USING SATURATION TO STABILIZE A CLASS OF SINGLE-INPUT PARTIALLY LINEAR COMPOSITE SYSTEMS

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Using Saturation to Stabilize a Class of Single-input Partially Linear Composite Systems *

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Abstract

We are interested in globally (semi-globally) stabilizing single-input nonlinear systems that cannot be globally full-state linearized. We focus on partially linear composite systems where the dynamics of the nonlinear subsystem are *not* zero-input asymptotically stable. We specify a class of such systems where a linear plus (small) bounded control can be used to stabilize the composite system. A subset of systems in this class can be globally (semi-globally) stabilized using only a bounded control.

We propose algorithms that use our recent result for stabilizing a (linear) chain of integrators with bounded controls [14]. In the nonlinear setting, the success of our algorithms depend only on the general properties of the nonlinear terms and not on their explicit form. Consequently the stability property is robust to unmodeled nonlinear terms that satisfy the general properties as well as unknown (possibly time-varying) bounded parameters.

Keywords. Saturation, composite systems, stabilization, global, semi-global.

1 Introduction

We will consider partially linear single-input composite systems of the form

$$\begin{aligned} \dot{\eta} &= f(\eta, z, u, t) \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_n &= u \end{aligned} \tag{1}$$

where $\eta \in \mathbb{R}^p$ and f is smooth with $f(0,0,0,t) = 0$ for all $t \geq t_o$.

Interest in such systems has been driven by input-output linearization theory [4] which allows partial linearization for systems that cannot be full-state linearized. There have been many recent global stabilization results for such composite systems ([1], [7], [8], [9], [10], [11], [12], [13]). In

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general these results either assume that the nonlinear subsystem is zero-input asymptotically stable or that f depends only on η and z_1. In the latter case it is also assumed that a smooth "input" z_1 is known which globally stabilizes the nonlinear subsystem.

The approach presented in this paper aims at globally (semi-globally) stabilizing a subclass of systems described by (1) where the nonlinear subsystem is *not* zero-input asymptotically stable and where f can depend on the complete state vector z as well as the input u. We will rely heavily on the "converging input bounded state" property of [12] and incorporate a recent result for stabilizing a (linear) chain of integrators with bounded controls [14] to achieve nonlinear stabilization. Interestingly, our design will provide intuition for determining coordinates and a feedback that yield a composite system of the form (1) where the nonlinear subsystem *is* zero-input globally asymptotically stable. More importantly, the approach outlined here depends only on the general properties of the nonlinear terms and not on their explicit form. Consequently, our approach is robust in the presence of a class of unmodeled nonlinear terms and in the presence of unknown (possibly time-varying) bounded parameters.

The assumptions we impose are not generic, but do allow us to handle systems that do not satisfy the conditions of existing methods. In this sense, our method presents a specialized tool intended to complement other existing methods in the nonlinear stabilization tool box.

Section 2 will define the general concepts used throughout the paper and will review the work of [12] as it applies to our problem. In section 3 we describe the class of systems for which our algorithm is applicable. In section 4 we state and prove our main results for global stabilization. Section 5 contains our main results for semi-global stabilizability. Finally, in section 6 we provide examples for both global and semi-global stabilization. In the global case, we show that our algorithm provides a solution to a previously unsolved benchmark problem [6]. In the semi-global case, we show that our algorithm provides a solution to the popular "ball and beam" example [3]. To our knowledge, the only existing stabilizing solutions to this problem were local in nature.

2 Preliminaries

We begin by defining what is meant by global and semi-global stabilizability. Consider a general finite-dimensional nonlinear system of the form

$$\dot{x} = f(x, u, t) \tag{2}$$

where $f(0, 0, t) = 0$ for all $t \geq t_o$. We then make the following definitions.

Definition 1 *A nonlinear system (2) is* globally stabilizable *by state feedback if there exists a control $u(x)$ such that $|x| \to 0$ as $t \to \infty$ from any initial condition.*

Definition 2 *A nonlinear system (2) is* semi-globally stabilizable *by a class of state feedback laws if, for initial conditions in any bounded subset X of the state space, there exists a control $u(x)$ belonging to the class such that $|x| \to 0$ as $t \to \infty$.*

Our objective is to globally stabilize a given nonlinear system. In the event that the assumptions we impose for global stabilizability do not hold, we are willing to modify our assumptions to achieve semi-global stabilizability. In either case we are not limited to initial conditions in a small neighborhood of the origin.

The following definition will play a prominent role in characterizing the class of systems to which our method applies.

Definition 3 *A function* $g : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ *denoted* $g(v, w)$ *is said to be* higher order in w uniformly in v *if* $\exists$ *positive constants* ϵ_o, C *such that* $\forall \epsilon < \epsilon_o$,

$$||w|| < \epsilon \Rightarrow |g(v, w)| < C\epsilon^2$$

Finally, we define the class of saturation functions that are fundamental to our control approach.

Definition 4 *Given two positive constants* δ_i *and* ϵ_i, *a function* $\sigma_i : \mathbf{R} \to \mathbf{R}$ *is said to be a* simple saturation *if it is a continuous, nondecreasing function satisfying*

1. $s\sigma_i(s) > 0$ *for all* $s \neq 0$
2. $\sigma_i(s) = s$ *when* $|s| \leq \delta_i$
3. $|\sigma_i(s)| = \epsilon_i$ *when* $|s| \geq \epsilon_i$.

In the literature for composite nonlinear systems, several different concepts are relied upon to demonstrate global stability. The most convenient concept for our purposes is the "converging input bounded state" condition presented in [12]. The nature of our problem requires this condition extended to allow for certain time-varying dynamics. In this framework, we recall the main result of [12]. Consider a finite-dimensional composite nonlinear system

$$\dot{y} = f(y, z, t) \tag{3}$$

$$\dot{z} = g(z, t) \tag{4}$$

where f and g are smooth and $f(0, 0, t) = 0$ and $g(0, t)$=0 for all $t \geq t_o$. Assume the composite system has the following properties:

Property 1 1. *The equilibrium point* $y = 0$ *of*

$$\dot{y} = f(y, 0, t)$$

is uniformly globally asymptotically stable.

2. *The equilibrium point* $z = 0$ *of (4) is uniformly globally asymptotically stable and locally exponentially stable.*

Property 2 *For some positive constants* K, α *and* $T > t_0$ *and for each bounded "control"* $z(\cdot)$ *on* $[t_0, \infty)$ *such that* $|z(t)| \leq Ke^{-\alpha t}$ *for all* $t > T$ *and for each initial state* y_o, *the solution of (3) with* $y(t_0) = y_o$ *exists for all* $t \geq t_0$ *and is bounded uniformly in* t_0.

Under these conditions there is the following result:

Theorem 2.1 ([12]) *If properties 1 and 2 are satisfied then the equilibrium* $(0, 0)$ *of (3),(4) is globally asymptotically stable.*

Proof. See [12].

3 Assumptions

We now specify the subclass of systems of the form (1) for which our control approach is suited. First we decompose the state vector η:

$$\eta = \begin{bmatrix} x \\ y \end{bmatrix}$$

where $x \in \mathbf{R}^k$ and $y \in \mathbf{R}^m$. We write

$$\begin{array}{rcl} \dot{x} & = & \bar{f}(x,y,z,u,t) \\ \dot{y} & = & g(x,y,z,u,t) \end{array} \tag{5}$$

We then make the following assumptions:

Assumption 1 *The dynamics of*

$$\dot{x} = \bar{f}(x,y,z,u,t) \tag{6}$$

with $y(\cdot)$,$z(\cdot)$,$u(\cdot)$ considered as "controls" satisfy property 1.1 and property 2.

Assumption 2 (Global) *The dynamics of y have the form*

$$\dot{y}_i = g_i(x,y,z,u,t) = y_{i+1} + h_i(x,y,z,u,t) \tag{7}$$

for $i = 1,\ldots,m$ ($y_{m+1} \equiv z_1$) where

1. *h_i is higher order in $y_i,\ldots,y_m,z,u$ uniformly in $x,y_1,\ldots,y_{i-1},t$.*
2. *$h_i(x,y_1,\ldots,y_{i+1},0,\ldots,0,t) = h_i^a + h_i^b$ where*
   (a) *h_i^a is higher order in y_{i+1} uniformly in $x,y_1,\ldots,y_i,t$*
   (b) *$y_i h_i^b \leq 0$ for all $x,y_1,\ldots,y_{i+1},t$.*
3. *for some $\epsilon_o > 0$, $h_i(x,y_1,\ldots,y_{i-1},y_i,0,\ldots,0,t) = 0$ for all y_i such that $|y_i| < \epsilon_o$.*
4. *For each finite $c > 0$, $\exists \epsilon_o > 0$ such that for all $\epsilon_i < \epsilon_o$ the dynamics of y_i satisfy property 2 with $[c\sigma_i(y_i) + y_{i+1}], y_{i+2},\ldots,y_m$, z,u as "controls" uniformly in $x,y_1,\ldots,y_i,t$. The function σ_i is a simple saturation with positive constants δ_i,ϵ_i.*

Remarks.

1. For examples of systems satisfying this assumption, see section 6.
2. The most difficult requirement to check in the above assumption is point 4. Sufficient conditions to guarantee point 4 is satisfied are either
   (a) the dynamics of y_i are "bounded input bounded state" uniformly in $x,y_1,\ldots,y_{i-1},t$ or
   (b) $\bar{h}_i$ defined by

$$\bar{h}_i \equiv h_i(x,y,z,u,t) - h_i(x,y_1,\ldots,y_i,-c\sigma_i,0,\ldots,0,t)$$

   can be bounded as

$$|\bar{h}_i| \leq \kappa_1(|\zeta|) + \kappa_2(|\zeta|)|y_i|$$

   where $\zeta = (c\sigma_i + y_{i+1}, y_{i+2},\ldots,y_m,z,u)^T$ and $\kappa_i(\cdot)$ are strictly increasing functions such that $\kappa_i(0) = 0$ and for some $k,\epsilon_o > 0$ $\kappa_i(\epsilon) \leq k\epsilon$ for $0 \leq \epsilon \leq \epsilon_o$.

The latter follows from a simple application of the Bellman-Gronwall lemma.

