MAPPING NONLINEAR LATTICE EQUATIONS ONTO CELLULAR NEURAL NETWORKS

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S. Paul *, K. Hüper *, J. A. Nossek * and L. O. Chua †

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Abstract

In the last years completely integrable Hamiltonian systems were of great interest because of their physical nature, e.g. the existence of soliton solutions, and their relation to eigenvalue and sorting problems. But until recently, they found little interest among electrical engineers because lossless circuits are difficult to realize as physical systems. However, if we are only interested in the "signals" associated with Hamiltonian systems, and not in conserving the energy in the individual circuit elements (nonlinear inductors and capacitors), then such systems can be built as analog circuits which implement some signal flow graphs.

Under certain restrictions, cellular neural networks (CNN) come very close to some Hamiltonian systems, therefore they are potentially useful for simulating or realizing such systems. In this paper, we will show how to map two one-dimensional nonlinear lattices, the Fermi-Pasta-Ulam lattice and the Toda lattice, onto a CNN. We demonstrate for the Toda lattice, what happens, if the signals are driven beyond the linear region of the output function. Though the system is no longer Hamiltonian, numerical experiments reveal the existence of solitons for special initial conditions. This interesting phenomenon is due to a special symmetry in the CNN system of ODE's.

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1 Introduction

In recent years, completely integrable Hamiltonian systems has attracted a great deal interest. Closely related to them is the existence of soliton solutions. These are nonlinear waves that can pass one another without changing their shape in the most simple case (elastic collision). The Fermi-Pasta-Ulam lattice and the Toda lattice, both discrete in space and continuous in time, are classic examples of such systems [1] [2].

The Hamiltonian structure of such systems implies, among other things, the physical property losslessness with respect to energy. Since such systems can never be built exactly as electrical circuits using lossless inductors and capacitors, and since no useful system with real application in signal processing or other areas was known, little attention was devoted to Hamiltonian systems in the electrical engineering literature.

This situation is currently changing due to the discovery of systems, like the Toda lattice, which are capable of solving eigenvalue problems [3]. Consequently, any task that can be formulated as an eigenvalue problem is a potential application. One of them is rank filtering [4], [5], [6].

A lot of current interest in electrical engineering has been focussed on regular architectures of analog nonlinear processing cells with local connections, called cellular neural networks (CNN). Chua and Yang proposed the basic concept in 1987 [7], [8]. These systems make extensive use of stability properties of equilibrium points. In general, they are not lossless and so soliton solutions do not seem to exist. Recently, Roska et. al. generalized the original CNN structure to allow fairly general nonlinear and delay-type feedback and control [9].

In this paper we demonstrate two things. First we show under what conditions the Fermi-Pasta-Ulam (FPU) lattice can be realized exactly by a conventional CNN. Afterwards we map a transformed version of the Toda lattice [10], [11], [3] onto the current framework of CNN.

The original CNN equations contain a piecewise-linear (PWL) map of the states with a saturation characteristic. This nonlinearity can destroy the integrability of the Toda lattice equations. But surprisingly under certain conditions it is still possible to find soliton solutions. The "invariants" of motion are no longer invariant, i.e. the system is not lossless but they exhibit an interesting recurrent phenomenon. These recurent phenomenon still allows sorting, subject to some additional interpretations.

In Sect. 2 the recently generalized definition of CNN is reviewed. Whenever possible, we refer to the literature for the stability proofs [9]. Sect. 3 is dedicated to a review of the lattice equations, including the FPU lattice and the Toda lattice, with some of their important properties. In Sect. 4 we discuss how they can be mapped onto the CNN. The behaviour of the ODE of the CNN with their PWL output maps, and their major properties are presented in Sect. 5.

2 Cellular Neural Network

The CNN is a nonlinear dynamical system of autonomous ordinary differential equations (ODE) [7], [8]. If the state variables $v_{xij}(t)$ are arranged in an $M \times N$ rectangle, then only the states in a local neighbourhood of any particular state v_{xij} are coupled to the state v_{xij}. Consequently, CNNs are well-suited for VLSI implementation.

Using the current definition of CNN nonlinear templates $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ which are functions of internal states and input signals, as well as time delays are allowed. In this paper, however, only delay free polynomial type nonlinearities are admitted. This restriction seems reasonalble for VLSI-realizations, since analog multipliers, which are building blocks for polynomial nonlinearities, are quite simple to implement as VLSI circuits.

The equations describing the most recent class of CNN without time delay with $1 \leq i \leq M; 1 \leq j \leq N$ are as follows:

State Equations:

$$
\begin{aligned}
C\frac{dv_{xij}(t)}{dt} \quad = \quad & -\frac{1}{R_x}v_{xij}(t) + \sum_{C(k,l)\in N_r i,j} A(i,j;k,l)v_{ykl}(t) + \sum_{C(k,l)\in N_r i,j} B(i,j;k,l)v_{ukl}(t) + \\
& \sum_{C(k,l)\in N_r i,j} \hat{A}(i,j;k,l)\left(v_{yij}(t), v_{ykl}(t)\right) + \sum_{C(k,l)\in N_r i,j} \hat{B}(i,j;k,l)\left(v_{yij}(t), v_{ukl}(t)\right) \\
& +I, \qquad (1)
\end{aligned}
$$

Output Equations:

$$v_{yij}(t) = \frac{1}{2}\left(|v_{xij}(t)+1| - |v_{xij}(t)-1|\right), \tag{2}$$

Input Equations:

$$v_{uij} = E_{ij}, \tag{3}$$

Constraint Equations:

$$|v_{xij}(0)| \leq 1, \qquad |v_{uij}| \leq 1, \tag{4}$$

Parameter Assumptions:

$$A(i,j:k,l) = A(k,l;i,j), \tag{5}$$

$$C > 0, \quad R_x > 0. \tag{6}$$

For the purpose of this paper we will relax some of the assumptions to tailor to our needs. In particular, the class of CNN considered in this paper is assumed to possess the following additional characteristics:

1. The initial state $v_{xij}(0)$ can be any finite real number. This relaxation has the effect of increasing the upper bound of the states $|v_{xij}|$.

2. No offset current: $I = 0$

3. The system has no inputs: $v_{ukl}(t) = 0$. This requirement reduces the problem to an initial value problem.

4. Ideal integrator: $R_x \to \infty$. In practical circuits the integrator will necessarily be leaky, thereby making R_x large but finite. The ideal integrator assumption invalidates our original proof on the boundedness of the states for general symmetric templates [7]. However, for the special templates proposed below; we can still prove the boundedness property holds.

5. The symmetry assumption (5) is dropped.

6. For the terms under the third summations sign in (1) we introduce the nonlinear template

$$\hat{A}(i,j;k,l)\,(v_{yij}v_{ykl}) = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix} \tag{7}$$

$m, n \in N_r(i,l)$ where r denotes the neighbourhood size. The functions f_{kl} are defined as

$$f_{kl} = \hat{a}_{kl} \bullet v_{yij} \bullet v_{ykl}^{\alpha}, \qquad \alpha \in \mathcal{N}. \tag{8}$$

Note that a nonlinear output function is included in the nonlinear self feedback signal.

