ON CHAOTIC SYNCHRONIZATION IN A LINEAR ARRAY OF CHUA'S CIRCUITS

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Abstract

In this paper the sufficient conditions that each cell in a one-dimensional array of Chua's circuits has identical, synchronized behavior are given.

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1. Introduction

Recently, a significant increase in the number of publications on synchronization of Chua's circuit and arrays of Chua's circuits have appeared[1-6]. Because of its simplicity, robustness and low cost, the Chua's circuit[7] has become a tool for analytical, numerical and experimental study of nonlinear phenomena. In this paper we present a rigorous analysis of synchronization in a one-dimensional array of Chua's circuits.

2. Main result

The basic cell of our one-dimensional array is a three-dimensional dynamical system described by the following dimensionless equations:

$$\left.\begin{aligned} \dot{x} &= -f(x) + a_1 y + a_2 z \\ \dot{y} &= b_{11} y + b_{12} z + b_1 x \\ \dot{z} &= b_{21} y + b_{22} z + b_2 x \\ & x,\ y,\ z \in \mathbb{R} \end{aligned}\right\} \tag{1}$$

where a_1, a_2, b_{11}, b_{12}, b_{21}, b_{22}, b_1 and b_2 are parameters, and $f(x)$ is a nonlinear function. The set of equations representing the array are, in its dimensionless form, :

$$\left.\begin{aligned} \dot{x}_k &= -f(x_k) + a_1 y_k + a_2 z_k + D(\, x_{k-1} - 2x_k + x_{k+1}) \\ \dot{y}_k &= b_{11} y_k + b_{12} z_k + b_1 x_k \\ \dot{z}_k &= b_{21} y_k + b_{22} z_k + b_2 x_k \\ & x_k,\ y_k,\ z_k \in \mathbb{R} \qquad k = 1,\ 2,\ \ldots,\ N, \end{aligned}\right\} \tag{2}$$

where $D > 0$ represents the diffusion coefficient of the variable x. In addition to (2), we impose the following zero-flux boundary conditions:

$$\partial x_1(t)/\partial s = 0,\ \partial x_N(t)/\partial s = 0$$

where s is the direction of the diffusion. This is equivalent to assuming that:

$$x_0(t) = x_1(t),\ x_{N+1}(t) = x_N(t), \text{ for all } t \geq 0 \tag{3}$$

Note that if $\Delta s \to 0$, then a continuous model is obtained, where the "diffusion" term of the first equation in (2) represents the Laplacian of x, $\partial^2 x/\partial s^2$.

Example 1. Eqs.(1) represent Chua's circuit, if:

$$a_1 = \alpha,\ a_2 = 0,\ b_{11} = -1,\ b_{12} = 1 = b_1,\ b_{21} = -\beta,\ b_{22} = b_2 = 0,\ f(x) = \alpha\ h(x)$$

where $h(x)$ describes a continuous three-segment piecewise-linear curve of Chua's diode[8,9]. In this case, Eqs.(2) represent an array of Chua's circuit. Recently, traveling wave fronts are investigated in an array of Chua's circuit[5,6]. For diffusion coefficient less than some nonzero critical value it has been observed numerically that the traveling front fail to propagate. A continuous model, the so called time-delayed Chua's circuit, is considered in Ref.10. It has been proven rigorously that the time-delayed Chua's circuit exhibits the period-adding phenomenon.

Example 2. Eqs.(2) represent an array of the unfolded canonical Chua's circuit[11-14], if:

$$a_1 = \alpha,\ a_2 = 0,\ b_{11} = -1,\ b_{12} = 1 = b_1,\ b_{21} = -\beta,\ b_{22} = -\gamma,\ b_2 = 0,\ f(x) = \alpha\ h(x).$$

Theorem 1. Assume that the following conditions are satisfied:

(i) There exists a positive definite Lyapunov function

$$V(x,\ y,\ z) = \frac{1}{2}\ x^2 + \frac{1}{2}\ (\ A\ y^2 + 2B\ yz + C\ z^2\ )$$

such that the trajectory derivative of $V(x, y, z)$, $\dot{V}(x, y, z) \overset{\Delta}{=} -Q(x, y, z)$, with respect to the linear system:

$$\left.\begin{aligned} \dot{x} &= -ax + a_1 y + a_2 z \\ \dot{y} &= b_{11} y + b_{12} z + b_1 x \\ \dot{z} &= b_{21} y + b_{22} z + b_2 x \end{aligned}\right\} \tag{4}$$

is negative definite.

(ii) There exists a real number

$$\nu \overset{\Delta}{=} \frac{\lambda_0 - a + 2D}{2D}, \tag{5}$$

such that $0 < \nu < 1$, where $\lambda_0 \overset{\Delta}{=} \min_{\xi \in \mathbb{R}} f'(\xi)$.

Then there exists an integer $N_0 = \text{Int}\,(\,\pi/\arccos \nu\,)$, defined to be the integer part of the argument, such that in the phase space of the array, described by Eqs. (2), of N cells, $N \le N_0$, the manifold:

$M \overset{\Delta}{=} \{ (x_1, y_1, z_1, \ldots x_N, y_N, z_N): x_k = x_{k+1}, y_k = y_{k+1}, z_k = z_{k+1}, k = 1, .., N-1\}$

is asymptotically stable.

Proof. See Appendix.

