PROPERTIES OF ADMISSIBLE SYMBOLIC SEQUENCES IN A SECOND ORDER DIGITAL FILTER WITH OVERFLOW NONLINEARITY

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Abstract

In this paper we present further properties of admissible sequences in a second order digital filter with overflow nonlinearity. We also present an alternative test for periodic admissible sequences for the case $a = \frac{1}{2}$. Some properties of sets which produce aperiodic admissible sequences are also presented.

1 Introduction

In [1], a simple second-order digital filter with overflow nonlinearity was introduced which exhibits rather complicated behaviors. The periodic behavior of the system was analyzed using symbolic dynamics. For a particular filter, all admissible periodic sequences of period less than 14 was found. In [2], all admissible periodic sequences of period less than 22 was found and some properties of admissible periodic sequences were presented.

In this paper, we present additional properties of admissible sequences. In section 2 the digital filter with overflow nonlinearity is defined along with other preliminary definitions. In section 3 properties of admissible sequences are presented. In section 4 an alternative condition is presented for finding all admissible periodic sequences for the case $a = \frac{1}{2}$. In section 5 we present some results on points which generate aperiodic behavior. In particular, we show that the set of points which generate a particular aperiodic sequence is a line-segment in the state space.

2 Digital Filter with Overflow Nonlinearity

The digital filter we consider is a second-order discrete-time dynamical system with the following state equations [1]:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ f[bx_1(k) + ax_2(k)] \end{bmatrix} = \mathbf{F}\left(\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}\right) \tag{1}$$

where

$$f(x) = x - 2n \qquad \text{for } -1 + 2n \le x < 1 + 2n, n \text{ an integer} \tag{2}$$

A graph of $f(\cdot)$ is shown in figure 1. The state space is $I^2 = \{(x_0, x_1) : -1 \le x_0 < 1, -1 \le x_1 < 1\}$.

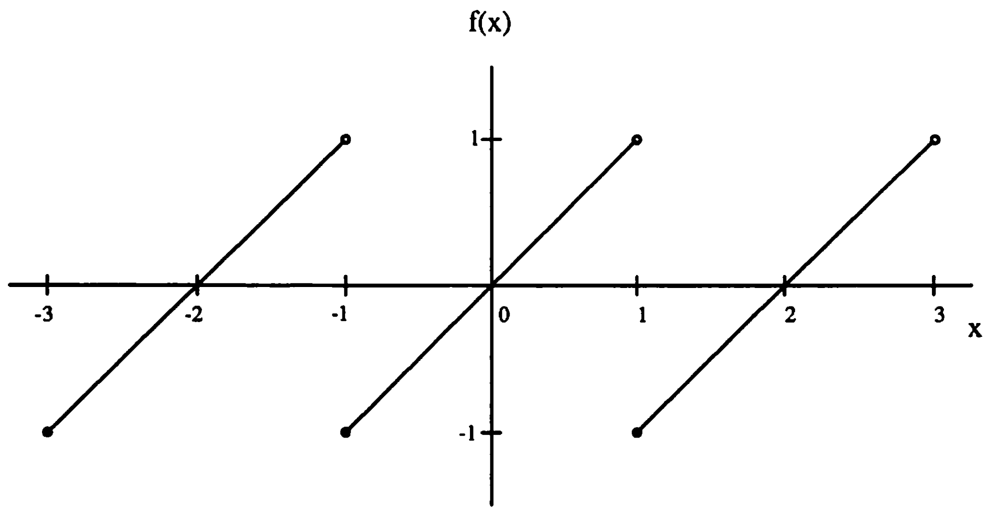

Figure 1: The overflow nonlinearity $f(x)$

We denote the closure of I^2 by $\overline{I^2}$. Throughout this paper, b is assumed to be -1 and $|a| < 2$.