3. The structure imposed by assumption 2 is, in a sense, complementary to the structure imposed by the *pure-feedback* conditions presented in [5]. For h_i, the *pure-feedback* conditions allow unrestricted dependence on $x, y_1, \ldots, y_{i+1}$ but disallow dependence on $y_{i+2}, \ldots, y_m, z, u$. Conversely, our assumption allows dependence on $y_{i+1}, \ldots, y_m, z, u$ but greatly restricts the allowed dependence on $x, y_1, \ldots, y_i$.

If the dynamics of y do not satisfy assumption 2 it is still possible to state a semi-global result under the following (recursive) assumption:

Assumption 3 (Semi-Global) *Let $i = m$ and consider the dynamics of y_i:*

1. *if assumption 2 holds then let $i = i - 1$.*
2. *otherwise, the dynamics of y_{i-1}, y_i have the form*

$$\begin{aligned} \dot{y}_{i-1} &= y_i + h_{i-1}(x, y, z, u, t) \\ \dot{y}_i &= y_{i+1} + h_i(x, y, z, u, t) \end{aligned} \tag{8}$$

   where for $j = i - 1, i$

   (a)

$$|h_j - h_j(x, y_1, \ldots, y_{i+1}, 0, \ldots, 0, t)| \leq (|y_{i-1}| + |y_i| + 1)|\bar{h}_j|$$

   where $\bar{h}_j$ is higher order in $y_{i+2}, \ldots, y_m$, z, u for bounded y_{i+1} uniformly in $x, y_1, \ldots, y_i, t$. Further, $\bar{h}_j$ is bounded for bounded $y_{i+1}, \ldots, y_m$, z, u.

   (b) i.

$$|h_{i-1}(x, y_1, \ldots, y_{i+1}, 0, \ldots, 0, t)| \leq (|y_i| + 1)|\hat{h}_{i-1}|$$

   where $\hat{h}_{i-1}$ is higher order in y_{i+1} uniformly in $x, y_1, \ldots, y_i, t$. Further, $\hat{h}_{i-1}$ is bounded for bounded y_{i+1}.

   ii. $h_i(x, y_1, \ldots, y_{i+1}, 0, \ldots, 0, t)$ depends only on y_{i+1}. Further it is higher order in y_{i+1} and bounded for bounded y_{i+1}.

   (c) For some $\epsilon_o > 0$, $h_j(x, y_1, \ldots, y_{i-1}, 0, \ldots, 0, t) = 0$ for all $|y_{i-1}| < \epsilon_o$.

   Let $i = i - 2$.

In the next sections we present our main results and illustrate the design procedure with examples.

4 Global Results

Our nonlinear global stabilizability result relies on a recent linear result for stabilizing a chain of integrators with saturation functions [14]. As in that work, for the nonlinear composite system (1) we employ nested saturations with linear arguments to achieve global stabilization. The proof of the following theorem will be constructive yielding an algorithm for generating the proposed globally stabilizing control law.

Theorem 4.1 (Global Stabilizability) *If assumptions 1 and 2 are satisfied, then there exists a time independent control of the form*

$$u = Kz - \sigma_m(T_m(y,z) + \sigma_{m-1}(T_{m-1}(y,z) + \cdots + \sigma_1(T_1(y,z)))\cdots)$$

which globally asymptotically stabilizes the origin of (1) where σ_i is a simple saturation for δ_i, ϵ_i, T_i is a linear function and the gains K are the coefficients of a Hurwitz polynomial.

We have the following corollary regarding globally stablizing (1) with a bounded control which follows by simply redefining the dynamics of y to include the dynamics of z also.

Corollary 4.1 (Global Stabilizability with bounded controls) *If assumptions 1 and 2 are satisfied, then there exists a time independent bounded control of the form*

$$u = -\sigma_{m+n}(T_{m+n}(y,z) + \sigma_{m+n-1}(T_{m+n-1}(y,z) + \cdots + \sigma_1(T_1(y,z)))\cdots)$$

which globally asymptotically stabilizes the origin of (1) where σ_i is a simple saturation and T_i is a linear function.

4.1 Proof of theorem 4.1

The proof is constructive and divides into three major parts. First we develop a convenient linear coordinate change that will simplify our analysis. Then we develop how the conditions of assumption 2 translate in the new coordinates. Finally, we show how these conditions allow for a globally stabilizing control law.

4.1.1 Coordinate change

Our first step in developing our coordinate change is to choose the input as $u = Kz + v$ where the gains K are the coefficients of a Hurwitz polynomial. We then have

$$\begin{aligned} \dot{x} &= \bar{f}(x,y,z,u,t) \\ \dot{y} &= g(x,y,z,u,t) \\ \dot{z} &= Az + Bv \end{aligned} \tag{9}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ k_1 & \cdots & \cdots & \cdots & k_n \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

The additional control v will be bounded and chosen to stabilize the y states. We proceed to make a linear coordinate change to achieve a convenient form for our approach. We choose

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= T_1 y + T_2 z \\ \tilde{z} &= z \end{aligned} \tag{11}$$

where T_1 and T_2 are constructed below. For purposes of compact notation, we employ the following selection operators:

$$\begin{aligned} S_i &: \mathbb{R}^{m+n} \to \mathbb{R}^n \\ S_i(w) &= [w_i, \ldots, w_{i+n-1}]^T \end{aligned}$$

and

$$\begin{aligned} P_i &: \mathbf{R}^{m+n} \to \mathbf{R} \\ P_i(w) &= w_i \end{aligned}$$

S_i is defined for $i = 1, \ldots, m$ and P_i is defined for $i = 1, \ldots, m+n$. Both operate on the concatenation of y and z:

$$w = [y^T, z^T]^T$$

We choose $\tilde{y}$ to have the following recursive construction:

$$\begin{aligned} \tilde{y}_m &= -KS_m(w) + P_{m+n}(w) \\ \tilde{y}_{m-1} &= \tilde{y}_m - KS_{m-1}(w) + P_{m+n-1}(w) \\ &\vdots \\ \tilde{y}_1 &= \tilde{y}_2 - KS_1(w) + P_{n+1}(w) \end{aligned}$$

It is apparent from this construction that T_1 has the form

$$T_1 = \begin{bmatrix} -k_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & -k_1 \end{bmatrix} \quad T_1^{-1} = \begin{bmatrix} -1/k_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & -1/k_1 \end{bmatrix} \tag{12}$$

(T_1 is invertible because $k_1 < 0$ for A to be Hurwitz.) In the new coordinates, the dynamics of (9) are given by

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}, T_1^{-1}(\tilde{y} - T_2\tilde{z}), \tilde{z}, u, t) \\ \dot{\tilde{y}} &= \tilde{g}(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{z}} &= A\tilde{z} + Bv \end{aligned} \tag{13}$$

It is obvious that the dynamics of $\tilde{x}$ satisfy assumption 1 with $\tilde{y}(\cdot), \tilde{z}(\cdot), u(\cdot)$ as "controls".

For the dynamics of $\tilde{y}$ we have

$$\begin{aligned} \dot{\tilde{y}}_i &= \tilde{g}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \cdots + \tilde{y}_m + v + \textstyle\sum_{j=i}^{m} T_{1_{ij}} h_j(\tilde{x}, T_1^{-1}(\tilde{y} - T_2\tilde{z}), \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \cdots + \tilde{y}_m + v + \tilde{h}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \end{aligned} \tag{14}$$

We proceed to determine the relevant properties of $\tilde{h}_i$.

4.1.2 Properties of Perturbation Terms

Define $\tilde{y}_{m+1} \equiv -k_1\tilde{z}_1$. The following properties of $\tilde{h}_i$ follow from assumption 2:

1. $\tilde{h}_i$ is higher order in $\tilde{y}_i, \ldots, \tilde{y}_m, \tilde{z}, u$ uniformly in $\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i-1}, t$
2. $\tilde{h}_i(\tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t) = \tilde{h}_i^a + \tilde{h}_i^b$ where
   (a) $\tilde{h}_i^a$ is higher order in $\tilde{y}_{i+1}$ uniformly in $\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_i, t$
   (b) for some $\epsilon_o, d > 0$ and $\forall \epsilon < \epsilon_o$, $\tilde{y}_i\tilde{h}_i^b \leq 0$ for all $\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i-1}$ and $|\tilde{y}_{i+1}| < \epsilon$ and $|\tilde{y}_i| > d\epsilon$.
3. for some $\epsilon_o > 0$, $\tilde{h}_i(\tilde{y}_1, \ldots, \tilde{y}_{i-1}, \tilde{y}_i, 0, \ldots, 0, t) = 0$ for all $\tilde{y}_i$ such that $|\tilde{y}_i| < \epsilon_o$.

4. $\exists\epsilon_o > 0$ such that for all $\epsilon_i < \epsilon_o$ the dynamics of $\tilde{y}_i$ satisfy property 2 with $[\sigma_i(\tilde{y}_m) + \tilde{y}_{i+1}], \tilde{y}_{i+2}, \ldots, \tilde{y}_m, \tilde{z}, u$ as "controls" uniformly in $\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_i, t$. The function σ_i is a simple saturation with positive constants δ_i, ϵ_i.

Consider point 1. For some $\epsilon_o > 0$ and any $\epsilon < \epsilon_o$ assume that $|\tilde{y}_j| < \epsilon$ for $j = i, \ldots, m$ and $||\tilde{z}|| < \epsilon$, $|u| < \epsilon$. From T_1^{-1} this implies, for some constant D, $|y_j| < D\epsilon$ for $j = i, \ldots, m$. Further $||z|| < \epsilon$. By assumption 2.1 this implies, for some constants C_j, $|h_j| < C_j\epsilon^2$, $j = i, \ldots, m$. Finally, from (14), for some constant $\tilde{C}$, $|\tilde{h}_i| < \tilde{C}\epsilon^2$.

