CONFIRMATION OF THE AFRAIMOVICH-SHILNIKOV TORUS-BREAKDOWN THEOREM VIA CHUA'S TORUS CIRCUIT

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Abstract

The Afraimovich-Shilnikov theorem on two-dimensional torus breakdown is formulated and used to carry out a detailed numerical investigation of the bifurcation routes from the torus to chaos in the three-dimensional Chua's torus circuit. Three scenarios of transition to chaos due to torus breakdown take place in this circuit in complete agreement with the theorem: (1) period-doubling bifurcations of the phase-locked limit cycles, (2) saddle-node bifurcation in the presence of a homoclinic structure, and (3) soft transition due to the loss of torus smoothness.

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1.Introduction

The appearance of chaos following the breakdown of a two-dimensional torus is a typical and interesting scenario of transitions to chaos. Ruelle and Takens[1], and later, Newhouse, Ruelle and Takens (NRT)[2] had provided the seminal results on this subject. Compared with the Landau-Hopf scenario[3,4], the NRT-scenario specifies a finite bifurcation sequence in the route to chaos: a stable equilibrium point (FP) $\Rightarrow$ a stable limit cycle (LC) $\Rightarrow$ a stable two-dimensional torus (T^2) $\Rightarrow$ chaos:

$$\mathrm{FP} \Rightarrow \mathrm{LC} \Rightarrow T^2 \Rightarrow \mathrm{Chaos} \tag{1}$$

The sequence of attractors (1) has been confirmed both by computer simulations of many simple models and by physical experiments on Couette flow, hydrodynamical instabilities, etc.[5-17]. However, the NRT-scenario does not give a full description of all possible bifurcation scenarios leading to the destruction of the torus T^2. In particular, it does not address the following questions. Does the T^2-breakdown always lead to chaos? If not, what additional conditions are required? What is the structure of the resulting chaotic attractor?

Many authors have investigated the phenomena of T^2-breakdown theoretically. One such results which will be applied in this paper is a theorem by Afraimovich and Shilnikov on two-dimensional tori breakdown[18,19].

In this paper we will use a member of the Chua's circuit family, namely, the Chua's torus circuit[10,20] as a vehicle to illustrate the main bifurcation phenomena, which cause transitions to chaos, as predicted in the Afraimovich-Shilnikov theorem.

Chua's torus circuit[10] was chosen for the following reasons. The phase space of the associated state equations has the dimension necessary for the transition $T^2 \Rightarrow$ Chaos. Moreover, the circuit is autonomous and the T^2-attractor arises without any external periodic excitations. Since the state equation is piecewise linear, some important results can be obtained analytically. Besides, it would be interesting to investigate whether the bifurcation scenarios predicted by the Afraimovich-Shilnikov theorem, which was proved only for smooth dynamical systems, also occur in piecewise-linear systems.

2.The Afraimovich-Shilnikov torus breakdown theorem

Consider the dynamical system:

$$\dot{x} = F(x,\mu) \qquad (2)$$

in $\mathbb{R}^N$, where $N \geq 3$. Here $x \in \mathbb{R}^N$, $\mu \in \mathbb{R}^K$ and $F(x,\mu)$ is assumed to be a sufficiently smooth function of x and of the parameter μ. Let us suppose that a smooth attracting torus $T^2(\mu_0)$ exists in some region G of the state space of system (2) at $\mu=\mu_0$ and assume that the torus $T^2(\mu_0)$ is formed by the closure of the flow of (2) originating from the unique stable limit cycle Γ^+, the saddle-type limit cycle Γ^- (which can be proved to always exist), and the unstable manifold W^u of Γ^-. Assume further that the multiplier of the stable limit cycle Γ^+ on the torus $T^2(\mu_0)$ which has the smallest absolute value is simple and real.

Consider a continuum set of continuous curves $H=\{\mu(s)\colon \mu \Rightarrow \mathbb{R} \Rightarrow \mathbb{R}^K\}$ in the parameter space $\mathbb{R}^K$ of system (2), $0 \leq s \leq 1$. Assume that at $\mu(0)$ a smooth stable torus corresponding to $\mu(0) \in H$ exists in the region G for system (2), but at $\mu(1)$ the torus does not exist. Then the Afraimovich-Shilnikov theorem asserts the following three distinct breakdown scenarios for the torus:

1. For some intermediate parameter value s_1, where $0<s_1<1$, the torus exists, however, either $T^2[\mu(s)]$ loses its smoothness due to the nonsmooth behavior of the unstable manifold W^u in the vicinity of the stable limit cycle Γ^+ for $s>s_1$, or the pair of multipliers $\rho_{1,2}$ of the stable limit cycle Γ^+ becomes complex-conjugate inside the unit circle at $s=s_1$.

2. There exists a value s^*, $s^*>s_1$, such that the attracting torus $T^2(\mu)$ no longer exist for $s>s^*$. In this case, there are three possible routes leading to the destruction of the torus:

A. At $s=s^*$, the stable limit cycle $\Gamma^+(\mu)$ loses its stability via some typical scenarios in the bifurcation of periodic solutions.

B. A structurally unstable homoclinic trajectory of the saddle-type limit cycle Γ^-occurs due to the presence of a tangency between the stable manifild W^S the an unstable manifold W^u of the saddle cycle.

C. At $s=s^*$ the stable and the unstable (saddle type) limit cycles on the torus merge into a saddle-node periodic solution and then disappear. The torus is nonsmooth at the bifurcation parameter $s=s^*$.

Fig.1 shows a sketch of the qualitative bifurcation diagram of phase-locking on the torus T^2 on the two-parameter μ_1-μ_2 plane. The direction of the paths on the parameter plane corresponding to routes A, B, and C of the Afraimovich-Shilnikov theorem is shown in Fig.1.

The phase-locked region is formed by the two bifurcation curves l_1, corresponding to the merging and annihilation of the saddle-type limit cycle Γ^- and the stable limit cycle Γ^+ on the torus $T^2(\mu)$. The phase-locked region originates from the codimension-2 bifurcation point K on the bifurcation curve l_0. The curve l_0 corresponds to the bifurcation of a torus T^2 spawned from a limit cycle Γ_0 having a pair of complex-conjugate multipliers on the unit circle.

Consider first the route PA of the diagram in Fig.1. The limit cycle Γ^+ becomes unstable upon crossing the curve l_2, via one of the several possible bifurcation routes. The torus $T^2(\mu)$ does not exist above the curve l_2. A loss of smoothness of the torus precedes the torus breakdown as we approach l_2 from the lower side. For example, the transition to chaos along the route PA can come from a period doubling bifurcation process. If a new two-dimensional torus occurs from the cycle Γ^+ on the curve l_2, which is possible in some systems[10,11,16], then the above scenario of torus breakdown will occur again.