Remarks:

1. If the manifold M is asymptotically stable, then all cells in our one-dimensional array are synchronized, i.e. each coupled subsystem has identical behavior, when $t \to \infty$:

$$x_k(t) = x_{k+1}(t),\; y_k(t) = y_{k+1}(t),\; z_k(t) = z_{k+1}(t),\; k = 1, .., N-1$$

2. The condition $\nu = 0$ determines the critical value of the diffusion coefficient:

$$D^* = \inf_a \frac{-\lambda_0 + a}{2}.$$

Because $\nu < 1$, it follows that $a > \lambda_0$, and consequently $D^* \ge 0$. If D is close to D^* ($D > D^*$), then $N_0 = 2$, while when $D \to \infty$, then $N_0 \to \infty$, as well. Let $\nu_i = \cos \frac{\pi}{i}$, $i = 2, 3, \ldots$, then from (5) we get $D_i = \frac{D^*}{(1 - \nu_i)}$. Hence, for fixed a and λ_0 (note that a and λ_0 depend only on the parameters of the system (1)), we have the following interpretation of the theorem 1: if $D_i < D \le D_{i+1}$, $i = 2, 3, \ldots$, then the array with " i " cells is synchronized.

3. Suppose a can be chosen such that it is arbitrarily close to λ_0, i.e. $a = \lambda_0 + \varepsilon$, where ε is a small positive number. Then, $D^* = 0$. In a similar way, $D_i = 0$ for all i. Hence,

if $D > 0$, the array with an arbitrary number of cells is synchronized.

4. The above theorem gives only sufficient conditions for asymptotic stability of the manifold M. As a consequence, this theorem does not give the minimal value of the critical diffusion coefficient, except in the case which is considered in the Remark 3.

5. Let the basic cell of our one-dimensional array be an n-dimensional dynamical system described by the following dimensionless equations:

$$\begin{aligned} \dot{x} &= -f(x) + \mathbf{d}^T \mathbf{y} \\ \dot{\mathbf{y}} &= \mathbf{D}\,\mathbf{y} + \mathbf{e}\,x \\ & x \in \mathbb{R},\ \mathbf{y} \in \mathbb{R}^{n-1} \end{aligned}$$

where $\mathbf{d}$, $\mathbf{e}$, $\mathbf{y}$ are (n-1)-dimensional vectors and $\mathbf{D}$ is an (n-1)x(n-1) matrix. An example of such a system is the n-dimensional canonical Chua's circuit[15]. The generalization of the given theorem in this case is straightforward.

3. Application to Chua's circuit

Chua's circuit

The state equations of Chua's circuit are:

$$\left.\begin{aligned} \dot{x} &= \alpha(\, y - h(x)\,) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y \end{aligned}\right\}$$

where $h(x) = m_1 x + (m_0 - m_1)[\ |x+1| - |x-1|\ ]/2$. The system is described by four parameters $\{\ \alpha,\ \beta,\ m_0,\ m_1\ \}$, with the double scroll chaotic attractor occurring in a neighborhood of $\{\ 9,\ 14(2/7),\ -1/7,\ 2/7\ \}$. In what follows we assume that $\alpha > 0$, $\beta > 0$ and we fix the values of m_0 and m_1 to -1/7 and 2/7, respectively. Consider the Lyapunov function defined by:

$$V(x,\ y,\ z) = \frac{1}{2}\, x^2 + \frac{1}{2}\,(\ A\ y^2 + 2B\ yz + C\ z^2\ )$$

where $A = \alpha$, $B = -\frac{\alpha}{\beta}\frac{\varepsilon}{\varepsilon + 1}$, $C = \frac{A - B}{\beta}$, and $\varepsilon > 0$ is a small parameter. It is easy to show that its trajectory derivative with respect to the linear system (4) is negative definite, if

$$a > (\varepsilon + 1)\,\alpha + \frac{\alpha\,\varepsilon}{4\,\beta\,(\varepsilon + 1)} \tag{6}$$

On the other hand, since $\lambda_0 = m_0$, the inequality $\nu < 1$ is satisfied if

$$a > m_0$$

The critical value of the diffusion coefficient is obtained when $\varepsilon \to 0$ in Eq.(6). Hence, it is given by:

$$D^* = \frac{\alpha - m_0}{2} \tag{7}$$

Thus, if $D > D^*$, the array of N Chua's circuit, $N \le N_0$, is synchronized. For the double scroll chaotic attractor, the critical value is:

$$D^* = \frac{9 + 1/7}{2} = 4.5714.$$

The first four values of the diffusion coefficient D_i are:

$$D_2 = D^*,\ D_3 = 9.1428,\ D_4 = 15.6077 \text{ and } D_5 = 23.9362.$$

So, if $D_i < D \le D_{i+1}$, $i = 2, 3, \dots$, then $N_0 = i$, and the array with $N \le N_0$ cells is synchronized. Figure 1 shows the number of cells N in the array of Chua's circuits as a function of the diffusion coefficient D. We stress that the above analysis with the same critical coefficient D^* given by (7) is still valid for all $m_0 < m_1$ and $m_0 < \alpha$.

Since the theorem 1. gives only sufficient conditions for asymptotic stability of the manifold M, the above values of D^* and D_i, $i = 2, 3, \dots$, are not the minimal values the diffusion coefficient. Numerically we found that D^* is less than 3.5. Figure 2 shows a double scroll chaotic attractor associated with the first and the last cell in a linear array of Chua's circuits with five cells and $D = 20$. As shown in Fig.3 the array is synchronized, i.e. each coupled subsystem has identical behavior, when $t \to \infty$. If we add just one more cell to the array while retaining the value of D ($D = 20$), we found that the array is no longer synchronized. In fact, all trajectories become unbounded and no attractor is found numerically. This example demonstrates that our *sufficient* condition for synchronization is quite sharp.