7. More complicated PWL-functions, such as

$$v_{yij} = \frac{1}{l+1} \sum_{k=1}^{l} |v_{xij} + v_k| - |v_{xij} - v_k|\,, \tag{9}$$

or

$$v_{yij} = \tanh(v_{xij}) \tag{10}$$

may be used for the output function, provided the function is odd-symmetric:

$$v_y\left(v_{xij}\right) = -v_y\left(-v_{xij}\right). \tag{11}$$

The above class of nonlinear transfer functions will be allowed because they arise naturally in opamp circuits driven beyond the linear operation range. They are therefore relatively simple to realize as integrated circuits.

3 Lattice Equations

In general, one-dimensional lattices are described by a set n of structurally identical ODEs (first or second order) with local coupling between the states

$$\Delta x_k = f\left(x_{k-1}, x_k, x_{k+1}\right), \tag{12}$$

where Δ denotes the differential operator. Some prominent examples of such equations are [13]:

- discrete Korteweg-de Vries equation $\frac{dx_k}{dt} = \exp\left(x_{k+1}\right) - \exp\left(x_k\right)$,
- modified Korteweg-de Vries equation $\frac{dx_k}{dt} = \left(1 \pm h^2 x_k^2\right)\left(x_{k+1} - x_{n-1}\right)$, $h \in \mathcal{R}$ or,
- the self-dual network equation

$$\begin{aligned} \frac{dx_k}{dt} &= \left(1 \pm x_k^2\right)\left(\hat{x}_k - \hat{x}_{n-1}\right) \\ \frac{d\hat{x}_k}{dt} &= \left(1 \pm \hat{x}_k^2\right)\left(x_{k+1} - x_n\right). \end{aligned}$$

A special case of these lattices are one-dimensional mechanical system consisting of n mass points connected by nonlinear springs, i.e. a system of coupled oscillators. It is governed by the equations of motion:

$$\ddot{x}_k = F\left(x_k - x_{k-1}\right) - F\left(x_{k+1} - x_k\right) \qquad k = 1, 2, \ldots, n. \tag{13}$$

with boundary conditions at both ends and initial conditions for all mass points.

Quadratic, cubic or piece-wise linear spring characteristics as shown in Fig. 1 were investigated in the fourties and early fifties by Fermi, Pasta and Ulam [1]. They observed that a sinusoidal initial condition for x_k is repeated periodically, though the system is nonlinear. Later, Zabusky and Kruskal found an explanation for this phenomenon [12]. Related to this recurrence phenomenon is the existence of soliton solutions of the lattice equations and the corresponding partial differential equations.

Ablowitz gives the following definition of a soliton [13]:

Definition 1 *A soliton is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with solitary waves, or more generally, with another (arbitrary) localized disturbance.*

Toda investigated the discrete one-dimensional lattice with the nonlinear springs characterized by an exponential force vs. r relationship, where r denotes the distance between the mass points [2]

$$F(r) = \exp(-r) - 1 \tag{14}$$

and where the outer mass points are fixed at $x_0 \to -\infty$ and $x_{n+1} \to \infty$. It can be shown both analytically and experimentally that solitons exist [14].

The equations of Toda were the origin of a series of mathematical developments after Symes' discovery of an intimate relationship between the Toda lattice and the eigenvalue problems of symmetric tridiagonal matrices [3].

Using the coordinate transformation

$$a_k = -\frac{1}{2}\dot{x}_k, \qquad b_k = \frac{1}{2}\exp\left(\frac{x_k - x_{k+1}}{2}\right) \tag{15}$$

and the boundary conditions $b_0 = b_n = 0$, eq. (13) with (14) can be written as

$$\dot{a}_k = 2\left(b_k^2 - b_{k-1}^2\right) \tag{16}$$

$$\dot{b}_k = b_k\left(a_{k+1} - a_k\right). \tag{17}$$

In the following we will call these equations the Toda lattice equations too since they are homeomorphic to the Toda lattice by (15) up to a constant. If the initial conditions for a_k, b_k are specified as the entries of a tridiagonal matrix

$$\mathbf{A}(0) = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & \ddots & 0 \\ 0 & \ddots & \ddots & b_{n-1} \\ 0 & 0 & b_{n-1} & a_n \end{bmatrix}, \tag{18}$$

then the fixed point of (16), (17) is characterised by

$$a_k = \lambda_k, \qquad b_k = 0 \tag{19}$$

where λ_k are the eigenvalues of $\mathbf{A}(0)$ with the additional property that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \tag{20}$$

i.e. they come out sorted. Observe that a fixed point analysis alone does not give any information on a_k.

As is well known, a property of completely integrable Hamiltonian systems is the existence of as many invariants of motion I_k as the degrees of freedom. Hence the trajectories of (17) flow on the manifold $\mathcal{M}$, defined by the initial conditions, or the invariants respectively, into the fixed points. These invariants deliver the information on the a_k at the fixed points.

The meaning of the invariants for the eigenvalue problem becomes clear, if it is written as the characteristic polynomial of $\mathbf{A}(0)$

$$p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0. \tag{21}$$

The coefficients are polynomials of a_k, b_k:

$$\alpha_m = f_m\left(a_1, \ldots, a_n, b_1, \ldots, b_{n-1}\right), \qquad m = 0, 1, \ldots, n-1. \tag{22}$$

The n expressions define the manifold $\mathcal{M}$ of all matrices with equal eigenvalues for a given set of α_m. These invariants can be carried over to another set [15]

$$H_k = \frac{1}{k}\text{trace } \mathbf{A}^k. \tag{23}$$

The system posesses $n!$ fixed points according to the $n!$ permuations of λ_k. The manifold $\mathcal{M}^l$ of the linearization about such a fixed point consists of the union of 3 different linear manifolds $\mathcal{W}^s$, $\mathcal{W}^u$, $\mathcal{W}^c$, depending on the eigenvalues

$$\left.\begin{array}{lcll} \mathrm{Re}(\lambda_i) & > & 0 & \mathcal{W}^u \text{ unstable} \\ \mathrm{Re}(\lambda_i) & < & 0 & \mathcal{W}^s \text{ stable} \\ \lambda_i & = & 0 & \mathcal{W}^c \text{ center} \end{array}\right\} \text{manifold.}$$

Only one of the fixed points is characterised by

$$\mathcal{M}^l = \mathcal{W}^s \bigcup \mathcal{W}^c, \quad \mathcal{W}^u = \emptyset, \tag{24}$$

i.e. after perturbation in the orthogonal complement of $\mathcal{W}^c$, the system returns to this fixed point. This stability property of fixed points can be used to sort real numbers given as initial conditions $a_1, \ldots, a_n$. A small perurbation that does not lie completely in $\mathcal{W}^c$ forces the system to the fixed point with sorted diagonal entries.