Next, let us move along the route PB. The phenomenon of homoclinic tangency of the stable and the unstable manifolds of the saddle cycle Γ^- on $T^2(\mu)$ takes place on the bifurcation curve l_h. In this case, the torus $T^2(\mu)$ is destroyed, but the stable cycle Γ^+ remains as an attractor. A structurally stable homoclinic structure emerges above the curve l_h and between the curves l_1 and l_2, but it is not an attractor. In this region the system (2) exhibits a metastable chaos having a finite life time[10,16]. A transition to a true dynamical chaos can be realized by moving along the route PB'. In this case a chaotic attractor emerges abruptly the moment we cross the saddle-node bifurcation curve l_1. In this case, a subset of hyperbolic trajectories is transformed into a chaotic attractor, and the stable limit cycle Γ^+ disappears.

Finally, moving along the route PC results in a saddle-node bifurcation upon crossing the curve l_1. Two cases can be realized here. Along the route PC' a transition from the phase-locked torus to an ergodic one (i.e., quasi-periodic solution) takes place. Although the structure of the trajectories on T^2 is changed upon crossing l_1, in this case, the torus remains an attractor. If we cross the curve l_1 in the region Chaos$_2$ of the nonsmooth torus $T^2(\mu)$, then a transition to chaos through intermittency will take place.

The formulation of the Afraimovich-Shilnikov theorem, the bifurcation diagram of Fig.2, and the above remarks are all concerned with these bifurcation phenomena which lead to the destruction of $T^2(\mu)$. It is obvious from Fig.1 that the phenomena associated with the transition $T^2 \Rightarrow$ Chaos can be investigated only by carrying out a two-parameter analysis.

3.Computer-aided two-parameter analysis of Chua's torus circuit

Consider the following dimensionless form of Chua's torus circuit[10]:

$$
\begin{aligned}
\frac{dx}{dt} &= -\alpha f(y-x) \\
\frac{dy}{dt} &= -z-f(y-x) \\
\frac{dz}{dt} &= \beta y
\end{aligned}
\tag{3}
$$

The piecewise-linear function describing the nonlinear resistor (Chua's diode[21]) in the circuit is described by:

$$f(\xi) = -a\xi+0.5(a+b)(|\xi+1|-|\xi-1|) \tag{4}$$

where as in reference 10, we choose the parameters

$$a = 0.07, \quad b = 0.10 \tag{5}$$

for comparison purposes.

We will use a computer to analyze the bifurcation phenomena of (3) over some strategically selected region in the α-β parameter plane.

First, a cycle Γ_0 of system (3) which gives birth to a torus $T^2(\alpha,\beta)$ will be found. Then the torus bifurcation curve l_0 will be obtained via a continuation method. On this curve the pair of complex-conjugate multipliers of Γ_0 lie on a unit circle; namely, $\rho_{1,2}= \exp(\mp j\varphi)$, $|\rho_{1,2}| = 1$. There is a point K on the curve l_0 corresponding to the phase-locked region having a rotation number 1:6. By finding a stable limit cycle Γ^+ on $T^2(\alpha,\beta)$ and by calculating the multipliers $\rho(\alpha,\beta)$ under various bifurcation conditions, we can construct the bifurcation diagram of system (3) in the region of interest.

To identify the type of the attractors, we take a Poincare section on the plane z=0 and calculate the corresponding power spectra $S(\omega)$ associated with the variable $x(t)$. To analyze the phenomenon occurring during the loss of torus smoothness, we introduce additive sources of white noise into equation (3). The full spectrum of Lyapunov exponents and the dimension D_L of the attractors were computed for diagnosing of the phenomenon of homoclinic tangency (curve l_h).

4. Bifurcation diagram in the region with 1:6 rotation number

The bifurcation diagram for system (3) in the region corresponding to a rotation number of 1:6 is shown in Fig.2. The curve l_0 (α=1) corresponds to the bifurcation which gives birth to the torus $T^2(\alpha,\beta)$ associated with the limit cycle Γ_0. This result was derived theoretically in reference 10. The two curves labelled l_1 are boundaries of the 1:6 phase-locked region and correspond to a saddle-node bifurcation of the stable limit cycle, henceforth denoted by Γ^+, on $T^2(\alpha,\beta)$. They were calculated by imposing the condition $\rho_1=+1$. On the curve l_2 one of the multipliers ρ_1 of the cycle Γ^+ is equal to -1. A "soft" period-doubling bifurcation of the limit cycle Γ^+ takes place on this curve. The curve l^1_{cr} corresponds to a transition to chaos due to the period-doubling bifurcation of the limit cycle Γ^+. On the curve l^2_{cr} a smooth transition into chaos resulting from the loss of smoothness of the torus $T^2(\alpha,\beta)$ is initiated. The bifurcation curve l_h in Fig.2 corresponds to a homoclinic trajectory spawned by an intersection of the stable and the unstable manifolds of the saddle-type limit cycle Γ^-. To the right of the curve l_h, the torus $T^2(\alpha,\beta)$ does not exist.

There is some peculiarity in the birth of the torus $T^2(\alpha,\beta)$ in the vicinity of the bifurcation curve l_0; namely, upon crossing the curve l_0 ($\alpha=1$) from the left to the right in Fig.2 an ergodic torus is born abruptly, i.e. the associated waveform $x(t)$ suddenly changes from a periodic to a quasi-periodic function defined by two independent and incommensurable frequencies ω_1 and ω_2 with finite amplitudes. The phase portrait and the power spectrum of a typical torus $T^2(\alpha,\beta)$ near the curve l_0 are shown in Fig.3. The oscillation dynamics is nonlinear in principle even in the vicinity of the torus birth curve l_0, as verified by the power spectrum of $x(t)$. Observe that besides the two basic frequencies $\omega_1=0.05$ and $\omega_2=1.00$, various harmonics $m\omega_1$ and $n\omega_2$ and combination frequencies $\omega_{m,n}=m\omega_1\pm n\omega_2$ are clearly seen in the spectrum at least for $n\leq3$, $m\leq6$ (in Fig.3 only the part of the spectrum up to the second harmonics of ω_1 is shown).

Observe that the phenomenon of an abrupt appearance of both quasi-periodic and periodic oscillations is a typical property of piecewise-linear systems.

There is another peculiarity in the bifurcation diagram of system (3). As shown in the diagram in Fig.2, the 1:6 phase-locked region does not originate from a single point K, but rather over an interval $[K_1,K_2]$. Such a situation can not take place in smooth dynamical systems. Hence, this phenomenon is also a consequence of the piecewise-linear nature of $f(\xi)$ in (4). Figure 4 shows a typical picture associated with a periodic limit cycle Γ^+ having a Poincare rotation number of 1:6. Observe that there are 6 points in the Poincare map corresponding the cross section at z=0. Observe that only harmonics $n\omega_2$ of the minimal frequency ω_2=0.22 exist in the power spectrum. The phase-locked condition is $\omega_1=6\omega_2$.