Canonical Chua's circuit

The state equations of the canonical Chua's circuit are:

$$\left.\begin{aligned} \dot{x} &= \alpha\,(\,y - h(x)\,) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta\, y - \gamma\, z \end{aligned}\right\} \tag{8}$$

where $h(x) = m_1 x + (m_0 - m_1)[\,|x+1| - |x-1|\,]/2$. The system (8) is described by five parameters $\{\alpha, \beta, \gamma, m_0, m_1\}$. Let us consider the Lyapunov function:

$$V(x,\ y,\ z) = \frac{1}{2}\,x^2 + \frac{1}{2}\,(\,A\,y^2 + 2B\,yz + C\,z^2\,)$$

where $A > 0$, $AC - B^2 > 0$. Assume that one can find the constants A, B, and C such that all of the main determinants of the matrix:

$$\begin{bmatrix} 2a & -(\alpha + A) & -B \\ -(\alpha + A) & 2(A + B\beta) & -A + B(\gamma + 1) + C\beta \\ -B & -A + B(\gamma + 1) + C\beta & 2(C\gamma - B) \end{bmatrix}$$

are positive. On the other hand, $\nu < 1$ implies $\lambda_0 < a$, where $\lambda_o = \min\{\,m_0,\ m_1\,\}$. Now, we can apply the above theorem. Of course, for fixed values of the parameters $\{\alpha, \beta, \gamma, m_0, m_1\}$, it is not always easy or possible to find such constants A, B and C. Here, we shall give some examples.

Example 3. If $\alpha > 0$, $\beta > 0$, $\gamma > 0$, then using:

$$A = \alpha,\ B = 0,\ C = \alpha/\beta$$

we obtain that the corresponding array is synchronized if $a > \max\{\ \alpha,\ \lambda_0\ \}$ and the critical value of the diffusion coefficient is given by:

$$D^* = \frac{\max\{\ 0,\ \alpha,\ \min\{\ m_0,\ m_1\}\ \} - \min\{\ m_0,\ m_1\}}{2} \tag{9}$$

For the case of the double scroll chaotic attractor occurring in a neighborhood of $\alpha = 9$, $\beta = 14(2/7)$, $m_0 = -1/7$, $m_1 = 2/7$ and small γ ($\gamma > 0$), (9) becomes (7).

Example 4. If $\alpha < 0$, $\beta > 0$, $\gamma > 0$, then using:

$$A = -\alpha,\ B = 0,\ C = -\alpha/\beta$$

the array is synchronized if $a > \max\ \{\ 0,\ \lambda_0\ \}$, and the critical value of the diffusion coefficient is again given by (9). Let us consider in more detail the following case: $0 < m_0 < m_1$. Then, both conditions of the theorem 1, are satisfied if $a > \lambda_0 = m_0$. Thus, a can be chosen arbitrarily close to λ_0: Eq.(9) gives $D^* = 0$, and for $D > 0$, the array of canonical Chua's circuits with an arbitrary number of cells is synchronized.

4. CONCLUSIONS

In this paper we have analyzed rigorously the synchronization in a one-dimensional array of Chua's circuits. We have proved that under some conditions given in theorem 1, there exists an integer $N_0 = \mathrm{Int}\ (\ \pi/\arccos \nu\ )$, where ν is given by (5), such that all cells in the one-dimensional array of N cells, $N \le N_0$, are synchronized, i.e. each coupled subsystem has identical behavior, when $t \to \infty$.

We close this paper with the following questions for further study:

(i) generalize theorem 1 to two- and three-dimensional arrays of Chua's circuit;

(ii) find the conditions such that in an array of N-cells, the first " i " cells , $i < N$, are synchronized, while the remaining ($N - i$) cells are not synchronized.

Acknowledgements

This work is supported in part by the National Science Foundation under Grant MIP 89-12639, by the Office of Naval Research under Grant N00014-89-J-1402, and by the Macedonian Ministry of Sciences.

APPENDIX. **Proof of Theorem 1.**

Using the notation :

$$X_k = x_k - x_{k+1}, \quad Y_k = y_k - y_{k+1}, \quad Z_k = z_k - z_{k+1}$$

we obtain:

$$\left.\begin{aligned} \dot{X}_k &= -\lambda(x_k, x_{k+1})X_k + a_1 Y_k + a_2 Z_k + D(X_{k-1} - 2X_k + X_{k+1}) \\ \dot{Y}_k &= b_{11}Y_k + b_{12}Z_k + b_1 X_k \\ \dot{Z}_k &= b_{21}Y_k + b_{22}Z_k + b_2 X_k \\ & k= 1, 2, \dots, N - 1, \end{aligned}\right\} \tag{A.1}$$

with boundary conditions

$$X_0 = X_N = 0.$$

where

$$\lambda(x_k, x_{k+1}) \overset{\Delta}{=} \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} .$$

Applying the Mean-value theorem, we can write $\lambda(x_k, x_{k+1}) = f'(\xi)$, $\xi \in (x_k, x_{k+1})$, and hence

$$\lambda(x_k, x_{k+1}) > \lambda_0$$

Consider the Lyapunov function

$$W = \sum_{k=1}^{N-1} V(X_k, Y_k, Z_k)$$

and its trajectory derivative with respect to (A.1):

$$\dot{W} = \sum_{k=1}^{N-1} \{ -(\lambda +2D -a)X_k^2 + 2DX_kX_{k+1} + 2DX_kX_{k-1} - Q(X_k, Y_k, Z_k) \}$$

