# DISCONTINUOUS ONE-DIMENSIONAL MAPS FROM THE GENERALIZED CHUA EQUATIONS

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Memorandum No. UCB/ERL M92/80

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# Discontinuous One-dimensional Maps from the Generalized Chua Equations

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September 15, 1992

**Abstract**

We present fifteen examples of one-dimensional maps that are derived from the generalized Chua equations. These maps illustrate the diversity of one-dimensional maps that arise from the Chua equations.

## 1 Introduction

This paper derives one-dimensional maps from a class of generalized Chua equations we call the restricted double scrolls. Section 2 defines the restricted double scroll and derives a canonical form that will be used to derive the one-dimensional maps. Section 3 discusses the single scroll mechanism. Section 4 illustrates fifteen one-dimensional maps derived on the basis of the analysis in Sec. 2.

## 2 The Restricted Double Scroll

The restricted double scroll is a subset of the type I generalized Chua equations, [Brown, 1992]. We add the modifier "restricted" to indicate that the dynamics of this class contains only the double scroll dynamics and not the broader class of dynamics that arises from Chua's circuit equations, [Chua, 1992]. The general form of the restricted double scroll is given by the equation:

$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \\ a_{21} & a_{22} & a_{23} \\ \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x - (q_1 + r_1 \operatorname{sgn}(u)) \\ \\ y - (q_2 + r_2 \operatorname{sgn}(u)) \\ \\ z - (q_3 + r_3 \operatorname{sgn}(u)) \end{pmatrix} \tag{1}$$

where the matrix $M$, defined by

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \\ a_{21} & a_{22} & a_{23} \\ \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

has eigenvalues $-\gamma$, and $\alpha_1 \pm j\,\alpha_2$, where $\gamma > 0$, and $\alpha_1 > 0$, and $j = \sqrt{-1}$; $q_i, r_i$ are any real numbers. The variable $u$ defines a plane in three space:

$$u = V \bullet (X - P)$$

where

$$X = \begin{bmatrix} x \\ \\ y \\ \\ z \end{bmatrix}$$

and $V$ and $P$ are any three-dimensional vectors, and "$\bullet$" indicates vector inner product. We can write this more briefly as

$$\dot{X} = M\,(X - (Q + R \operatorname{sgn}(V \bullet (X - P))))$$

where

$$Q = \begin{bmatrix} q_1 \\ \\ q_2 \\ \\ q_3 \end{bmatrix}$$

$$R = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

and

$$P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

We require that the vector $Q$ must lie in the plane defined by $u$, i.e., $V \bullet (Q - P) = 0$. By a change of coordinates as done in [Brown,1992] we can bring $M$ into the form:

$$M = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & -\gamma \end{bmatrix}$$

By a rotation in the $x - y$ plane we can change the vector $V$ to

$$V = \begin{bmatrix} m_1 \\ 0 \\ m_3 \end{bmatrix}$$

and by factoring $m_1$ out of $V$ and including it in $R$ we may assume that $V$ has the following form:

$$V = \begin{bmatrix} -1 \\ 0 \\ m \end{bmatrix}$$

After these changes we have the following

**Canonical Restricted Double Scroll:**

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & -\gamma \end{bmatrix} \begin{pmatrix} x - (q_1 + r_1 \operatorname{sgn}(u)) \\ y - (q_2 + r_2 \operatorname{sgn}(u)) \\ z - (q_3 + r_3 \operatorname{sgn}(u)) \end{pmatrix} \tag{2}$$

where $u = m(z - p_2) - (x - p_1)$, and the transformed vector $Q$ also lies in this plane. Of course the constants in this transformed equation are different from those of the previous equation, but, for convienence we use the same set of notation.

Also note that the canonical equation is invariant under the transformation $F(X) = 2Q - X$, which is a flip through the point $Q$ which lies in the plane defined by $u$, called the transfer plane. This is essential if we are to derive the one-dimensional maps we seek. This symmetry amounts to saying that the vector field on one side of the plane $u$ is topologically conjugate to the vector field on the other side by the conjugacy $F$.

Clearly this decoupled set of equations may be solved on each side of the plane $u = 0$, and the actual solution can be generated from these two separate sets of solutions. However, since this equation has a symmetry, we can do more, just as in [Brown, 1992]. We may study the dynamics of Eq.(1) by using only the vector field on one side, say where $u > 0$, and then applying the conjugacy to obtain the component of the solution on the other side. Following [Brown, 1992] we have the following situation:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & -\gamma \end{bmatrix} \begin{pmatrix} x - (q_1 + r_1) \\ y - (q_2 + r_2) \\ z - (q_3 + r_3) \end{pmatrix} \tag{3}$$

where we apply the map $F(X) = 2Q - X$ when the solution reaches the plane $u = 0$.

We may write down the solution of Eq.(3) as follows:

$$
\begin{aligned}
x &= q_1 + r_1 + \exp(s\,t)\,((x_0 - (q_1 + r_1))\cos(\omega\,t) + A\sin(\omega\,t)) && (4)\\
y &= q_2 + r_2 + \exp(s\,t)\,((y_0 - (q_2 + r_2))\cos(\omega\,t) - B\sin(\omega\,t)) && (5)\\
z &= q_3 + r_3 + (z_0 - (q_3 + r_3))\exp(-\gamma\,t) && (6)
\end{aligned}
$$

where,

$$A = \frac{a_{12}\,(y_0 - (q_2 + r_2)) + (a_{11} - s)(x_0 - (q_1 + r_1))}{\omega}$$

$$B_0 = (a_{12}(/y_0 - (q_2 + r_2)) - (s - a_{11})(x_0 - (q_1 + r_1)))(a_{11} - s)$$

$$B = \frac{\omega^2\,(x_0 - (q_1 + r_1)) - B_0}{a_{12}\omega}$$

$$s = \mathrm{tr}(M)/2$$

$$\omega = \sqrt{\det(M) - s^2}$$

The single scroll equations needed to obtain the one-dimensional maps we will illustrate are obtained by letting $\gamma \to \infty$. When $\gamma \to \infty$, $z \to q_3 + r_3$ and $u \to m(q_3 + r_3 - p_3) - (x - p_1)$. The line in the plane where we apply the translated flip map, $X \to 2Q - X$, is given by-,

$$x = m(q_3 + r_3 - p_3) + p_1$$

# 3 The Mechanism of the Single Scroll

The general single scroll equations are given by:

$$
\begin{aligned}
x &= q_1 + r_1 + \exp(s\,t)\,((x_0 - (q_1 + r_1))\cos(\omega\,t) + A\sin(\omega\,t)) && (7)\\
y &= q_2 + r_2 + \exp(s\,t)\,((y_0 - (q_2 + r_2))\cos(\omega\,t) - B\sin(\omega\,t)) && (8)
\end{aligned}
$$

where the curve at which the general flip is applied is given by:

$$x = m(q_3 + r_3 - p_3) + p_1$$

The genreal flip, $X \to 2Q - X$, reduces to $(x, y) \to (2q_1 - x, 2q_2 - y)$.

## 3.1 Misiurewicz's Single Scroll

The single scroll of Misiurewicz is obtained from the general restricted double scroll equations by setting:
$a_{11} = a_{22} = s$, $a_{21} = -a_{12} = 1$, $q_i = 0$, $r_3 = 1$, $p_i = 0$, $m = 1$, and $r_1 = a$, $r_2 = b, r_3 = 1$, where $a, b$ are arbitrary; $u = z - x$, and $Q$ is the origin. As a result we have the following restricted double scroll determined by Misiurewicz's form of the single scroll:

$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} s & -1 & 0 \\ \\ 1 & s & 0 \\ \\ 0 & 0 & -\gamma \end{bmatrix} \begin{pmatrix} x - a\,\mathrm{sgn}(u) \\ \\ y - b\,\mathrm{sgn}(u) \\ \\ z - \mathrm{sgn}(u) \end{pmatrix} \tag{9}$$

where $u = (z - x)$.

To obtain the solution of these equations from Eqs. (4),(5),(6) we need only set:

$$\begin{aligned} A &= -(y_0 - b) \\ B &= (x_0 - a) \\ \omega &= 1 \end{aligned}$$

The single scroll is obtained by letting $\gamma \to \infty$ in which case, $z \to 1$. Doing this we have the following single scroll equations:

$$x(\tau) = a + \exp(s\,\tau)\,((x_0 - a)\cos(\tau) - (y_0 - b)\sin(\tau)) \tag{10}$$
$$y(\tau) = b + \exp(s\,\tau)\,((y_0 - b)\cos(\tau) + (x_0 - a)\sin(\tau)) \tag{11}$$

Since, as $\gamma \to \infty$, $z \to 1$, the flip map is now applied when,

$$u = 1 - x = 0$$

or, in other words, when $x = 1$. The general flip, $X \to 2Q - X$, reduces to standard flip $(1, y) \to (-1, -y)$ for this case.

## 3.2 Invariant Regions and Fixed Points

The role of the $y$ coordinate $b$ for a fixed value of the $x$ coordinate $a$ is to define an invariant region. Invariant regions, as shown by Misiurewicz, are determined by the point where the single scroll is tangent to the line $x = m(q_3 + r_3 - p_3) + p_1$. We obtain this point by solving

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x - (q_1 + r_1) \\ y - (q_2 + r_2) \end{pmatrix} \tag{12}$$

for $dy/dx$.

Doing this gives the equation:

$$\frac{dy}{dx} = \frac{a_{21}(x - (q_1 + r_1)) + a_{22}(y - (q_2 + r_2)}{a_{11}(x - (q_1 + r_1)) + a_{12}(y - (q_2 + r_2)}$$

To obtain the vertical tangent to the line $u = 0$ we need to evaluate

$$\frac{dx}{dy} = 0$$

at $x = m(q_3 + r_3 - p_3) + p_1$. This requires that

$$a_{11}(x - (q_1 + r_1) + a_{12}(y - (q_2 + r_2) = 0$$

for this value of $x$. To simplify the illustration we choose $q_i = p_i = 0$, $r_1 = a, r_2 = b, r_3 = 1$ to get the equation for $y$,

$$y = b - (a_{11}(m - a)/a_{12})$$

This determines the point, $(m, y)$ at which the tangent line is $x = m$. If we take this as an initial condition and integrate forward in time to the place, $y_1$, where this curve next strikes the line $x = -m$ and then where it goes on to strike $x = m$, say as $y_2$, it defines the potential invariant region. In particular, if we choose $b$ so that $|y_2| \leq |y_1|$, we will have an invariant region.

We may make this invariant region maximal in some sense if we require that $|y_2| = |y_1|$. This means that the point $(-1, y_1)$ must be a fixed point for the one-dimensional map. We examine the fixed points for the special case where $a_{11} = a_{22}$, $a_{12} = 1 = -a_{21}$. We have the following equation for fixed points in this case:

$$\cos(\tau) - b\sin(\tau) = \cosh(s\,\tau) - a\sinh(s\,\tau)$$

When $a = -1$ we get the equation

$$\cos(\tau) - b\sin(\tau) = -\exp(s\,\tau)$$

And so there are infinitely many fixed points.

### 3.3 A Special Case: Exact Solutions

In some special cases we can obtain closed form equations for the inverse of the one-dimensional map. We begin with the equations of Misiurewicz:

$$\begin{aligned} x(\tau) &= a + \exp(s\,\tau)\,((x_0 - a)\cos(\tau) - (y_0 - b) + \sin(\tau)) && (13) \\ y(\tau) &= b + \exp(s\,\tau)\,((y_0 - b)\cos(\tau) + (x_0 - a)\sin(\tau)) && (14) \end{aligned}$$

In the special case where $a = x_0$ these equations reduce to

$$\begin{aligned} x(\tau) &= a + \exp(s\,\tau)(-1)(y_0 - b)\sin(\tau) && (15) \\ y(\tau) &= b + \exp(s\,\tau)(y_0 - b)\cos(\tau) && (16) \end{aligned}$$

When $x(\tau) = 1$, $a = -1$ we apply the flip map and so we have the equations for the one-dimensional map:

$$\begin{aligned} 1 &= -1 + \exp(s\,\tau)(-1)(y_0 - b)\sin(\tau) && (17) \\ y_f &= b + \exp(s\,\tau)(y_0 - b)\cos(\tau) && (18) \end{aligned}$$

Solving these equations for $y_0$, $y_f$ we have

$$\begin{aligned} y_0 &= b - 2\exp(-s\,\tau)\csc(\tau) && (19) \\ y_f &= b - 2\cot(\tau) && (20) \end{aligned}$$

This equation can be solved for $y_0$ to give

$$y_0 = b - 2\csc(\operatorname{arccot}(2(b - y_f)))\exp(-s\operatorname{arccot}(2(b - y_f)))$$

or more simply

$$y_0 = b - 2\sqrt{1 + 4(b - y_f)^2}\exp(-s\operatorname{arccot}(2(b - y_f)))$$

In this case we also have the following equation for the derivative:

$$\frac{dy_f}{dy_0} = \frac{4 + (b - y_f)^2}{(b - y_0)(2s + b - y_f)}$$

### 3.4 The Transfer Surface

The transfer surface in the restricted double scroll was taken as a plane in our opening discussion. There is nothing in that discussion that prevents us from taking this surface to be an arbitrary manifold. In the case of the single scroll this reduces to taking the transfer curve to be defined by a nonlinear equation, such as a polynomial or as a series of connecetd line segments having different slopes. In fact, the line segments need not be connected. In the figures presented, we use both linear and nonlinear transfer curves in order to illustrate the diversity of one-dimensional maps that are possible.