To investigate the main points of the Afraimovich-Shilnikov torus breakdown theorem it is sufficient to investigate the dynamics of system (3) in the neighborhood of the 1:6 phase-locked region.

5. Torus breakdown and routes of transition to chaos

Let us investigate in detail the evolution of the oscillatory regimes in system (3) along the routes A,B,C predicted by the Afraimovich-Shilnikov theorem and determine the conditions where chaos originates from a torus breakdown.

A. *Torus breakdown due to the period-doubling bifurcation of phase-locked limit cycle* Γ^+ *on the torus* $T^2(\alpha,\beta)$. Our computer simulation in Fig.2 shows that if we move along the direction PA, we would cross the period-doubling bifurcation curve l_2 of the limit cycle Γ^+. The torus is destroyed on the curve l_2. This destruction is preceded by a loss of smoothness as evidenced from the distortion of an invariant curve in the Poincare map. The loss of smoothness is caused by oscillations of the unstable separatrix of the saddle point as it approaches the stable node point. We should note that a curve l exists in the phase-locked region at a very short distance from l_1 inside of the Arnold tongue, where the multipliers of cycle Γ^+ change from real to complex-conjugate on this curve[1]. A rotation leads to the loss of smoothness, thereby resulting in the destruction of the torus on the curve l_2. We shall come back to this phenomenon in the next section.

The cascade of period-doubling bifurcations of the stable limit cycle Γ^+ and the transition to chaos occur above the curve l_2. But the route of period-doubling sequence is finite in system (3), where the transition $T_0 \Rightarrow 2T_0 \Rightarrow 4T_0 \Rightarrow \text{Chaos}_1$ takes place (here T_0 is the period of the limit cycle Γ^+). Some results are presented in Fig.5 to illustrate this transition. The Poincare sections and the corresponding power spectra clearly show the typical scenario as we move and cross the curve l_2 transversally into the Chaos$_1$ region. Observe that the finite number of period-doubling bifurcations is not due to any computation difficulties, but is a genuine phenomenon caused by the piecewise-linear character of the function $f(\xi)$ in system (3). The Lyapunov exponents in the Chaos$_1$ region in the vicinity of curve l^1_{cr} in Fig.2 for α=23.06, β=2.25 are:

$$L_1 = +0.021,\ L_2 = -0.00009,\ L_3 = -0.200 \tag{6}$$

This results in a corresponding value of the Lyapunov dimension $D_L = 2.104$.

B. *Torus breakdown due to the appearance of a homoclinic trajectory of the saddle -type limit cycle Γ^-: the abrupt transition to chaos*. Our goal here is to construct

---

[1]This curve is not plotted in Fig.2 because it practically coincides with the curve l_1 inside the phase-locked region.

the bifurcation curve corresponding to a tangency of the stable and the unstable manifolds of the saddle-type limit cycle Γ^- on the torus. This problem can be solved directly by calculating the unstable and the stable separatrices of the corresponding saddle points in the Poincare section, as it was done, for example, in Ref.13. However, this method is rather time consuming and we therefore devise a different method. Our method consist of adding an additive source of white noise to Eq.(3); namely,

$$\frac{dx}{dt} = -\alpha f(y-x) + \xi_1(t)$$

$$\frac{dy}{dt} = -z-f(y-x) + \xi_2(t) \qquad (7)$$

$$\frac{dz}{dt} = \beta y + \xi_3(t)$$

where $<\xi_i(t)>=0$, $<\xi_i(t)\xi_i(t+\tau)>=D\delta(t-\tau)$.

Let us analyze the Poincare maps and the power spectra of the stable limit cycles Γ^+ in the phase-locked region under a small noise perturbation of intensity D=0.01. This noise excitation causes the unstable manifolds to become visible[12,13]. Moreover, as a criterion for the existence of a homoclinic trajectory we can calculate the condition which gives rise to a positive Lyapunov exponent of a solution of the perturbed system (7)[22]. So, starting from the inside of the phase-locked region and identifying the transition of the maximal Lyapunov exponent from a negative to a positive value, it is possible to construct the curve l_h[22,23].

Figure 6 shows the computed results for increasing values of the parameter α inside the phase-locked region but in the vicinity of l_1. Using the calculated maximal Lyapunov exponent, the curve l_h was obtained and plotted in the diagram of Fig.2. Evidently, results of such computations depend, to some extent, on the noise intensity *D*. However, the results derived from this calculation can be used for a qualitative interpretation of the dynamics.

Hence, the torus $T^2(\alpha,\beta)$ does not exist in the region bounded by the curves l_h and l_1. It was destroyed because of the emergence of a stable homoclinic structure, which is not a part of the attractor. The stable limit cycle Γ^+ is the only attractor here. Its Poincare section and power spectrum are shown in Fig.4. However, if we leave this region in the direction B' (see diagram of Fig.2), then the limit cycle Γ^+ merges with the limit cycle Γ^- and disappears upon crossing the curve l_1 thereby giving birth to chaos abruptly in the region labelled Chaos_2. Fig.7 illustrates the phenomenon of the abrupt appearance of chaos in this region. The result of our computations illustrating the exit from the phase-locked region in the neighborhood of corresponding points from Fig.6 are presented here. Hence, the abrupt transition to chaos, corresponding to the scenario B' of the Afraimovich-Shilnikov theorem on torus breakdown, also takes place in the system (3).

C. *Torus breakdown due to the loss of smoothness: soft transition to chaos.* In the above scenarios A and B, the torus is destroyed before chaos is born. In case A, the torus is destroyed on the curve l_2 and the transition to chaos is connected with the period doubling bifurcations of a stable limit cycle, which no longer lie on the torus. In case B, the torus is destroyed on the curve l_h. The transition to chaos in this case occurs by crossing the curve l_1, when the stable limit cycle Γ^+ disappears, and the homoclinic structure forms a quasi-attractor. In case C we have a different situation. A direct transition $T^2(\alpha,\beta) \Rightarrow \text{Chaos}_2$ takes place here! The loss of smoothness of the torus $T^2(\alpha,\beta)$ is a necessary condition for this direct transition.

As it is seen from the diagram in Fig.2, the bifurcation curve l_h starts near a point $\alpha \cong 5.2$, $\beta \cong 0.38$. For $\alpha \leq 5$ the torus exist. The torus $T^2(\alpha,\beta)$ loses its smoothness on its approach to the curve l_1 inside of the 1:6 phase-locked region. The exit from the 1:6 phase-locked region through the curve l_1 leads to a soft transition to chaos here. Figure 8 illustrates this mechanism. The exit from the 1:6 phase-locked region near the homoclinic curve l_h, but below it, results in a "soft" birth of Chaos$_2$. Although the maximal Lyapunov exponent $L>0$ in this case, it is less than the Lyapunov exponent in case B. The Poincare map of the attractor shown in Fig.8 is displayed by an almost smooth invariant closed curve. The mechanism of destruction is the loss of smoothness here.