For $n = 2$, the system of ODEs (16), (17) can be solved explicitly [16]:

$$\begin{aligned} a_1(t) &= \beta + \frac{\alpha}{2}\tanh(\alpha t + \gamma) && (25) \\ a_2(t) &= \beta - \frac{\alpha}{2}\tanh(\alpha t + \gamma) && (26) \\ b_1(t) &= \frac{\alpha}{2}\frac{1}{\cosh(\alpha t + \gamma)} && (27) \end{aligned}$$

with α, β, γ determined by the initial conditions. Eq. (27) shows already the typical bell-shaped solution of solitons.

4 CNNs and Lattice Equations

After presenting the equations for CNN with unspecified matrices $\mathbf{A}$, $\mathbf{B}$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and the equations for FPU and the Toda lattice, we are now ready to relate both systems. In the case of a first order ODE lattice equation, the states of the lattice equation are identical to the CNN states. The lattice equations from mass spring models are second-order ODEs in time with one spatial dimension, while the CNN equations are first-order ODEs with two spatial dimensions. Hence, we have the choice to use either a double-layer CNN or a $2 \times n$ cell single-layer CNN. A special choice for the template matrices $\mathbf{A}$ and $\hat{\mathbf{A}}$ make the structure absolutely regular (Fig. 2 for Toda lattice). This works well under the condition that two rows and columns (shaded area) of cells deliver zero signals to the cells.

4.1 Fermi-Pasta-Ulam Lattice

The FPU lattice with the PWL spring characteristic fits perfectly into the original CNN structure with the coordinate transformation $v_{xij} = x_i - x_{i-1}$ and the template

$$\mathbf{A} = \begin{bmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ \frac{1}{K} & -\frac{2}{K} & \frac{1}{K} \end{bmatrix}, \tag{28}$$

where all other templates are zero. Written in the form of state equations, we have

$$C\dot{v}_{x1j} = Kv_{y2j} \tag{29}$$

$$C\dot{v}_{x2j} = \frac{1}{K}\left(v_{y1j+1} - 2v_{y1j} + v_{y1j-1}\right). \tag{30}$$

The scaling factor K has to be adjusted for the condition

$$|v_{x2j}| < 1 \Rightarrow v_{x2j} = v_{y2j}, \tag{31}$$

i.e. the first equation (29) is purely linear. Only the states v_{x2j} are forced to the linear region of the output map. Since the FPU lattice conserves the energy given by the initial conditions, the state voltage is bounded by

$$|v_{xij}| < \sqrt{\sum_{i,j} v_{xij}^2(0)}. \tag{32}$$

In this way simple experiments of the FPU lattice can be performed on this CNN. This applies to more complicated PWL output maps too. The mapping of the FPU lattice onto the CNN is exact.

4.2 Toda Lattice

The Toda lattice equations can be further simplified by a transformation $c_k = b_k^2$, proposed in [6], and a linear time scaling to drop the factor 2.

In detail the following template matrices are to be implemented:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/\hat{R}_x & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} 0 & f_{12} & f_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{33}$$

$$f_{12} = -v_{yij}v_{ykl}, \qquad f_{13} = v_{yij}v_{ykl}.$$

We call this system the modified Toda lattice.

The term $1/\hat{R}_x$ is useful only in the presence of $R_x < \infty$. It makes the system less dissipative, but the choice of $\hat{R}_x$ is very restrictive ($\hat{R}_x > R_x$) for stability reasons.

As opposed to the FPU lattice, the mapping of the Toda lattice is exact if all states v_{xij} utilize only the linear part of the output function. Suppose this function is linear for $|v_{xij}| < v_{max}$, then the states are bounded by the following inequalities ($t \geq 0$, $a_{max} = \max_{k=1,\ldots,n} a_k(0)$, $b_{max} = \max_{k=1,\ldots,n-1} b_k(0)$, [5]:

$$b_k(t) \;\leq\; \sqrt{\frac{1}{2}\left(n\left(a_{max}^2(0) - a_{min}^2(0)\right) + 2(n-1)b_{max}^2(0)\right)} \tag{34}$$

$$a_k(t) \leq \sqrt{H_2}, \tag{35}$$

i.e. the CNN is equal to the Toda lattice if the initial conditions fulfill:

$$\sqrt{\frac{1}{2}\left(n\left(a_{max}^2(0) - a_{min}^2(0)\right) + 2(n-1)b_{max}^2(0)\right)} \leq v_{max} \tag{36}$$

$$\sqrt{H_2} \leq v_{max}. \tag{37}$$

4.3 Modified Toda Lattice

Because of the nonlinear output map, the CNN does not exactly behave like a Toda lattice for initial conditions that does not fulfill (36), (37). The saturation characteristics bounds the derivatives, namely

$$|\dot{a}_k| \leq 2, \qquad \left|\dot{b}_k\right| \leq 2. \tag{38}$$

This implies bounded acceleration and propagation speed in terms of the Toda lattice.

Due to the boundary conditions $b_0 = b_n = 0$, the fixed point is still given by

$$b_k = 0.$$

This follows by induction from the boundary conditions and the equations for $\dot{a}_k$. As for the Toda lattice, no information on the a_k is available.

The invariant H_1 is quite simple to calculate. Differentiation of

$$H_1 = a_1 + a_2 + \ldots + a_n \tag{39}$$

and substitution of all $\dot{a}_k$ by the right side of (16) lead to the invariance property. This is equivalent to impulse or charge conservation.

Several questions arise after such a crude modification of the Toda lattice.

- Can this equation still calculate eigenvalues, or at least sort numbers?

- How about the other invariants?

- And most interesting, but related to the questions above, do soliton solutions exist?

These questions are the subject of the next Section, as far as it is possible to answer them by analytical solutions.

5 Numerical Experiments with Modified Toda Equations

This section summarizes numerical experiments on the modified Toda lattice (eqs. (1) – (6) and (33)) with a PWL output map

$$v_{yij}(t) = \frac{1}{2}\left(|v_{xij}(t)+1| - |v_{xij}(t)-1|\right) = g\left(v_{xij}\right). \quad (40)$$

The results reveal an interesting behaviour that was not expected. The FPU lattice is not considered here because we only wanted to show that a CNN could be used for an electrical circuit realization of this lattice.

To demonstrate the effect of the PWL map, the Toda lattice and the modified Toda lattice for the same initial conditions are shown, whenever it seems helpful.

If the transients for given initial conditions remain in the linear part of (40) the modified Toda lattice behaves like the Toda lattice. This case is explicitly excluded here.

5.1 The Most Simple System

At first 3 examples for a 2×2 matrix are investigated.

Example 1:

$$\mathbf{A}(0) = \begin{bmatrix} -2.0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$$

The transients for a_1, a_2, b_1 are shown in Fig. 3 a, b. The bounds of the derivatives stretch the transients, but the initial values $a_1(0)$, $a_2(0)$ are still sorted by value. A look at the invariant[1] H_2 (eq. (23)) indicates that the PWL map influences the behaviour of the ODE seriously. The system is no longer lossless but can be lossless, passive or active. The passive and active mode keep a balance such that the invariant H_2 is recovered after sorting. The invariant H_1 does not change its value.