Define

$$I_1 = \{(x_0, x_1) \in I^2 : -x_0 + ax_1 \ge 1\} \tag{3}$$

$$I_0 = \{(x_0, x_1) \in I^2 : -1 \le -x_0 + ax_1 < 1\} \tag{4}$$

$$I_{-1} = \{(x_0, x_1) \in I^2 : -x_0 + ax_1 < -1\} \tag{5}$$

Define Σ as the set of infinite sequences consisting of the symbols $-1, 0$ and 1. Let s be a sequence in Σ, then $-s$ is the sequence obtained by replacing -1 by 1 and vice versa. If s is $(s_0 s_1 s_2 \cdots)$, then $\sigma(s)$ is the sequence $(s_1 s_2 \cdots)$. The period of a periodic sequence $s \in \Sigma$ is the smallest positive integer n such that s repeats itself after n symbols. Given an initial condition $(x_0, x_1) \in I^2$ we can generate a symbolic sequence $s \in \Sigma$ corresponding to the trajectory of the system by the map $S : I^2 \to \Sigma$

$$s = S((x_0, x_1)) = (s_0 s_1 s_2 \cdots s_i \cdots) \tag{6}$$

$$s_i = \begin{cases} 1 & \mathbf{F}^i\left(\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}\right) \in I_1 \\ 0 & \mathbf{F}^i\left(\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}\right) \in I_0 \\ -1 & \mathbf{F}^i\left(\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}\right) \in I_{-1} \end{cases} \tag{7}$$

A sequence in Σ is admissible if it is generated by some initial conditions in I^2. In other words, the set of admissible sequences is $\Sigma_F = S(I^2)$. Sufficient and necessary conditions for a sequence to be admissible are given in [1, 2].

We partition Σ into three subsets:

$$\Sigma_\alpha = \{s \in \Sigma : s \text{ is periodic}\} \tag{8}$$

$$\Sigma_\beta = \{s \in \Sigma : s \text{ is eventually periodic, i.e. } s \notin \Sigma_\alpha \text{ and } \sigma^p(s) \in \Sigma_\alpha \text{ for some integer } p > 0\} \tag{9}$$

$$\Sigma_\gamma = \Sigma \setminus (\Sigma_\alpha \cup \Sigma_\beta) \tag{10}$$

which corresponds to the following partition of I^2:

$$I_\alpha = S^{-1}(\Sigma_\alpha \cap \Sigma_F) \tag{11}$$

$$I_\beta = S^{-1}(\Sigma_\beta \cap \Sigma_F) \tag{12}$$

$$I_\gamma = S^{-1}(\Sigma_\gamma \cap \Sigma_F) \tag{13}$$

3 Properties of Admissible Sequences

Remark 1: Because of symmetry s is admissible if and only if $-s$ is admissible. It is clear that a periodic sequence s is admissible if and only if $\sigma(s)$ is admissible.

Theorem 1 *If $0 < a < 2$, then a sequence with* 11 *or* $-1-1$ *as subsequences is not admissible.*

Proof: suppose that there is an admissible sequence with 11 as a subsequence. Then there exists $x_0, x_1 \in [-1, 1)$ such that

$$x_2 = -x_0 + ax_1 \geq 1 \tag{14}$$

$$-x_1 + a(x_2 - 2) \geq 1 \tag{15}$$

Thus

$$-x_1 + a(-x_0 + ax_1 - 2) \geq 1 \tag{16}$$

$$(a^2 - 1)x_1 - ax_0 - 2a \geq 1 \tag{17}$$

It is easy to show that when $a \in (0,2)$,

$$\left|a - \frac{1}{a}\right| - 1 - \frac{1}{a} < 0 \tag{18}$$

This implies that

$$\begin{aligned} x_0 &\leq -2 - \frac{1}{a} + (a - \frac{1}{a})x_1 && (19)\\ &\leq -1 + \left|a - \frac{1}{a}\right| - 1 - \frac{1}{a} && (20)\\ &< -1 && (21) \end{aligned}$$

which is a contradiction. The case of $-1\ -1$ as a subsequence follows from remark 1. ■

Remark 2: This theorem extends corollary 2 in [2].

Lemma 1 *Let $-\sqrt{2} < a < \sqrt{2}$. If $x_0, x_1 \in [-1,1)$ satisfy*

$$x_2 = -x_0 + ax_1 \in [-1,1) \tag{22}$$

$$-x_1 + ax_2 \geq 1 \tag{23}$$

then $ax_0 < 0$. If in addition, $-1 < a < 1$, then $ax_0 < 0$ and $x_1 < 0$.