## 4 One-dimensional Maps

We now investigate the one-dimensional maps that can arise from the general single scroll. In all figures $Q = P = 0$. In each example we show two figures. The first figure is the one-dimensional map and the second figure is the restricted double scroll associated to the one-dimensional map. A complete explanation of the manner in which the one-dimensional maps are produced can be found in Brown [1992]. The equation appearing with each one-dimensional map is the equation for the single scroll. The equation appearing with the double scroll figures is the equation used to obtain the figure. For the single scroll, the general form of the initial conditions is needed to write the computer program for the one-dimensional map explaned in Brown [1992]. These are provided with each one-dimensional map figure. Also provided is the equation for the curve at which the flip map must be used.

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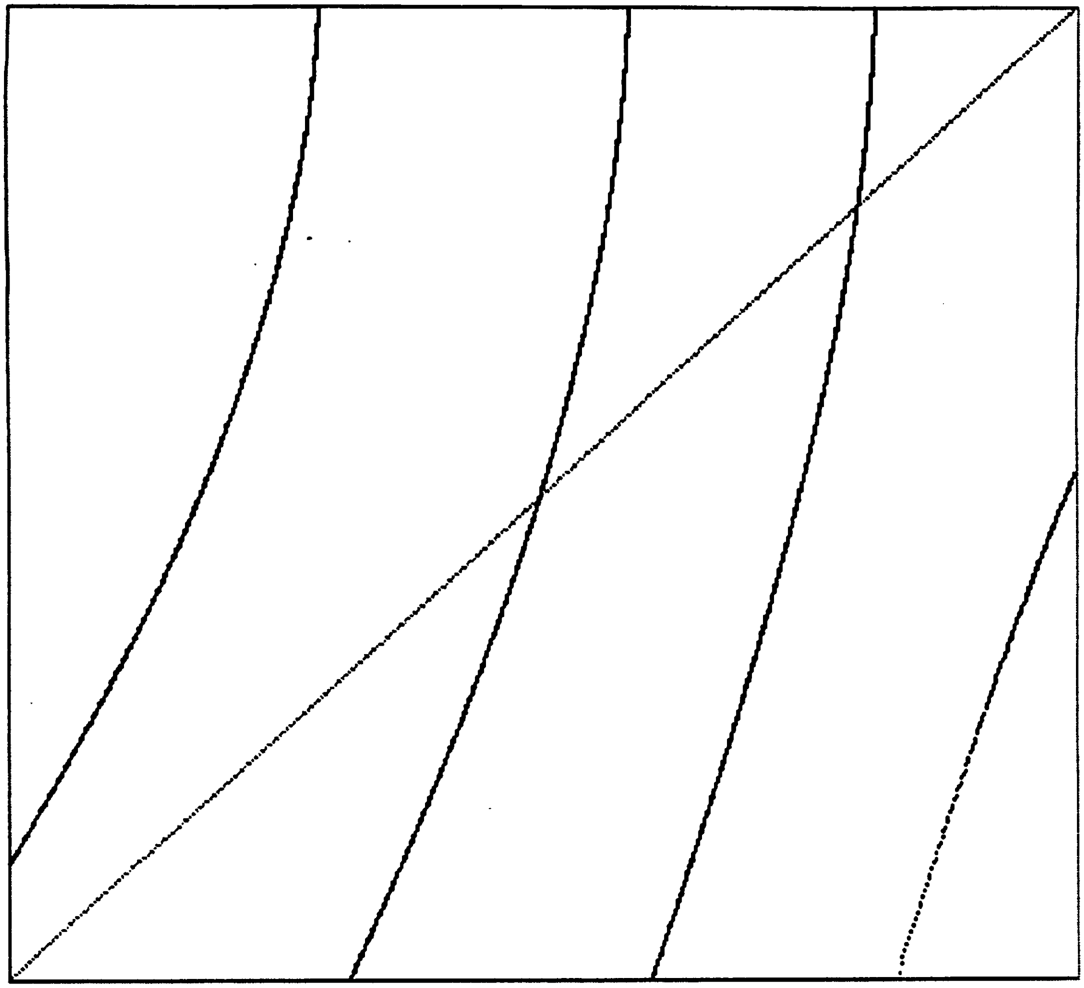
[1] Brown, R. [1992] "Generalizations of the Chua Equations," International Journal of Bifurcation and Chaos 2(4),

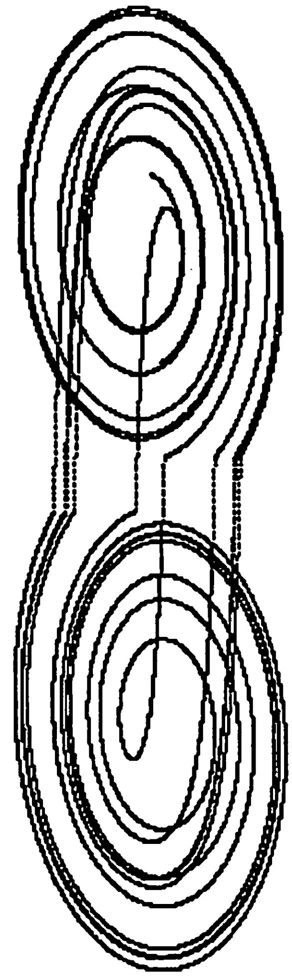
[2] Brown, R. & Chua, L. [1991] "Generalizing the Twist-and-Flip Paradigm," International Journal of Bifurcation and Chaos 1(2), 385-416.

[3] Misiurewicz, M. [1992] "Unimodal interval maps obtained from the modified Chua equations", International Journal of Bifurcation and Chaos 2(4),

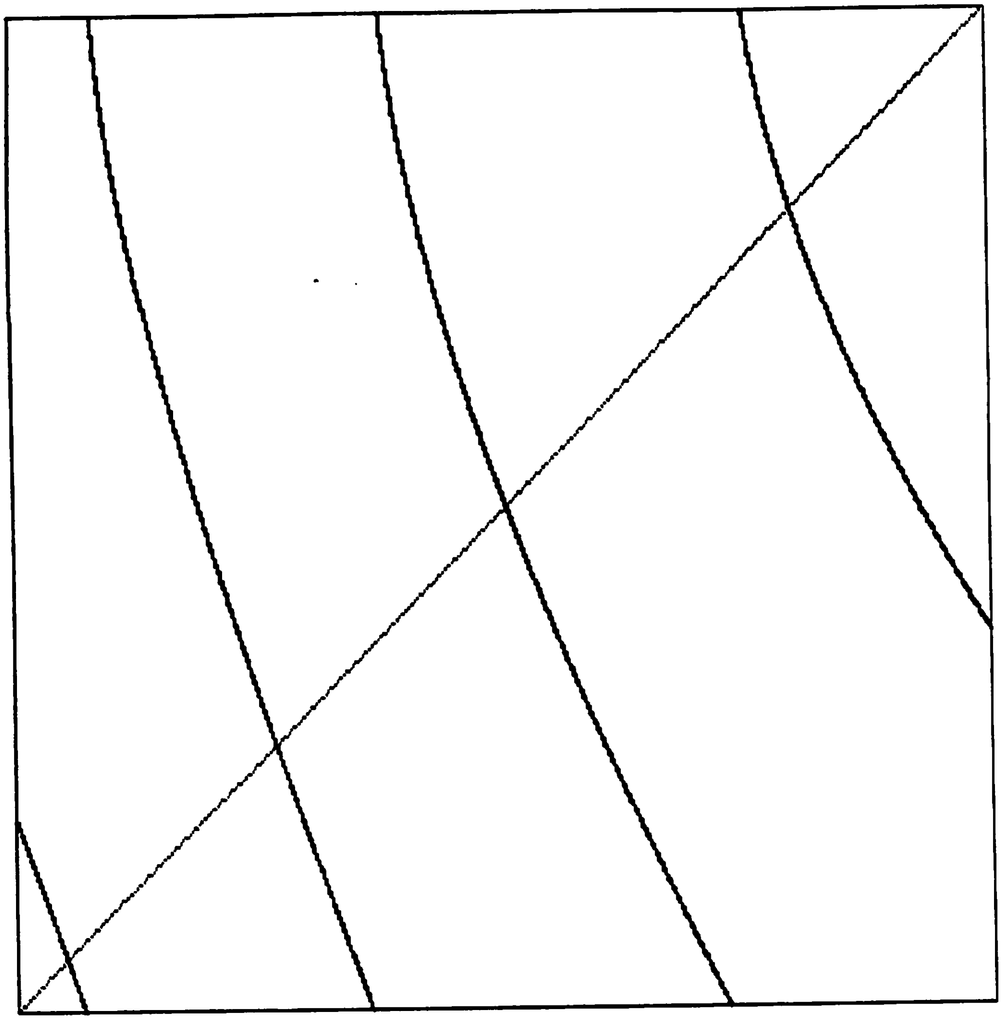


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0 & -8 \\ \\ 1 & 1/3 \end{bmatrix} \begin{pmatrix} x - 0.5 \\ \\ y + .05 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\mathrm{sgn}(x - y + 1) < 0$.

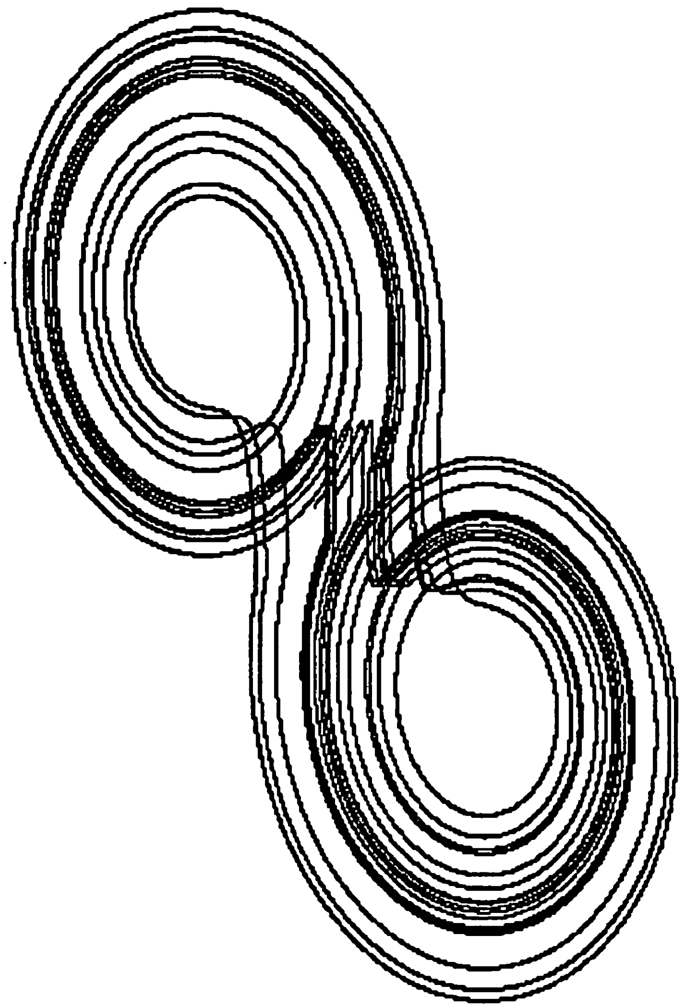


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0 & -8 & 0 \\ 1 & 1/3 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.5\,\mathrm{sgn}(x-y+z) \\ y + .05\,\mathrm{sgn}(x-y+z) \\ z - \mathrm{sgn}(x-y+z) \end{pmatrix}$$

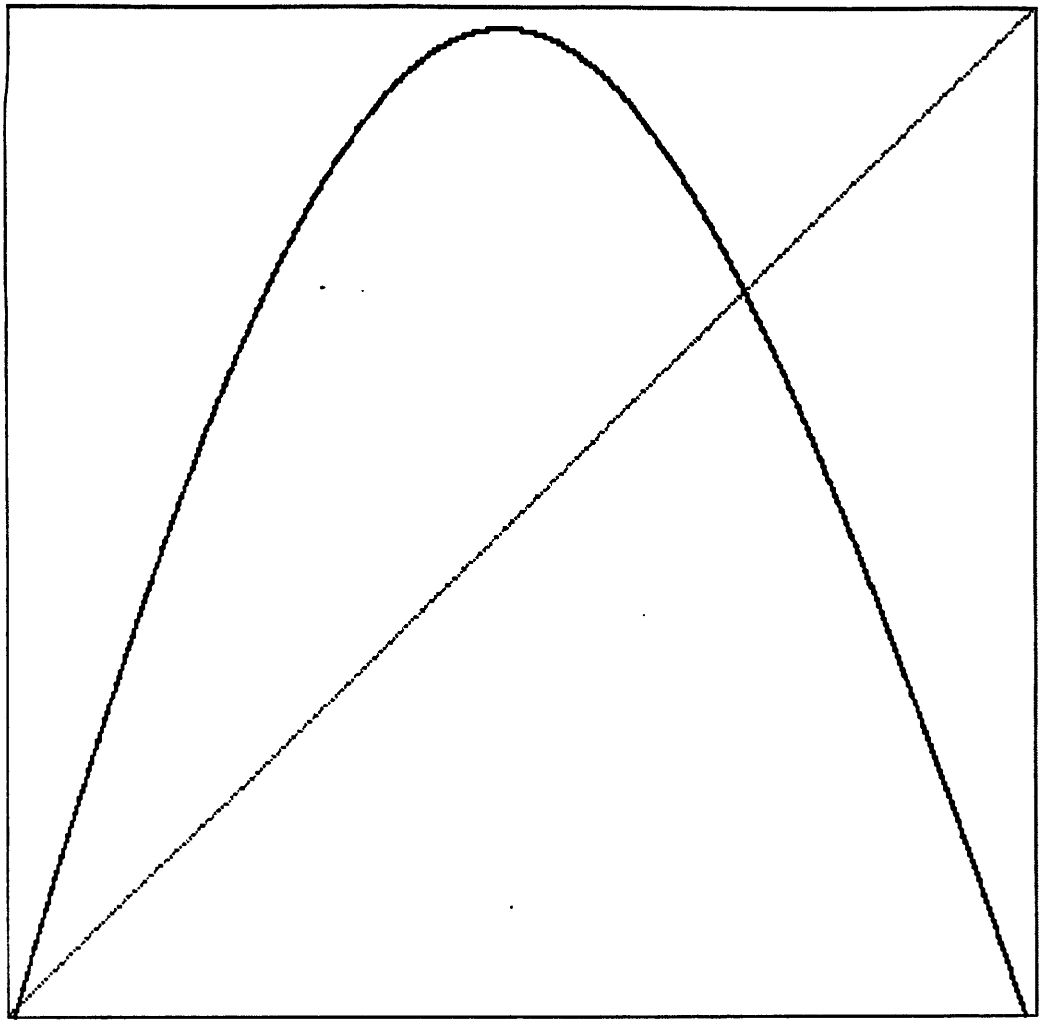


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0 & -5 \\ 1 & 0.3 \end{bmatrix} \begin{pmatrix} x-1 \\ y+1 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\mathrm{sgn}(x - y + 1) < 0$.