Consider point 2. Decompose $\tilde{h}_i(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t)$ as $\tilde{h}_i = \tilde{h}_i^a + \tilde{h}_i^b$ where

$$\begin{aligned} \tilde{h}_i^a &= T_{1_{ii}} h_i^a + \textstyle\sum_{j=i+1}^m T_{1_{ij}} h_j \\ \tilde{h}_i^b &= T_{1_{ii}} h_i^b \end{aligned}$$

where h_i^a and h_i^b are defined by assumption 2.2. Consider point 2a above. Assume that $|\tilde{y}_{i+1}| < \epsilon$ and $\tilde{y}_j = 0$ for $j = i+2, \ldots, m$ and $\tilde{z} = 0$, $u = 0$. From T_1^{-1} this implies $|y_{i+1}| < D\epsilon$, $y_j = 0$ for $j = i+2, \ldots, m$ and $z = 0$. By assumption 2.2a this implies $|h_i^a| < C\epsilon^2$. Further, assumption 2.3 implies $h_j = 0$ for $j = i+1, \ldots, m$. Hence, for some constant $\tilde{C}$, $|\tilde{h}_i^a(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t)| < \tilde{C}\epsilon^2$. Consider point 2b above. Again assume $|\tilde{y}_{i+1}| < \epsilon$ and $\tilde{y}_j = 0$ for $j = i+2, \ldots, m$ and $\tilde{z} = 0$, $u = 0$. It follows that

$$\tilde{y}_i \tilde{h}_i^b(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t) = (T_{1_{ii}} y_i + T_{1_{i,i+1}} y_{i+1}) T_{1_{ii}} h_i^b$$

From assumption 2.2.b, it follows that $\tilde{y}_i \tilde{h}_i^b \leq 0$ for $|y_i| > |\frac{T_{1_{i,i+1}}}{T_{1_{ii}}} y_{i+1}|$ and all $\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i-1}$.

Consider point 3. For some $\epsilon_o > 0$ assume that $|\tilde{y}_i| < \epsilon_o$. Further, assume $\tilde{y}_j = 0$ for $j = i+1, \ldots, m$ and $\tilde{z} = 0$, $u = 0$. From T_1^{-1} this implies, for some constant D, $|y_i| < D\epsilon_o$ $y_j = 0$ for $j = i+1, \ldots, m$ and $z = 0$. By assumption 2.3, for ϵ_o small enough, $h_j = 0$ for $j = i, \ldots, m$. Finally, from (14), $\tilde{h}_i(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i-1}, \tilde{y}_i, 0, \ldots, 0, t) = 0$.

Consider point 4. Let $[\sigma_i(\tilde{y}_i) + \tilde{y}_{i+1}], \tilde{y}_{i+2}, \ldots, \tilde{y}_m$, $\tilde{z}, u$ converge to zero with an exponential tail. From T_1^{-1}, we have $[-\frac{1}{k_1}\sigma(\tilde{y}_i) + y_{i+1}], y_{i+2}, \ldots, y_m$, z, u converge to zero with an exponential tail. Note that, for any bounded $y_{i+1}, \ldots, y_m, z$ and sufficiently large y_i, $\sigma(\tilde{y}_i) = \sigma(y_i)$. Since we are trying to establish the boundedness of y_i we can, without loss of generality, assume $|y_i|$ is sufficiently large. Then we have that $[-\frac{1}{k_1}\sigma(y_i) + y_{i+1}], y_{i+2}, \ldots, y_m$, z, u converge to zero with an exponential tail. Hence, from assumption 2.4, y_i is bounded. Hence, by T_1, $\tilde{y}_i$ is bounded.

4.1.3 Stability Analysis

Throughout our analysis we will rely on lemmas taken from [2] which apply to the finite-dimensional unperturbed differential equation

$$\dot{x} = f(x, t) \tag{15}$$

with f satisfying certain smoothness assumptions and such that $f(0, t) = 0$ for $t \geq t_o$, and the perturbed differential equation

$$\dot{x} = f(x, t) + g(x, t) \tag{16}$$

Lemma 4.1 *If the equilibrium of (15) is exponentially stable and if $g(x, t)$ satisfies an estimate $g(x, t) = o(||x||)$ then the equilibrium of (16) is also exponentially stable, in fact with the same exponent.*

Lemma 4.2 *Let the equilibrium of (15) be (locally) exponentially stable. Then (for sufficiently small $||x||$) there exists a Lyapunov function $V(x,t)$ which satisfies estimates of the form*

$$\begin{aligned} a_1||x||^2 &\leq V(x,t) \leq a_2||x||^2 \\ \dot{V} &\leq -a_3||x||^2 \\ ||\frac{\partial V}{\partial x}|| &\leq a_4||x|| \end{aligned} \tag{17}$$

for certain positive constants a_1, a_2, a_3, a_4.

Lemma 4.3 *If the equilibrium of (15) is exponentially stable and if $g(x,t)$ satisfies an estimate $||g(x,t)|| \leq \epsilon$ for ϵ sufficiently small then for sufficiently small $||x(t_\circ)||$, $||x(t)||$ satisfies an estimate of the form $||x(t)|| \leq a\epsilon$ for all $t > T$ for some $T \geq t_\circ$ and for some positive constant a which depends on a_1, a_2, a_3, a_4.*

We now propose the following for the remaining control v:

$$v = -\sigma_m(\tilde{y}_m + \sigma_{m-1}(\tilde{y}_{m-1} + \cdots + \sigma_1(\tilde{y}_1)))\cdots) \tag{18}$$

where σ_i is a simple saturation for δ_i, ϵ_i and we show that the values δ_i, ϵ_i can be chosen to yield global asymptotic stability.

The first thing to observe is that, with all of the saturating limits removed, the dynamics of $(\tilde{y}, \tilde{z})$ are of an asymptotically stable linear system perturbed by higher order terms. Hence, from lemma 4.1, for the system with the saturating limits removed, there is an open neighborhood $U \subset \mathbb{R}^{m+n}$ of 0 such that if $(\tilde{y}_0, \tilde{z}_0) \in U$ then the equilibrium $(\tilde{y}, \tilde{z}) = (0,0)$ is exponentially stable. It follows that, if we can show from any initial condition the states $(\tilde{y}, \tilde{z})$ enter and remain in a small neighborhood $V \subset U$ in which the functions σ_i for $i = 1, \ldots, m$ operate in their linear region, lemma 4.1 allows us to conclude global asymptotic stability and local exponential stability for the dynamics of $\tilde{y}, \tilde{z}$ with the saturating limits included. Finally, by assumption 1 and theorem 2.1, the complete composite system has $(0,0,0)$ as a G.A.S. equilibrium.

We set out to establish that all of the states $(\tilde{y}, \tilde{z})$ can be steered to the set V in finite time by judicious choice of δ_i, ϵ_i.

Observe that the dynamics of $\tilde{z}$ are given by an asymptotically stable linear system perturbed by a small disturbance (with maximum absolute amplitude of ϵ_m). Here the estimates of lemma 4.2 apply globally. Hence lemma 4.3 applies for any initial condition $\tilde{z}(0)$. This leads to a bound $|\tilde{z}(t)| \leq a\epsilon_m$ for all $t > T_{m+1}$ for some $T_{m+1} > t_\circ$. Observe that $|u(t)| \leq a_K\epsilon_m$ for all $t > T_{m+1}$ where a_K depends on a and the feedback gains K. We define $a_m = max\{a, a_K, k_1 a\}$.

With this bound on $\tilde{z}$ we define $\tilde{y}_{m+1} = -k_1\tilde{z}_1$ and proceed by induction showing that given ϵ_{i-1} sufficiently small, $\exists\epsilon_i$ sufficiently small such that if

$$\begin{aligned} |\tilde{y}_{j+1}(t)| &\leq a_i\epsilon_i \quad j = i, \ldots, m \\ ||\tilde{z}(t)|| &\leq a_i\epsilon_i \\ |u(t)| &\leq a_i\epsilon_i \end{aligned}$$

for all $t > T_{i+1}$, then

$$\begin{aligned} |\tilde{y}_j(t)| &\leq a_{i-1}\epsilon_{i-1} \quad j = i, \ldots, m \\ ||\tilde{z}(t)|| &\leq a_{i-1}\epsilon_{i-1} \\ |u(t)| &\leq a_{i-1}\epsilon_{i-1} \end{aligned}$$

for all $t > T_i > T_{i+1}$.

Assume that ϵ_i is chosen sufficiently small such that σ_{i+1} operates in its linear region for all $t > T_{i+1}$. (σ_{m+1} can be considered a globally linear function.) Consider the dynamics for $\tilde{y}_i, \ldots, \tilde{y}_m, \tilde{z}$ after time T_{i+1}:

$$\begin{aligned} \dot{\tilde{y}}_i &= -\sigma_i + \tilde{h}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{y}}_{i+1} &= -\tilde{y}_{i+1} - \sigma_i + \tilde{h}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{y}}_{i+2} &= -\tilde{y}_{i+2} - \tilde{y}_{i+1} - \sigma_i + \tilde{h}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ &\vdots \\ \dot{\tilde{z}} &= A\tilde{z} - B(\tilde{y}_m + \cdots + \tilde{y}_{i+1} + \sigma_i) \end{aligned} \tag{19}$$

We show that, for ϵ_i sufficiently small $\tilde{y}_i$ becomes small and after some finite time $T_i > T_{i+1}$ remains in a region such that σ_i is linear. Consider $\tilde{y}_i$ such that $|\tilde{y}_i| > \epsilon_i + \epsilon_{i-1}$ and make the coordinate change

$$\begin{aligned} \bar{y}_{i+1} &= \tilde{y}_{i+1} + \sigma_i \\ \bar{y}_j &= \tilde{y}_j \qquad\qquad j = i+2, \ldots, m \\ \bar{z} &= \tilde{z} \end{aligned}$$

Then the dynamics of $\tilde{y}_i, \bar{y}_{i+1}, \ldots, \bar{y}_m, \bar{z}$ are

$$\begin{aligned} \dot{\tilde{y}}_i &= -\sigma_m(\tilde{y}_i) + \tilde{h}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\bar{y}}_{i+1} &= -\bar{y}_{i+1} + \bar{h}_{i+1}(\tilde{x}, \bar{y}, \bar{z}, u, t) \\ \dot{\bar{y}}_{i+2} &= -\bar{y}_{i+2} - \bar{y}_{i+1} + \bar{h}_{i+2}(\tilde{x}, \bar{y}, \bar{z}, u, t) \\ &\vdots \\ \dot{\bar{z}} &= A\bar{z} - B(\bar{y}_m + \cdots + \bar{y}_{i+1}) \end{aligned} \tag{20}$$

since, when $|\tilde{y}_i| > \epsilon_i + \epsilon_{i-1}$,

$$\begin{aligned} \dot{\sigma}_i &= 0 \\ \sigma_m(\tilde{y}_i) &= \sigma_i(\tilde{y}_i + \sigma_{i-1}(\cdot)) \end{aligned}$$

Observe that, for the dynamics of $\bar{y}_{i+1}, \ldots, \bar{y}_m, \bar{z}$, lemma 4.1 applies so that, for ϵ_i sufficiently small, $\bar{y}_{i+1}, \ldots, \bar{y}_m, \bar{z}$ converge exponentially toward zero. Point 3 above is crucial for the perturbations $\bar{h}_j$ for $j = i+1, \ldots, m$ to remain higher order in the $\bar{y}_{i+1}, \ldots, \bar{y}_m, \bar{z}$ coordinates. Note that, since the control is $u = K\bar{z} - \bar{y}_m - \cdots - \bar{y}_{i+1}$, u also converges exponentially toward zero. We assert that, for small enough ϵ_i, and with these "controls" set to zero, the set $M = \{\tilde{y}_i : |\tilde{y}_i| \leq \epsilon_i + \epsilon_{i-1}\}$ is attractive. Further, since the dynamics of $\tilde{y}_i$ satisfy property 2, by theorem 2.1, at some finite time $T > T_{i+1}$, $\tilde{y}_i$ will enter M.