Proof:

$$-x_1 - ax_0 + a^2x_1 \geq 1 \tag{24}$$

$$(a^2 - 1)x_1 - ax_0 \geq 1 \tag{25}$$

If $-\sqrt{2} < a < \sqrt{2}$, then $|(a^2-1)x_1| < |x_1| < 1$, so $ax_0 < 0$. If $-1 < a < 1$, then $-1 < (a^2 - 1) < 0$ and $-1 < -a < 1$, so from equation (25) $x_1 < 0$. ■

Theorem 2 *Let $-\sqrt{2} < a < 2$. Then any admissible sequence cannot have the following subsequences:*

101 -10-1

Proof: suppose 101 is a subsequence of an admissible sequence s. Then there exist $x_0, x_1 \in [-1,1)$ such that

$$x_2 = -x_0 + ax_1 \geq 1 \tag{26}$$

$$-1 \leq x_3 = -x_1 + a(x_2 - 2) < 1 \tag{27}$$

$$-(x_2 - 2) + ax_3 \geq 1 \tag{28}$$

First let us take $-\sqrt{2} < a < \sqrt{2}$. From lemma 1, $ax_1 < 0$. But $x_2 = -x_0 + ax_1 \geq 1$, which implies that $ax_1 \geq 1 + x_0 \geq 0$, a contradiction.

Expanding x_2 and x_3 in equation (28),

$$(a^3 - 2a)x_1 + (1 - a^2)x_0 + 1 - 2a^2 \geq 0 \tag{29}$$

If $\sqrt{2} \leq a < 2$, then $a^3 - 2a \geq 0$ and $1 - a^2 < 0$. Thus

$$(a^3 - 2a) + (a^2 - 1) + 1 - 2a^2 \geq 0 \tag{30}$$

$$a^3 - a^2 - 2a \geq 0 \tag{31}$$

which contradict the fact that $a^3 - a^2 - 2a < 0$ on $\sqrt{2} \leq a < 2$. The other case follows from remark 1. ∎

Corollary 1 *Let* $0 < a < 2$ *and let* s *be an admissible sequence. Then* s *has no zeros if and only if* s *is a periodic sequence of period 2.*

Proof: if s is a periodic sequence of period 2, then the only possible admissible periodic sequences permitted by theorem 1 are $1{-}1$, -11, 10,-10,01 and $0{-}1$. To show that s has no zeros, it suffices to show that 10 is not an admissible periodic sequence. By theorem 2, $1010101010\cdots$ is not admissible because it contains 101 as a subsequence.

Now suppose that s has no zeros. The only sequences permitted by theorem 1 are $1-11-1\cdots$ and $-11-11\cdots$, which are periodic sequences of period 2. ∎

Theorem 3 *Let* $-1 < a < 1$. *Then any admissible sequence cannot have the following subsequences:*

101 -10-1 -101 10-1

Proof: suppose -101 is a subsequence of an admissible sequence s. Then there exist $x_0, x_1 \in [-1, 1)$ such that

$$x_2 = -x_0 + ax_1 < -1 \tag{32}$$

$$-1 \leq x_3 = -x_1 + a(x_2 + 2) < 1 \tag{33}$$

$$-(x_2 + 2) + ax_3 \geq 1 \tag{34}$$

By lemma 1, $x_2 + 2 < 0$,

$$-x_0 + ax_1 + 2 = x_2 + 2 < 0 \tag{35}$$

$$-x_0 + ax_1 < -2 \tag{36}$$

which is a contradiction since $-1 < a < 1 \Rightarrow -1 < ax_1 < 1 \Rightarrow -2 < -x_0 + ax_1 < 2$. The other cases follow from theorem 2 and remark 1. ∎

Corollary 2 *Let* $0 < a < 1$. *Then any admissible sequences cannot have the following subsequences:*

11 -1-1 101 -10-1 -101 10-1

Proof: follows from theorem 1 and theorem 3. ∎

Remark 3: corollary 2 implies that when $0 < a < 1$ and s is an admissible periodic sequence that is not of period 1 or 2, then s has 100 or -100 as a subsequence.

4 Periodic admissible sequences for the case $a = \frac{1}{2}$

In [1, 2], conditions were given for finding periodic admissible sequences. The number of cases that needs to be tested for finding all admissible sequences of period k is 3^k, which limits the size of the period that we are able to analyze. In this section we will give a simple condition for finding all admissible periodic sequences of a certain period for the case $a = \frac{1}{2}$.