$\dot{W}$ is negative definite if the quadratic form:

$$P \overset{\Delta}{=} \sum_{k=1}^{N-1} \{ (\lambda_0 + 2D - a)X_k^2 - 2DX_kX_{k+1} - 2DX_kX_{k-1} \}$$

is positive definite. This is true if all main determinants Δ_k, $k = 1, \dots, N$ of the matrix:

$$\begin{pmatrix} 2\nu & -1 & 0 & . & 0 & 0 \\ -1 & 2\nu & -1 & . & 0 & 0 \\ 0 & -1 & 2\nu & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 2\nu & -1 \\ 0 & 0 & 0 & . & -1 & 2\nu \end{pmatrix}$$

are positive, where ν is given by (5). One can easily verify that the following recurrent equation is valid:

$$\Delta_k = 2\nu\Delta_{k-1} - \Delta_{k-2}, \ \Delta_0 = 1, \ \Delta_{-1} = 0, \ k = 1, \ \ldots, \ N-1.$$

If $\nu < 1$, then the last equation has a solution:

$$\Delta_k = \frac{\sin (k+1)\phi}{\sin \phi}, \ k = 1, \ \ldots. \ N-1$$

where $\phi = \arccos \nu$. For $\nu > 0$ and $N \le N_0$, $N_0 = \mathrm{Int}\ (\pi/\arccos \nu)$, all Δ_k, $k = 1, \ldots, N-1$ are positive. Hence, the origin of the system (A.1) is asymptotically stable, and consequently, the manifold M is asymptotically stable as well. ■

References

1. L.O.Chua, Lj.Kocarev, K.Eckert, and M.Itoh, "Experimental Chaos Synchronization in Chua's Circuit", *Int. J. of Bifurcation and Chaos*, 2(3), (1992), 705 - 708.

2. Lj.Kocarev, K.S.Halle, K.Eckert, L.O.Chua, and U.Parlitz, "Experimental demonstration of secure communications via chaotic synchronization", *Int. J. of Bifurcation and Chaos*, 2(3), (1992), 709 - 713.

3. U.Parlitz, L.O.Chua, Lj.Kocarev, K.S.Halle, and A.Shang, "Transmission of Digital Signals by Chaotic Synchronization", *Int. J. of Bifurcation and Chaos*, 2(4), (1992) in press.

4. V.Perez-Munuzuri, V.Perez-Villar, and L.O.Chua, "Propagation Failure in Linear Arrays of Chua's Circuits", *Int. J. of Bifurcation and Chaos*, 2(2), (1992), 403-406.

5. V.Perez-Munuzuri, V.Perez-Villar, and L.O.Chua, "Traveling Wave Front and its Failure in a One-dimensional Array of Chua's Circuits" to appear in *J. of Circuits, Systems and Computers*, vol.3, No.1, (1993).

6. V.Perez-Munuzuri, V.Perez-Villar, and L.O.Chua, "Autowaves for Image Processing on a Two-dimensional CNN Array of Chua's Circuits: Flat and Wrinkled Labyrinths", Technical Report M92/75, Electronic Research Laboratory, 1992.

7. L. O. Chua, "The Genesis of Chua's Circuit", *Archiv fur Elektronik und Ubertragungstechnik*, 46 (1992), 250-257.

8. M.P.Kennedy, "Robust op amp Realization of Chua's Circuit", *Frequenz*, 46 (1992) , 60-80.

9. Cruz M.J. and L.O.Chua [1992] "A CMOS IC nonlinear resistor for Chua's circuit", *IEEE Transactions on Circuits and Systems*, vol.39, N.9, in press.

10. A.N.Sharkovsky, Yu.Maistrenko, Ph.Deregel, and L.O.Chua, "Dry Turbulence from a Time-Delayed Chua's Circuit", *J. of Circuits, Systems and Computers*, Vol.3, No.2, (1993).

11. Lj.Kocarev, Lj.Karadzinov, Z.Tasev and P.Janjic, "New Canonical Realization of Chua's

Circuit Family", *Proc.of Yug.Soc.of Elec.Eng.- ETAN*, vol.25 (1991) 75-82.

12. A.S.Huang, "Two New Canonical Circuits of Realizing Every Member of Chua's Circuit Family", *J.Shanghai Ins. Mech. Eng.*, 13 (3), (1991), 885-902.

13. L.O.Chua, " A Zoo of Strange Attractors from the Canonical Chua's Circuit", *Proc. of the 35th Midwest Symposium on Circuit and Systems*, August 10-12, 1992.

14. L.O.Chua, "Global Unfolding of Chua's Circuit", *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, May 1993.

15. Lj.Kocarev, Lj.Karadzinov, and L.O.Chua, "N-Dimensional Canonical Chua's Circuit", *J. of Circuits, Systems and Computers*, Vol.3, No.1, (1993).

Figure captions:

Fig.1 The number of cells N in a linear array of Chua's circuits as a function of the diffusion coefficient: if $D_i < D \leq D_{i+1}$, $i = 2, 3, \ldots$, then $N_0 = i$, and the array with $N \leq N_0$ cells is synchronized.

Fig.2 Double scroll chaotic attractor observed from (a) the first cell and (b) the fifth cell of the array of Chua's circuit with five cells.

Fig.3 Synchronization in a linear array of Chua's circuits with five cells:

(a) x_1 vs. x_5, (b) y_1 vs. y_5 , and (c) z_1 vs. z_5.