Consider the dynamics of $\tilde{y}_i$ with the "controls" $\bar{z}, \bar{y}_m, \ldots, \bar{y}_{i+1}, u$ set to zero:

$$\begin{aligned} \dot{\tilde{y}}_i &= -\sigma_i(\tilde{y}_i) + \tilde{h}_i(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_i, -\sigma_i, 0, \ldots, 0, t) \\ &= -\sigma_i(\tilde{y}_i) + \tilde{h}_i^a + \tilde{h}_i^b \end{aligned} \tag{21}$$

With regard to point 2 above we have $|\tilde{y}_{i+1}| = \epsilon_i$. Consider the time derivative of the Lyapunov function $V_i = \tilde{y}_i^2$ along the trajectories of (21):

$$\begin{aligned} \dot{V}_i &= 2\tilde{y}_i[-\sigma_i(\tilde{y}_i) + \tilde{h}_i^a + \tilde{h}_i^b] \\ &\leq 2|\tilde{y}_i|[-\epsilon_i + C_1\epsilon_i^2 + C_2\epsilon_i^2] \end{aligned}$$

(Note that the term $\tilde{y}_i\tilde{h}_i^b \leq 0$ for $|\tilde{y}_m| > d\epsilon_m$ from point 2, and is uniformly higher order for $|\tilde{y}_m| \leq d\epsilon_i$ from point 1.) It follows that we must choose ϵ_i such that

$$\epsilon_i - (C_1 + C_2)\epsilon_i^2 < 0$$

to insure that the set M is attractive with the controls $\tilde{y}_{i+1}, \ldots, \tilde{y}_m, \tilde{z}, u$ set to zero.

We show now that for ϵ_{i-1} sufficiently small, $\tilde{y}_i$ enters and stays in a region where $\sigma_i(\cdot)$ is linear. (Note that $\epsilon_0 \equiv 0$.) Again consider the dynamics of $\tilde{y}_i$ beginning at the time when $\tilde{y}_i$ enters M:

$$\dot{\tilde{y}}_i = -\sigma_m(\tilde{y}_i + \sigma_{i-1}) + \tilde{h}_i(\tilde{y}, \tilde{z}, u, t) \tag{22}$$

We take the derivative of the Lyapunov function $V_i = \tilde{y}_i^2$ along the trajectories of (22) and employ point 1 from above:

$$\begin{aligned} \dot{V}_i &= 2\tilde{y}_i[-\sigma_i(\tilde{y}_i + \sigma_{i-1}) + \sigma_i(\tilde{y}_i) - \sigma_i(\tilde{y}_i) + \tilde{h}_i] \\ &\leq 2|\tilde{y}_i|[-|\sigma_i(\tilde{y}_i)| + \epsilon_{i-1} + \tilde{C}\epsilon_i^2] \end{aligned}$$

First note that if $\epsilon_i - \epsilon_{i-1} - C_3\epsilon_i^2 > 0$ then the set M is invariant. Second, observe that given δ_i, ϵ_i, if ϵ_{i-1} satisfies

$$\epsilon_{i-1} < \frac{\delta_i - C_3\epsilon_i^2}{2}$$

then $\tilde{y}_i$ will enter the set $Q_i = \{\tilde{y}_i : |\tilde{y}_i| \leq \frac{\delta_i + C_3\epsilon_i^2}{2}\}$ in finite time and remain in Q_i thereafter. With $\tilde{y}_i \in Q_i$ the argument of σ_i is bounded by

$$\begin{aligned} |\tilde{y}_i + \sigma_{i-1}| &\leq |\tilde{y}_i| + |\sigma_{i-1}| \\ &\leq \frac{\delta_i + C_3\epsilon_i^2}{2} + \frac{\delta_i - C_3\epsilon_i^2}{2} \\ &\leq \delta_i \end{aligned}$$

Hence $\sigma_i(\cdot)$ enters in finite time and thereafter remains in its linear region.

Note that after this finite time the dynamics of $(\tilde{y}_i, \ldots, \tilde{y}_m, \tilde{z})$ are of an asymptotically stable linear system perturbed by higher order terms as well as a perturbation of maximum amplitude ϵ_{i-1}. Combining lemma 4.1 and lemma 4.3, if ϵ_i is sufficiently small (to start in a small neighborhood of the origin) then we can establish bounds $|\tilde{y}_j| < a_{i-1}\epsilon_{i-1}$ for $j = i, \ldots, m$ and $||\tilde{z}|| < a_{i-1}\epsilon_{i-1}$ and $|u| < a_{i-1}\epsilon_{i-1}$ for all $t > T_i > T_{i+1}$. □

5 Semi-Global Results

In this section we state our main results for semi-global stabilizability. Again the proof of this theorem will be constructive yielding an algorithm for generating the proposed class of semi-globally stabilizing control laws.

Theorem 5.1 (Semi-Global Stabilizability) *If assumptions 1 and 3 are satisfied, then there exists a family of time independent control laws of the form*

$$u = Kz - \sigma_m(T_m(y,z) + \sigma_{m-i}(T_{m-i}(y,z) + \cdots + \sigma_1(T_1(y,z)))\cdots)$$

which semi-globally stabilizes the origin of (1) where σ_i is a simple saturation for δ_i, ϵ_i, T_i is a linear function and the gains K are the coefficients of a Hurwitz polynomial.

We have the following corollary regarding semi-globally stabilizing (1) with a bounded control which follows by simply redefining the dynamics of y to include the dynamics of z as well.

Corollary 5.1 (Semi-Global Stabilizability with bounded controls) *If assumptions 1 and 3 are satisfied, then there exists a family of time independent bounded control laws of the form*

$$u = -\sigma_{m+n}(T_{m+n}(y,z) + \sigma_{m+n-i}(T_{n+m-i}(y,z) + \cdots + \sigma_1(T_1(y,z)))\cdots)$$

which semi-globally stabilizes the origin of (1) where σ_i is a simple saturation for δ_i, ϵ_i and T_i is a linear function.

5.1 Proof of theorem 5.1

The proof is again constructive. We employ the same convenient coordinate change as the the case of global stabilization. We will develop how the conditions of assumption 3 translate in these new coordinates. Most of the work then lies in showing how these conditions allow for a semi-globally stabilizing class of control laws.

5.1.1 Coordinate change

As in the case for global stabilization we begin by choosing the input as $u = Kz + v$ where the gains K are the coefficients of a Hurwitz polynomial. In addition, we add the condition that the gains K are such that Re $\sigma(A) \le -1$ where A is defined in (10). The coordinate change then proceeds in the same way as in the global case (see section 4.1.1.) Once again we have

$$\begin{aligned} \dot{\tilde{y}}_i &= \tilde{g}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \cdots + \tilde{y}_m + v + \textstyle\sum_{j=i}^m T_{1_{ij}} h_j(\tilde{x}, T_1^{-1}(\tilde{y} - T_2\tilde{z}), \tilde{z}, u, t) \\ &= \tilde{y}_{i+1} + \cdots + \tilde{y}_m + v + \tilde{h}_i(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \end{aligned} \tag{23}$$

We proceed to determine the relevant properties of $\tilde{h}_i$.

5.1.2 Properties of Perturbation Terms

Define $\tilde{y}_{m+1} \equiv -k_1\tilde{z}_1$. Next we establish the properties of $\tilde{h}_i$ that follow from assumption 3. First observe that if assumption 3.1 applies to $\tilde{h}_i$ then the four points established in section 4.1.2 apply. Otherwise we establish the following properties for $\tilde{h}_i$ and $\tilde{h}_{i-1}$ that follow from assumption 3.2:

for $j = i-1, i$

1. $$|\tilde{h}_j - \tilde{h}_j(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t)| \le (|\tilde{y}_{i-1}| + |\tilde{y}_i| + 1)|\bar{h}_j|$$

   where $\bar{h}_j$ is bounded for bounded $\tilde{y}_{i+1}, \ldots, \tilde{y}_m$, z, u and higher order in $\tilde{y}_{i+2}, \ldots, \tilde{y}_m$, z, u for bounded $\tilde{y}_{i+1}$ uniformly in $\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_i$, t.

2. (a) $$|\tilde{h}_{i-1}(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t)| \le (|y_i| + 1)|\bar{h}_{i-1}|$$

   where $\bar{h}_{i-1}$ is higher order in $\tilde{y}_{i+1}$ uniformly in $\tilde{y}_1, \ldots, \tilde{y}_i$, t and bounded for bounded $\tilde{y}_{i+1}$.

   (b) $\tilde{h}_i(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t)$ depends only on $\tilde{y}_{i+1}$. Further it is higher order in $\tilde{y}_{i+1}$ and is bounded for bounded $\tilde{y}_{i+1}$.

3. For some $\epsilon_o > 0$, $\tilde{h}_j(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i-1}, 0, \ldots, 0, t) = 0$ for $|\tilde{y}_{i-1}| < \epsilon_o$.