As a final remark, although the transition from the phase-locked torus to an ergodic one (the route C' in the diagram of Fig.1) exists in the 1:6 phase-locked region in Fig.2, it is not easy to pinpoint the exact parameters (α,β) for this situation in practice. The reason is as follows. First, inside the 1:6 phase-locked region the multipliers of the limit cycle Γ^+ are complex-conjugate numbers practically everywhere along the boundary of the tongue near the curve l_1. The rotation of the invariant manifold of Γ^- in the vicinity of Γ^+ caused by the complex-conjugate multipliers led to the loss of smoothness of the torus. Second, in view of the location of the curve l_h, the critical curve l^2_{cr} representing the boundary between the ergodic torus and chaos is very near to the curve l_1, but is located outside of the phase-locked region. Therefore, the probability of a transition from the phase-locked region to chaos, or from the same regime into another phase-locked region is very high (see the region near l^2_{cr} in Fig.2).

6. Concluding remarks

The main result of this paper is as follows. All three mechanisms responsible for the breakdown of the two-dimensional torus are realized in the autonomous three-dimensional system (3) with piecewise-linear characteristics $f(\xi)$. The first mechanism is the destruction of the torus due to the loss of stability of the phase-locked limit cycle Γ^+ via period doubling on curve l_2 (Fig.2). The second mechanism is the destruction of the torus $T^2(\alpha,\beta)$ caused by the effect of the homoclinic tangency of the stable and the unstable manifolds of the saddle-type limit cycle Γ^- on the torus (the curve l_h in Fig.2). The third mechanism is the destruction of the torus due to the loss of the smoothness on the curve l_1 below the point of intersection between the curves l_h and l_1 (see route C in Fig.2).

All three routes to chaos are realized due to torus breakdown: (1) transition via a cascade of period-doubling bifurcations (route A), (2) abrupt transition to chaos in the homoclinic region via saddle-node bifurcation of the limit cycle Γ^+ (route B') and (3) soft transition to chaos due to the loss of torus smoothness (route C).

These results give evidence that, at least from the experimental point of view, all conclusions of the Afraimovich-Shilnikov theorem on torus breakdown proved for smooth dynamical systems are also applicable to the piecewise-linear system (3). The non-smoothness of the function $f(\xi)$ in (3) generates some pecularities in the system dynamics, which were identified here. However, these pecularities do not have any significant influence on the basic aspects of the bifurcation phenomena.

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Figure captions

Fig.1. The qualitative illustration of bifurcations of the two-dimensional phase-locked torus breakdown. The region of the torus regime is bounded by curves l_1, l_h and l_2 (the shaded region). Routes related to transitions to chaos on the parameter plane are labelled A, B' and C.

Fig.2. The experimental bifurcation diagram of the system (3) illustrating the Afraimovich-Shilnikov theorem: (a) the complete diagram, (b) the fragment of the diagram for $\alpha\leq 6$, $\beta\leq 0.6$. The curve l_2' is related to the period-doubling bifurcation of the phase-locked double cycle, $[K_1,K_2]$ is the segment of the curve l_0 related to the resonance 1:6 on the torus. All other notations are the same as in Fig.1.

Fig.3. The torus $T^2(\mu)$ in the system (3) near its birth bifurcation and the power spectrum calculated for the variable $x(t)$.

Fig.4. The Poincare section and the power spectrum of the phase-locked cycle Γ^+ on the torus $T^2(\mu)$ in the phase-locked 1:6 region.

Fig.5.Transition to Chaos_1 due to period-doubling bifurcations under motion in the direction A of the bifurcation diagram of the system (3) (Fig.2,a).

Fig.6. Poincare sections and corresponding power spectra of the phase-locked cycle Γ^+ of the system (3) in the presence of noise. Visualization of homoclinic structures.

Fig.7. An illustration of the abrupt transition to Chaos_2 in the system (3) (route B' in Fig.2,a).

Fig.8. An illustration of the soft transition to Chaos_2 due to the loss of the torus $T^2(\mu)$ smoothness (route C in Fig.2,b).