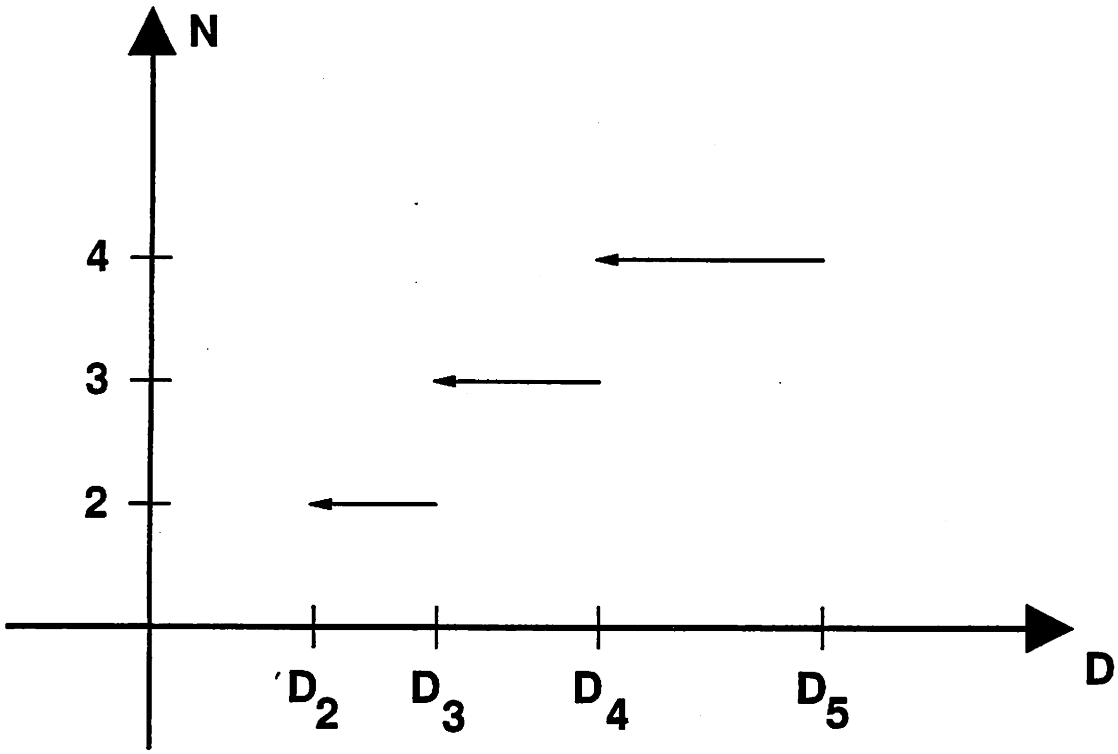

Fig. 1

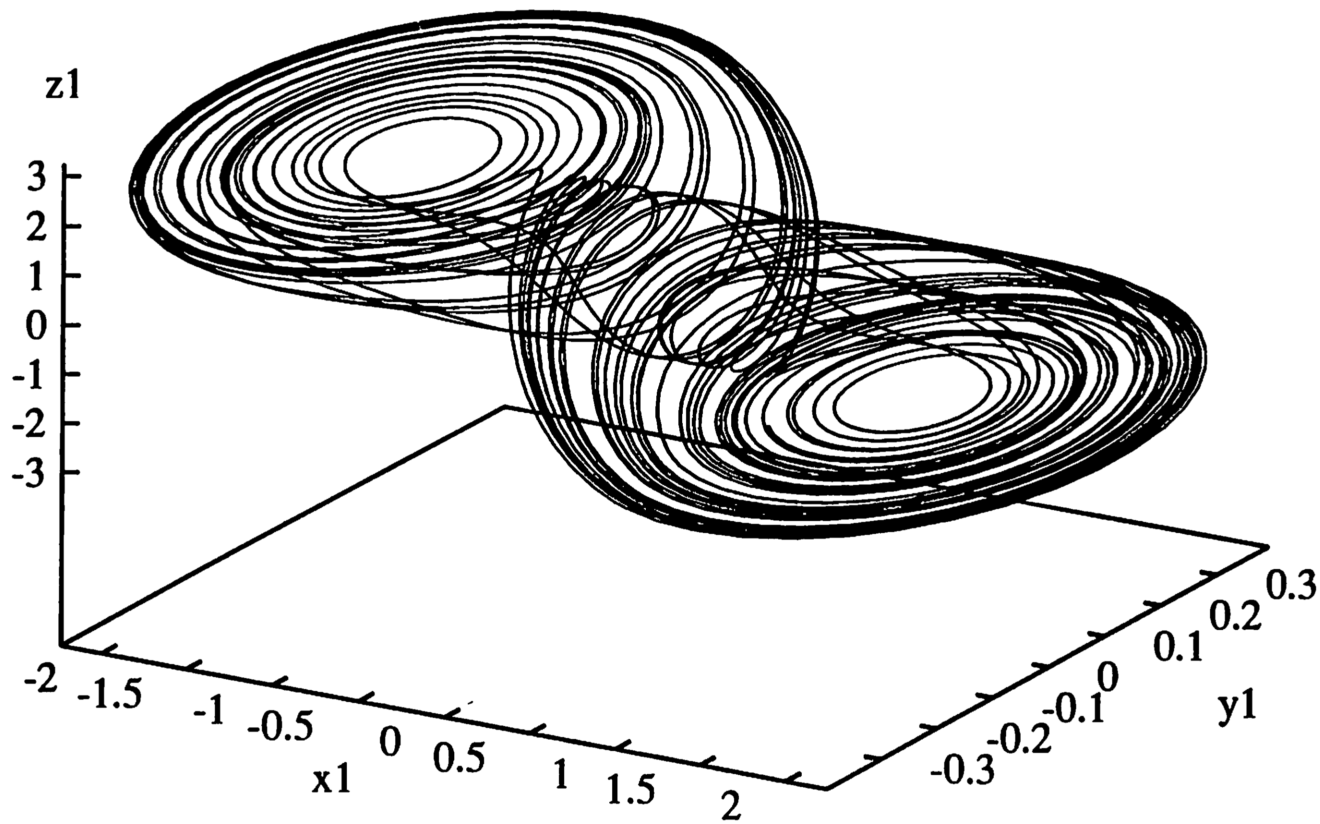

