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CHAOS SYNCHRONIZATION IN CHUA'S CIRCUIT

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Leon O. Chua* Makoto Itoh[†]

Ljupco Kocarev ‡

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Kevin Eckert[§]

Abstract

A number of recent papers have investigated the feasibility of synchronizing chaotic systems. Experimentally one of the easiest systems to control and synchronize is the electronic circuit. This paper examines synchronization in Chua's Circuit, proven to be the simplest electronic circuit to exhibit chaotic behavior.

^{*}Electronics Research Laboratory, University of California, Berkeley, CA 94720 Nagasaki University, Japan

Electronics Research Laboratory, University of California, Berkeley, CA 94720

SElectronics Research Laboratory, University of California, Berkeley, CA 94720

1 Introduction

The essential property of a chaotic signal is that it is not asymptotically stable; closely correlated initial conditions have trajectories which quickly become uncorrelated. Recently it was shown that it is possible to construct a set of chaotic systems so that their common signals will have identical, or *synchronized*, behavior. Afraimovich, Verichev, and Rabinovich [1] consider two identical oscillators having chaotic behavior with linear diffusion coupling:

$$\dot{x} = y \dot{y} = -ky - x(1 + q\cos\theta + x^2) + \delta(y' - y) \dot{x}' = y' \dot{y}' = -ky' - x'(1 + q\cos\theta + x'^2) + \delta(y - y')$$
 (1)

where δ is a coupling parameter. They have shown that there exists a critical value δ^* , such that for all $\delta > \delta^*$ the two oscillators have identical chaotic behavior independent of initial conditions: x(t) = x'(t) and y(t) = y'(t).

The basic construction of Pecora and Carroll [2, 3], can be described as follows: Consider the autonomous n-dimensional dynamical system

$$\frac{du}{dt} = F(u) \tag{2}$$

Now divide the system into two subsystems (u = (v, w))

$$\frac{dv}{dt} \doteq G(v, w) \quad \frac{dw}{dt} = H(v, w) \tag{3}$$

where $v = (u_1, ..., u_m)$, $g = (F_1, ..., F_m)$, $w = (u_{m+1}, ..., u_n)$ and $H = (F_{m+1}, ..., F_n)$

Next, create a new subsystem w' identical to the w subsystem. This yields a (2n - m)-dimensional system:

$$\frac{dv}{dt} = G(v, w) \quad \frac{dw}{dt} = H(v, w) \quad \frac{dw'}{dt} = H(v, w') \tag{4}$$

The v - w system is called the drive system, and the w' subsystem the response system. If w'(t) converges asymptotically to w(t) and continues to remain in step with w(t) the two subsystems are said to have synchronized. The Lyapunov exponents of the response subsystem for a particular driven trajectory are called conditional Lyapunov exponents (hereafter referred to as CLE). Pecora and Carroll have shown that the necessary and sufficient condition for the chaotic trajectory w(t) to be asymptotically stable is for all of the CLE to be negative.

Since its discovery in 1984 [4, 5] the Chua circuit has been studied extensively; see references in [6]. Its unique advantage over other chaotic systems lies in the fact that it is an extremely simple system yet it exhibits the complex dynamics of bifurcation and chaos. In this paper we shall show experimentally, numerically and theoretically (in some cases) that it is possible to synchronize Chua's circuit.

2 The Chua Circuit

Shown in Fig. 1a the circuit consists of a linear inductor L, a linear resistor R, two linear capacitors C_1 and C_2 and a nonlinear resistor N_R . The circuit equations can be written as

$$C_{1} \frac{dv_{C_{1}}}{dt} = \frac{1}{R} (v_{C_{2}} - v_{C_{1}}) - g(v_{C_{1}})$$

$$C_{2} \frac{dv_{C_{2}}}{dt} = \frac{1}{R} (v_{C_{1}} - v_{C_{2}}) + i_{L}$$

$$L \frac{di_{L}}{dt} = -v_{C_{2}}$$
(5)

where $g(\cdot)$ is a *piecewise-linear* function defined by:

$$g(v_R) = G_b v_R + \frac{1}{2} (G_a - G_b) \left[|v_R + B_p| - |v_R - B_p| \right]$$
(6)

and is shown in FIG. 1b. The slopes of the inner and outer regions are G_a and G_b , while B_p indicates breakpoints.

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Most of the analytical studies of the circuit have focused on a dimensionless form of the equations which is obtained by rescaling the parameters of the system:

$$\begin{array}{lll} x = v_{C_1}/B_p & y = v_{C_2}/B_p & z = i_L/(B_pG) & \tau = tG/C_2 \\ a = RG_a & b = RG_b & \alpha = C_2/C_1 & \beta = C_2R^2/L \end{array}$$

which gives the state equations

$$\dot{x} = \alpha(y - x - f(x)) \dot{y} = x - y + z \dot{z} = -\beta y$$

$$(7)$$

where

$$f(x) = bx + \frac{1}{2}(a-b)[|x+1| - |x-1|]$$
(8)

and $\dot{x} = dx/d\tau$.