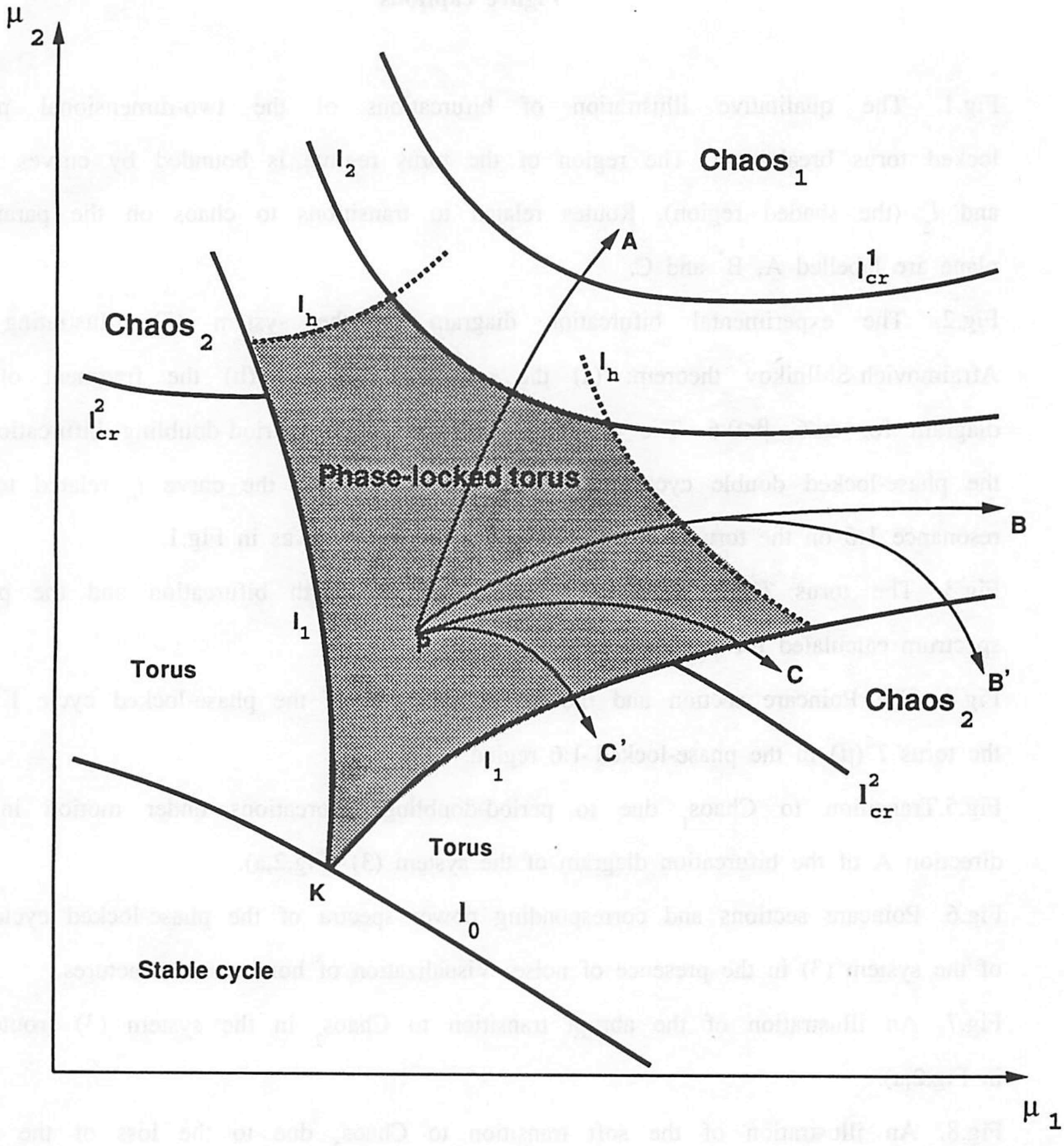

μ 2
μ 1
Chaos 1
Chaos 2
Chaos 2
l 2
l h
l h
l cr 1
l cr 2
l cr 2
l 1
l 1
l 0
A
B
B'
C
C'
K
Phase-locked torus
Torus
Torus
Stable cycle

Fig.1

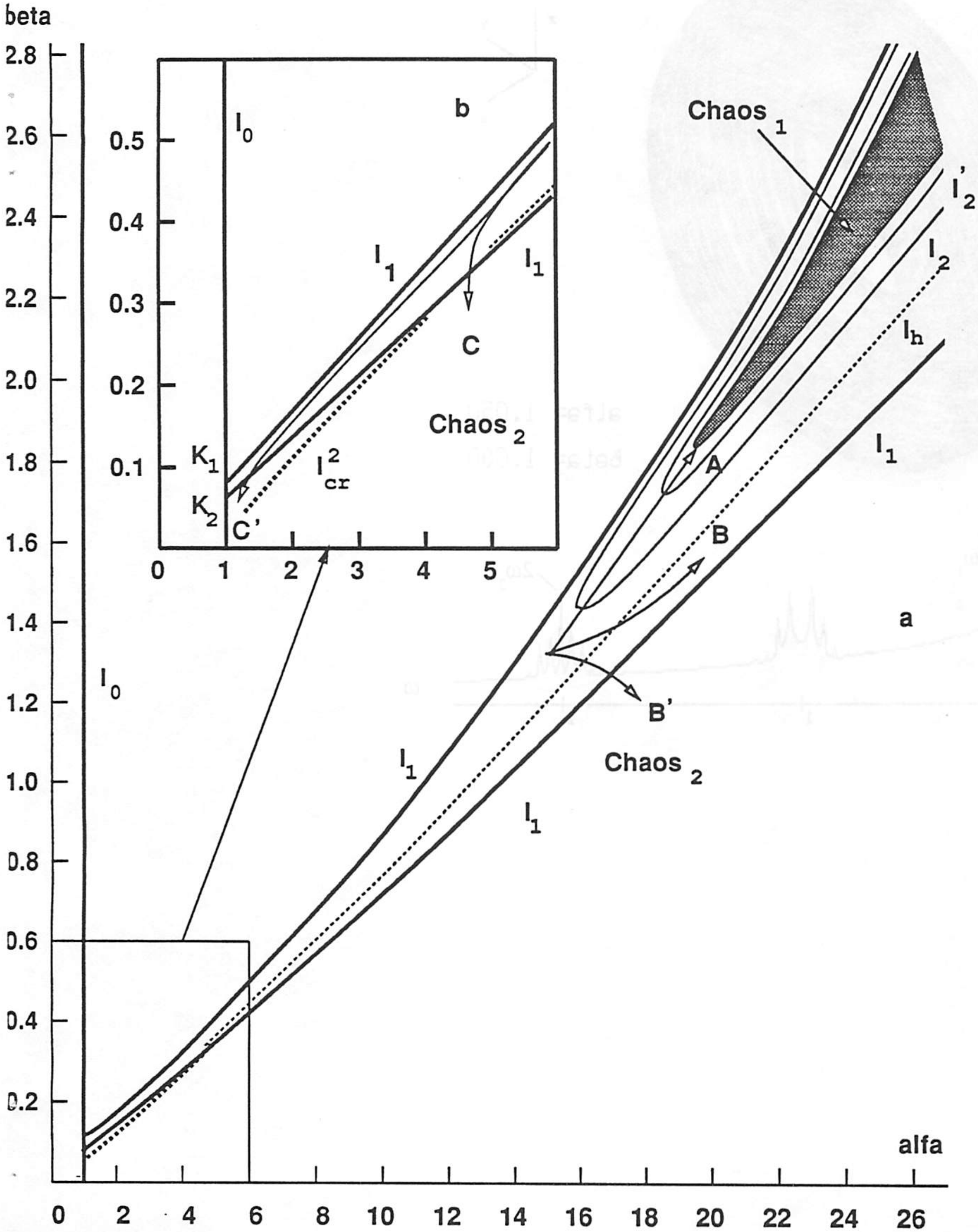
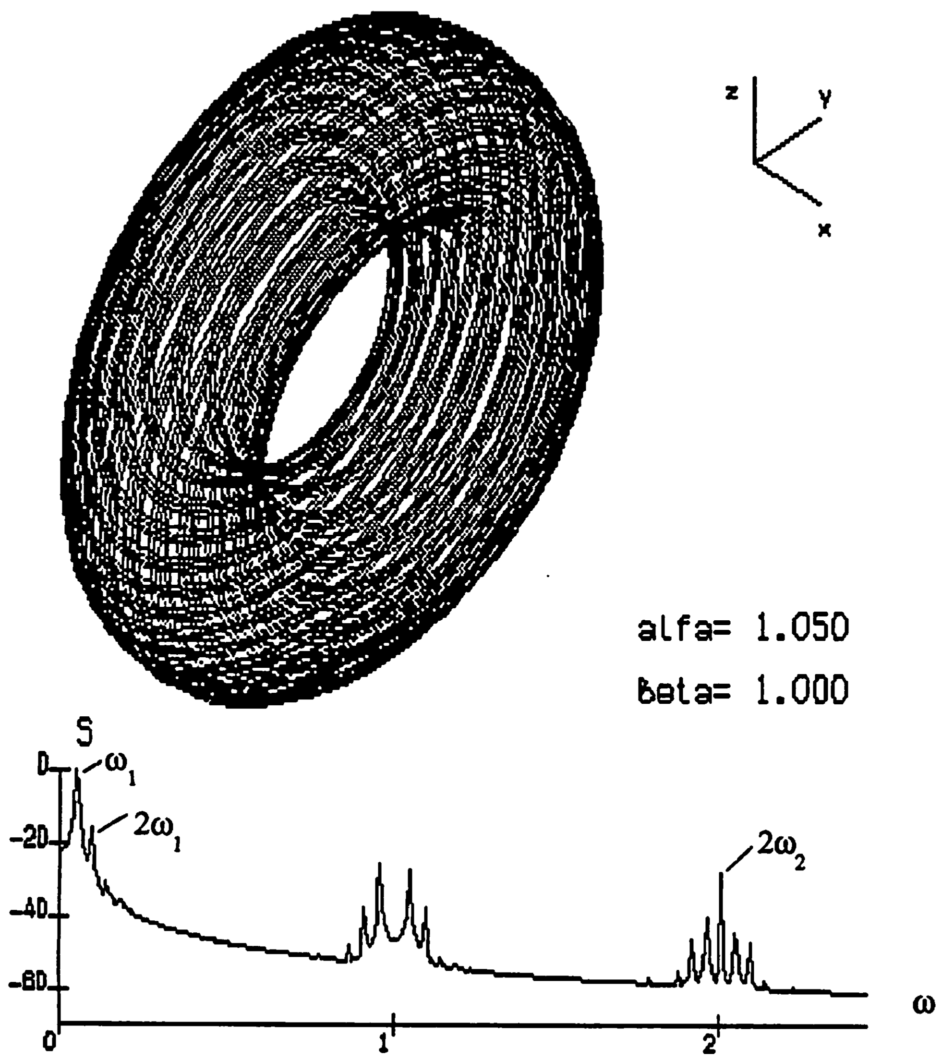
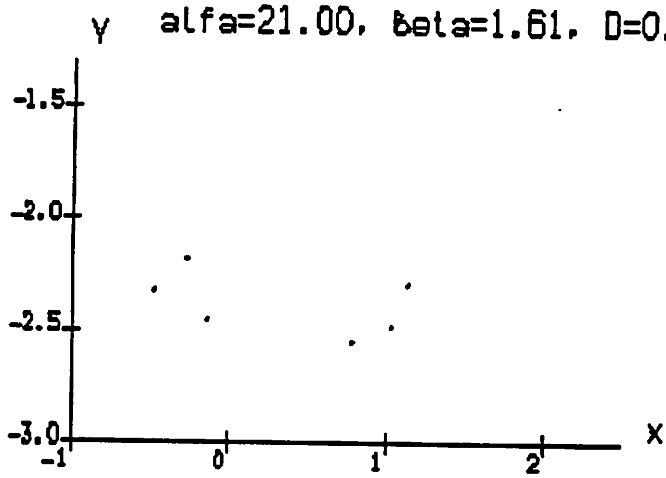