Consider point 1. Denote $h_j(x, y_1, \ldots, y_{i+1}, 0, \ldots, 0, t)$ by h_j° and likewise for $\tilde{h}_j$. From (23),

$$|\tilde{h}_j - \tilde{h}_j^\circ| \le \sum_{k=j}^{m} T_{1_{jk}} |h_k - h_k^\circ|$$

Point 1 then follows from apply assumptions 3.2.a and 2.1 to the appropriate terms in the summation and then using T_1 to return to the $\tilde{y}$ coordinates.

Consider point 2a. Assume $\tilde{y}_k = 0$ for $k = i+2, \ldots, m$ and $\tilde{z} = 0$ and $u = 0$. From T_1^{-1} this implies $y_k = 0$ for $k = i+2, \ldots, m$ and $z = 0$. By assumption 3.2.b this implies $|h_{i-1}| < (|y_i| + 1)|\hat{h}_{i-1}|$ where $\hat{h}_{i-1}$ is higher order in y_{i+1} uniformly in $x, y_1, \ldots, y_i, t$ and is bounded for bounded y_{i+1}. Also h_i is higher order in y_{i+1} uniformly in $x, y_1, \ldots, y_i, t$ and bounded for bounded y_{i+1}. Further, assumptions 3.2.c and 2.3 imply $h_k = 0$ for $k = i+1, \ldots, m$. Hence, from (23) and T_1, $|\tilde{h}_{i-1}(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t)| < (|\tilde{y}_i| + 1)|\bar{h}_{i-1}|$ where $\bar{h}_{i-1}$ is higher order in $\tilde{y}_{i+1}$ uniformly in $\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_i, t$. Consider point 2b. Since $h_k = 0$ for $k = i+1, \ldots, m$, $\tilde{h}_i = T_{1_{ii}} h_i$. Now h_i depends only on y_{i+1} and is higher order in y_{i+1} and bounded for bounded y_{i+1}. Finally, since $\tilde{y}_j = 0$ for $j = i+2, \ldots, m$, it follows that $\tilde{y}_{i+1} = T_{1_{ii}} y_{i+1}$. Hence, $\tilde{h}_i(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0, t)$ depends only on $\tilde{y}_{i+1}$ and is higher order in $\tilde{y}_{i+1}$ and bounded for bounded $\tilde{y}_{i+1}$.

Consider point 3. For some $\tilde{\epsilon}_o > 0$, assume that $|\tilde{y}_{i-1}| < \tilde{\epsilon}_o$ for some $\tilde{\epsilon}_o > 0$. Further, assume $\tilde{y}_j = 0$ for $j = i, \ldots, m$ and $\tilde{z} = 0$ and $u = 0$. From T_1^1 this implies $|y_{i-1}| < \tilde{\epsilon}_o / c$ for some constant $c > 0$ and $y_j = 0$ for $j = i, \ldots, m$ and $z = 0$. Define $\tilde{\epsilon}_o \equiv c\epsilon_o$. By assumptions 3.2.c and 2.3 this implies $h_j = 0$ for $j = i-1, \ldots, m$. Finally, from (23), $\tilde{h}_j(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{i-2}, \tilde{y}_{i-1}, 0, \ldots, 0, t) = 0$ for $j = i-1, i$.

5.1.3 Stability Analysis

Again we will rely on lemmas taken from [2] which are stated in section 4.1.3. In addition will will use the following lemma in our proof.

Lemma 5.1 *Consider the n-dimensional nonlinear system*

$$\dot{x} = f(x, t)$$

where $|f_i(x,t)| \leq q_i(t) + \sum_{j=1}^{n} a_{ij}|x_j|$ *for all* $t \geq t_0$. *Define the constant matrix* $\bar{A}$ *by* $\bar{A}_{ij} = a_{ij}$. *Consider the vectors*

$$\begin{aligned} \bar{x}(t) &= [|x_1(t)|, \ldots, |x_n(t)|]^T \\ \bar{q}(t) &= [|q_1(t)|, \ldots, |q_n(t)|]^T. \end{aligned}$$

Then $\bar{x}(t)$ *is bounded as*

$$\bar{x}(t) \leq e^{\bar{A}(t-t_0)}\bar{x}(t_0) + \int_{t_0}^{t} e^{\bar{A}(t-\tau)}\bar{q}(\tau)d\tau$$

We begin by formulating a bounded control v to stabilize $\tilde{y}$ using the following algorithm:

1. let $k = m$ and let $v = -\sigma_m$ where σ_m is a simple saturation for ϵ_m, δ_m to be specified.

2. if assumption 3.1 applies to $\tilde{h}_k$ then

   (a) let the argument of σ_k be $\tilde{y}_k + \sigma_{k-1}(\cdot)$ where σ_{k-1} is a simple saturation for $\epsilon_{k-1}, \delta_{k-1}$ to be specified.

   (b) let k=k-1

   (c) return to step 2.

3. if assumption 3.2 applies to $\tilde{h}_k$ (and hence $\tilde{h}_{k-1}$) then

   (a) let the argument of σ_k be $\tilde{y}_k + \tilde{y}_{k-1} + \sigma_{k-2}(\cdot)$ where σ_{k-2} is a simple saturation for $\epsilon_{k-2}, \delta_{k-2}$ to be specified.

   (b) let k=k-2

   (c) return to step 2.

We show that, given initial conditions in some bounded set X, the values ϵ_i, δ_i can be chosen to yield asymptotic stability.

The proof proceeds in the same manner as the proof for global stabilizability. Again we define $\tilde{y}_{m+1} \equiv -k_1\tilde{z}_1$. In addition, we define k to be the largest index such that assumption 3.2 applies to $\tilde{h}_i$ rather than assumption 3.1. It follows then from the proof of the global result that, $\exists \epsilon_j$ for $j = k, \ldots, m$ sufficiently small such that σ_j for $j = k+1, \ldots, m$ operate in their linear region for all $t > T_{k+1}$ (σ_{m+1} can be considered as a globally linear function.)

We now show that $\exists \epsilon_k$ sufficiently small such that for ϵ_{k-2} sufficiently small σ_k operates in its linear region for all $t > T_k > T_{k+1}$. (For $k = 2$, observe that $\epsilon_{k-2} \equiv 0$.)

First, we know that $u, \tilde{z}$ and $\tilde{y}_i$ for $i = k+1, \ldots, m$ are bounded for all $t > 0$. Then since $\tilde{h}_{k-1}, \tilde{h}_k$ are globally Lipschitz in $\tilde{y}_{k-1}, \tilde{y}_k$ for bounded $u, \tilde{z}, \tilde{y}_i$ for $i = k+1, \ldots, m$, $\exists R$ which depends on the initial conditions of $\tilde{y}_i(t_0)$ for $i = k-1, \ldots, m$ and $\tilde{z}(t_0)$ and on ϵ_i for $i = k+1, \ldots, m$ such that for $j = k-1, k$

$$|\tilde{y}_j(T_{k+1})| \le R.$$

Consider the dynamics for $\tilde{y}_{k-1}, \ldots, \tilde{y}_m, \tilde{z}$ for $t > T_{k+1}$:

$$\begin{aligned} \dot{\tilde{y}}_{k-1} &= \tilde{y}_k - \sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_{k-1}(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{y}}_k &= -\sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_k(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ \dot{\tilde{y}}_{k+1} &= -\tilde{y}_{k+1} - \sigma_k + \tilde{h}_{k+1}(\tilde{x}, \tilde{y}, \tilde{z}, u, t) \\ &\vdots \\ \dot{\tilde{z}} &= A\tilde{z} - B(\tilde{y}_m + \cdots + \tilde{y}_{k+1} + \sigma_k) \end{aligned} \tag{24}$$

Again from the proof of global stabilizability we know that, for all $t > T_{k+1}$,

$$\begin{aligned} |\tilde{y}_i(t)| &\le a_{k+1}\epsilon_{k+1} \quad i = k+1, \ldots, m \\ ||\tilde{z}(t)|| &\le a_{k+1}\epsilon_{k+1} \\ |u(t)| &\le a_{k+1}\epsilon_{k+1}. \end{aligned}$$

Then, since $\tilde{h}_{k-1}, \tilde{h}_k$ satisfy points 1 and 2 of section 5.1.2, the dynamics of $\tilde{y}_{k-1}, \tilde{y}_k$ are of a 2-dimensional nonlinear system satisfying the conditions of lemma 5.1 with

$$\bar{A} = \begin{bmatrix} C\epsilon_{k+1}^2 & 1 + C\epsilon_{k+1}^2 \\ C\epsilon_{k+1}^2 & C\epsilon_{k+1}^2 \end{bmatrix}$$

and

$$\bar{q}(t) = \begin{bmatrix} \epsilon_{k+1} + C\epsilon_{k+1}^2 \\ \epsilon_{k+1} + C\epsilon_{k+1}^2 \end{bmatrix}$$

(since we will choose $\epsilon_k < \epsilon_{k+1}$.) A simple calculation using lemma 5.1 shows that for some $\bar{K}$ depending on ϵ_{k+1}, for $j = k-1, k$ and $\forall t > T_{k+1}$

$$|\tilde{y}_j(t)| \le R\bar{K}e^{\alpha(t - T_{k+1})} \tag{25}$$

where

$$\alpha = C\epsilon_{k+1}^2 + \sqrt{C\epsilon_{k+1}^2(1 + C\epsilon_{k+1}^2)}$$

For convenience, we choose ϵ_{k+1} such that $\alpha < 0.5$.

Now since the linear approximation at the origin of the dynamics of $\tilde{y}_{k+1}, \ldots, \tilde{y}_m, \tilde{z}$ has eigenvalues with real part less than or equal to -1, we can conclude from the lemmas 4.1 and 4.3 that

$$\begin{array}{lll} |\tilde{y}_{i+1}(t)| & \leq a_k\epsilon_k + a_{k+1}\epsilon_{k+1}e^{-(t-T_{k+1})} & i = k, \ldots, m \\ ||\tilde{z}(t)|| & \leq a_k\epsilon_k + a_{k+1}\epsilon_{k+1}e^{-(t-T_{k+1})} & \\ |u(t)| & \leq a_k\epsilon_k + a_{k+1}\epsilon_{k+1}e^{-(t-T_{k+1})} & \end{array}$$

for all $t > T_{k+1}$.