FIG. 2 (a)

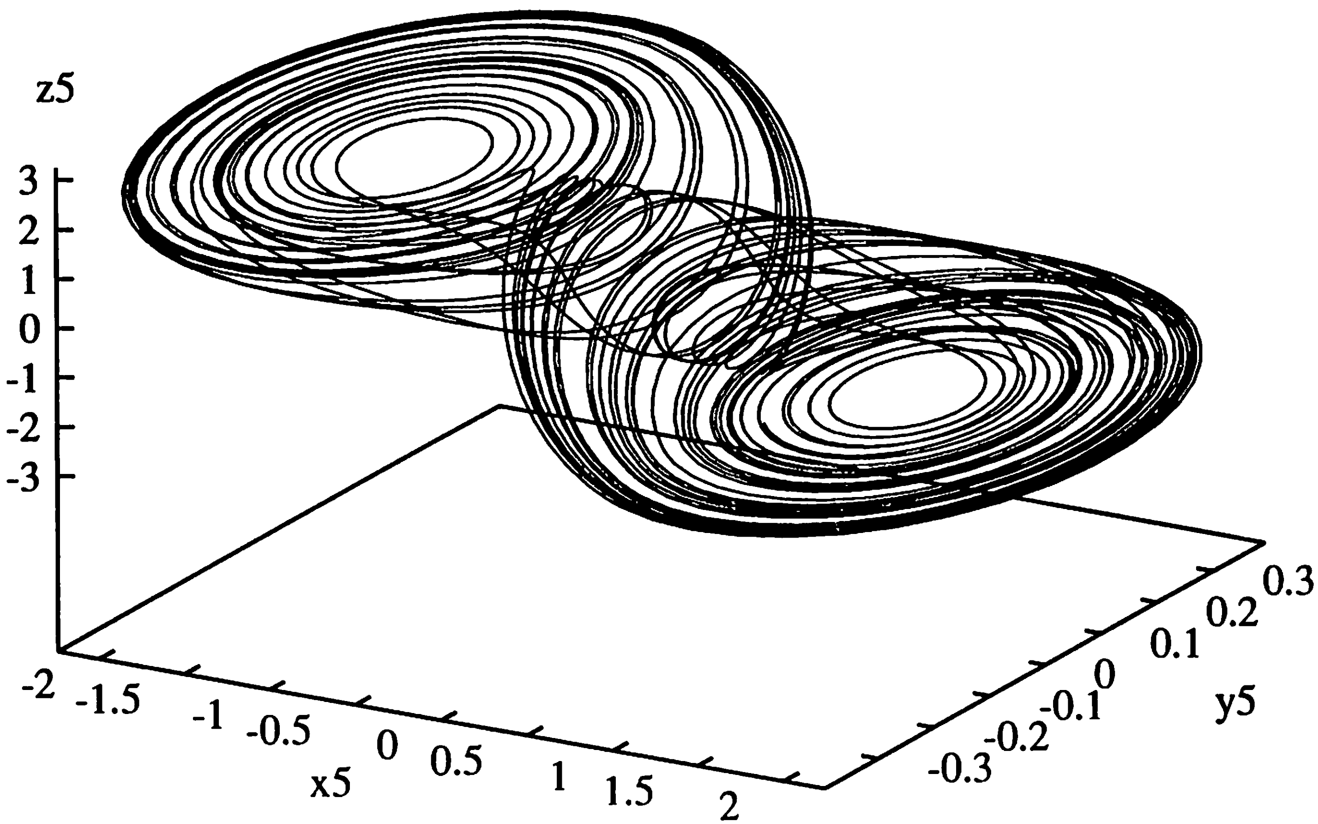

FIG. 2 (B)

