\mathrm{ad}_X^k Y = (-1)^k \frac{\partial}{\partial x_k}$. Then

$$\mathrm{ad}_X^{k+1} Y = [X, \mathrm{ad}_X^k Y] = (-1)^k \left(X(\frac{\partial}{\partial x_k}) - \frac{\partial}{\partial x_k}(X) \right) = (-1)^{k+1} \frac{\partial}{\partial x_{k+1}}$$

The proof for $\mathrm{ad}_Y^k X$ is identical using the facts $[Y, X] = -[X, Y]$ and $y_1 := -x_1$. □

Proposition 8 (Controllability of the two-chain system).
The two-chain system (13) *is maximally nonholonomic (controllable).*

Proof. There are $2n-1$ coordinates in (13) and the $2n-1$ Lie products

$$\{X, Y, \mathrm{ad}_X^i Y, \mathrm{ad}_Y^j X\} \qquad i \geq 1, \quad j \geq 2$$

are independent using Lemma 7. We require $j \geq 2$ since $\mathrm{ad}_Y X = -\mathrm{ad}_X Y$ and hence those Lie products can never be independent. □

To steer this system, we use sinusoids at integrally related frequencies. Roughly speaking, if we use $u_1 = \sin t$ and $u_2 = \cos kt$ then $\dot{x}_1$ will have components at frequency $k-1$, $\dot{x}_2$ at frequency $k-2$, etc. $\dot{x}_k$ will have a component at frequency zero and when integrated we get motion in x_k while all previous variables return to their starting values. In the y variables, all frequency components will be of the form $m \cdot k \pm 1$ and hence we get no motion for $k > 1$. (For $k = 1$, y_1 and x_1 are the same variable). We make this precise with the following algorithm.

Algorithm 3.

1. Steer x_0 and y_0 to their desired values.
2. For each x_k, $k \geq 1$, steer x_k to its final value using $u_1 = a \sin t$, $u_2 = b \cos kt$, where a and b satisfy

$$x_k(2\pi) - x_k(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi$$

3. For each y_k, $k \geq 2$, steer y_k to its final value using $u_1 = b \cos kt$, $u_2 = a \sin t$, where a and b satisfy

$$y_k(2\pi) - y_k(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi$$

Proposition 9. *Algorithm* 3 *can steer* (13) *to an arbitrary configuration.*

Proof. The proof is constructive. It suffices to consider only step 2 since step 3 can be proved by switching x and y in what follows. We must show 2 things:

(1) moving x_k does not affect x_j, $j < k$
(2) moving x_k does not affect y_j, $j = 1, \cdots, n_y$

To verify that using $u_1 = a \sin t$, $u_2 = b \cos kt$ produces motion only in x_k, we integrate the x states. If x_{k-1} has terms at frequency ω_i, then x_k has corresponding terms at $\omega_i \pm 1$ (by expanding products of sinusoids as sums of sinusoids). Since the only way to have $x_i(2\pi) \neq x_i(0)$ is to have x_i have a component at frequency zero, it suffices to keep track only of the lowest frequency component in each variable;

higher components will integrate to zero. Direct computation starting from the origin yields

$$
\begin{aligned}
x_0 &= a(1-\cos t)\\
x_1 &= \int \frac{ab}{k}\sin kt \sin t = \frac{1}{2}\frac{ab}{k(k-1)}\sin(k-1)t + \frac{1}{2}\frac{ab}{k(k+1)}\sin(k+1)t\\
x_2 &= \tfrac{1}{2^k}\tfrac{a^2 b}{k(k-1)(k-2)}\sin(k-2)t + \cdots\\
&\vdots\\
x_k &= \int\left(\frac{a^k b}{2^{k-1}k!}\sin^2 t + \cdots\right)dt = \frac{a^k b}{2^{k-1}k!}\frac{t}{2} + \cdots
\end{aligned}
$$

$x_k(2\pi) = x_k(0) + \frac{(a/2)^k b}{k!}\pi$ and all earlier x_i's are periodic and hence $x_i(2\pi) = x_i(0)$, $i < k$. If the system does not start at the origin, the initial conditions generate extra terms of the form $x_{i-1}(0)u_2$ in the i^{th} derivative and this integrates to zero, giving no net contribution.