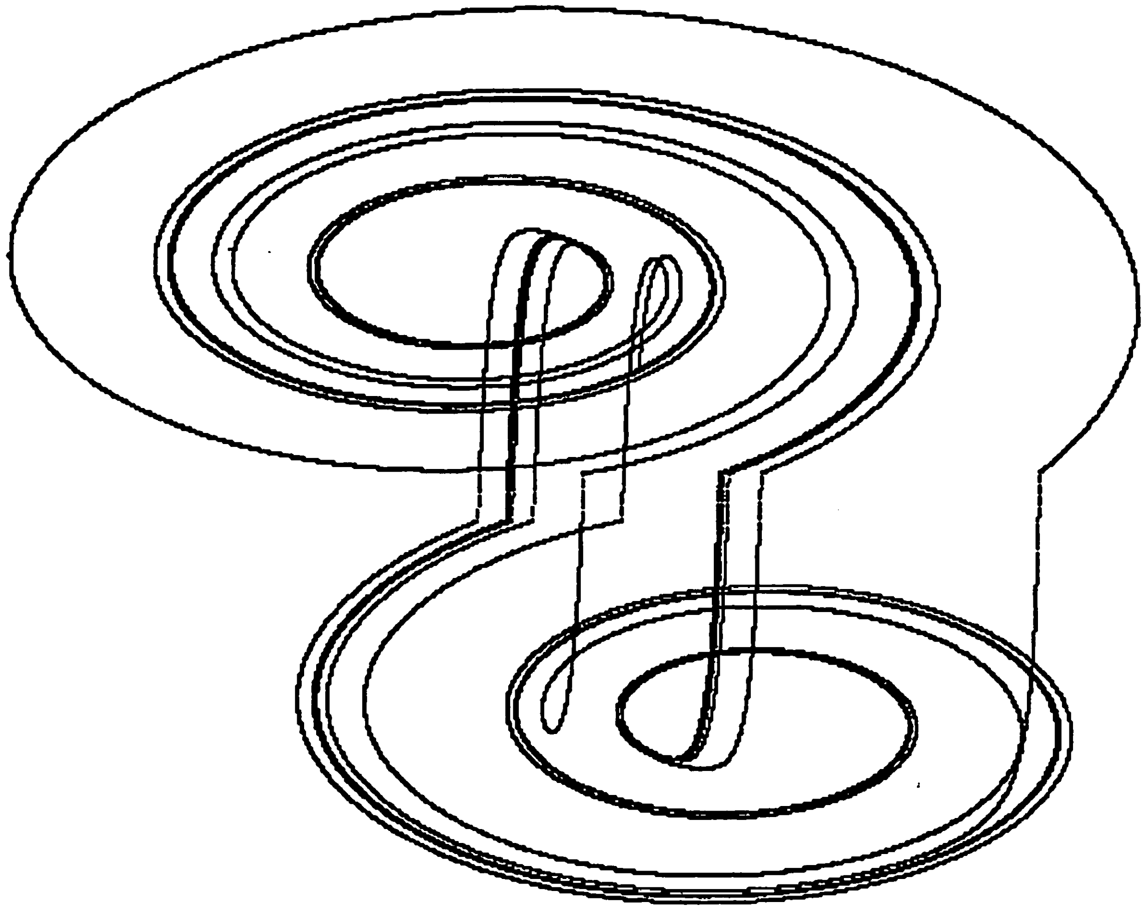


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0 & -5 & 0 \\ \\ 1 & 0.3 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - \mathrm{sgn}(x - y + z) \\ \\ y + \mathrm{sgn}(x - y + z) \\ \\ z - \mathrm{sgn}(x - y + z) \end{pmatrix}$$

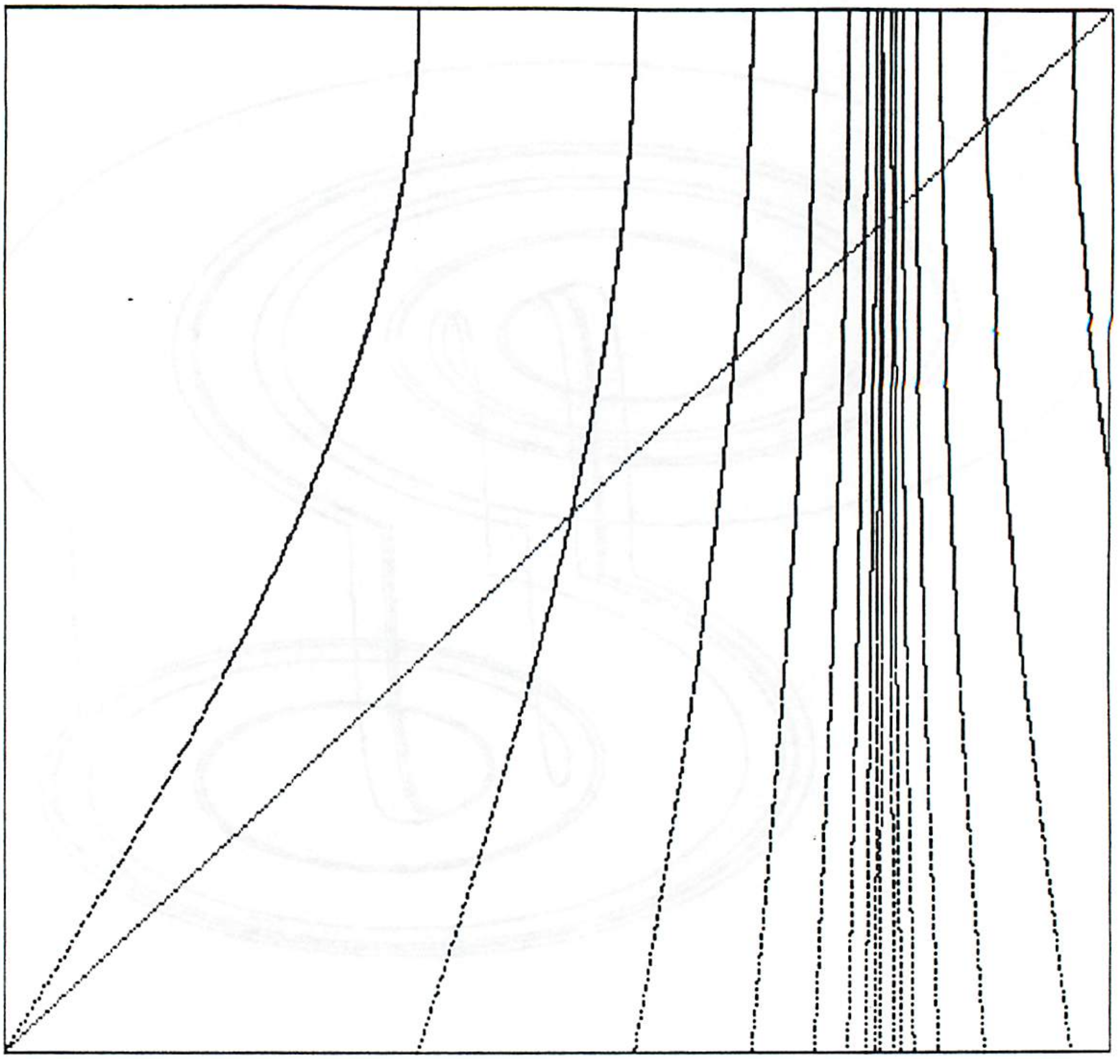


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 \\ \\ 1 & 0.1 \end{bmatrix} \begin{pmatrix} x - 0.613 \\ \\ y + 0.5 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\text{sgn}(x + 1) < 0$.

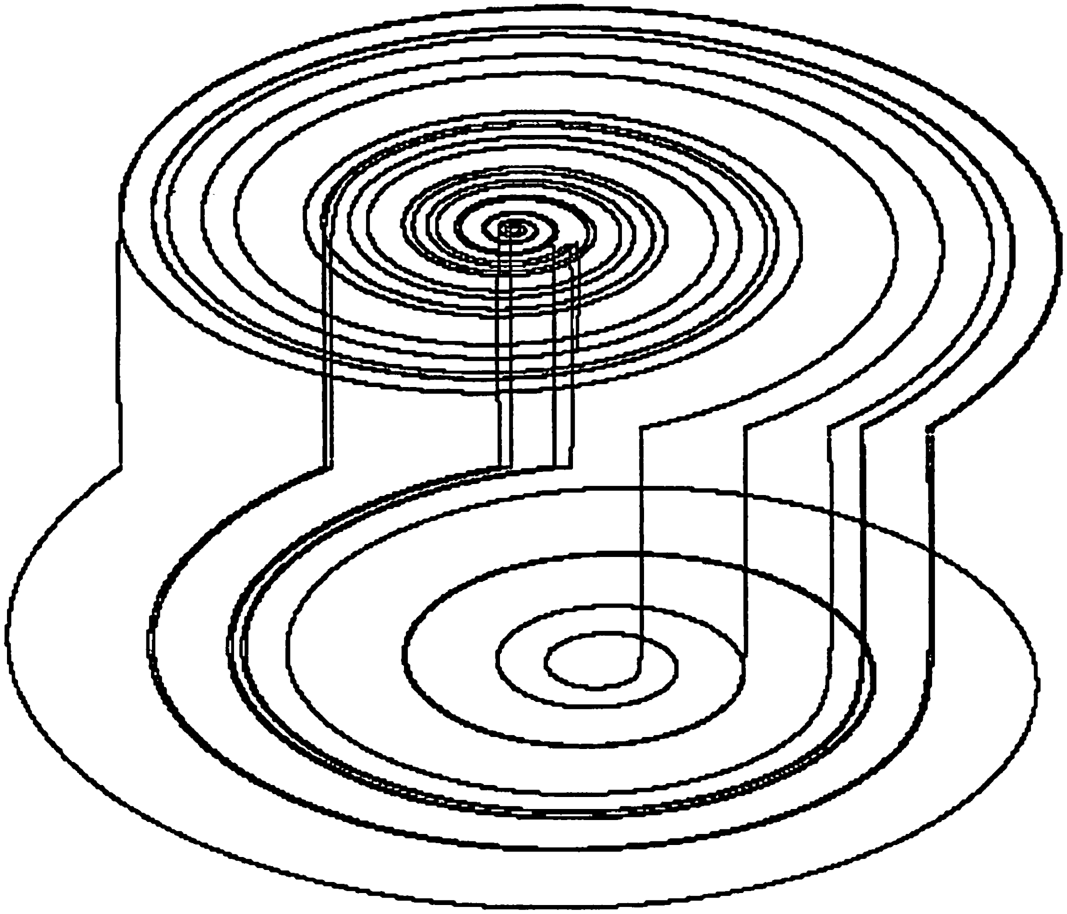


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 & 0 \\ \\ 1 & 0.1 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.613\,\mathrm{sgn}(x+z) \\ \\ y + 0.5\,\mathrm{sgn}(x+z) \\ \\ z - \mathrm{sgn}(x+z) \end{pmatrix}$$