beta
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
alfa
a
b
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
l_0
l_1
l_2
l_2'
l_h
l_{cr}^2
K_1
K_2
Chaos $_1$
Chaos $_2$
A
B
B'
C
C'

Fig. 2

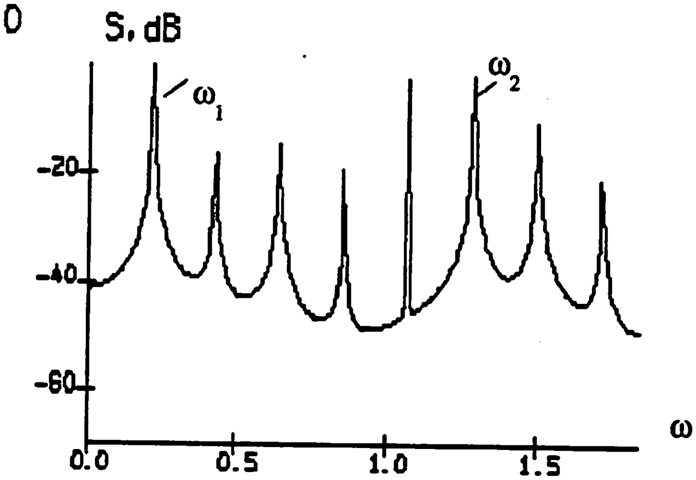


z
y
x
alfa= 1.050
beta= 1.000
S
0
-20
-40
-60
ω_1
$2\omega_1$
$2\omega_2$
ω
0
1
2