Example 2:

$$\mathbf{A}(0) = \begin{bmatrix} -4.0 & 2.1 \\ 2.1 & 0 \end{bmatrix}$$

This example for eigenvalue calculation (Fig. 4) proves that in general the modified Toda lattice can not compute eigenvalues, i.e. it does not perform a similarity transform. So we restrict to the case of sorting.

In both examples, the ODE is still nonlinear for all $t > 0$. The difference between example 1 and 2 can be seen form the eigenstructure of the Jacobian matrices of both systems (Fig. 3e, 4e). The modified Toda lattice preserves a symmetry for a time t_m such that the eigenstructure is odd-symmetric with respect to this point. In the second example such a point can not be found.

Example 3:

$$\mathbf{A}(0) = \begin{bmatrix} -4.0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$$

For this initial matrix, once again a sorting example (Fig. 5a), the system becomes linear

---

[1] In the following we use the term invariant for the expressions H_i, though they are no longer constant.

in a certain time intervall. This can be concluded from the constant eigenvalues of the Jacobian matrix (Fig. 5b).

The classification of the 26 (+ 1 for Toda lattice) possible sets of ODEs with respect to the values of the output function show that 20 solutions have transients that are linear or quadratic in time, 2 solutions are exponentially increasing or decreasing. These are all linear ODEs. The equations are nonlinear for : $|a_1| > 1$ and ($|a_2| < 1$ or $|b_1| < 1$), $|a_2| > 1$ and ($|a_2| < 1$ or $|b_1| < 1$), i.e. there exists 4 different sets of equations, e.g.

$$\begin{aligned} \dot{a}_1 &= 2b_1^2 \\ \dot{a}_2 &= -2b_1^2 \\ \dot{b}_1 &= b_1(a_2 - 1). \end{aligned}$$

The solution of these equations with $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ given by the initial conditions is:

$$a_1(t) = \tilde{\beta} + \frac{\tilde{\alpha}}{\sqrt{2}} \tanh(\frac{\tilde{\alpha}}{\sqrt{2}} t + \tilde{\gamma}) \quad (41)$$

$$a_2(t) = \tilde{\beta} - \frac{\tilde{\alpha}}{\sqrt{2}} \tanh(\frac{\tilde{\alpha}}{\sqrt{2}} t + \tilde{\gamma}) \quad (42)$$

$$b_1(t) = \frac{\tilde{\alpha}}{2} \frac{1}{\cosh(\frac{\tilde{\alpha}}{\sqrt{2}} t + \tilde{\gamma})}. \quad (43)$$

Note the solution of (25) - (27) of the Toda lattice and the above solution differ by a time scaling of $\sqrt{2}$. The invariant H_2 yields

$$H_2 = \tilde{\beta}^2 + \tilde{\alpha}^2 - \frac{\tilde{\alpha}^2}{2\cosh^2(\tilde{\alpha}t + \tilde{\gamma})}. \quad (44)$$

Hence, H_2 is not constant, but decreasing for $t_m < -\frac{\tilde{\gamma}}{\sqrt{2\tilde{\alpha}}}$, i.e. the system is passive, and increasing for $t_m > -\frac{\tilde{\gamma}}{\sqrt{2\tilde{\alpha}}}$, i.e. the system is active. So we have a symmetry with respect to t_m. Therefore in example 1 and 3 the diagonal entries in the fixed point are the same as the initial conditions but sorted. Additionally, in example 3 a linear system, symmetric around t_m, is included. In general the derivative of the stored energy can be either decreasing, zero or increasing.

In the modified Toda lattice for dimension $n = 2$ with initial conditions such that the ODE is still nonlinear, a recurrence phenomena can be observed. This is due to a symmetry in time, e.g. of the eigenvalues of the Jacobian matrix.

5.2 Sorting of Numbers

The modified Toda equations behave like a sorter, but a strict sorting order no longer exists. This is due to the limited propagation speed of the data and the stability properties of the fixed points. The linearization of the modified Toda lattice about a fixed point gives:

$$\begin{aligned} \dot{a}_k &= 0 \\ \dot{b}_k &= \left(g(\hat{a}_{k+1}) - g(\hat{a}_k)\right) g'(0) b_k. \end{aligned} \tag{45}$$

The variables $\hat{a}_k$ denote the values of the diagonal entries at the fixed point and $g'(0)$ denotes the derivative of the output function. A fixed point is called asymptotically stable[2] with respect to perturbations in b_k, if for all first sub- and superdiagonal entries, the following inequality holds ($g'(0) > 0$):

$$g(\hat{a}_{k+1}) - g(\hat{a}_k) < 0. \tag{46}$$

The inequality gives no information on the order for $\hat{a}_{k+1}$ and $\hat{a}_k$ for $\hat{a}_{k+1} > 1$ and $\hat{a}_k > 1$ (-1 equivalently). So their order at the fixed point is undecided. In these cases the center manifold dimension is increased by one.

Suppose $|\hat{a}_k| < 1$ for n_s values, $\hat{a}_k > 1$ for n_+ values and $\hat{a}_k < -1$ for n_- values. Then, for the modified Toda lattice the number of asymptotically stable fixed points n_{fp} is determined by the number of permutations of the elements with $\hat{a}_k > 1$ and with $\hat{a}_k < -1$, thus

$$n_{fp} = n_+! n_-!. \tag{47}$$

[2]Strictly speaking they are only stable.

If the output map is applied to the sorting result, a half order is realized, since all values $\hat{a}_k > 1$ ($\hat{a}_k < 1$) reduce to $\hat{a}_k = 1$ ($\hat{a}_k = -1$).

The multiplicity in the fixed points is caused by the bounded derivatives. Therefore the propagation speed for data $|\hat{a}_k| > 1$ is equal. So their final position depends on the initial position.

5.3 Soliton Solutions

The most interesting result of the modified Toda lattice equations is the existence of soliton solutions, in the sense of definition 1. Such a solution requires the initial conditions to be such that the ODEs operate in a nonlinear regime. Then a wave with constant shape is built up and two waves pass each other without changing their shape after the collision.

In Fig. 6 the evolution of (states vs. time) of the variables a_k for a lattice of dimension 7 with initial values

$$\begin{bmatrix} a_1, & \ldots, & n-1, & n \\ b_1, & \ldots, & b_{n-1} & \end{bmatrix} = \begin{bmatrix} -0.75 & 0 & 0 & 0 & 0 & 0 & 1.4 \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \end{bmatrix}$$

is shown. The little irregularity in the shape of the curves is due to a downsampling of the simulation result. These initial values exclude either a pure Toda lattice or a purely linear ODE. This picture reveals the typical behaviour of soliton solutions: their shape is constant over time and is not changed by head-on collisions.