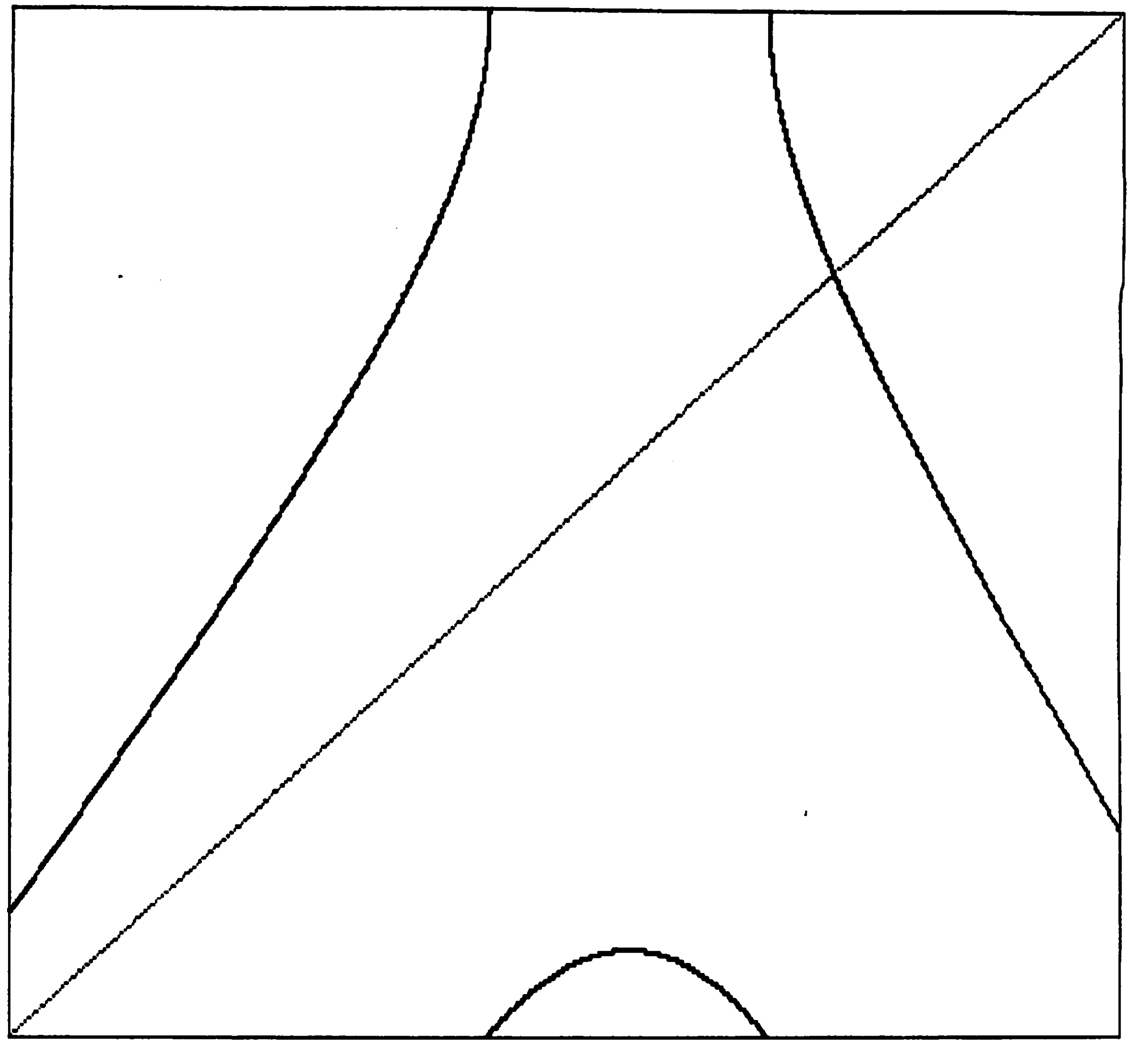


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 \\ \\ 1 & 0.1 \end{bmatrix} \begin{pmatrix} x - 1 \\ \\ y + 0.21 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\mathrm{sgn}(x+1) < 0$.

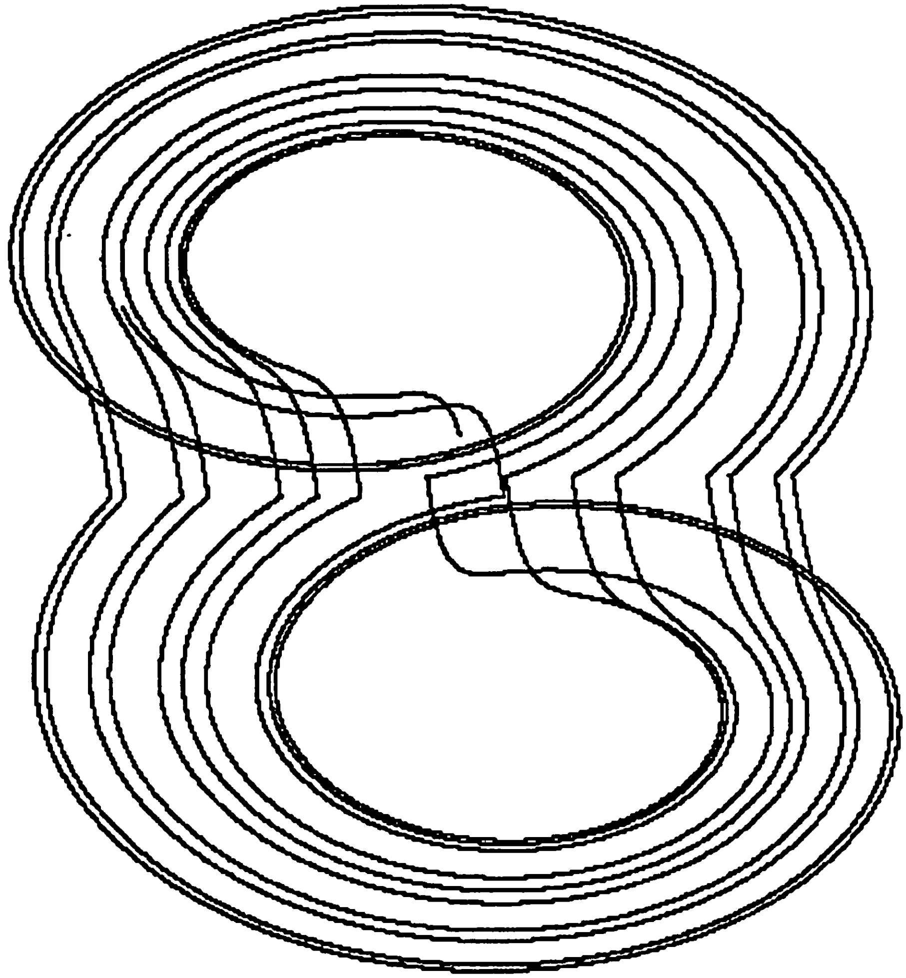


$$
\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 & 0 \\ \\ 1 & 0.1 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - \operatorname{sgn}(x+z) \\ \\ y + 0.21 \operatorname{sgn}(x+z) \\ \\ z - \operatorname{sgn}(x+z) \end{pmatrix}
$$

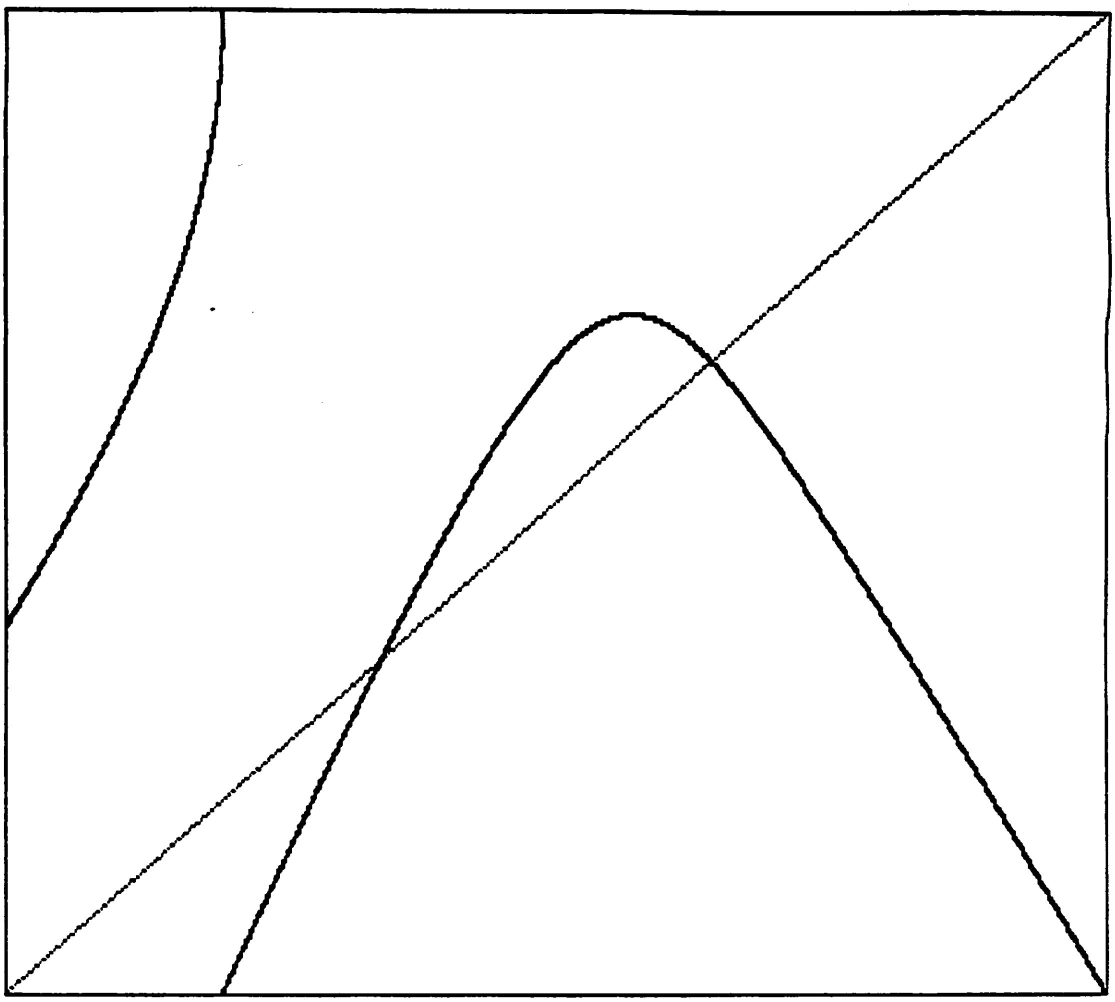


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 \\ \\ 1 & 0.1 \end{bmatrix} \begin{pmatrix} x-3 \\ \\ y+1 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\text{sgn}(x+1) < 0$.

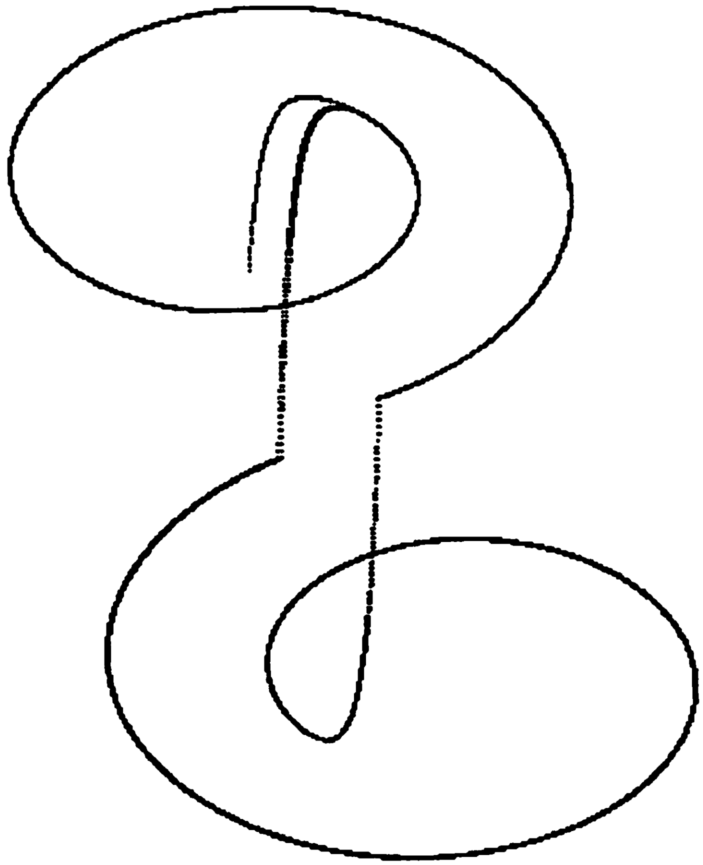


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 & 0 \\ 1 & 0.1 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 3\,\mathrm{sgn}(x+z) \\ y + \mathrm{sgn}(x+z) \\ z - \mathrm{sgn}(x+z) \end{pmatrix}$$