Fig.3



Y
alfa=21.00, beta=1.61, D=0.0
-1.5
-2.0
-2.5
-3.0
-1
0
1
2
X



S, dB
ω1
ω2
-20
-40
-60
0.0
0.5
1.0
1.5
ω

Fig. 4

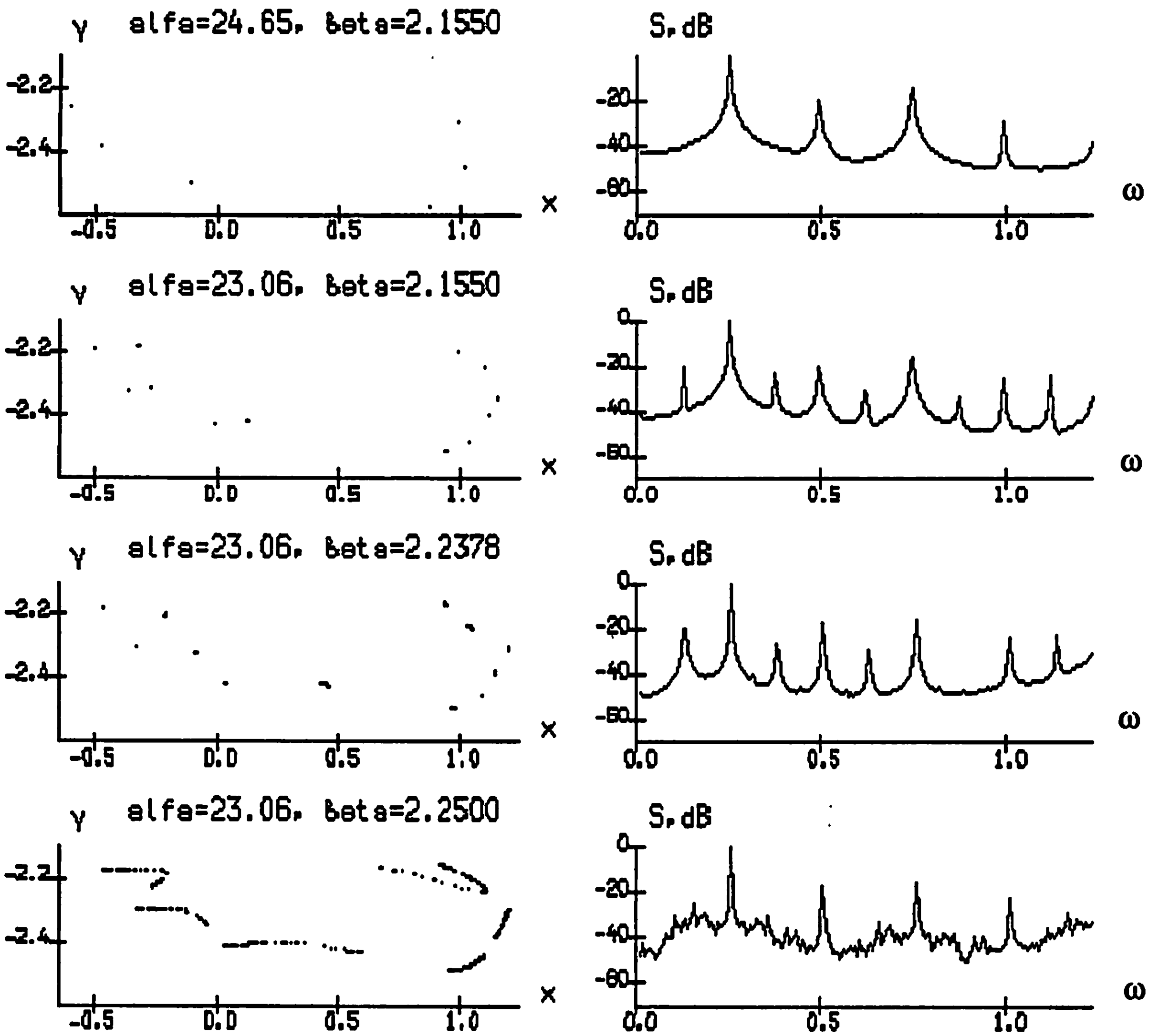

y
alfa=24.65, beta=2.1550
-2.2
-2.4
-0.5
0.0
0.5
1.0
x
S, dB
-20
-40
-60
0.0
0.5
1.0
ω
y
alfa=23.06, beta=2.1550
S, dB
y
alfa=23.06, beta=2.2378
S, dB
y
alfa=23.06, beta=2.2500
S, dB

Fig.5

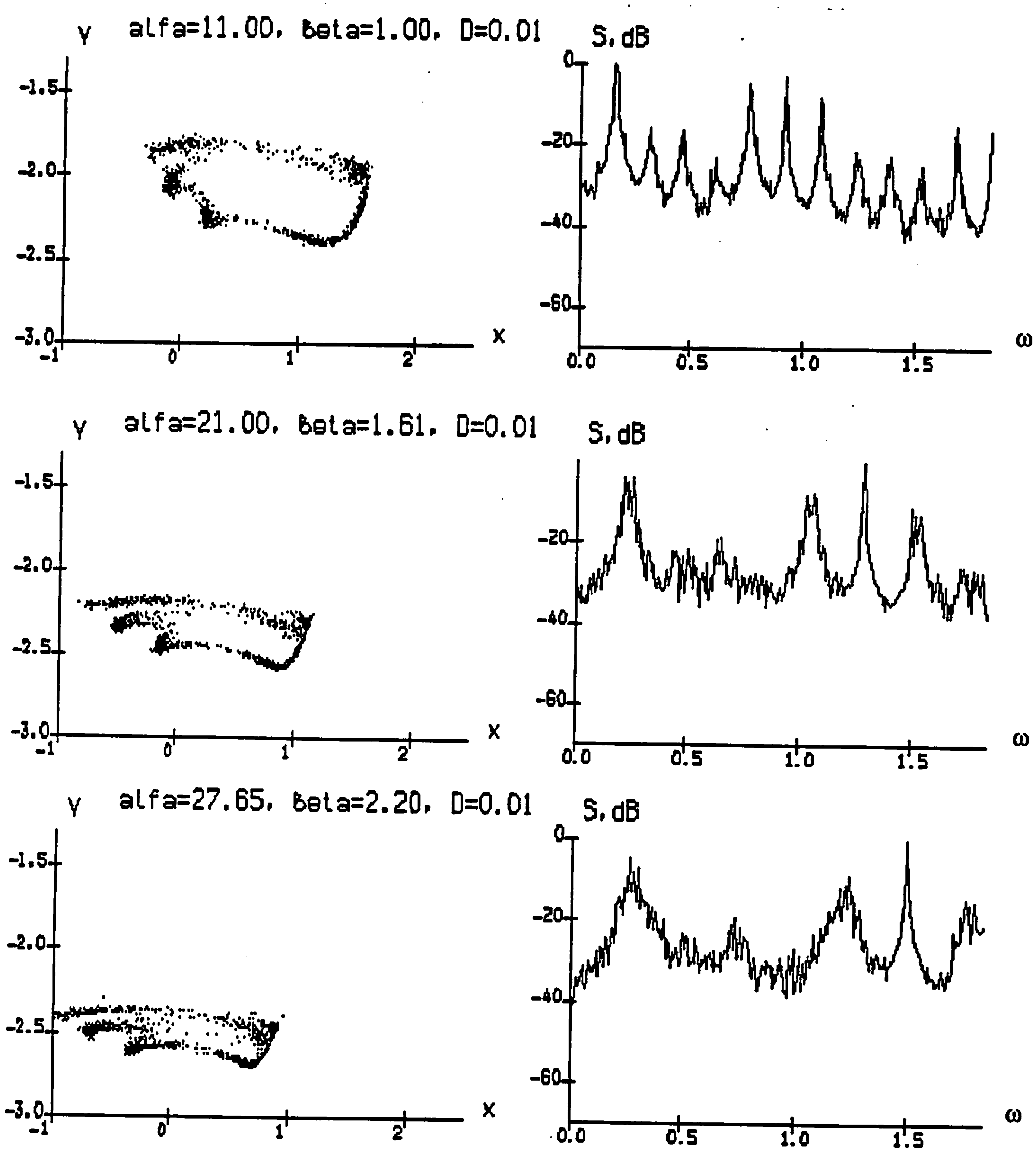
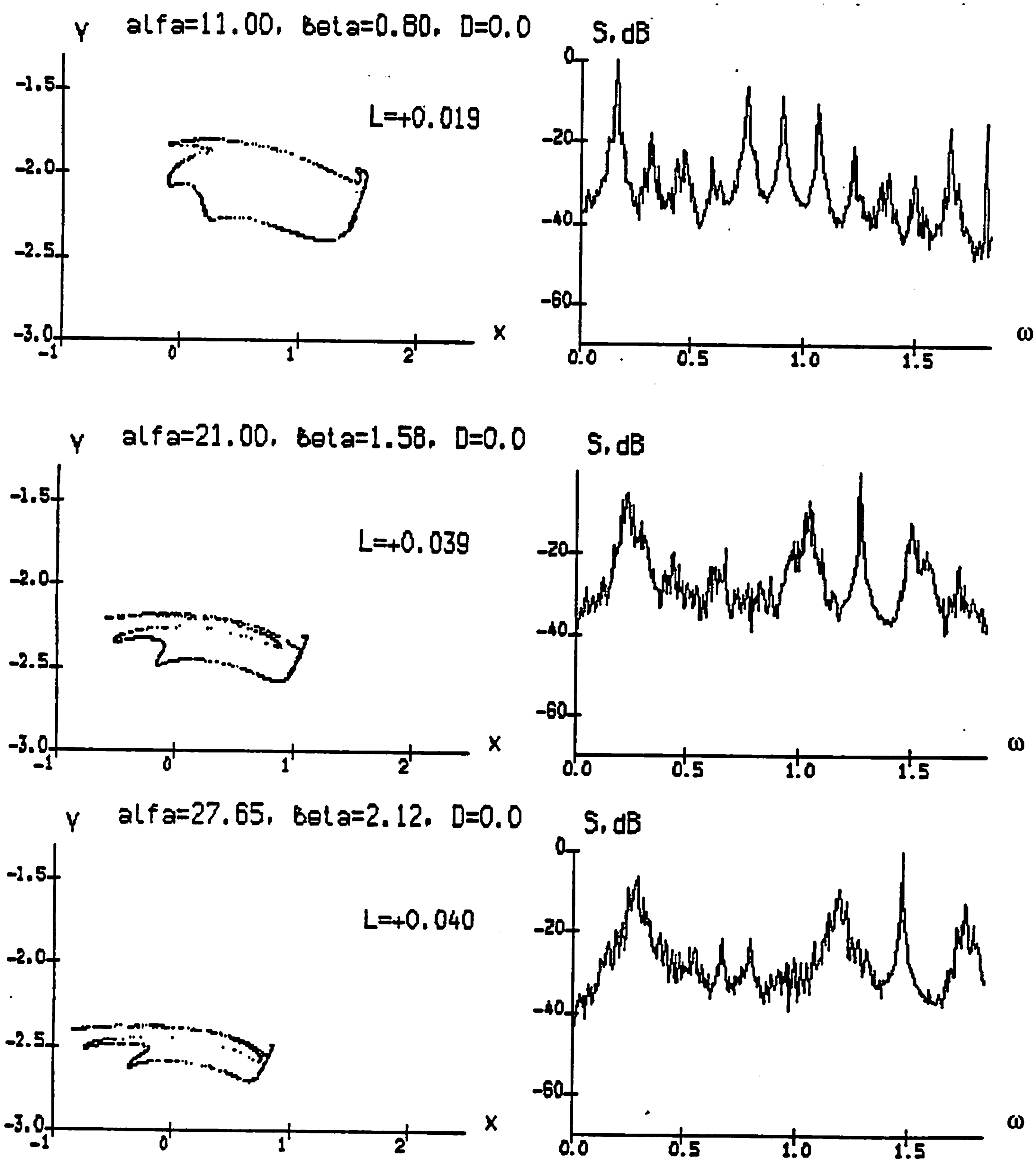