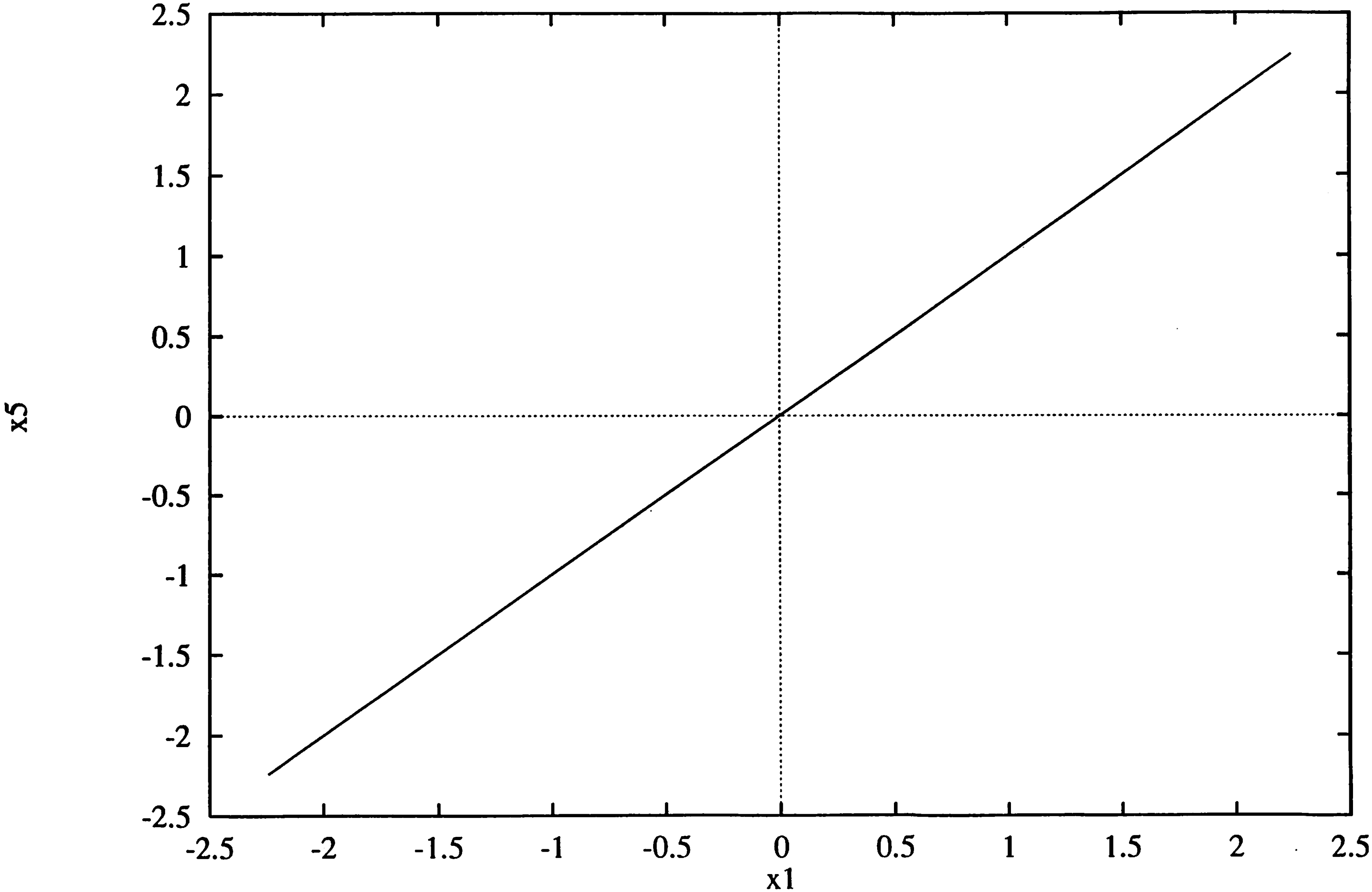

FIG. 3 (a)