In Fig. 7 the contour plot of the same lattice with initial conditions

$$\begin{bmatrix} a_1, & \ldots, & n-1, & n \\ b_1, & \ldots, & b_{n-1} & \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 & 1.5 & 0 & 0 & 2 \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \end{bmatrix}$$

is depicted. The bounded derivatives limit the propagation speed of the data 2 and 1.5. Thus the waves for these initial values travel in parallel from state 4, 7 to 1, 2 as can be seen in Fig. 7. The third soliton travels to the right.

Though these numerical experiments can not prove the existence of solitons solutions, they point out that soliton solutions seem to exist in nonintegrable systems, if some symmetry property is met.

6 Conclusion

In this paper we have shown how two famous nonlinear lattice equations, the FPU lattice and the Toda lattice can be mapped onto a CNN. For this purpose certain original restrictions on the templates, such as symmetry, were was dropped. Nevertheless the system is still stable, because the underlying mechanical models are stable. The mapping is exact under certain conditions for the state variables or initial conditions, respectively. Otherwise, the nonlinear output map of the CNN modifies the lattice equations. This was demonstrated for the Toda lattice. Numerical experiments and some analytical reasonings show that the sorting operations can be performed for certain initial conditions. Related to this property is the existence of soliton solutions. It has to be stressed that the system is not Hamiltonian, but possesses symmetries in time. Soliton solutions for the modified Toda lattice exists.

In practice there are always dissipative terms for the proposed CNN structure. They force the trajectories to zero and so a soliton in the strict sense does not exist. But if the dissipation is small, solitary wave can still be observed over a short time interval.

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Figure 1: Schematic spring characteristics considered by Fermi, Pasta and Ulam, a) quadratic, b) cubic, c) piecewise linear

Figure 2: Mapping of the Toda lattice equations onto a la CNN architecture. The variables in the cells denote the equation that is realized by the respective cell

Figure 3: Example 1, a), b) trajectories a_1 (—), a_2 (- - -), b_1 (......), c) invariants H_1 (—), H_2 (.....), d), e) eigenvalues of the Jacobian matrix

Figure 4: Example 2, a), b) trajectories a_1 (—), a_2 (- - -), b_1 (......), b) eigenvalues of the Jacobian matrix

Figure 5: Example 3, a), b) trajectories a_1 (—), a_2 (- - -), b_1 (......), b) eigenvalues of Jacobian

Figure 6: Soliton solution in modified Toda lattice with head on collision of two solitons

Figure 7: Effect of limited propagation speed of two solitons as a contour plot of the solution