To show that we get no motion in the y variables, we show that all frequency components in the y's have the form $mk \pm 1$ where m is some integer. This is true for $y_1 := -x_1$ from the calculation above. Assume it is true for y_i:

$$
\begin{aligned}
\dot{y}_{i+1} &= y_i u_2\\
&= \sum_m \alpha(m)\sin(mk \pm 1)t \cdot \cos kt\\
&= \sum_m \frac{\alpha(m)}{2}(\sin((m+1)k \pm 1)t + \sin((m-1)k \pm 1)t)
\end{aligned}
$$

Hence y_{i+1} only has components at non-zero frequencies $m'k \pm 1$ and therefore $y_i(2\pi) = y_1(0)$. □

To include systems with more than two inputs, we replicate the structure of (13) for each additional input. Let h^k_{ij} represent the motion corresponding to the Lie product $\mathrm{ad}^k_{X_i} X_j$. In the two input case, $x_k = h^k_{21}$ and $y_k = h^k_{12}$. The following system on $\mathbf{R}^n$ is the *m-chain system*:

$$
\begin{aligned}
\dot{h}^0_j &= u_j & j &= 1,\cdots,m\\
\dot{h}^1_{ij} &= h^0_i u_j & i &> j \text{ and } h^1_{ji} := -h^1_{ij}\\
\dot{h}^k_{ij} &= h^{k-1}_{ij} u_j
\end{aligned}
\tag{14}
$$

Proposition 10 (Multi-chain system controllability).
The multi-chain system of (14) *is maximally nonholonomic and can be steered using sinusoids.*

Proof. The system (14) can be rewritten

$$\dot{h} = X_1 u_1 + \cdots + X_m u_m$$

with

$$X_j = \frac{\partial}{\partial h_j^0} + \sum_{\substack{i=1\\i<j}}^{m} h_i^0 \frac{\partial}{\partial h_{ij}^1} + \sum_k \sum_i h_{ij}^{k-1} \frac{\partial}{\partial h_{ij}^k}$$

Given any two X_i, X_j, their Lie product expansions only involve terms of the form h_{ij}^k for some k. But this is precisely the vector fields from Lemma 7 and hence

$$\mathrm{ad}_{X_i}^k X_j = (-1)^k \frac{\partial}{\partial h_{ij}^k}$$

Taking these terms for all possible i, j, k we get a set of independent Lie products just as in the proof of Theorem 8.

To show that the system can be steered using sinusoids, pick any $i, j \in \{1, \cdots, m\}$, $i < j$. Fix $u_l = 0$ for all $l \neq i, j$. The resulting system is identical to (13) can be steered using algorithm 3. By choosing all possible combinations of i and j, we can move to any position. □

4.3. Non-canonical Chained Systems. We would like to extend the class of systems which we can steer by including systems which have similar structure to equation (13), but with additional nonlinearities. The following example illustrates the limitations of using sinusoidal inputs for this purpose. Consider the system

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= (x_2 + \epsilon x_2^2) u_1 \\
\dot{x}_4 &= (x_3 + \epsilon x_3^2) u_1 \\
\dot{x}_5 &= x_2 u_1
\end{aligned}$$

This satisfies our definition of a chained system with a single chain: $\mathrm{ad}_{g_1}^k g_2$, $k = 0, 1, 2, 3$ together with g_2 forms a basis for $\mathbf{R}^5$.

If we apply inputs $u_1 = \sin t$ and $u_2 = \cos 3t$, we get the following motion, starting from $x = 0$

$$\begin{aligned}
x_1(2\pi) &= 0 \\
x_2(2\pi) &= 0 \\
x_3(2\pi) &= 0 \\
x_4(2\pi) &= -\tfrac{7}{1440}\epsilon^2 \\
x_5(2\pi) &= \tfrac{\pi}{24} + 2.5 \times 10^5 \epsilon^2
\end{aligned}$$

The reason for this perturbation in x_2 is that the (small) nonlinear terms cause zero frequency components to appear in $\dot{x}_2$. Hence we cannot use simple sinusoids to steer this system as before.

Nonetheless, there are many special instances where sinusoids are an important tool. For example, we were able to steer the automobile with sinusoids, despite the nonlinearities. Since the automobile had degree of nonholonomy 2, the problems present in the previous example do not occur. Another example is a system which has the chained canonical form until the last coordinate. In this case, harmonic analysis is needed when finding the motion at the last step of the algorithm and zero frequency terms do not appear in any previous coordinates.