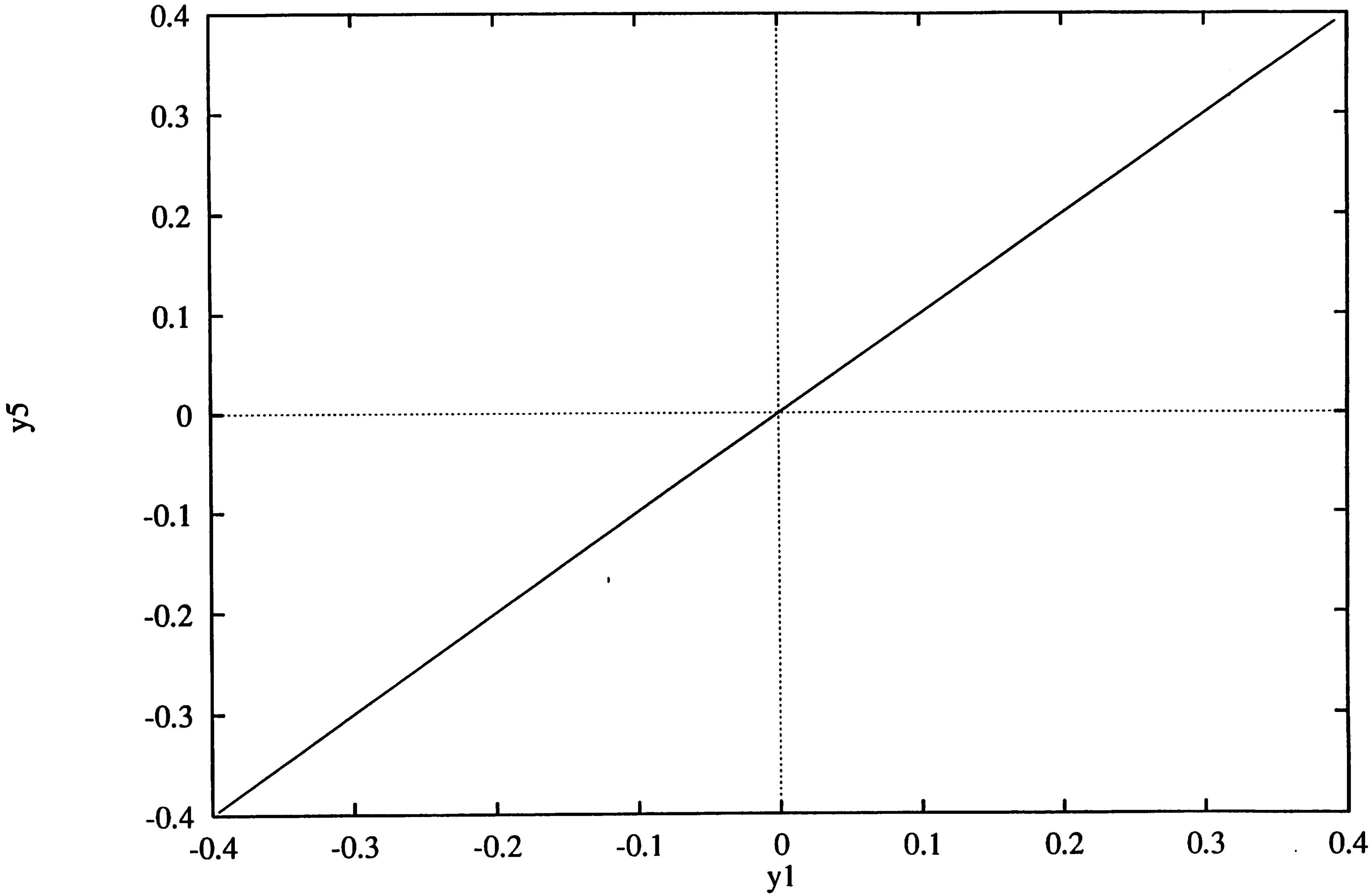

FIG. 3 (6)

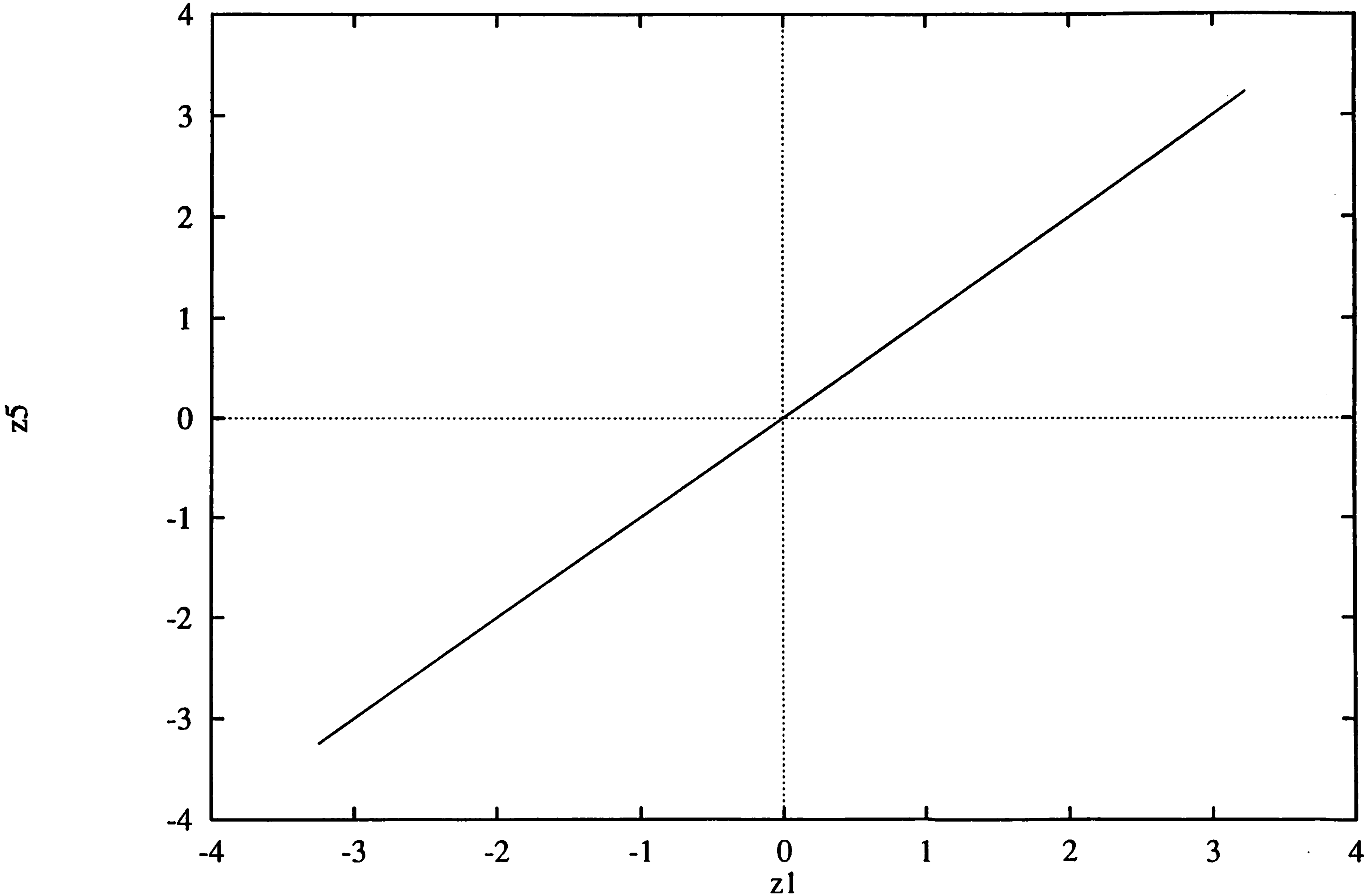

FIG. 3 (c)