We solve for the time t_ϵ such that, for all $t \geq t_\epsilon$

$$\begin{array}{lll} |\tilde{y}_{i+1}(t)| & \leq 2a_k\epsilon_k & i = k, \ldots, m \\ ||\tilde{z}(t)|| & \leq 2a_k\epsilon_k & \\ |u(t)| & \leq 2a_k\epsilon_k & \end{array}$$

We find

$$t_\epsilon = T_{k+1} - \ln \frac{a_k\epsilon_k}{a_{k+1}\epsilon_{k+1}}.$$

Further, from (25), we determine a bound on $\tilde{y}_j(t)$ for $T_{k+1} \leq t \leq t_\epsilon$ for $j = k-1, k$ to be

$$|\tilde{y}_j| \leq R\bar{K}\left(\frac{a_{k+1}\epsilon_{k+1}}{a_k\epsilon_k}\right)^\alpha \equiv R_{\epsilon_k} \tag{26}$$

So then it remains to determine whether ϵ_k can be chosen sufficiently small such that a small neighborhood of the origin is attractive for the dynamics

$$\begin{array}{lll} \dot{\tilde{y}}_{k-1} & = & \tilde{y}_k - \sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_{k-1}(t) \\ \dot{\tilde{y}}_k & = & -\sigma_k(\tilde{y}_{k-1} + \tilde{y}_k + \sigma_{k-2}) + \tilde{h}_k(t) \end{array} \tag{27}$$

from initial conditions such that

$$|\tilde{y}_j| \leq R_{\epsilon_k}$$

for $j = k-1, k$ and where $\tilde{h}_j$ satisfy the properties of section 5.1.2. To show that this is possible we begin with the simple coordinate change

$$\begin{array}{lll} x_1 & = & \tilde{y}_{k-1} + \tilde{y}_k \\ x_2 & = & \tilde{y}_k \end{array}$$

yielding the dynamics

$$\begin{array}{lll} \dot{x}_1 & = & x_2 - 2\sigma_k(x_1 + \sigma_{k-2}) + f_1(t) \\ \dot{x}_2 & = & -\sigma_k(x_1 + \sigma_{k-2}) + f_2(t) \end{array} \tag{28}$$

It can be shown that for $j = 1, 2$

$$|f_j - f_j(\tilde{y}_1, \ldots, \tilde{y}_{i+1}, 0, \ldots, 0)| \leq (|x_1| + |x_2| + 1)\bar{f}_j \tag{29}$$

where $\bar{f}_j$ is higher order in $\tilde{y}_{k+2}, \ldots, \tilde{y}_m$, $\tilde{z}, u$ uniformly in $\tilde{y}_1, \ldots, \tilde{y}_{k+1}, t$ (since $\tilde{y}_{k+1}$ is bounded.) Further, it can be shown that

$$|f_1(\tilde{y}_1, \ldots, \tilde{y}_{k+1}, 0, \ldots, 0)| \leq (|x_2| + 1)D\epsilon_k^2 \tag{30}$$

and $f_2(\tilde{y}_1, \ldots, \tilde{y}_{k+1}, 0, \ldots, 0)$ depends only on $\tilde{y}_{k+1}$ and is higher order.

Observe that if $|x_1(t_b)| \geq \epsilon_k + \epsilon_{k-2}$, then for all $t \geq t_b$ such that $|x_1(t)| \geq \epsilon_k + \epsilon_{k-2}$ we have

$$\begin{array}{rcl} |\tilde{y}_{i+1}(t)| & \leq & \epsilon_k + 2a_k\epsilon_k e^{-(t-t_b)} \\ |\tilde{y}_{i+2}(t)| & \leq & 2a_k\epsilon_k e^{-(t-t_b)} \qquad i = k,\ldots,m \\ ||\tilde{z}(t)|| & \leq & 2a_k\epsilon_k e^{-(t-t_b)} \\ |u(t)| & \leq & 2a_k\epsilon_k e^{-(t-t_b)} \end{array} \tag{31}$$

This follows from point 3 of section 4.1.2. Hence, for $t \geq t_b$ and such that $|x_1(t)| \geq \epsilon_k + \epsilon_{k-2}$,

$$\begin{array}{rcl} |f_1| & \leq & (|x_2|+1)D\epsilon_k^2 + (|x_1|+|x_2|+1)D\epsilon_k^2 e^{-(t-t_b)} \\ f_2 & = & C + \hat{f}_2 \end{array}$$

where C is a constant and

$$\begin{array}{rcl} |\hat{f}_2| & \leq & (|x_1|+|x_2|+1)D\epsilon_k^2 e^{-(t-t_b)} \\ |C| & \leq & D\epsilon_k^2 \end{array}$$

Then from (25) and (26), we have the bound

$$\begin{array}{rcl} |f_1| & \leq & |x_2|D\epsilon_k^2 + D\epsilon^{2-\alpha} \\ |\hat{f}_2| & \leq & D\epsilon_k^{2-\alpha} e^{-(1-\alpha)(t-t_b)} \end{array} \tag{32}$$

These bounds on the nonlinear terms when $|x_1| \geq \epsilon_k + \epsilon_{k-2}$ will play a crucial part in our analysis.

The remainder of the proof consists of three points. Consider the set

$$Q = \{(x_1, x_2) : |x_1| \leq \epsilon_k + \epsilon_{k-2}\}$$

Point 1 will be to show that if $x_1(t_\epsilon) \notin Q$ then $\exists t_c > t_\epsilon$ which is finite such that $x_1(t_c) \in Q$ and we establish a worst case value for $|x_2(t_c)|$. Point 2 will be to consider the "Lyapunov-like" function

$$W = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}x_2^2 \tag{33}$$

which we will demonstrate is uniformly decreasing when $x_1(t) \in Q \cap U^c$. Here U is a neighborhood of the origin depending on ϵ_{k-2} and such that $\sigma_k(x_1 + \sigma_{k-2}) = x_1 + \sigma_{k-2}$ for all $x_1 \in U$. (U^c is the complement of U.) Point 3 will be to show that whenever the trajectory leaves Q it returns to Q and when it does it returns at a lower energy level for W.

For point 1, define the set

$$Q_r = \{(x_1, x_2) : x_1 > \epsilon_k + \epsilon_{k-2}\}$$

and without loss of generality assume $x_1(t_\epsilon) \in Q_r$. We demonstrate that for ϵ_k sufficiently small and for

$$2\epsilon_k + D\epsilon^{2-\alpha} < x_2(t_0) \leq R_{\epsilon_k}$$

$\exists t_d > t_\epsilon$ such that $x_1(t_d) = x_1(t_\epsilon)$ and further,

$$|x_2(t_d)| \leq x_2(t_0)$$

In the set Q_r the dynamics of (x_1, x_2) are given by

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 - 2\epsilon_k + f_1(t) \\ \dot{x}_2 & = & -\epsilon + C + \hat{f}_2(t) \end{array} \tag{34}$$

From the bounds on C and $\hat{f}_2(t)$, x_2 is monotonically decreasing for sufficiently small ϵ_k. We now consider the forms of $f_1, \hat{f}_2$ which will maximize $|x_2(t_d)|$. We do this by considering the instantaneous slope of the trajectory in the (x_1, x_2) plane. The instantaneous slope is given by

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{-\epsilon_k + C + \hat{f}_2(t)}{x_2 - 2\epsilon_k + f_1(t)} \tag{35}$$

Observe that since x_2 is monotonically decreasing, the actual trajectory will be bounded by two curves. The outer curve is produced by flowing along the vector field that minimizes the magnitude of the negative instantaneous slope (x_1 increasing) and maximizes the positive instantaneous slope (x_1 decreasing.) The inner curve is produced by flowing along the vector field that maximizes the magnitude of the negative instantaneous slopes and minimizes positive instantaneous slopes. It is straightforward to see that, as long as $|x_2| \leq R_{\epsilon_k}$, the outer curve is generated by setting $f_1(t) = D\epsilon_k^{2-\alpha}$ and

$$\hat{f}_2(t) = \left\{ \begin{array}{ll} D\epsilon_k^{2-\alpha}e^{-(t-t_b)} & x_2(t) \geq 2\epsilon_k - D\epsilon_{2-\alpha} \\ -D\epsilon_k^{2-\alpha}e^{-(t-t_b)} & x_2(t) < 2\epsilon_k - D\epsilon_{2-\alpha} \end{array} \right\}$$

Likewise, the inner curve is generated by setting $f_1(t) = -D\epsilon_k^{2-\alpha}$ and

$$f_2(t) = \left\{ \begin{array}{ll} -D\epsilon_k^{2-\alpha}e^{-(t-t_b)}) & x_2(t) \geq 2\epsilon_k + D\epsilon_{2-\alpha} \\ D\epsilon_k^{2-\alpha}e^{-(t-t_b)} & x_2(t) < \epsilon_k + D\epsilon_{2-\alpha} \end{array} \right\}$$

(The value of C is fixed as a function of ϵ_k.) The outer curve gives us a least upper bound on $|x_2(t_d)|$. To compute this bound, we first calculate the time t_i on the outer curve such that $x_2(t_i) = 2\epsilon_k - D\epsilon_k^{2-\alpha}$. The value of $x_2(t)$ for $t_\epsilon \leq t \leq t_i$ along the outer curve is given by

$$x_2(t) = x_2(t_\epsilon) - (\epsilon_k + C)t - \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}[e^{-(1-\alpha)t} - 1] \tag{36}$$

(We have temporarily reinitialized $t_\epsilon = 0$ for convenience.) Thus, we have (implicitly)

$$t_i = \frac{1}{\epsilon_k - C}x_2(t_\epsilon) - 2\epsilon_k - \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}[e^{-(1-\alpha)t_i} - 1] \tag{37}$$

The value of $x_1(t)$ for $t_\epsilon \leq t \leq t_i$ along the outer curve is given by

$$\begin{aligned} x_1(t) \;=\; & x_1(t_\epsilon) - \tfrac{1}{2}(\epsilon_k - C)t^2 + [x_2(0) - 2\epsilon_k + \tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha} + D\epsilon_k^{2-\alpha}]t \\ & + \tfrac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}[e^{-(1-\alpha)t} - 1] \end{aligned} \tag{38}$$