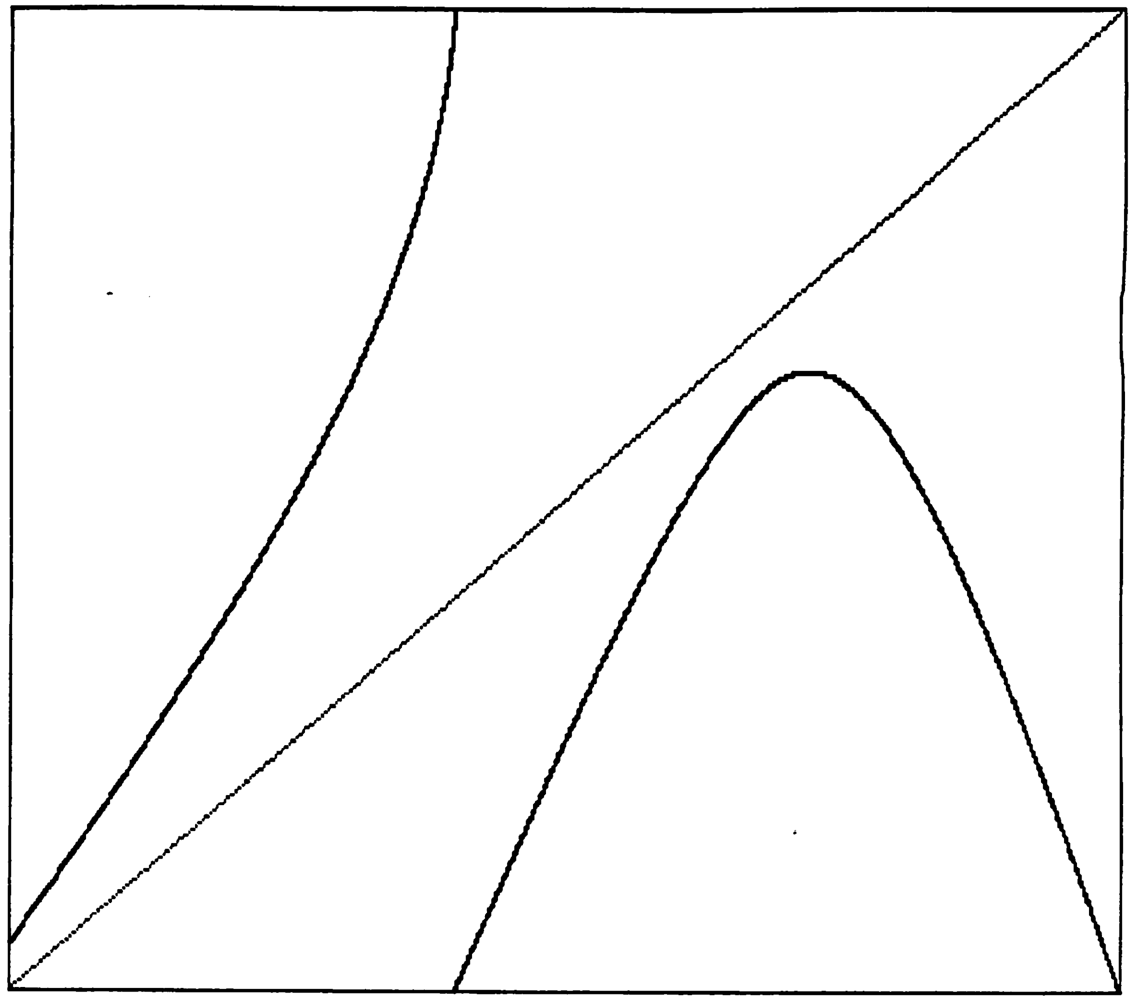


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 \\ \\ 1 & 0.1 \end{bmatrix} \begin{pmatrix} x - 0.4 \\ \\ y + 0.4 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\mathrm{sgn}(x+1) < 0$.

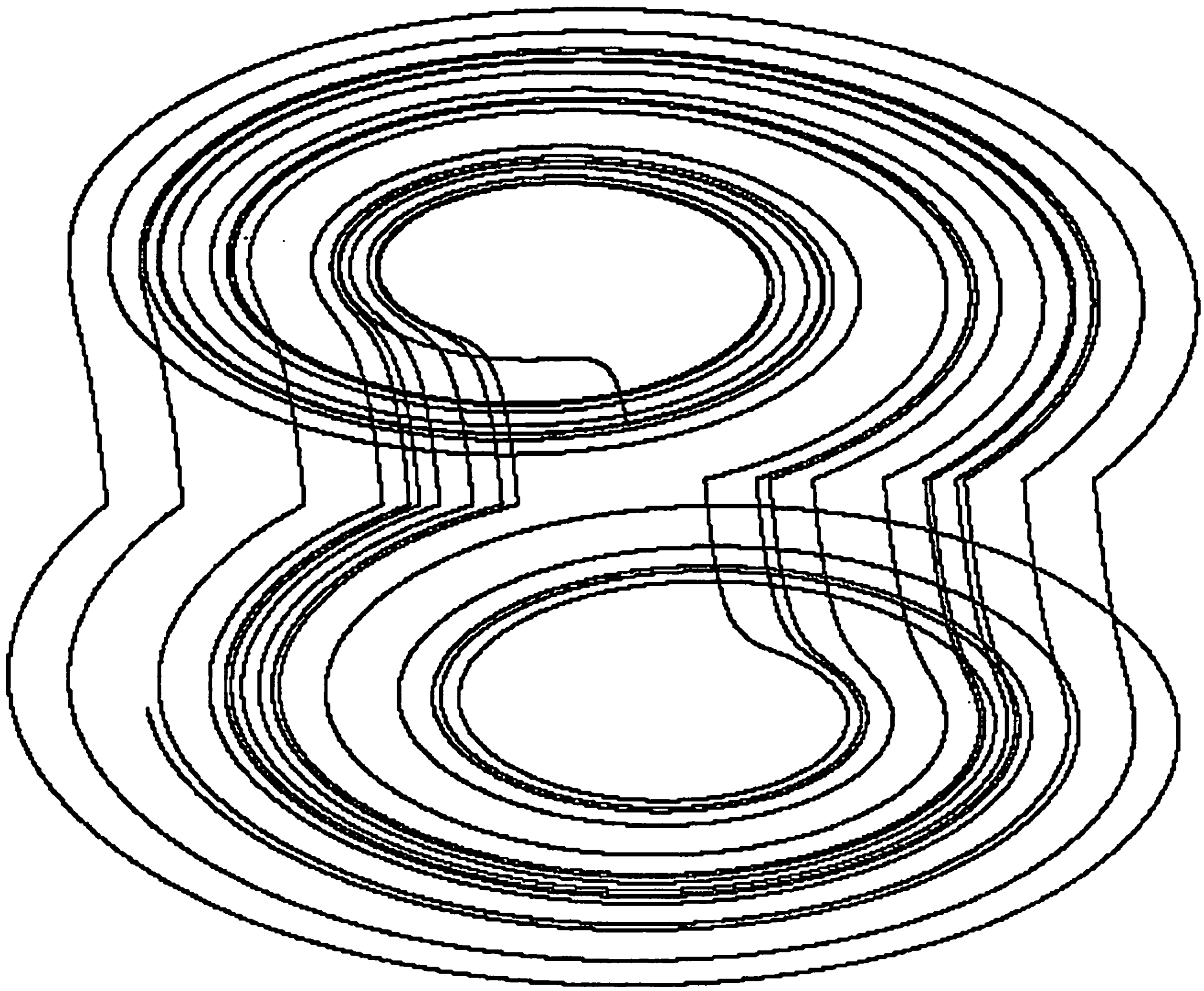


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 & 0 \\ \\ 1 & 0.1 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.4\,\mathrm{sgn}(x+z) \\ \\ y + 0.4\,\mathrm{sgn}(x+z) \\ \\ z - \mathrm{sgn}(x+z) \end{pmatrix}$$

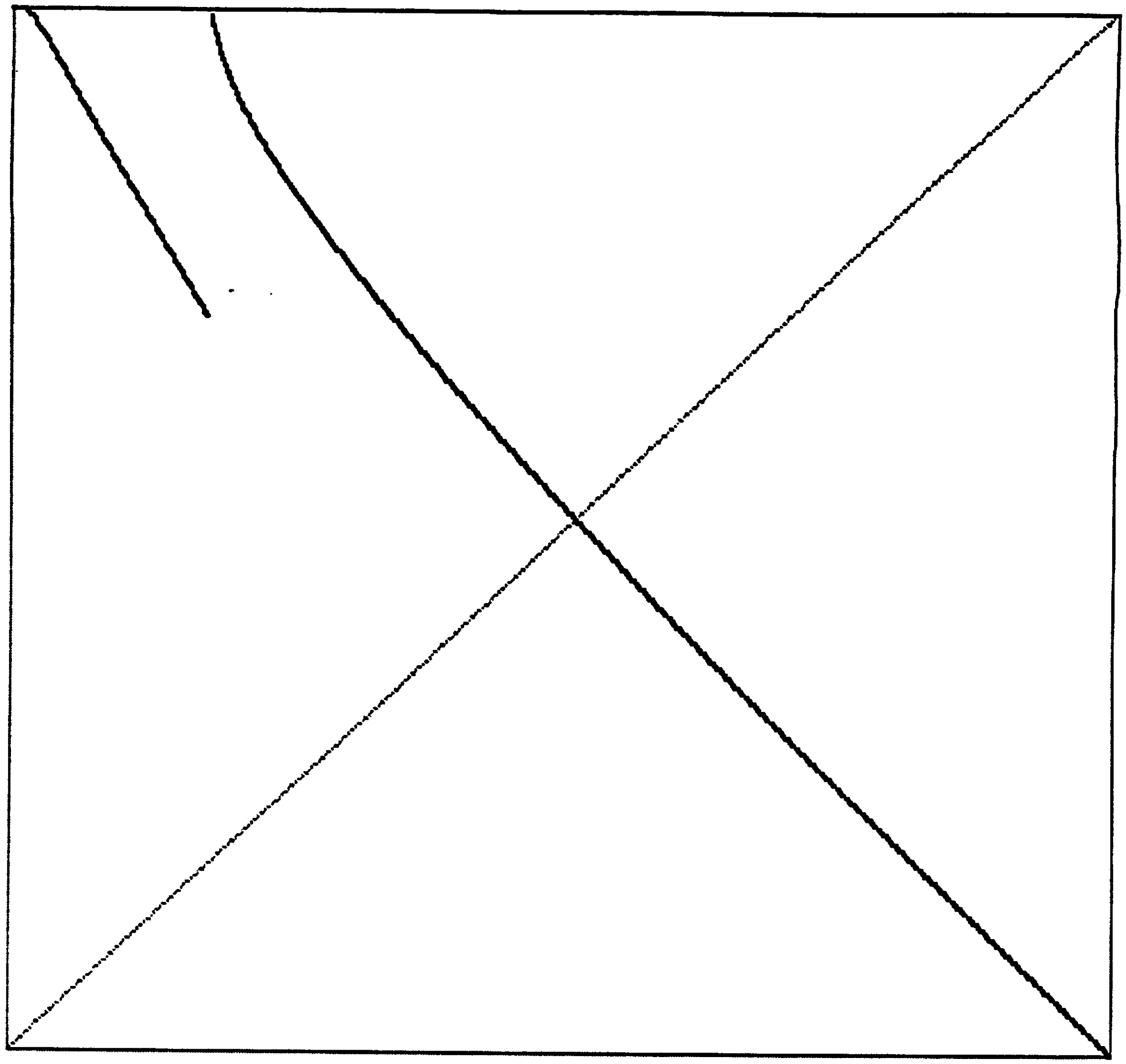


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} .1 & -1 \\ 1 & .1 \end{bmatrix} \begin{pmatrix} x-2 \\ y+.5 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\mathrm{sgn}(x+1) < 0$.

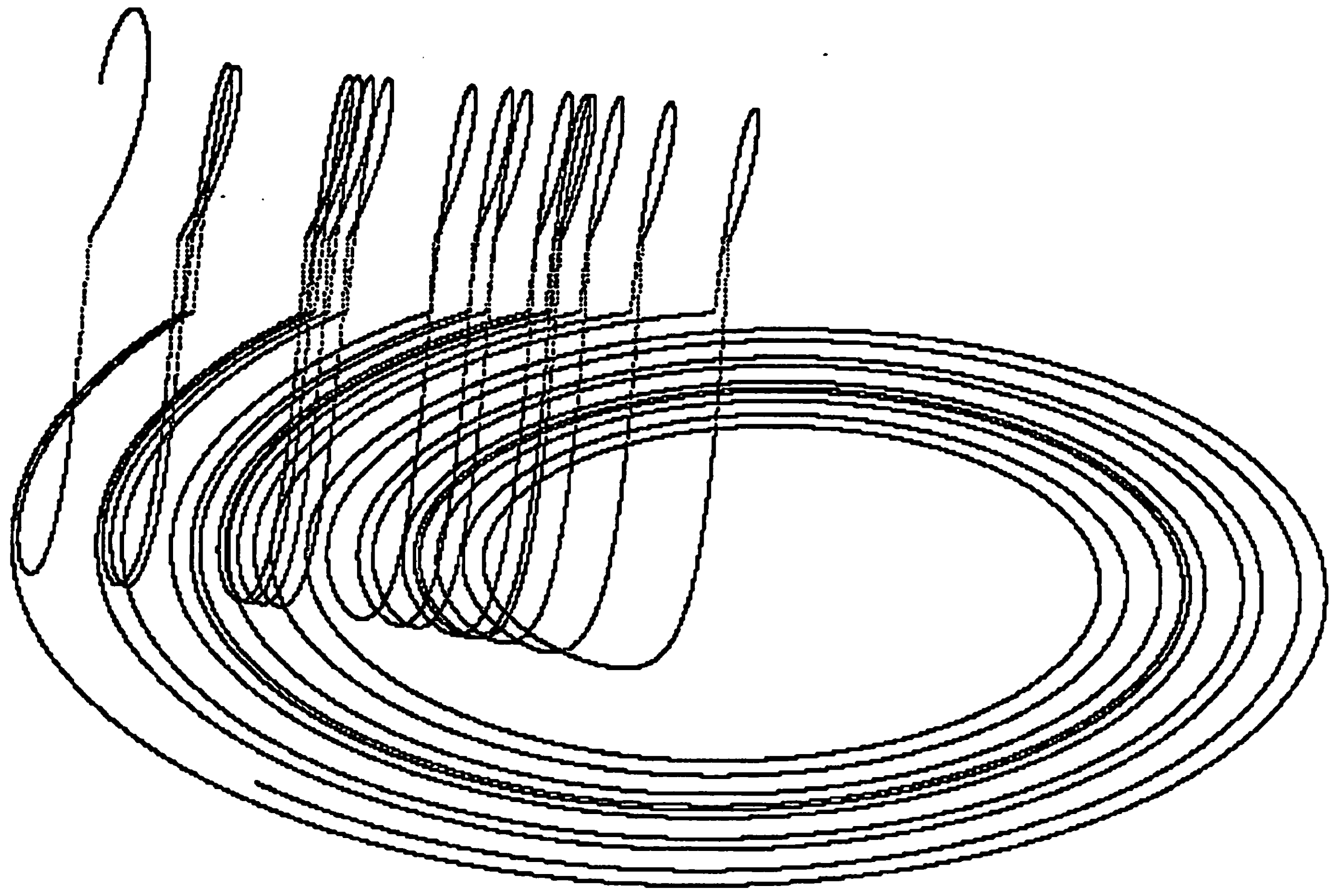


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} .1 & -1 & 0 \\ \\ 1 & .1 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 2\,\mathrm{sgn}(x+z) \\ \\ y + .5\,\mathrm{sgn}(x+z) \\ \\ z - \mathrm{sgn}(x+z) \end{pmatrix}$$