First, let us introduce some conventions, notations and definitions. When we talk about the numerator and denominator of a rational number, we assume that the rational has already been reduced to its lowest terms and that the denominator is positive.

For a vector $\mathbf{x}$ of rational numbers (elements of $\mathbb{Q}^n$), denote $\mu(\mathbf{x})$ as the lowest common multiple of the denominators of each element of $\mathbf{x}$. We use a similar definition for matrices of rational numbers. For integers n and m, we write $n|m$ if n divides m. Denote $\nu(n)$ as the nonnegative exponent of the prime 2 in the prime factorization of the integer n.

By theorem 3 in [1], searching for periodic admissible sequences is equivalent to searching for periodic points of the discrete system (1). In [1] it is shown that if $\mathbf{z}_0$ is a periodic point of period k, then $\mathbf{z}_0$ must satisfy

$$\mathbf{z}_0 = (\mathbf{E} - \mathbf{A}^k)^{-1}(s_0\mathbf{A}^{k-1}\mathbf{b} + \cdots + s_{k-2}\mathbf{A}\mathbf{b} + s_{k-1}\mathbf{b}) \tag{37}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

for some $s_i \in \{-1, 0, 1\}$. Since $\mathbf{A}$ and $\mathbf{b}$ are both rational matrices, $\mathbf{z}_0$ is a rational vector. In fact, we can compute a bound on $\mu(\mathbf{z}_0)$. Let us denote $D = \mu((\mathbf{E} - \mathbf{A}^k)^{-1})$. Then by equation (37) $\mu(\mathbf{z}_0)|2^{k-2}D$. We can express $2^{k-2}D$ as $T = 2^{k-2}D = 2^l R$ where R is odd and $\nu(T) = l \geq k-2$.

Express $T\mathbf{z}_0$ as (p, q), where p and q are integers. We can then iterate $\mathbf{F}$ in $\mathbb{Z}^2$.

Remark 4: since $\mathbf{F}((p,q)) = (q, 2sT - p + q/2), s \in \{-1, 0, 1\}$ is a pair of integers, it follows that q must be even.

Lemma 2 *If* $0 \leq \min(\nu(p), \nu(q) - 1) < l$, *then* $\nu(p) = \nu(q) - 1$.

Proof: Let $(p_i, q_i) = \mathbf{F}^i((p,q))$. Suppose $\nu(p) < \nu(q) - 1$. We then have

$$\begin{array}{rcl} (p_0, q_0) & = & (2^{\nu(p)}t_0, 2^{\nu(q)}t_1) \\ (p_1, q_1) & = & (2^{\nu(q)}t_1, -2^{\nu(p)}t_0 + 2^{\nu(q)-1}t_1 + 2^l u_1 = 2^{\nu(p)}t_2) \\ (p_2, q_2) & = & (2^{\nu(p)}t_2, -2^{\nu(q)}t_1 + 2^{\nu(p)-1}t_2 + 2^l u_2 = 2^{\nu(p)-1}t_3) \\ \vdots & & \vdots \\ (p_{\nu(p)+1}, q_{\nu(p)+1}) & = & (2t_{\nu_p}, t_{\nu_p+1}) \end{array} \tag{39}$$

where the t_i's are odd and $2^l u_1$ and $2^l u_2$ equal to $\pm 2T$. Since t_{ν_p+1} in (39) is odd, this contradicts remark 4.

Suppose $\nu(p) > \nu(q) - 1$.

$$\begin{array}{rcl} (p_0, q_0) & = & (2^{\nu(p)}t_0, 2^{\nu(q)}t_1) \\ (p_1, q_1) & = & (2^{\nu(q)}t_1, -2^{\nu(p)}t_0 + 2^{\nu(q)-1}t_1 + 2^l u_1 = 2^{\nu(q)-1}t_2) \\ (p_2, q_2) & = & (2^{\nu(q)-1}t_2, -2^{\nu(q)}t_1 + 2^{\nu(q)-2}t_2 + 2^l u_2 = 2^{\nu(q)-2}t_3) \\ \vdots & & \vdots \\ (p_{\nu(q)}, q_{\nu(q)}) & = & (2t_{\nu_q}, t_{\nu_q+1}) \end{array} \tag{40}$$