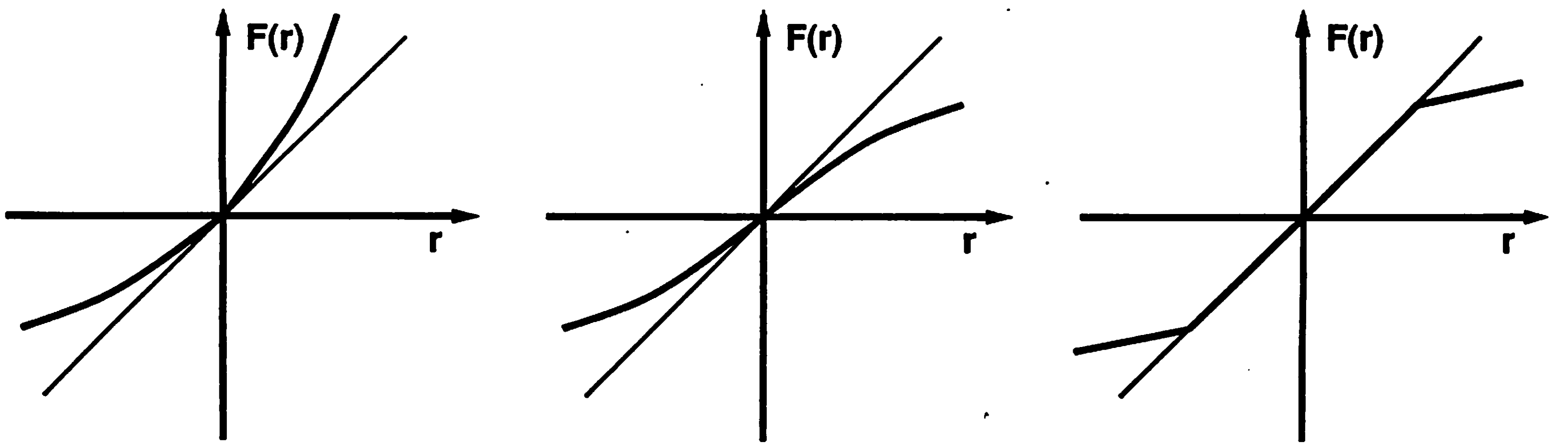


Figure 1

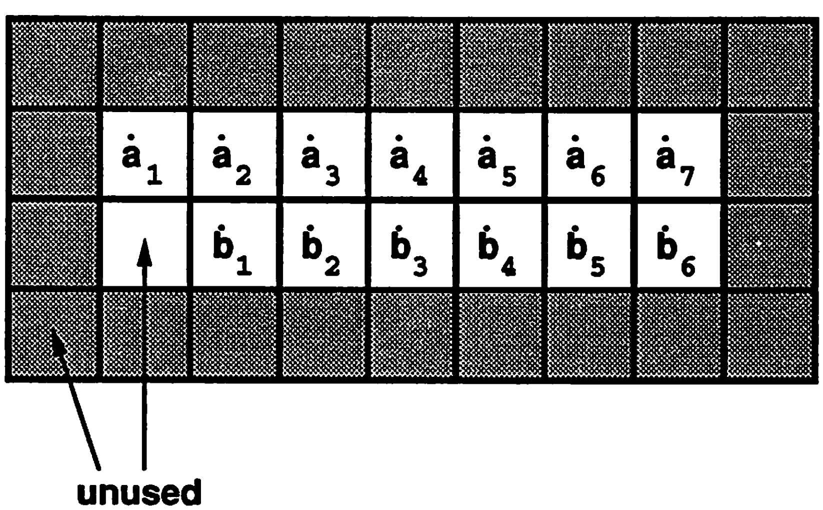


Figure 2

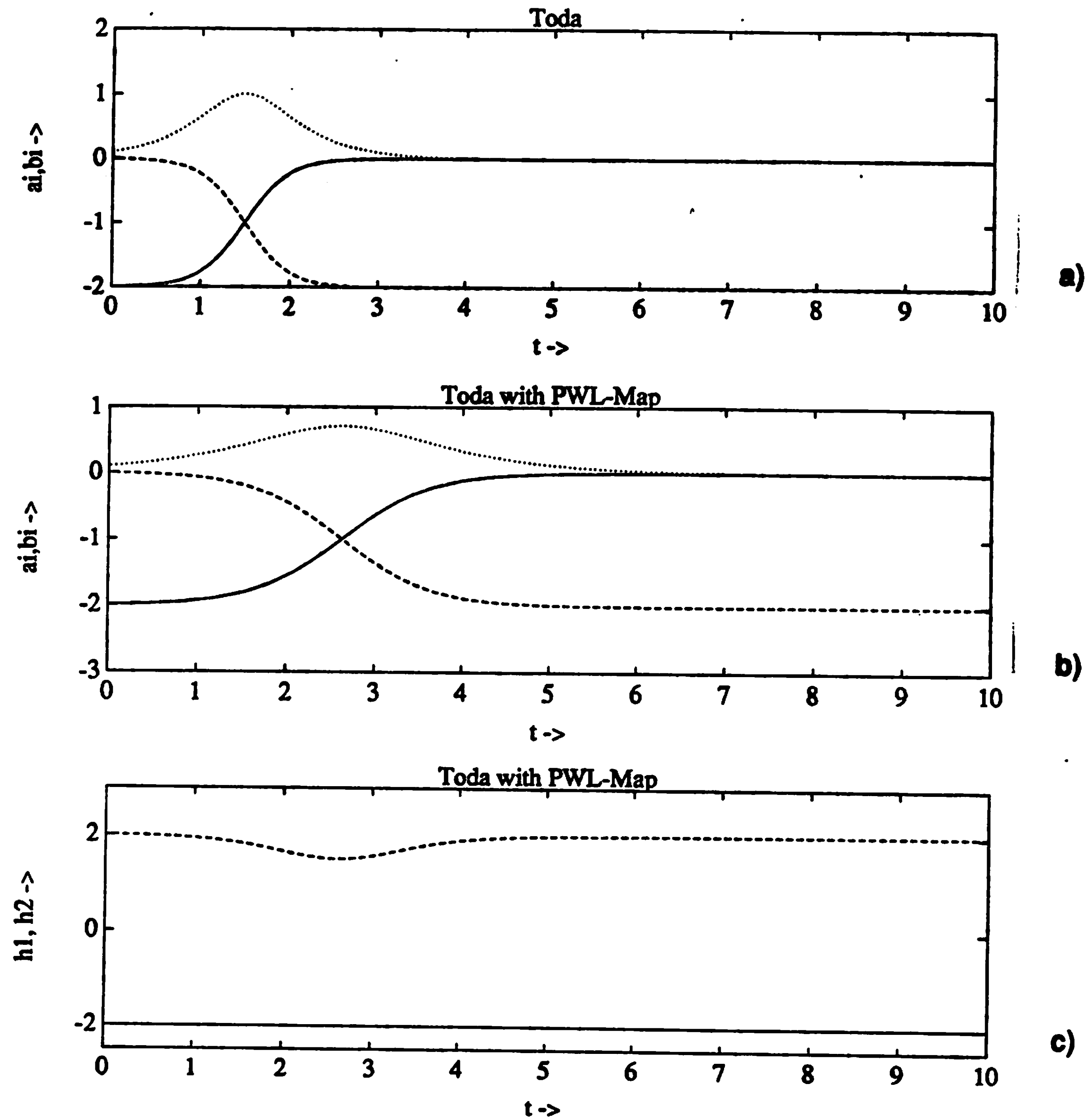
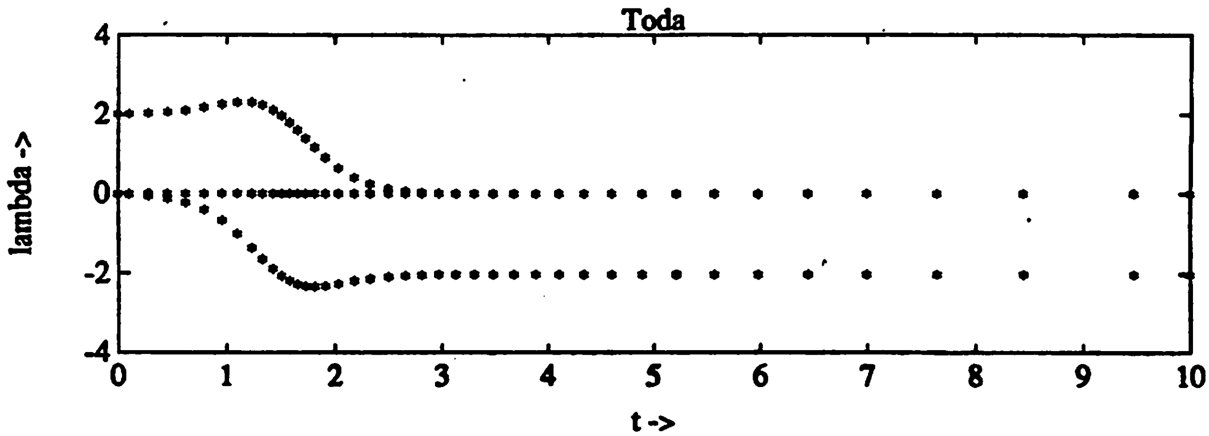
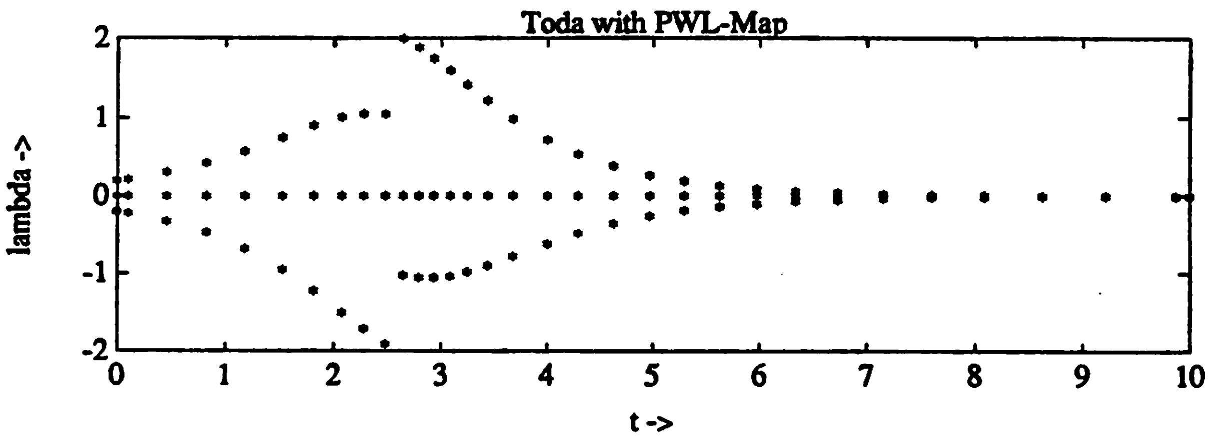

Toda
ai,bi ->
2
1
0
-1
-2
0 1 2 3 4 5 6 7 8 9 10
t ->
a)
Toda with PWL-Map
ai,bi ->
1
0
-1
-2
-3
0 1 2 3 4 5 6 7 8 9 10
t ->
b)
Toda with PWL-Map
h1, h2 ->
2
0
-2
0 1 2 3 4 5 6 7 8 9 10
t ->
c)

Figure 3


Toda
lambda ->
t ->
4
2
0
-2
-4
0
1
2
3
4
5
6
7
8
9
10

d)