We need two identical chaotic systems; let the second system be $x'(\tau), y'(\tau), z'(\tau)$ with the same state equations as (7). Establish a difference system $\xi(\tau) = p(\tau), q(\tau), r(\tau)$ with

$$p(\tau) = x(\tau) - x'(\tau)$$

$$q(\tau) = y(\tau) - y'(\tau)$$

$$r(\tau) = z(\tau) - z'(\tau)$$
(9)

To investigate the synchronization of identical chaotic systems we would like to fix the parameters of the Chua circuit so that the system exhibits a chaotic attractor; specifically the so-called *double scroll* attractor. The following nominal values produce the double scroll:

$$\begin{array}{ll} C_1 = 10nF & B_p = 1V \\ C_2 = 100nF & G_b = -0.41 \\ L = 18.75mH & G_a = -0.76 \\ G = 0.599mS \end{array}$$

Which correspond to the rescaled parameters

$$\alpha = 10.00$$
 $\beta = 14.87$
 $a = -1.27$ $b = -0.68$

These values will be fixed throughout the following discussion. This realization of the Chua circuit is taken from [7]. Fig. 2 shows the double scroll attractor.

3 Coupled Systems

We consider a simple coupling of two Chua circuit systems:

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$$\dot{x} = \alpha(y - x - f(x)) + \delta_x(x' - x)
\dot{y} = x - y + z + \delta_y(y' - y)
\dot{z} = -\beta y + \delta_z(z' - z)
\dot{x}' = \alpha(y' - x' - f(x')) + \delta_x(x - x')
\dot{y}' = x' - y' + z' + \delta_y(y - y')
\dot{z}' = -\beta y' + \delta_z(z - z')$$
(10)

where for the x(resp. y, z) coupled system only $\delta_x(\text{resp. } \delta_y, \delta_z)$ is different from zero.

Proposition 1 Suppose that there exist values (δ_1, δ_2) such that for $\delta_1 < \delta < \delta_2$, where δ denotes δ_x , δ_y or δ_z , the real part of all eigenvalues of the matrices

$\begin{bmatrix} -\alpha - a\alpha - 2\delta_x \\ 1 \\ 0 \end{bmatrix}$	$\begin{array}{c} \alpha \\ -1 - 2\delta_y \\ -\beta \end{array}$	$\begin{array}{c} 0 \\ 1 \\ -2\delta_z \end{array}$	
$\begin{bmatrix} -\alpha - b\alpha - 2\delta_x \\ 1 \\ 0 \end{bmatrix}$	$\alpha \\ -1 - 2\delta_y \\ -\beta$	0 - 1 -2δ .	

and

are negative. Then the submanifold $\{x, y, z, x', y', z' : x = x', y = y', z = z'\}$ in the phase space \Re^6 of (10) is stable in the sense that all trajectories in the two systems approach each other asymptotically, regardless of initial conditions. It is assumed that the initial conditions are in the basin of attraction.

Proof:

From (9) and (10) we have

$$\dot{p} = \alpha q - \alpha - \alpha [f(x) - f(x')] - 2\delta_x p$$

$$\dot{q} = p - q + r - 2\delta_y q$$

$$\dot{r} = -\beta q - 2\delta_z r$$
(11)

Since $f(x) - f(x') = f'(\eta)(x - x')$ and $f'(\eta)$ takes two values a and b, (11) reduces to the linear system

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = A \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

where the matrix A is given by either of the matrices in Proposition 1. If the real part of all eigenvalues of these matrices are negative it follows that the equilibrium of (11) is stable. Remarks:

- 1. This theorem gives only necessary conditions for synchronization; in other words if we cannot find δ_1 and δ_2 it does not mean that the system cannot be synchronized (see section 3.2, 3.3).
- 2. All computer simulations were done using software package INSITE [8]. In all the computer simulations the following identifications hold: x[1] = x, x[2] = y, x[3] = z. x[4,5,6] will vary according to individual simulation. All transients were allowed to die out; only the steady state behavior is shown.