Thus, we have

$$\begin{aligned} x_1(t_i) \;=\; & x_1(t_\epsilon) + \tfrac{1}{2(\epsilon_k - C)}(x_2(t_\epsilon) - 2\epsilon_k + D\epsilon_k^{2-\alpha} + \tfrac{D\epsilon_k^{2-\alpha}}{(1-\alpha)})^2 \\ & - \tfrac{1}{2(\epsilon_k - C)}[\tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha}e^{-(1-\alpha)t_i}]^2 - \tfrac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}[1 - e^{-(1-\alpha)t_i}] \end{aligned} \tag{39}$$

We now continue the flow beginning at the point $(x_1(t_i), x_2(t_i))$. Then for $t_i < t \leq t_d$ the flow along the outer curve is given by

$$\begin{aligned} x_1(t) \;=\; & x_1(t_i) + [x_2(t_i) - 2\epsilon_k + D\epsilon_k^{2-\alpha} - \tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha}e^{-(1-\alpha)t_i}]t - \tfrac{1}{2}(\epsilon_k - C)t^2 \\ & - \tfrac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}e^{-(1-\alpha)t_i}[e^{-(1-\alpha)t} - 1] \\ x_2(t) \;=\; & x_2(t_i) - (\epsilon_k - C)t + \tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha}e^{-(1-\alpha)t_i}[e^{-(1-\alpha)t} - 1] \end{aligned} \tag{40}$$

(We have reinitialized $t_i = 0$ for convenience.) We are now interested in determining $x_2(t_d)$ where t_d is such that $x_1(t_d) = x_1(t_0)$. Since we are interested in a worst case bound for $|x_2(t_d)|$ and x_2 is monotonically decreasing in the region we are considering, it suffices to determine a least upper bound for t_d. We find that

$$t_d \leq t_i + \frac{1}{\epsilon_k - C}\sqrt{[x_2(t_\epsilon) - 2\epsilon_k + D\epsilon_k^{2-\alpha} + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}]^2 + 2(\epsilon_k - C)\frac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}} \tag{41}$$

Consequently, we can conclude that

$$\begin{aligned} x_2(t_d) \;\geq\; & 2\epsilon_k - D\epsilon_k^{2-\alpha} - (x_2(t_0) - 2\epsilon_k + D\epsilon_k^{2-\alpha} + \tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha}) \\ & -\sqrt{2(\epsilon_k - C)\tfrac{D\epsilon_k^{2-\alpha}}{(1-\alpha)}} \end{aligned} \tag{42}$$

It is readily apparent that ϵ_k can be chosen sufficiently small so that $|x_2(t_d)| \leq |x_2(t_\epsilon)|$ since it was assumed that $x_2(t_\epsilon)$ is positive. In fact, for later purposes it is important to note that ϵ_k can be chosen sufficiently small so that $x_2(t_d) \geq -x_2(t_0) + 3\epsilon_k$.

We continue now with point 1 and, without loss of generality, assume that the trajectory of (x_1, x_2) begins at the point $(x_1(t_d), x_2(t_d)) = (R_{\epsilon_k}, -R_{\epsilon_k})$. Again note that x_2 is monotonically decreasing. In (32) we will assume a bound on $|x_2|$ to be $|x_2| \leq aR_{\epsilon_k}$ (a constant and independent of ϵ_k) and hence $|f_1| \leq aD\epsilon_k^{2-\alpha}$. Then, since x_2 is monotonically decreasing from $-R_{\epsilon_k}$, if we can show that $|x_2(t_c)| \leq aR_{\epsilon_k}$ (where $x_1(t_c) = \epsilon_k + \epsilon_{k-2}$) then this is a worst case bound on $|x_2(t_c)|$. To maximize $|x_2(t_c)|$ we again flow along the outer curve described previously. The flow is given by

$$\begin{aligned} x_1(t) &= x_1(t_d) + [x_2(t_d) - 2\epsilon_k + aD\epsilon_k^{2-\alpha} - \tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha}]t - \tfrac{1}{2}(\epsilon_k - C)t^2 \\ &\quad - \tfrac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2}[e^{-(1-\alpha)t} - 1] \\ x_2(t) &= x_2(t_d) - (\epsilon_k - C)t + \tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha}[e^{-(1-\alpha)t} - 1] \end{aligned} \tag{43}$$

In this instance, a worst case bound on t_c is given by

$$t_c \leq \frac{1}{\epsilon_k - C}[b + \sqrt{b^2 + 2(\epsilon_k - C)c}] \tag{44}$$

where

$$\begin{aligned} b &= R_{\epsilon_k} - 2\epsilon_k + aD\epsilon_k^{2-\alpha} - \tfrac{D\epsilon_k^{2-\alpha}}{1-\alpha} \\ c &= R_{\epsilon_k} + \tfrac{D\epsilon_k^{2-\alpha}}{(1-\alpha)^2} - \epsilon_k - \epsilon_{k-2} \end{aligned} \tag{45}$$

Then $|x_2(t_c)|$ is bounded by

$$|x_2(t_c)| \leq R_{\epsilon_k} + \frac{D\epsilon_k^{2-\alpha}}{1-\alpha}b + \sqrt{b^2 + 2(\epsilon_k - C)c} \tag{46}$$

It is straightforward to see that, for ϵ_k sufficiently small, a worst case bound on $|x_2(t_c)|$ is given by

$$|x_2(t_c)| \leq aR_{\epsilon_k} \tag{47}$$

for $a \geq 3$.

We are now ready to move to point 2. Here we show that we can choose ϵ_k and ϵ_{k-2} sufficiently small, such that for W defined by (33), $\dot{W} \leq 0$ for all $x \in Q$. Consider $\dot{W}$ along the trajectories of (34):

$$\begin{aligned} \dot{W} &= (x_1 - x_2)[\dot{x}_1 - \dot{x}_2] + x_2\dot{x}_2 \\ &= (x_1 - x_2)[x_2 - 2\sigma_k(x_1 + \sigma_{k-2}) + f_1(t) + \sigma_k(x_1 + \sigma_{k-2}) - f_2(t)] \\ &\quad + x_2[-\sigma_k(x_1 + \sigma_{k-2}) + f_2(t)] \end{aligned} \tag{48}$$

Recall that, in Q, we have the bounds (for $j = 1, 2$):

$$|f_j| \leq (|x_1| + |x_2| + 1)D\epsilon_k^2$$

Hence,

$$\begin{aligned} \dot{W} &\leq -x_1\sigma_k(x_1 + \sigma_{k-2}) + x_1x_2 - x_2^2 \\ &\quad +(|x_1| + |x_2|)(|x_1| + |x_2| + 1)D\epsilon_k^2) \\ &\leq -0.5x_1^2 - 0.5x_2^2 - 0.5(x_1 - x_2)^2 + x_1(x_1 - \sigma_k(x_1 + \sigma_{k-2})) \\ &\quad +(|x_1| + |x_2|)(|x_1| + |x_2| + 1)D\epsilon_k^2 \end{aligned}$$

Consider the level set

$$W = \frac{1}{2}(\beta\epsilon_{k-2})^2$$

and define U to be the interior of this level set. On this level set, (a circle of radius $\beta\epsilon_k$ in the original y_{k-1}, y_k coordinates), it can be shown that

$$\beta\epsilon_{k-2} \leq |x_i| \leq \sqrt{2}\beta\epsilon_{k-2}$$

for $i = 1, 2$. Also notice that for $x_1 \in Q$,

$$|x_1 - \sigma_k(x_1 + \sigma_{k-2})| \leq \epsilon_{k-2}$$

Consequently, we have on this level set

$$\dot{W} \leq -0.5(\beta\epsilon_{k-2})^2 - 0.5(k\beta\epsilon_{k-2})^2 + k\beta\epsilon_{k-2}^2 + (2k\beta\epsilon_{k-2})(2k\beta\epsilon + 1)D\epsilon_k^2$$

where $k \in [1, \sqrt{2}]$. As a function of k we have

$$\dot{W} \leq -[(.5\beta^2 - 4\beta^2 D\epsilon_k^2)k^2 - (\beta + 2\beta D\epsilon_k^2)k + .5\beta^2]\epsilon_{k-2}^2$$

Then since $k \in [1, \sqrt{2}]$, we can choose β ($\beta > 2$ is sufficient) such that ϵ_k can be chosen sufficiently small such that $\dot{W} < 0$ on this level set. Since, for ϵ_k small enough, $\dot{W}$ is bounded by a quadratic negative definite function plus a linear perturbation in Q, $\dot{W} < 0$ in $Q \cap U^c$. Notice also, for ϵ_{k-2} small enough, σ_k operates in its linear region for all $x \in U$.

Finally, for point 3, we demonstrate that, for ϵ_k small enough, whenever the trajectory leaves Q it returns to Q at a lower energy level of W. We simply need to show that this is true for $|x_2| \leq aR_{\epsilon_k}$ where a comes from (47). We demonstrate that this follows from the first part of point 1. We can consider trajectories that enter Q_r from Q at some time t_0 without loss of generality. Consequently $x_1(t_0) = \epsilon_k + \epsilon_{k-2}$. From the first part of point 1, by incorporating the constant a into the constant D perhaps further decreasing ϵ_k, it follows that for each $x_1(t_0) \in Q_r$ and each $aR_{\epsilon_k} \geq x_2(t_0) \geq 2\epsilon_k + D\epsilon_k^{2-\alpha}$ there exists a t_d such that $x_1(t_d) = x_1(0)$ and

$$x_2(t_d) \geq -x_2(t_0) + 3\epsilon_k. \tag{49}$$

Consider

$$W(t_d) - W(t_0) = \frac{1}{2}[(x_1(t_d) - x_2(t_d))^2 + x_2(t_d)^2] - \frac{1}{2}[(x_1(t_0) - x_2(t_0))^2 - x_2(t_0)^2]$$

From (49) and the lower bound of $x_2(t_0)$ we can conclude that

$$\frac{1}{2}[x_2(t_d)^2 - x_2(t_0)]^2 < 0.$$

Also from (49) the remaining terms are bounded as

$$\begin{aligned}[x_1(t_d) - x_2(t_d)]^2 - [x_1(t_0) - x_2(t_0)]^2 &\leq [x_1(t_0) + x_2(t_0) - 3\epsilon_k]^2 - [x_1(t_0) - x_2(t_0)]^2 \\ &\leq [x_2(t_0 - x_1(t_0) - \epsilon_k + 2\epsilon_{k-2}]^2 - [x_2(t_0) - x_1(t_0)]^2\end{aligned}$$

If $\epsilon_{k-2} < \frac{1}{2}\epsilon_k$ then this quantity is also less than zero since $x_2(t_0) > 2\epsilon_k + D\epsilon_k^{2-\alpha}$ and $x_1(t_0) = \epsilon_k + \epsilon_{k-2}$.