Again t_{ν_q+1} is odd resulting in a contradiction. ■

Lemma 3 *$\nu(p) \geq l$ and $\nu(q) - 1 \geq l$.*

Proof: if either $\nu(p)$ or $\nu(q) - 1$ is less than l, then $\nu(p) = \nu(q) - 1$ by lemma 2.

$$\begin{array}{rcl} (p_0, q_0) & = & (2^{\nu(q)-1}t_0, 2^{\nu(q)}t_1) \\ (p_1, q_1) & = & (2^{\nu(q)}t_1, 2^{\nu(q)+1}t_2) \\ \vdots & & \vdots \\ (p_{l-\nu(q)}, q_{l-\nu(q)}) & = & (2^{l-1}t_{l-\nu(q)}, 2^l t_{l-\nu(q)}) \end{array} \tag{41}$$

where all the t_i's are odd integers. In the next iteration, $p_{l-\nu(q)+1} = 2^l t_{l-\nu(q)}$, and $q_{l-\nu(q)+1} = 2^{l+1}x$ to avoid contradicting lemma 2. A similar reasoning shows that from this point on, $\nu(p_i) \geq l$ and $\nu(q_i) \geq l$. Since this is a periodic point, p_0 must be equal to one of these p_i's resulting in a contradiction. ■

Our main result in this section is the following:

Theorem 4 *Let $\mathbf{z}_0$ be a periodic point. Then $\mu(\mathbf{z}_0)|R$.*

Proof: from lemma 3, the numerators of $\mathbf{z}_0$ each has a factor of 2^l, which we can cancel from the denominator $T = 2^l R$. ■

Remark 5: This analysis can be extended to the case when a is a rational number.

This theorem suggests an algorithm for finding all admissible periodic sequences: check whether any of the $4R^2$ rational points in I^2 with denominator R is a periodic point of period k. This method has the advantage of using only integers in the computation which does not introduce any roundoff errors. Furthermore, as $4R^2$ is generally much smaller than 3^k, the amount of computation is substantially reduced. To further reduce the amount of computation, we also use the following condition:

Theorem 5 *Suppose* $\mathbf{z}_0$ *is a periodic point of period* k. *Then the integers* p *and* q *obtained from* $R\mathbf{z}_0 = (p, q)$ *must satisfy the following congruences:*

$$2^k(\mathbf{E} - \mathbf{A}^k)\begin{bmatrix} p \\ q \end{bmatrix} = 0 \bmod R \tag{42}$$

Proof: multiply both sides of equation (37) by $2^k(\mathbf{E} - \mathbf{A}^k)$ to obtain

$$2^k(\mathbf{E} - \mathbf{A}^k)\mathbf{z}_0 = 2^k(s_0\mathbf{A}^{k-1}\mathbf{b} + \cdots + s_{k-2}\mathbf{A}\mathbf{b} + s_{k-1}\mathbf{b}) \tag{43}$$

The right hand side is a integer vector and the conclusion follows from $\mu(\mathbf{z}_0)|R$. ■

Using these conditions, we have found that there are no admissible sequences of period 23, 24 or 25.

5 Topology of I_γ

Lemma 4 *Let* $s \in \Sigma$ *be a sequence. Then* $S^{-1}(s)$ *is convex.*

Proof: suppose $\mathbf{x}, \mathbf{y}$ are in $S^{-1}(s)$, where $s = (s_0 s_1 s_2 \cdots)$. Let us define $\mathbf{x}_i = \mathbf{F}^i(\mathbf{x}), \mathbf{y}_i = \mathbf{F}^i(\mathbf{y})$ for all $i \geq 0$.

Then

$$\mathbf{x}_{i+1} = \mathbf{F}(\mathbf{x}_i) = \mathbf{A}\mathbf{x}_i + \mathbf{b}s_i = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}\mathbf{x}_i + \begin{bmatrix} 0 \\ 2 \end{bmatrix} s_i \tag{44}$$

Define $\mathbf{z}_i = \mathbf{x}_i + \alpha(\mathbf{y}_i - \mathbf{x}_i), 0 \leq \alpha \leq 1$. Then

$$\mathbf{z}_{i+1} = \mathbf{A}\mathbf{z}_i + \mathbf{b}s_i \tag{45}$$

Since I^2 is convex, $\mathbf{z}_i \in I^2$ for all $i \geq 0$. Therefore $S(\mathbf{z}_0) = s$. ■

Corollary 3 *If* $(1s_0s_1\cdots)$ *and* $(-1s_0s_1\cdots)$ *are both admissible, then so is* $(0s_0s_1\cdots)$.