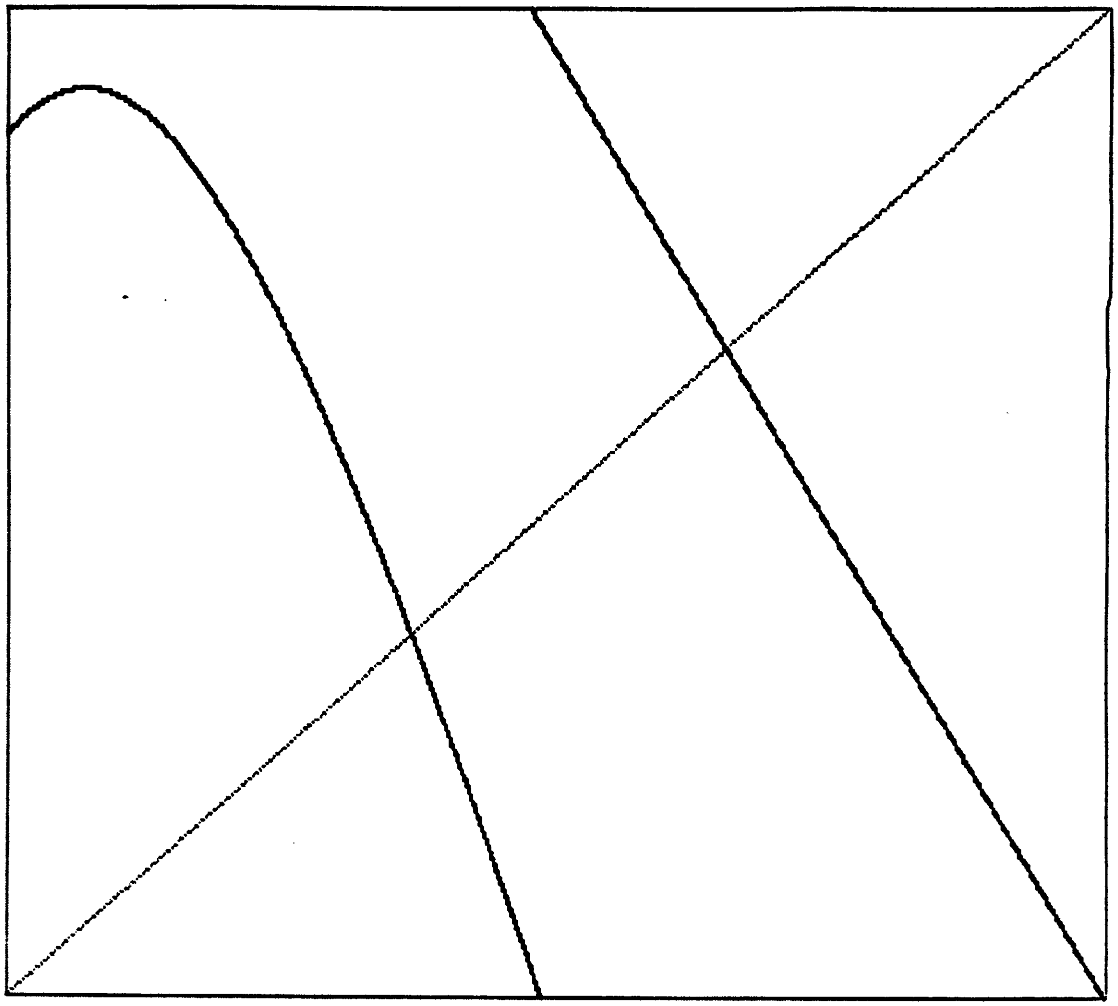


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.08 & -1 \\ 1 & 0.08 \end{bmatrix} \begin{pmatrix} x - 0.3575 \\ y + 2.64 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\operatorname{sgn}(x+1) < 0$.

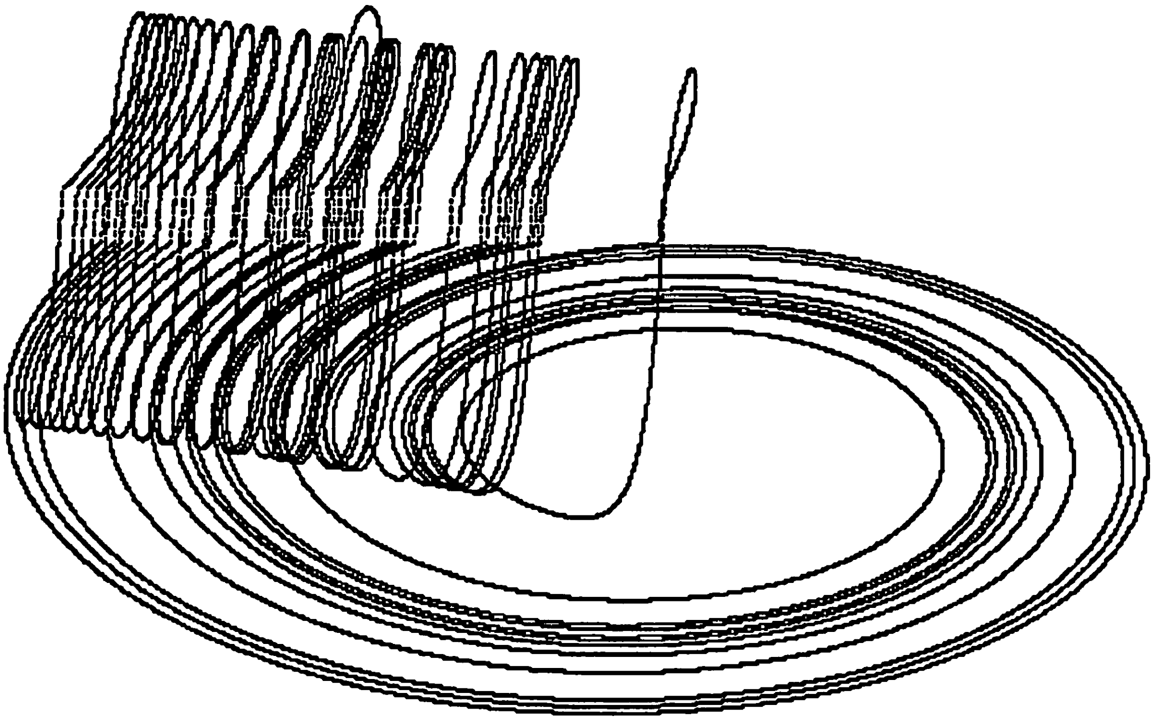


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.08 & -1 & 0 \\ 1 & 0.08 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.3575\,\mathrm{sgn}(x+z) \\ y + 2.64\,\mathrm{sgn}(x+z) \\ z - \mathrm{sgn}(x+z) \end{pmatrix}$$

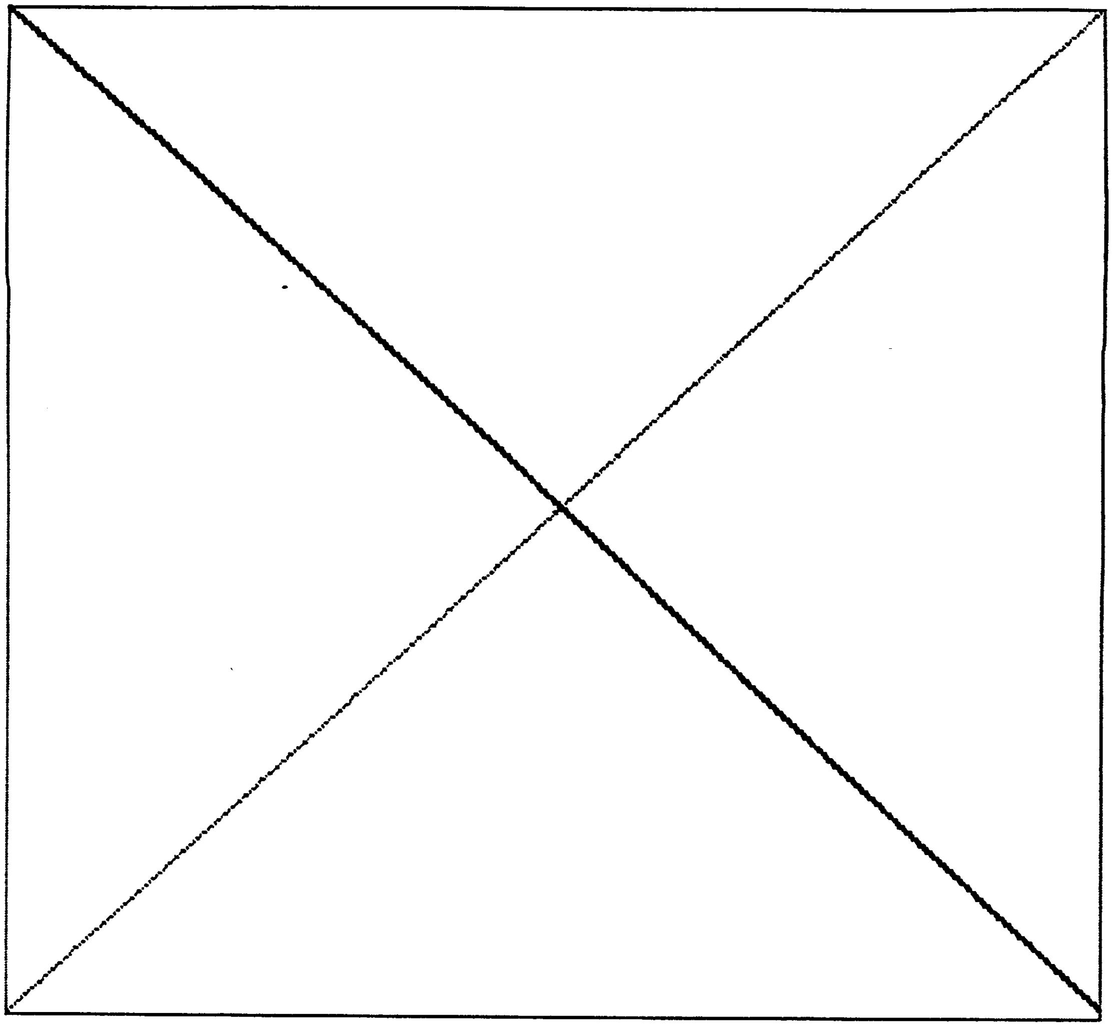


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 \\ \\ 1 & 0.1 \end{bmatrix} \begin{pmatrix} x - 0.4 \\ \\ y + 2.1 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\operatorname{sgn}(x+1) < 0$.