y alfa=11.00, beta=1.00, D=0.01
S, dB
-1.5
-2.0
-2.5
-3.0
-1
0
1
2
x
0
-20
-40
-60
0.0
0.5
1.0
1.5
ω
y alfa=21.00, beta=1.61, D=0.01
S, dB
y alfa=27.65, beta=2.20, D=0.01
S, dB

Fig. 6



Y
alfa=11.00, beta=0.80, D=0.0
L=+0.019
-1.5
-2.0
-2.5
-3.0
-1
0
1
2
X
S, dB
0
-20
-40
-60
0.0
0.5
1.0
1.5
ω
Y
alfa=21.00, beta=1.58, D=0.0
L=+0.039
-1.5
-2.0
-2.5
-3.0
-1
0
1
2
X
S, dB
-20
-40
-60
0.0
0.5
1.0
1.5
ω
Y
alfa=27.65, beta=2.12, D=0.0
L=+0.040
-1.5
-2.0
-2.5
-3.0
-1
0
1
2
X
S, dB
0
-20
-40
-60
0.0
0.5
1.0
1.5
ω

Fig. 7

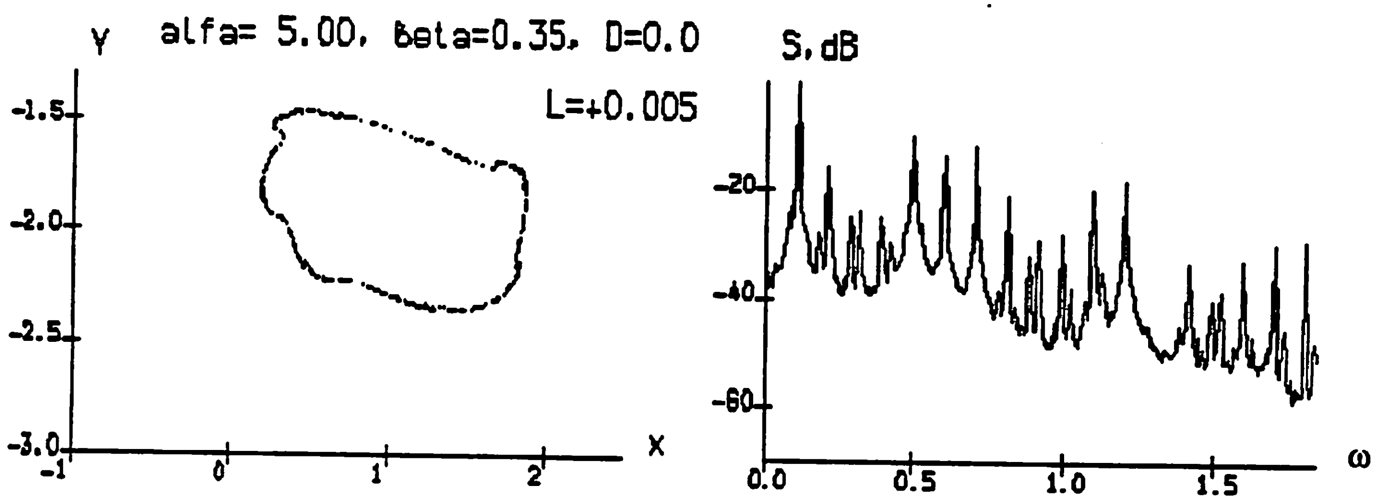

alfa= 5.00, beta=0.35, D=0.0
L=+0.005
Y
-1.5
-2.0
-2.5
-3.0
-1
0
1
2
X
S, dB
-20
-40
-60
0.0
0.5
1.0
1.5
ω

Fig. 8