It may also be possible to use feedback transformation to convert certain systems into chained canonical form. This is similar to the technique used in nonlinear control to convert a nonlinear system into a linear one by using a change of coordinates and state feedback. Similar efforts have been used by Lafferriere and Sussmann [23] to convert systems into nilpotent form for use with their planning algorithm. It is interesting to note that in several of their examples, the converted systems were also in chained canonical form. We study this possibility in detail in the next section.

Finally, sinusoids may be useful for steering systems which are not locally in canonical form. The minimal structure necessary to attempt motion generation using sinusoids is a *triangular system.* A system is triangular if we can find a set of coordinates $h = (h^0, h^1, \cdots, h^p) \in \mathbf{R}^{m_0 \times m_1 \times \cdots \times m_p} = \mathbf{R}^n$ such that

$$\begin{aligned} \dot{h}^0 &= v & v \in \mathbf{R}^{m_0} \\ \dot{h}^1 &= f^2(h^0)v \\ \dot{h}^2 &= f^3(h^0, h^1)v \\ &\vdots \\ \dot{h}^p &= f^p(h^0, \cdots, h^{p-1})v \end{aligned}$$

The triangular form was necessary in our examples to insure that the differential equations driven by sinusoidal inputs could be integrated in a stepwise fashion.

4.4. Converting Systems to Chained Form. In this section we introduce a set of sufficient conditions for determining if a system can be converted to chained form. This set of conditions gives a constructive method for building a feedback transformation which accomplishes the conversion. We concentrate on the two input case with a single chain.

Proposition 11 (Converting systems to two-chained form).
Consider a controllable system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2,$$

with g_1, g_2 linearly independent and smooth and having the special form

$$\begin{aligned} g_1(x) &= \tfrac{\partial}{\partial x_1} + \textstyle\sum_{i=2}^n g_1^i(x) \tfrac{\partial}{\partial x_i} \\ g_2(x) &= \textstyle\sum_{i=2}^n g_2^i(x) \tfrac{\partial}{\partial x_i} \end{aligned}$$

(by appropriate change of basis, if necessary). Define

$$\begin{aligned} \Delta_0 &:= \operatorname{span}\{g_1, g_2, \operatorname{ad}_{g_1} g_2, \cdots, \operatorname{ad}_{g_1}^{n-2} g_2\} \\ \Delta_1 &:= \operatorname{span}\{g_2, \operatorname{ad}_{g_1} g_2, \cdots, \operatorname{ad}_{g_1}^{n-3} g_2\} \end{aligned} .$$

If for some open set U, $\Delta_0(x) = \mathbf{R}^n$ for all $x \in U \subset \mathbf{R}^n$ and Δ_1 is involutive on U, then there exists a local feedback transformation

$$\xi = \phi(x) \quad u = \beta(x)v$$

such that the transformed system is in chained form:

$$\begin{aligned}\dot\xi_1 &= v_1\\ \dot\xi_2 &= v_2\\ \dot\xi_3 &= \xi_2 v_1\\ &\vdots\\ \dot\xi_n &= \xi_{n-1}v_1\end{aligned}$$

Proof. Since Δ_1 is an involutive distribution of dimension $n-2$, there exists a function h such that $dh\cdot\Delta_2 = 0$ and $dh\cdot \mathrm{ad}_{g_1}^{n-2} g_2 \neq 0$. Define the map $\phi : x \mapsto \xi$ as

$$\begin{aligned}\xi_1 &= x_1\\ \xi_2 &= L_{g_1}^{n-2}h\\ &\vdots\\ \xi_{n-1} &= L_{g_1}h\\ \xi_n &= h\end{aligned}$$

To verify that ϕ is a valid change of coordinates, we use the fact that

$$L_{[f,g]}h = L_f L_g h - L_g L_f h$$

so that

$$\begin{aligned}L_{ad_{g_1}^{n-2}g_2}h &= L_{g_1}L_{ad_{g_1}^{n-3}g_2}h - L_{ad_{g_1}^{n-3}g_2}L_{g_1}h\\ &= (-1)^{n-2}L_{g_2}L_{g_1}^{n-2}h \neq 0\end{aligned}$$

and $L_{ad_{g_1}^k g_2}h = 0$ for $k < n-2$ by the same reasoning. Using this calculation,

$$\frac{\partial\phi}{\partial x} = \begin{bmatrix} dh\\ dL_{g_1}^{n-2}h\\ \vdots\\ dL_{g_1}h\end{bmatrix} \qquad \frac{\partial\phi}{\partial x}\Delta_0 = \begin{bmatrix} 1 & 0 & \cdots & 0\\ * & \pm a(x) & & \vdots\\ \vdots & & \ddots & 0\\ * & \cdots & * & \pm a(x)\end{bmatrix}$$

where $a(x) = L_{g_2}L_{g_1}^{n-2}h \neq 0$.