In other words, if $(0s_0s_1\cdots)$ is not admissible, then either $(1s_0s_1\cdots)$ or $(-1s_0s_1\cdots)$ is not admissible.

Proof: follows from the fact that $\mathbf{F}(I_{-1}) \cup \mathbf{F}(I_1)$ is not connected and thus not convex. ■

Lemma 5 *Let* $s \in \Sigma_\gamma$. *Then* $S^{-1}(s)$ *has no interior points.*

Proof: denote $\Pi(\rho, \mathbf{z}_0)$ as the ellipse described by

$$\Pi(\rho, \mathbf{z}_0) = \{\mathbf{x} \in \mathbb{R}^2 : (\mathbf{x} - \mathbf{z}_0)^t \begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}^{-1} (\mathbf{x} - \mathbf{z}_0) \leq \rho^2\} \tag{46}$$

It was shown in [1] that for the linear system $\mathbf{G}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $\mathbf{G}(\Pi(\rho, 0)) = \Pi(\rho, 0)$ for all ρ. If $S^{-1}(s)$ has an interior point, then there exist $\mathbf{z}_0 \in I^2$, $\rho > 0$ such that $\Pi(\rho, \mathbf{z}_0) \subseteq S^{-1}(s)$. Since every point in $\Pi(\rho, \mathbf{z}_0)$ generate the same symbolic sequence,

$$\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{z}_0) = \mathbf{A}(\mathbf{x} - \mathbf{z}_0) \tag{47}$$

for all $\mathbf{x} \in \Pi(\rho, \mathbf{z}_0)$. This means that $\mathbf{F}(\Pi(\rho, \mathbf{z}_0)) = \Pi(\rho, \mathbf{F}(\mathbf{z}_0))$. Similarly $\mathbf{F}^k(\Pi(\rho, \mathbf{z}_0)) = \Pi(\rho, \mathbf{F}^k(\mathbf{z}_0))$ for all $k \geq 0$. This means that by iterating the map $\mathbf{F}$ on $\Pi(\rho, \mathbf{z}_0)$ we obtain other ellipses of the same size. These ellipses cannot intersect as the symbolic sequence s belongs to Σ_γ. This is impossible as I^2 has finite measure and thus cannot contain infinitely many nonintersecting ellipses of the same size. ■

Theorem 6 *Let $s \in \Sigma_\gamma \cap \Sigma_F$. Then $S^{-1}(s)$ is a line-segment in I^2, i.e. it has the following form:*

$$S^{-1}(s) = \{x \in I^2 : x = a + \mu(b - a), \quad \mu \in B\} \tag{48}$$

for some $a, b \in \overline{I^2}$, where B is $(0, 1)$, $(0, 1]$ or $[0, 1]$.

Proof: follows from lemma 4 and lemma 5 and the fact that the only nonempty convex sets in I^2 with no interior points are line segments. ■

Some immediate facts that follow from theorem 6 are:

- If $s \in \Sigma_\gamma$, then $\bigcup_{n=1}^{\infty} \mathbf{F}^n \left(S^{-1}(s)\right)$ has Lebesque measure zero.
- If there are finitely many sequences in $(\Sigma_\alpha \cup \Sigma_\beta) \cap \Sigma_F$, then there are uncountably many sequences in $\Sigma_\gamma \cap \Sigma_F$.

6 Conclusions

In this paper, further properties of a second order digital filter with overflow nonlinearities are presented. We show that the admissible symbolic sequences cannot contain certain subsequences. By transforming the system into the integer domain, we found an alternative necessary condition for a point to be periodic, which leads to a more efficient and robust algorithm for finding periodic points and periodic admissible sequences. Finally, some properties of the topology of I_γ are presented.

Acknowledgements

This work is supported in part by the Office of Naval Research under grant N00014-89-J-1402 and by the National Science Foundation under grant MIP 86-14000.

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