Toda with PWL-Map
lambda ->
t ->
2
1
0
-1
-2
0
1
2
3
4
5
6
7
8
9
10

e)

Figure 3

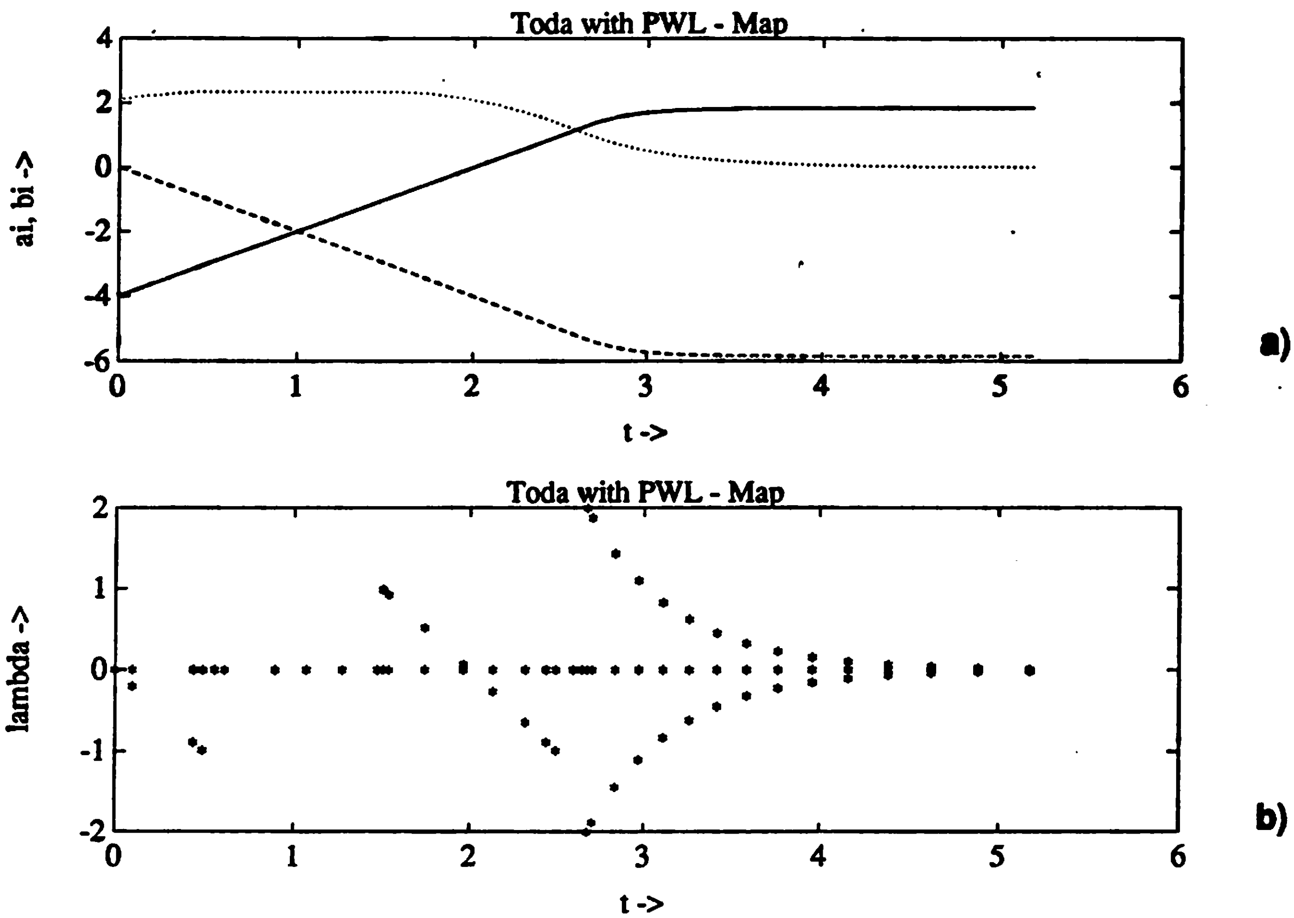


Figure 4

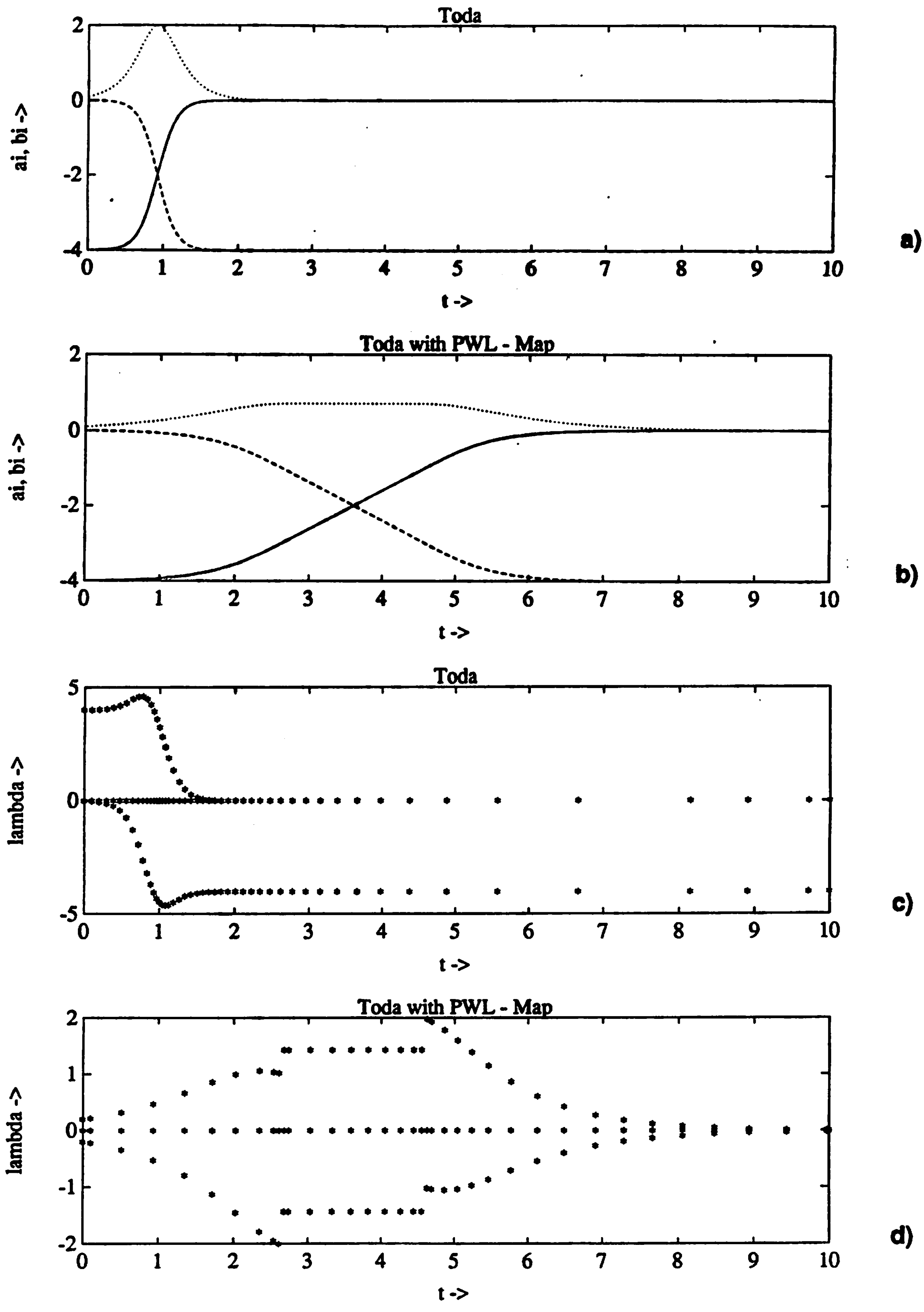

Toda
ai, bi ->
t ->
a)