Evaluating the derivatives of the coordinate transformation, we define

$$\begin{aligned}v_1 &= u_1\\ v_2 &= (L_{g_1}^{n-1}h)u_1 + (L_{g_2}L_{g_1}^{n-2}h)u_2\end{aligned}.$$

Since $L_{g_2}L_{g_1}^{n-2}h \neq 0$, this change of inputs is invertible and the resulting system is in chained form. □

This proposition gives a set of sufficient conditions for converting a system with relative growth vector $\sigma = (2,1,\cdots,1)$ into chained form. In order to apply the results, however, we must modify the original inputs to the system such that one of the states is controlled directly by the input. Such a change of input is always possible due to the assumption that the input vector fields are linearly independent. This change of input is not unique.

One corollary to Proposition 11 is that all systems with relative growth vector $\sigma = (2,1)$ can be converted to chained form. This is a direct consequence of the fact that all 1 dimensional distributions are involutive.

Example 9. Consider as our first example, the kinematic model of an automobile. The equations governing the motion of the system were derived in section 2.4:

$$\begin{aligned} \dot{x} &= \cos\theta\, u_1 \\ \dot{y} &= \sin\theta\, u_1 \\ \dot{\phi} &= u_2 \\ \dot{\theta} &= \tfrac{1}{l}\tan\phi\, u_1 \end{aligned} \tag{15}$$

To convert the system to chained form, we first scale the inputs so that u_1 enters $\dot{x}$ directly. Reusing the symbol u_1, the kinematics become:

$$\begin{aligned} \dot{x} &= u_1 \\ \dot{y} &= \tan\theta\, u_1 \\ \dot{\phi} &= u_2 \\ \dot{\theta} &= \tfrac{1}{l}\sec\theta\tan\phi\, u_1 \end{aligned}$$

Choose the y position of the car as the function h; it is easy to verify that this function satisfies the conditions of Proposition 11. The resulting change of coordinates is

$$\begin{aligned} &\xi_1 = x && u_1 = v_1 \\ &\xi_2 = \tfrac{1}{l}\sec^3\theta\tan\phi && u_2 = -\tfrac{3}{l}\sin^2\phi\sin\theta v_1 + \tfrac{1}{l}\cos^2\theta\cos^3\phi v_2 \\ &\xi_3 = \tan\theta && \\ &\xi_4 = y && \end{aligned}$$

And the transformed system has the form:

$$\begin{aligned} \dot{\xi}_1 &= v_1 \\ \dot{\xi}_2 &= v_2 \\ \dot{\xi}_3 &= \xi_2 v_1 \\ \dot{\xi}_4 &= \xi_3 v_1 \end{aligned}$$

This system can now be steered using the sinusoidal algorithm of the previous section or another method, such as Lafferriere and Sussmann's algorithm for generating motions for nilpotent systems. The motion is implemented as a feedback pre-compensator which converts the v inputs into the actual system inputs, u. This feedback transformation agrees the that used in Lafferriere and Sussmann to nilpotentize the kinematic car example. Their formulation of the feedback transformation was not presented, although it seems clear that a similar approach must have been used.

Figure 10 shows the results of using chained form to steer an automobile. These trajectories are qualitatively similar to those in Figure 9, but do not require the calculation of Fourier coefficients for determining open loop trajectories.