The above three points demonstrate that x_1, x_2 eventually enter U where σ_k is linear. So it follows that $\tilde{y}_{k-1}, \tilde{y}_k$ eventually enter a neighborhood of the origin where σ_k is linear. The size of this neighborhood is determined by ϵ_{k-2}. The remainder of the proof follows by induction using either the global or semi-global result when appropriate. (Point 3 of section 5.1.2 will be used to conclude (31) in the subsequent step of the induction.) □

6 Examples

We now present some examples to demonstrate the design procedure.

Example 6.1 Our first example is one that can be globally stabilized using the methods of [7], [9], [12] for example in the case where the constant parameter θ is known. In the case where the parameter θ is fixed but unknown, this system can be *locally* stabilized using the adaptive method of [5]. On the other hand, our method is able to yield global asymptotic stability in the presence of an unknown parameter θ which can be time-varying as long as a bound on $|\theta(t)|$ is known.

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta(t)x_2^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\end{aligned} \tag{50}$$

Assume $|\theta(t)| \leq K$. In the notation of section 3, we have

$$h_1 = \theta(t)x_2^2$$

It is easy to see that assumption 2 is satisfied. We choose

$$u = -x_2 - x_3 + v$$

where v will be specified to stabilize x_1. We form the coordinate transformation

$$\begin{aligned}y_1 &= x_1 + x_2 + x_3 \\ y_2 &= x_2 \\ y_3 &= x_3\end{aligned}$$

Then we let $v = -\sigma(y_1)$ where σ is a simple saturation for some ϵ, δ.

Example 6.2 This example has been mentioned in recent work as an unsolved problem, both in the adaptive and known parameter context (see [6].)

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta(t)x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \tag{51}$$

Again, for the sake of generality we allow θ to be time dependent but we will restrict it such that $|\theta(t)| \leq K$. In the notation of section 3, we have

$$h_1 = \theta(t)x_3^2$$

It is easy to see that assumption 2 is satisfied. The control is constructed in the same manner as in the previous example. We choose

$$u = -x_2 - x_3 + v$$

where v will be specified to stabilize x_1. We form the coordinate transformation

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}$$

Then we let $v = -\sigma(y_1)$ where σ is a simple saturation for some ϵ, δ.

Remarks.

1. With θ constant, the above example fails the well-known involutivity condition that is required for the system to be full-state linearizable. However, with the output

   $$h(x) = x_3 + x_2 + \sigma(x_1 + x_2)$$

   (where σ is a simple saturation with sufficiently small δ, ϵ) the system is relative degree one with globally asymptotically stable zero dynamics. Our design procedure provides the intuition for coming up with such an output.

2. It should also be noted that this system can be globally stabilized with the bounded control

   $$u = -\sigma_3(x_3 + \sigma_2(x_2 + x_3 + \sigma_1(x_1 + 2x_2 + x_3)))$$

   with each σ_i a simple saturation and ϵ_i, δ_i chosen appropriately.

Example 6.3 We add to the complexity of the previous example by adding nonlinear terms and an extra dimension. This is done to illustrate the kind of nonlinearities that are allowed by assumption 2.

$$\begin{aligned} \dot{x}_1 &= \sin(x_2) - x_1x_2^2 + x_1x_3\cos(u) \\ \dot{x}_2 &= x_3 + \theta(t)x_4^2 + \sin(x_1t)x_3^2e^u + u^2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \tag{52}$$

In the notation of section 3, we have

$$\begin{aligned} h_1 &= (\sin(x_2) - x_2) - x_1x_2^2 + x_1x_3\cos(u) \\ h_2 &= \theta(t)x_4^2 + \sin(x_1t)x_3^2e^u + u^2 \end{aligned}$$

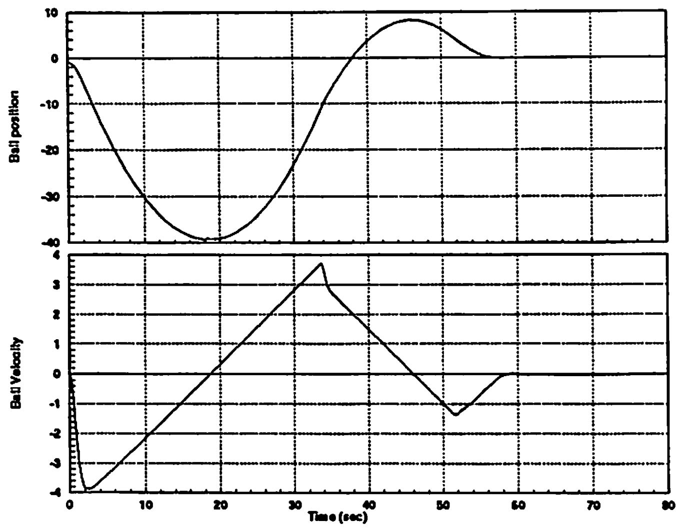

Figure 1: Ball position and velocity

Since $|\theta(t)| \leq K$ and $|\sin(x_1 t)| \leq 1$, it is obvious that h_2 is higher order in x_2, x_3, x_4, u uniformly in x_1, t. Likewise h_1 is higher order in x_1, x_2, x_3, x_4, u uniformly in t.

For point 2 of assumption 2,

$$\begin{aligned} h_2(x_1, x_2, x_3, 0, 0, t) &= \sin(x_1 t) x_3^2 \\ h_1(x_1, x_2, 0, 0, 0, t) &= (\sin(x_2) - x_2) - x_1 x_2^2 \end{aligned}$$

For h_2, $h_2^b \equiv 0$ and h_2^a is higher order in x_3 uniformly in x_1, x_2, t. For h_1, $h_1^b = -x_1 x_2^2$ and hence $x_1 h_1^b \leq 0$ for all x_1, t. Also h_1^a is higher order in x_2 uniformly in x_1, t.

For point 3, of assumption 2,

$$\begin{aligned} h_2(x_1, x_2, 0, 0, 0, t) &= 0 \\ h_1(x_1, 0, 0, 0, 0, t) &= 0 \end{aligned}$$

And finally, for point 4 of assumption 2, both h_1 and h_2 satisfy point (b) of remark 1 after assumption 2.

We choose

$$u = -x_3 - x_4 + v$$

We form the coordinate transformation

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 2x_3 + x_4 \\ y_2 &= x_2 + x_3 + x_4 \\ y_3 &= x_3 \\ y_4 &= x_4 \end{aligned}$$

Then $v = -\sigma_2(y_2 + \sigma_1(y_1))$ where σ_i is a simple saturation for some ϵ_i, δ_i sufficiently small.

Finally, we present a physical example to demonstrate the semi-global result.

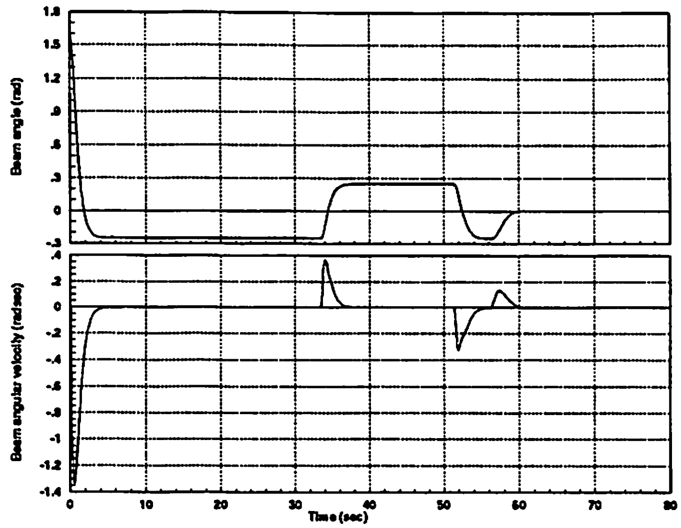

Figure 2: Beam angle and angular velocity

Example 6.4 ("the ball and beam") The dynamics for the "ball and beam" were derived in [3]. A globally invertible nonlinear transformation between torque and angular acceleration has been made. In the dynamics that follow, x_1 is the ball position, x_2 is the ball velocity, x_3 is the beam angle, and x_4 is the beam angular velocity.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -G\sin(x_3) + x_1x_4^2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \tag{53}$$

In the notation of section 3, we have

$$\begin{aligned} h_1 &= 0 \\ h_2 &= G(x_3 - \sin(x_3)) + x_1x_4^2 \end{aligned}$$

It is easy to see that assumption 3.2 is satisfied for h_1, h_2. We choose

$$u = -4x_3 - 4x_4 + v$$

We form the coordinate transformation

$$\begin{aligned} y_1 &= -\tfrac{4}{G}x_1 - \tfrac{8}{G}x_2 + 5x_3 + x_4 \\ y_2 &= -\tfrac{4}{G}x_2 + 4x_3 + x_4 \\ y_3 &= x_3 \\ y_4 &= x_4 \end{aligned}$$

Then $v = -\sigma(y_1 + y_2)$ where σ is a simple saturation for some ϵ, δ. ϵ will be inversely proportional to the bound on the set of initial conditions. Hence we have a semi-global stability result. To demonstrate the capability of such a control law we present, in figures 1 and 2, simulation results starting the beam at a 90° angle and the ball at a position below the pivot of the beam. The function σ was chosen to be C° with $\delta = \epsilon = 1$.

7 Conclusion

We have proposed a globally (semi-globally) stabilizing control approach for a class of single-input nonlinear systems that is especially useful for systems that cannot be globally full-state linearized. We employ saturation functions to systematically drive the state to the origin. In certain instances our control approach can be used to globally (semi-globally) stabilize a nonlinear system using a bounded control. An important feature of our approach is that it is robust to unknown (possibly time-varying) parameters as well as unmodeled nonlinear perturbations that satisfy certain general properties.

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