3.1 x-coupled system

Fig. 3 shows the experimental set-up. The state equations are:

$$\begin{aligned} \dot{x} &= \alpha (y - x - f(x)) + \delta_x (x' - x) & \dot{x}' &= \alpha (y' - x' - f(x')) + \delta_x (x - x') \\ \dot{y} &= x - y + z & \dot{y}' &= x' - y' + z' \\ \dot{z} &= -\beta y & \dot{z}' &= -\beta y' \end{aligned}$$

where $\delta_x = R\alpha/R_x$. The difference system is

$$\dot{p} = \alpha q - \alpha p - s_i \alpha - 2\delta_x p$$

$$\dot{q} = p - q + r$$

$$\dot{r} = -\beta q$$
(12)

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$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -\alpha - s_i \alpha - 2\delta_x & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

where $s_i = a, b; i = 1, 2$. The characteristic equation is

$$\lambda^3 + (\alpha + s_i\alpha + 2\delta_x + 1)\lambda^2 + (s_i\alpha + 2\delta_x + \beta)\lambda + \beta(\alpha + s_i\alpha + 2\delta_x) = 0$$
(13)

or

$$\lambda^3 + \kappa \lambda^2 + \rho \lambda + \sigma = 0 \tag{14}$$

If $\sigma > 0$, $\rho > 0$ and $\kappa \rho - \sigma > 0$ then p = q = r = 0 is a stable point and the subsystems will synchronize. In the present case the value of δ_1 was found to be 5.56, and thus for all $\delta_x > 5.56$ the subsystems will synchronize. For values of δ_x below this value the theorem makes no predictions; experimental and numerical evidence indicates that the circuit will synchronize for $\delta_x > 0.5$. See Fig. 4a-b.

3.2 y-coupled system

The experimental set-up is shown in Fig. 5. The state equations are

$$\begin{aligned} \dot{x} &= \alpha (y - x - f(x)) & \dot{x}' &= \alpha (y' - x' - f(x')) \\ \dot{y} &= x - y + z + \delta_y (y' - y) & \dot{y}' &= x' - y' + z' + \delta_y (y - y') \\ \dot{z} &= -\beta y & \dot{z}' &= -\beta y' \end{aligned}$$

where $\delta_y = R/R_x$.

Unfortunately we cannot apply Proposition 1 in this case because we found that at least one matrix in Proposition 1 has positive eigenvlues. However, this does not imply that the y-coupled system will never synchronize. Experimentally the system synchronized for $\delta_y > 5.5$ and numerically the system synchronized for $\delta_y > 1$; see Fig. 6a-b.

3.3 z-coupled system

We only examined the circuit by computer simulation.

$$\begin{aligned} \dot{x} &= \alpha (y - x - f(x)) & \dot{x}' &= \alpha (y' - x' - f(x')) \\ \dot{y} &= x - y + z & \dot{y}' &= x' - y' + z' \\ \dot{z} &= -\beta y + \delta_z (z' - z) & \dot{z}' &= -\beta y' + \delta_z (z - z') \end{aligned}$$

As in section (3.2) we cannot apply Proposition 1 to this system, for the same reason. Numerically synchronization was found for $0.7 < \delta_z < 2$ for a particular set of initial conditions; see Fig. 7.

4 Drive-Response Systems

Pecora and Carroll [2, 3] have shown that the subsystems will synchronize if the CLE of the response system are all negative. The CLE are found by calculating the Lyapunov exponents for the entire 2n-m dimensional system and comparing these to the Lyapunov exponents of the *n* dimensional drive system. The remaining n-m Lyapunov exponents will be the CLE of the response system. There is a relatively simple method to see whether the CLE will be negative, given that the subsystem is linear. Let $\xi(t) = w(t) - w'(t)$, and call ξ the difference system. Then we have

$$\xi(t) = \dot{w}(t) - \dot{w}'(t) = h(v, w) - h(v, w')$$
(15)

If the subsystem is linear, we have

$$\xi(t) = A\dot{\xi}(t) \tag{16}$$

where A is an $(n-m) \times (n-m)$ constant matrix. Let the eigenvalues of A be $(\lambda_1, \lambda_2, \dots, \lambda_{n-m})$. The real part of these eigenvalues are by definition the CLE we seek.

If all of the CLE are negative then $\lim_{t\to\infty} \xi(t) = 0$ and the subsystems will synchronize; if there is a positive CLE the subsystems will grow farther apart as $t \to \infty$ and thus will never synchronize. An intermediate case occurs if one or more of the CLE are zero but none are positive; as $t \to \infty$ the subsystems will be separated by a fixed distance R, dependent upon the initial conditions.

If the subsystems are linear circuits with passive elements it is trivial to calculate the CLE. If the subsystems are nonlinear the CLE are not so easily determined; we must resort to computer software such as INSITE.

In our paper the n-dimensional dynamical system of (2) will be the rescaled state equations (7). We will investigate three kinds of drive-response systems:

Drive	Response	Subsystem
x	(y,z)	linear
y	(x,z)	nonlinear
z	(x, y)	nonlinear

We examine each of these in turn.