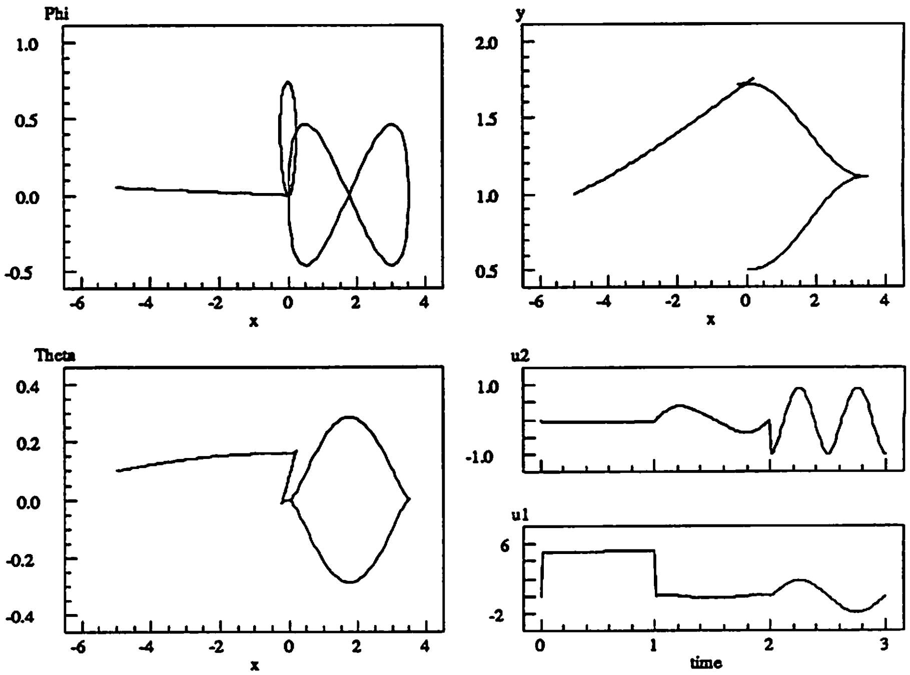

FIGURE 10. Sample trajectories for a car using chained form.

Example 10 (Car with N trailers). Consider first the case of a car pulling a single trailer. The equations of motion are identical to those of the car, with an additional equation specifying the motion of the attached trailer:

$$\dot{\theta}_1 = \sin(\theta_0 - \theta_1)u_1$$

By solving the partial differential equations in the statement of the proposition above, it can be shown that the function

$$h(y, \theta_1) = y - \log(\frac{1 + \sin\theta_1}{\cos\theta_1})$$

generates a chained set of coordinates. Again we can locally steer the trailer using sinusoidal inputs or other methods.

When additional trailers are added, the distribution Δ_1 is no longer involutive and hence the procedure outlined above does not apply. Since the conditions in the proposition are only sufficient conditions, this does not mean that a car with N trailers cannot be steered using sinusoids. But a more complicated change of basis would be required in order to convert the vector fields to the necessary form. This example points out the weaknesses of the theorem and provides directions for future research.

5. Discussion and Future Work

Most current nonholonomic motion planners rely on special system structure to generate efficient motions. In some cases the structure is very specific, as evidenced by the large number of path planners for car-like robots using the special form of the kinematics for that system. More general path planners, such as the one proposed by Lafferriere and Sussmann [23], require that either the system be nilpotent or that an iterative procedure be used. In the non-nilpotent case, the iterative algorithm generates very complex paths which can steer arbitrarily close to the goal only at the cost of additional complexity. The results of Section 3 are somewhat complimentary—the methods can easily be applied to certain systems which are not nilpotent, but the general case requires a restrictive canonical form.

Research in efficient motion planning for general nonholonomic systems can proceed in many ways. More general conditions under which a distribution can be represented by a nilpotent or chained basis would clarify the extent to which particular algorithms can be applied. On the other hand, new approaches using metric or other properties of nonholonomic distributions might lead to path planners which work for more general classes of systems. Computational approaches such as those proposed by Barraquand and Latombe [2] might also be extended to handle higher dimensional systems with very few structural requirements.

The work in nonholonomic motion planning thus far has been primarily in the generation of open loop trajectories. Closed loop control of nonholonomic systems is very difficult, in part because of fundamental restrictions which prohibit the existence of smooth feedback controllers which asymptotically stabilize a point. Indeed, one can show using the results of Brockett, Sontag [4, 38] that the class of nonholonomic systems is not stabilizable by smooth state feedback. Nonetheless, it is vital to introduce closed loop control for these systems to account for initial condition and modeling errors, noise, and other effects that are encountered in any real implementation. Figure 11 shows an example of the effects of initial condition errors on parallel parking maneuvers for an automobile.

A possible approach to the control of nonholonomic systems is the study of controllability along a reference trajectory. If we are given a desired state trajectory, we would like to construct a controller which stabilizes the system to this trajectory. The simplest example of such a controller is a control law for steering a car down the road. While the car is moving, it is quite easy to linearize the system and design linear feedback controllers which cause the car to stay aligned with a given trajectory. In fact, if the car is moving at a constant velocity, $u_1 = v_c$, then we can write

$$\begin{aligned} \dot{x} &= g_1(x)v_c + g_2(x)u_2 \\ &= f(x) + g_2(x)u_2 \end{aligned}$$

Furthermore, this system is completely controllable as a nonlinear system. Methods for extending these results to more complicated systems are currently being pursued.