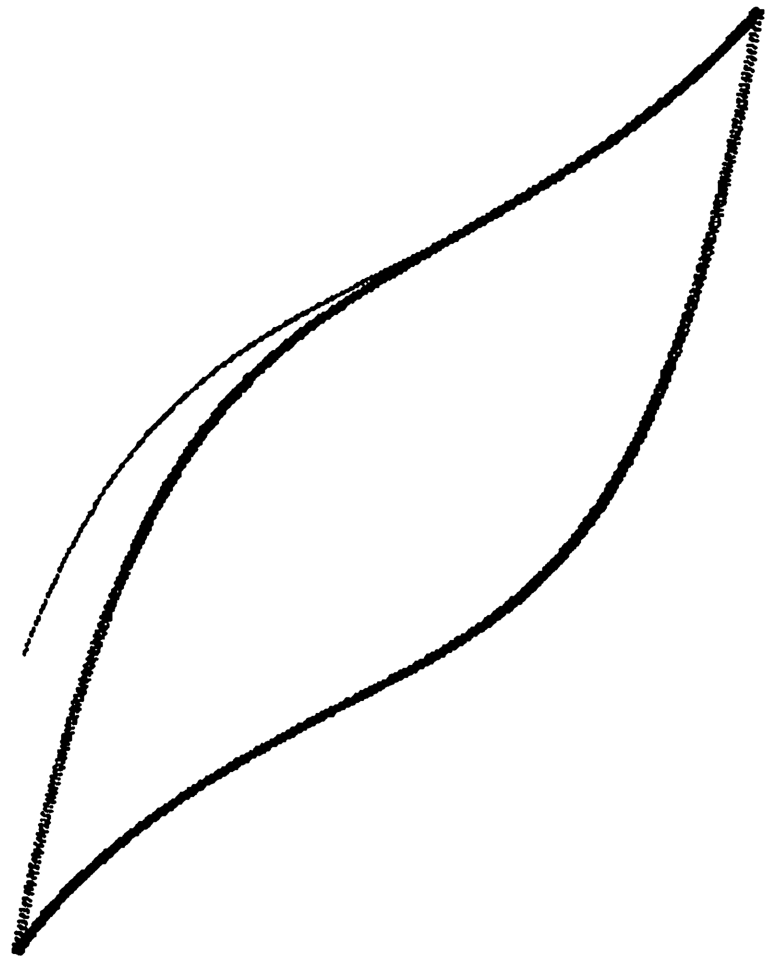


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 & 0 \\ 1 & 0.1 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.4\,\mathrm{sgn}(x+z) \\ y + 2.1\,\mathrm{sgn}(x+z) \\ z - \mathrm{sgn}(x+z) \end{pmatrix}$$

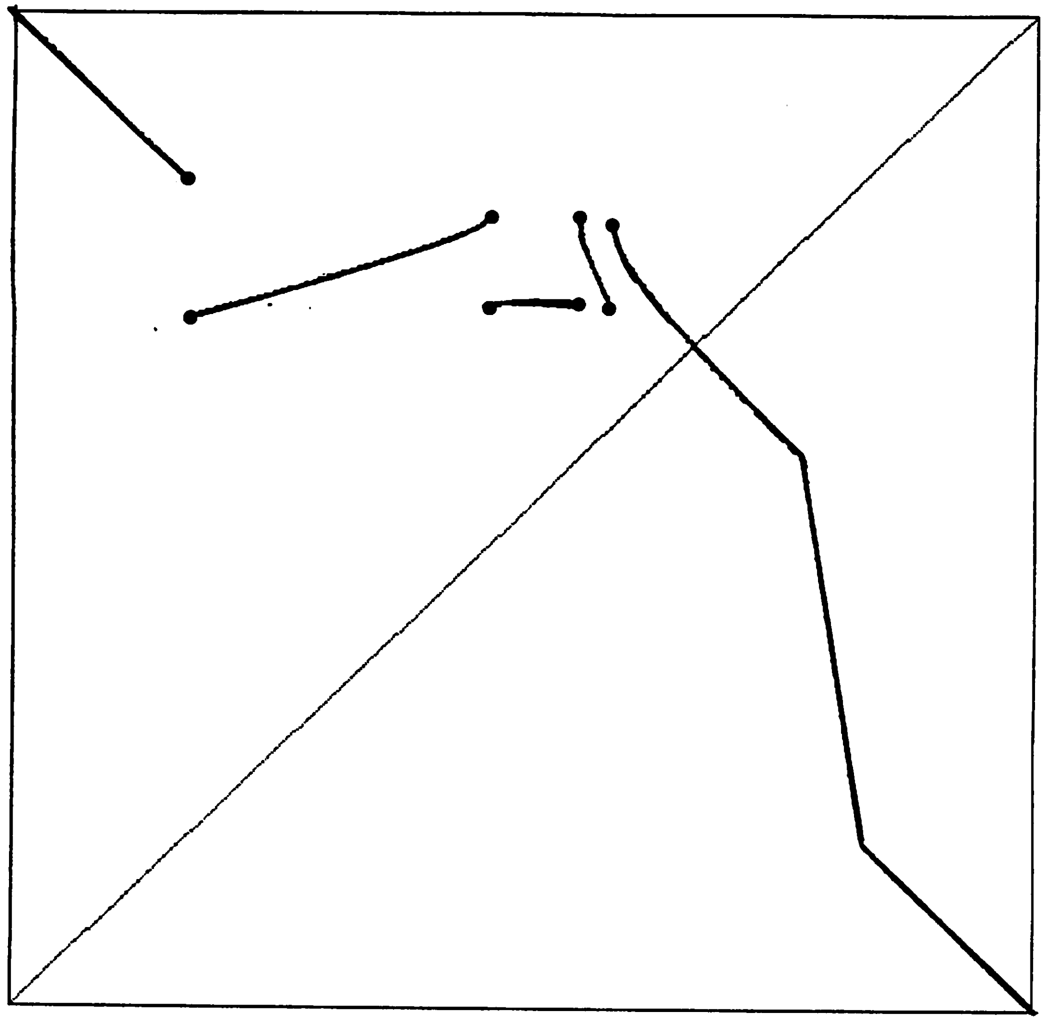


$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 \\ \\ 1 & 0.1 \end{bmatrix} \begin{pmatrix} x - 0.05 \\ \\ y + 2.1 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\operatorname{sgn}(x+1) < 0$.

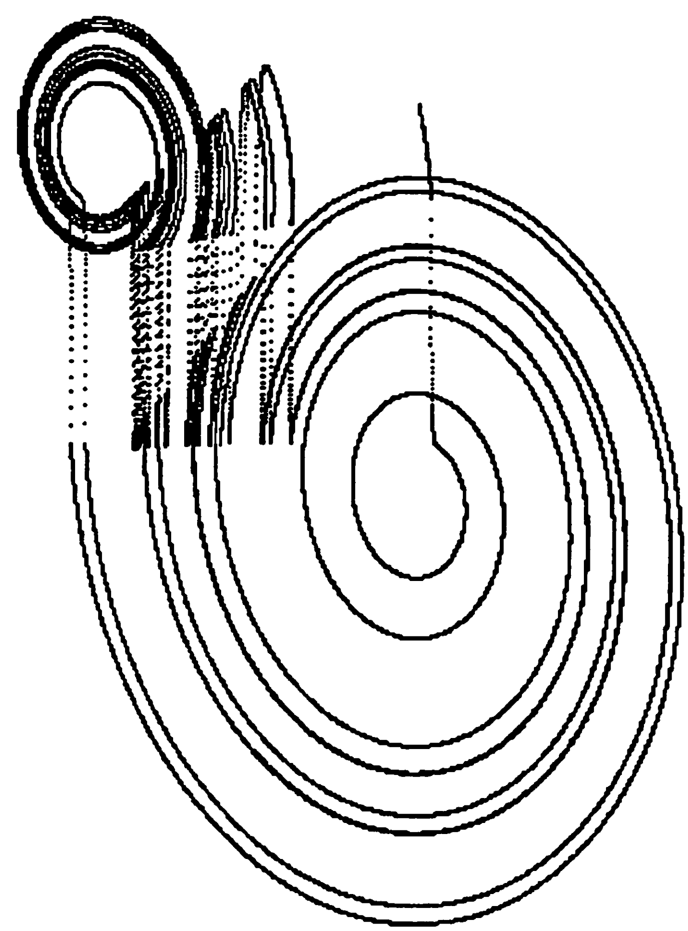


$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.1 & -1 & 0 \\ 1 & 0.1 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.05\,\mathrm{sgn}(x+z) \\ y + 2.1\,\mathrm{sgn}(x+z) \\ z - \mathrm{sgn}(x+z) \end{pmatrix}$$



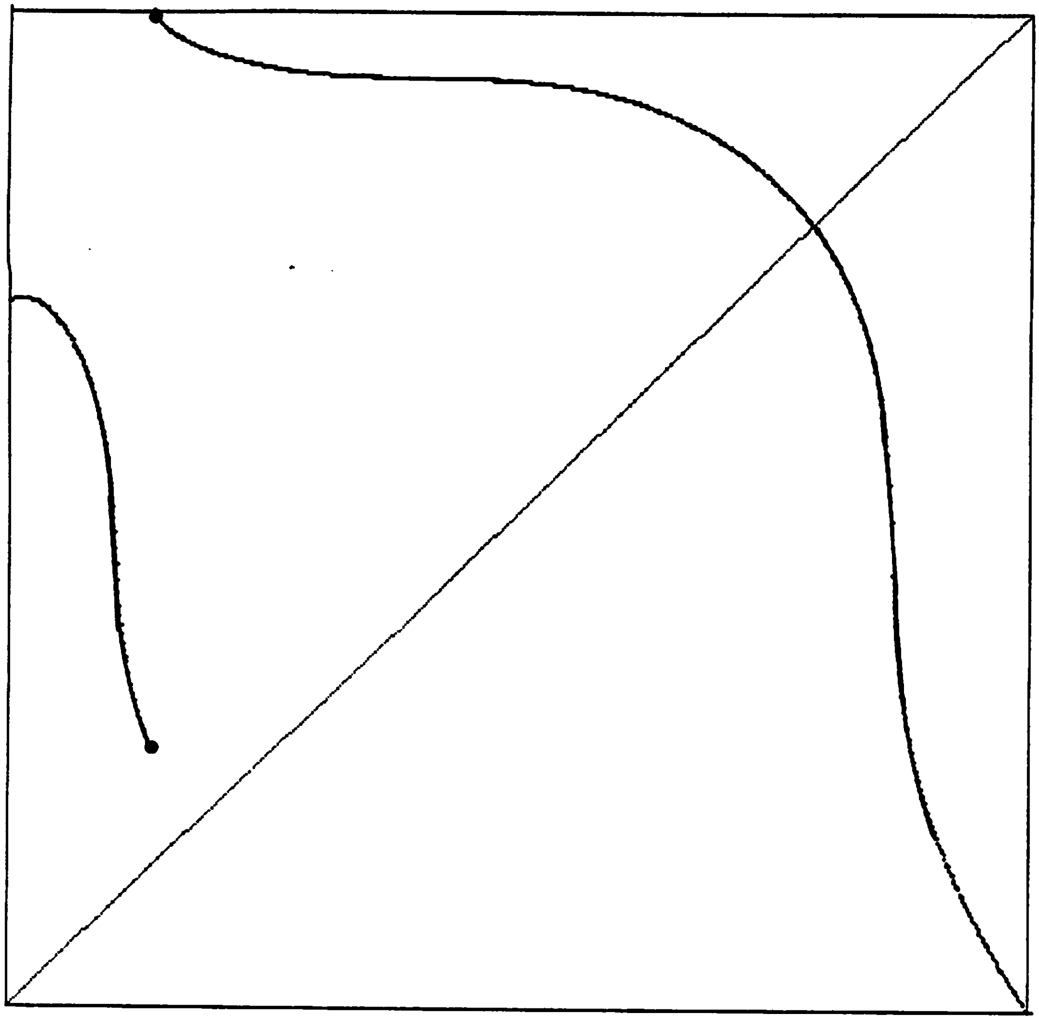
$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -5.0 \\ \\ 1 & 0.4 \end{bmatrix} \begin{pmatrix} x - 0.5 \\ \\ y + 1.0 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$.
Apply flip map when
$\mathrm{sgn}(2x - 1.5\,|0.1 - x| + 1 - y) < 0$.



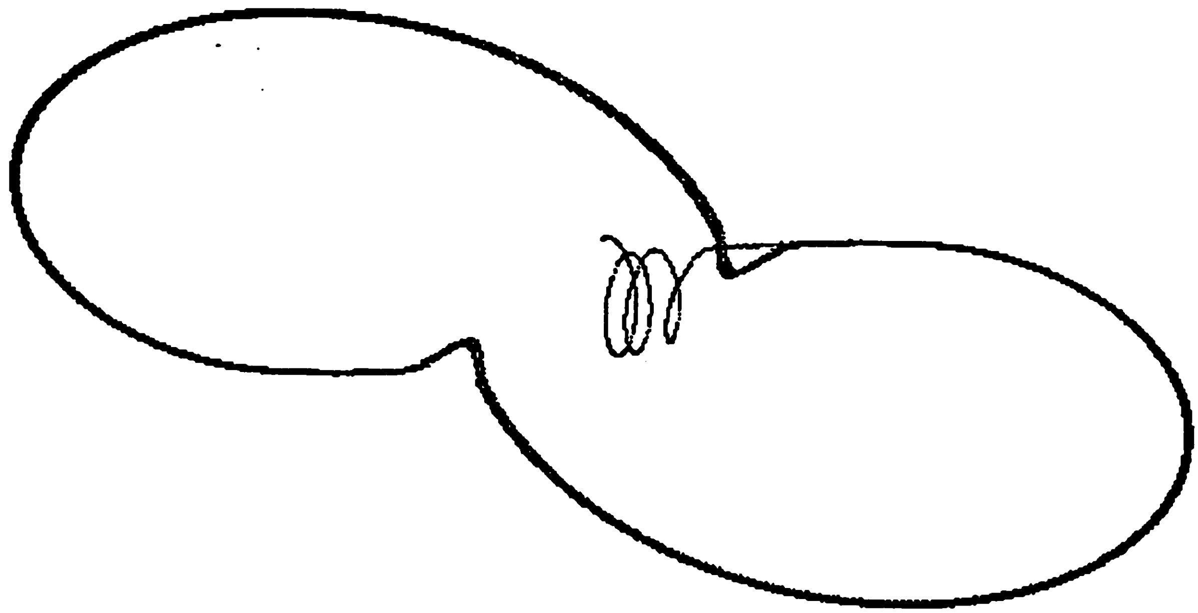
$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -5 & 0 \\ \\ 1 & 0.4 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.5\,\mathrm{sgn}(u) \\ \\ y + 1.0\,\mathrm{sgn}(u) \\ \\ z - \mathrm{sgn}(u) \end{pmatrix}$$

where $u = (2x - 1.5\,|0.1 - x| + z - y)$.