Toda with PWL - Map
ai, bi ->
t ->
b)
Toda
lambda ->
t ->
c)
Toda with PWL - Map
lambda ->
t ->
d)

Figure 5

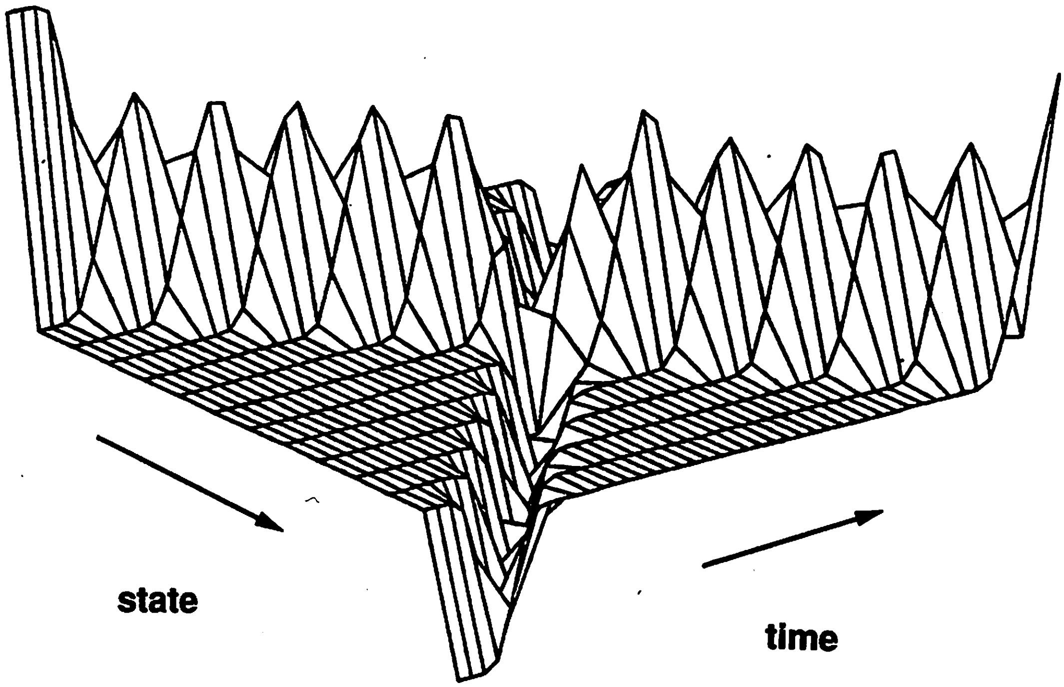


Figure 6

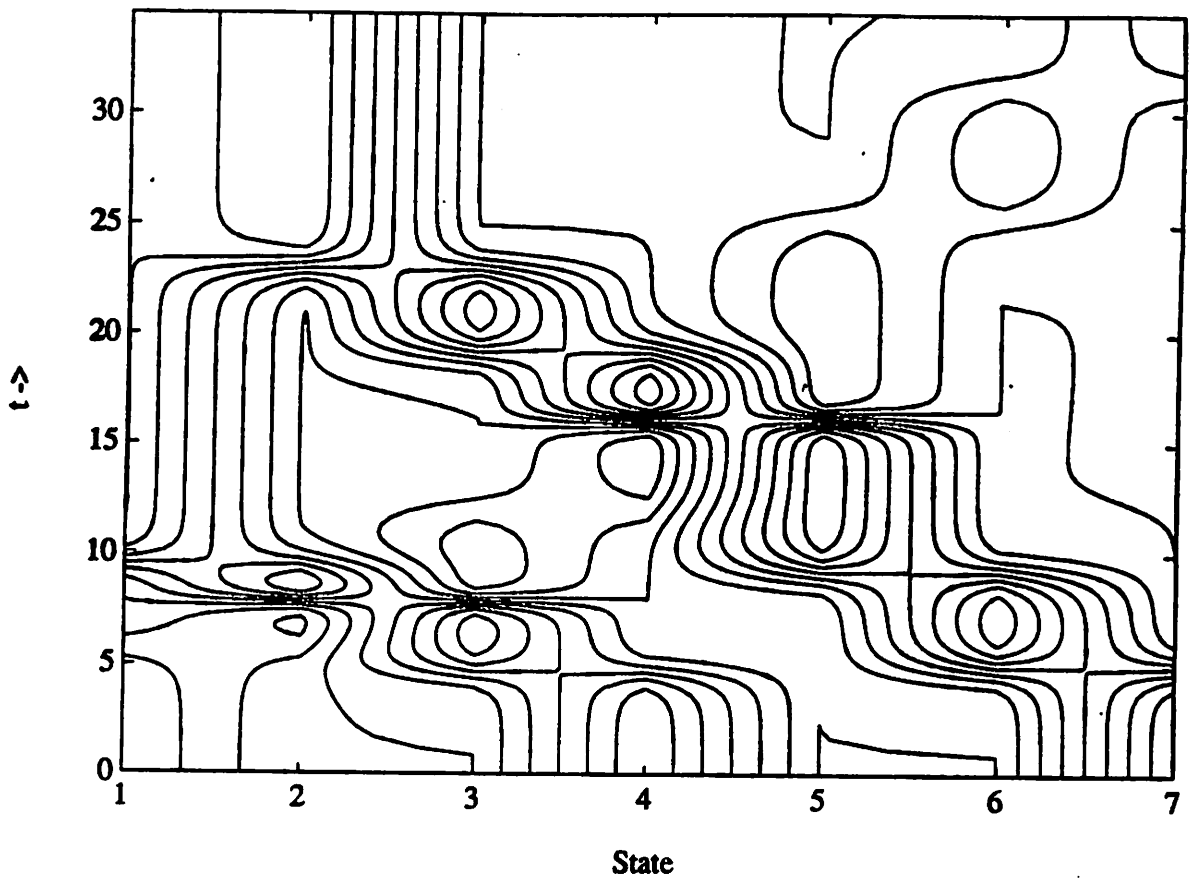


Figure 7