The development of closed loop controls may allow simplifications in planning for nonholonomic systems. Rather than attempt to find an input which steers us between the initial and desired locations, we might construct a piecewise feasible trajectory which connects the two points. We then apply a feedback controller about the piecewise feasible segments to implicitly define the input u. To illustrate this approach, we consider a parallel parking maneuver as shown in Figure 12. This

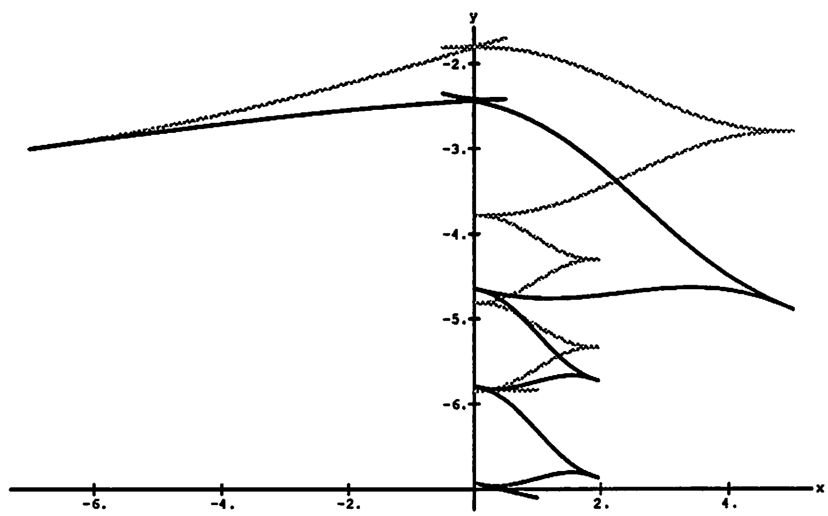

FIGURE 11. Affects of initial condition errors on open loop paths. The gray line shows a parking maneuver for an automobile. The solid path is the trajectory which is followed when the initial steering wheel angle of the car is off by 0.5 radians (approximately 3 degrees).

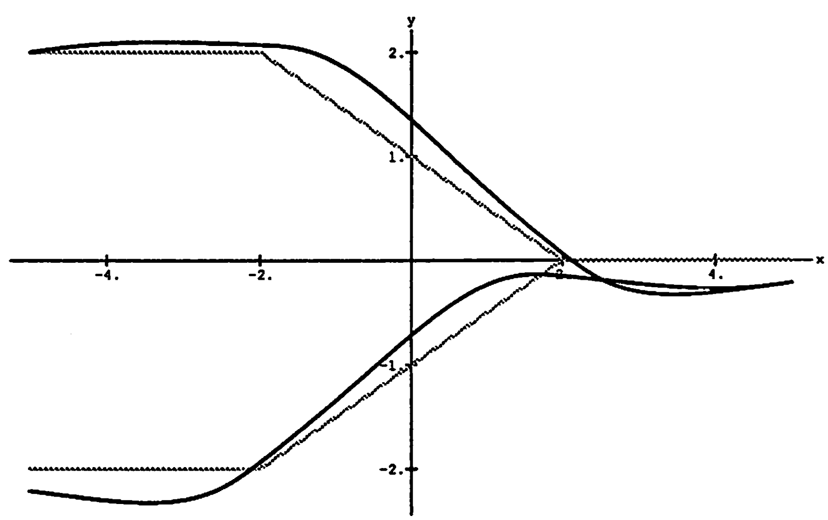

FIGURE 12. Parallel parking maneuver using piecewise feasible segments (gray lines) and closed loop control.

controller was constructed by using piecewise linear state feedback for each feasible segment.

Finally, we consider the problem of planning for systems with a nonzero drift vector field:

$$\dot{x} = f(x) + g(x)u$$

The planning problem for this system is to steer between two equilibrium points of the system using u. If the equilibrium points lie on a connected manifold and the system is controllable at each point along the manifold, this problem can be solved for very general systems (see [16] for a specific example). However, if the start and goal position are not connected by an equilibrium manifold, it is not clear how to proceed. Although the existence of a trajectory is guaranteed by the appropriate controllability conditions, construction of a trajectory for systems with drift is still an open problem.

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