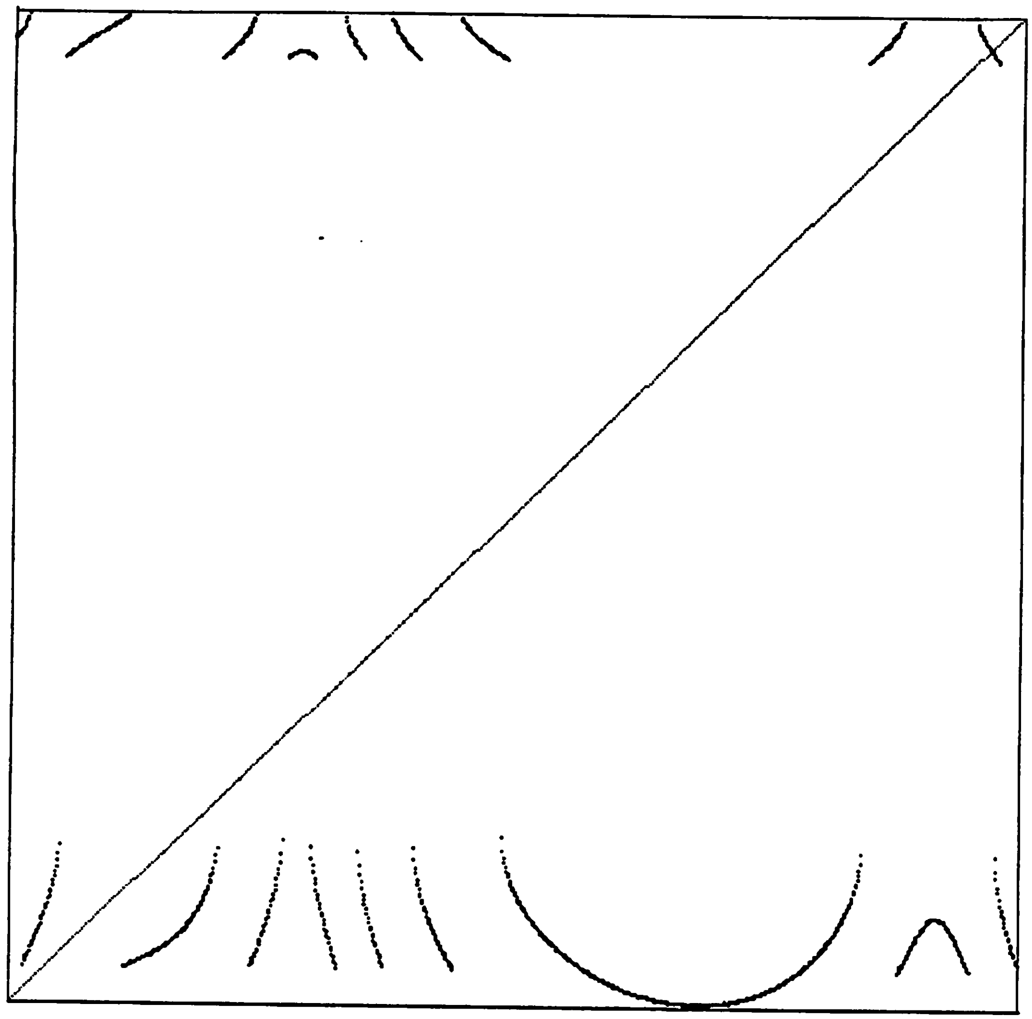
$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -1.0 \\ \\ 1.0 & 0.3 \end{bmatrix} \begin{pmatrix} x - 0.5 \\ \\ y + 3.0 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\mathrm{sgn}(x^3 + 1 - y) < 0$.



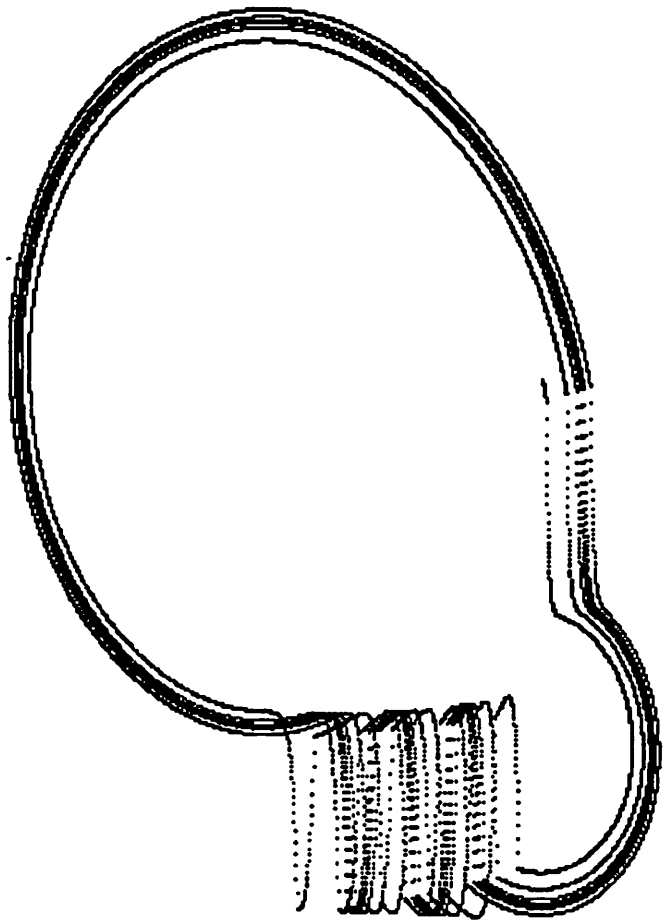
$$
\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -1 & 0 \\ \\ 1.0 & 0.3 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.5\,\mathrm{sgn}(u) \\ \\ y + 3.0\,\mathrm{sgn}(u) \\ \\ z - \mathrm{sgn}(u) \end{pmatrix}
$$

where $u = (x^3 + z - y)$.



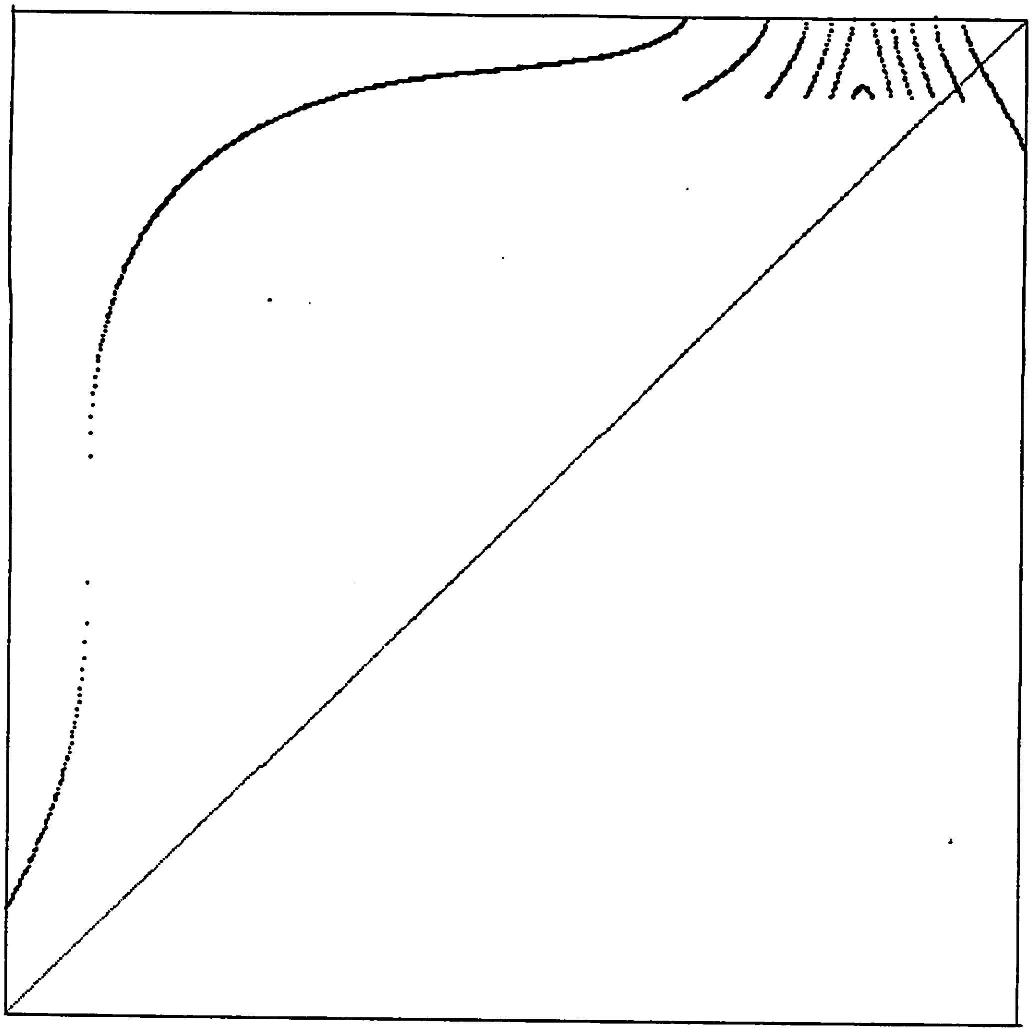
$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -5.0 \\ 1.0 & 0.4 \end{bmatrix} \begin{pmatrix} x - 0.5 \\ y + 1.0 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\operatorname{sgn}(0.5\,x^3 + x^2 - x + z - y) < 0$.



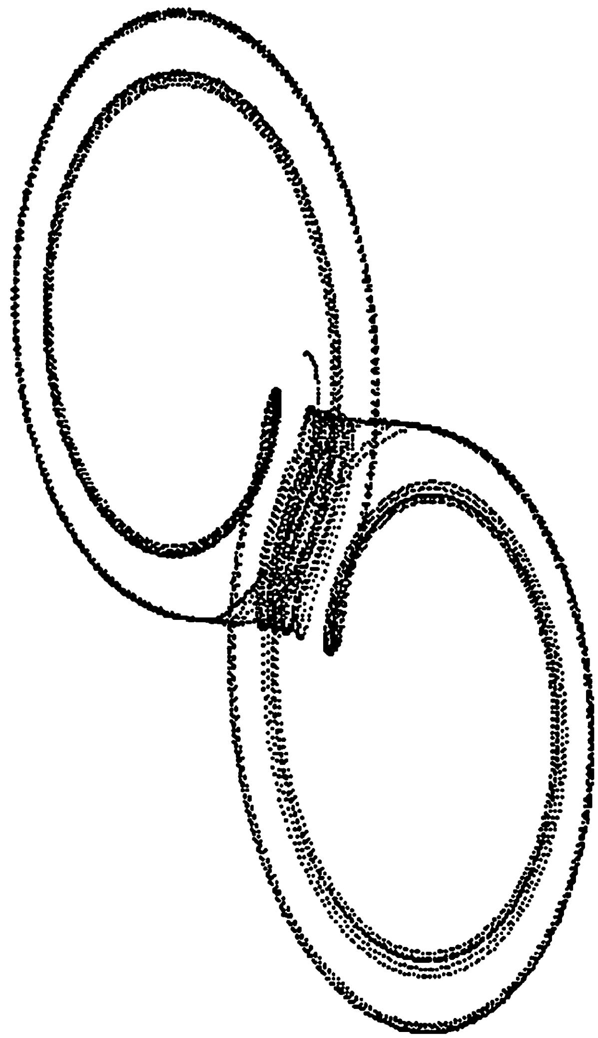
$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -5 & 0 \\ 1.0 & 0.4 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.5\,\mathrm{sgn}(u) \\ y + 1.0\,\mathrm{sgn}(u) \\ z - \mathrm{sgn}(u) \end{pmatrix}$$

where $u = 0.5\,x^3 + x^2 - x + z - y.$



$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -8.0 \\ \\ 1.0 & 1/3 \end{bmatrix} \begin{pmatrix} x - 0.5 \\ \\ y + 0.5 \end{pmatrix}$$

Where $x(0) = 1$ and $y(0) = y_0$. Apply flip map when $\mathrm{sgn}(x^3 + 1 - y) < 0$.



$$\begin{pmatrix} \dot{x}(t) \\ \\ \dot{y}(t) \\ \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -8 & 0 \\ \\ 1.0 & 1/3 & 0 \\ \\ 0 & 0 & -10 \end{bmatrix} \begin{pmatrix} x - 0.5\,\mathrm{sgn}(u) \\ \\ y + 0.5\,\mathrm{sgn}(u) \\ \\ z - \mathrm{sgn}(u) \end{pmatrix}$$

where $u = (x^3 + 1 - y)$.