4.1 x-drive configuration

Fig. 8 shows the experimental set-up. The state equations become

$$\dot{x} = \alpha(y - x - f(x)) \quad \dot{y}' = x - y' + z' \dot{y} = x - y + z \qquad \dot{z}' = -\beta y' \dot{z} = -\beta u$$

and the difference system in matrix form is

$$\begin{bmatrix} \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix}$$

The eigenvalues are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{4\beta - 1}i$ giving the solution

$$\xi(\tau) = e^{-\tau/2} (C \cos(\sqrt{4\beta - 1}/2)\tau + D \sin(\sqrt{4\beta - 1}/2)\tau)$$
(17)

C and D are constants of integration. The CLE are (-0.5, -0.5) and as expected $\lim_{\tau\to\infty} \xi(\tau) = 0$. The subsystems synchronize, see Fig. 9a-b.

4.2 y-drive configuration

Fig. 10 shows the experimental set-up. The state equations are

$$\dot{x} = \alpha(y - x - f(x)) \quad \dot{x}' = \alpha(y - x' - f(x'))$$

$$\dot{y} = x - y + z \qquad \dot{z}' = -\beta y$$

$$\dot{z} = -\beta y$$

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Note that the *LC*-tank circuit is redundant in view of the <u>substitution theorem</u> [9] and can be deleted. Using INSITE the CLE are found to be $(-2.5 \pm 0.05, 0)$ As expected the subsystems synchronize. Because the second eigenvalue is $0, z(\tau)$ and $z'(\tau)$ will remain apart by a constant distance B = |z(0) - z'(0)|. See Fig. 11a-b.

4.3 z-drive configuration

Here we only examined the circuit by computer simulation. The state equations become

$$\dot{x} = \alpha(y - x - f(x)) \quad \dot{x}' = \alpha(y' - x' - f(x'))$$

$$\dot{y} = x - y + z \qquad \dot{y}' = x' - y' + z$$

$$\dot{z} = -\beta y$$

Using INSITE the CLE are found to be $(-5.42 \pm 0.02, 1.23 \pm 0.03)$. As expected the subsystems do not synchronize.

5 Closing Remarks

This study raises some interesting questions, and encourages further investigation. How much can the parameters of the subsystems be varied before synchronization is lost completely, and can this be predicted theoretically? How does detuning of the basic frequencies affect the system? The simplicity and robustness of the Chua circuit makes it a convenient vehicle to investigate all of these topics.

6 Acknowledgements

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FIGURES

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1a) Chua's circuit

1b) Nonlinear resistor (N_R) v-i characterisitc

2) The double scroll attractor; parameters $C_1 = 10$ nF, $C_2 = 100$ nF, L = 18.75mH, R = 1.67 k Ω , $m_0 = -0.35$, $m_1 = =0.76$; horizontal axis (V_{C_1}) 1V/div, vertical axis (V_{C_2}) 500mV/div

3) x-coupled configuration; 10k potentiometer used

4a) x-coupled synchronization, $\delta = 0.5$; initial conditions x[1] = -2, x[2] = 0.02, x[3] = 4, x[4] (x') = 0.7, x[5] (y') = 0.4, x[6] (z') = -0.8

4b) V_{C_1} -coupled synchronization, $\delta = 0.5$ ($R_x = 3.34 \text{ k}\Omega$; horizontal axis: V_{C_2} , vertical axis: V'_{C_2} ; both axis 200mV/div

5) y-coupled configuration; 10k potentiometer used

6a) y-coupled synchronization, $\delta = 1$; initial conditions x[1] = -2, x[2] = 0.02, x[3] = 4, x[4](x') = 0.7, x[5](y') = 0.4, x[6](z') = -0.8

6b) V_{C_2} -coupled synchronization, $\delta = 5.8$ ($R_x = 290 \Omega$; horizontal axis: V_{C_1} , vertical axis: V'_{C_1} ; both axis 1V/div

7) z-coupled synchronization, $\delta = 1$; initial conditions x[1] = 0.01, x[2] = 0.01, x[3] = 0.01, x[4] (x') = 0.04, x[5] (y') = 0.01, x[6] (z') = 0.01

8) x-drive configuration; op-amp from AD712 chip

9a) x-drive synchronization; initial conditions: x[1] = -2, x[2] = 0.02, x[3] = 4, x[4](y') = 0.4, x[5](z') = -0.8

9b) V_{C_1} -drive synchronization; horizontal axis: V_{C_2} , vertical axis: V'_{C_2} ; both axis 200mV/div

10) y-drive configuration; op-amp from AD712 chip

11a) y-drive synchronization; initial conditions: x[1] = -2, x[2] = 0.02, x[3] = 2, x[4](x') = 0.7, x[5](z') = -0.8

11b) V_{C_2} -drive synchronization; horizontal axis: V_{C_1} , vertical axis: V'_{C_1} ; both axis 1V/div

Note difference in z - z' offset in Fig. 11a due to initial conditions



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