SELF-SIMILARITY AND UNIVERSALITY IN CHUA'S CIRCUIT VIA THE APPROXIMATE CHUA'S 1-D MAP

by

A. P. Kuznetsov, S. P. Kuznetsov, I. R. Sataev,
and L. O. Chua

Memorandum No. UCB/ERL M93/15

25 February 1993

SELF-SIMILARITY AND UNIVERSALITY IN CHUA'S CIRCUIT VIA THE APPROXIMATE CHUA'S 1-D MAP

by

A. P. Kuznetsov, S. P. Kuznetsov, I. R. Sataev,
and L. O. Chua

Memorandum No. UCB/ERL M93/15

25 February 1993

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

This work is supported in part by the Office of Naval Research under grant N00014-89-J-1402 and the National Science Foundation under grant MIP-9001336.

SELF-SIMILARITY AND UNIVERSALITY IN CHUA'S CIRCUIT VIA THE APPROXIMATE CHUA'S 1-D MAP

by

A. P. Kuznetsov, S. P. Kuznetsov, I. R. Sataev,
and L. O. Chua

Memorandum No. UCB/ERL M93/15

25 February 1993

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

This work is supported in part by the Office of Naval Research under grant N00014-89-J-1402 and the National Science Foundation under grant MIP-9001336.

SELF-SIMILARITY AND UNIVERSALITY IN CHUA'S CIRCUIT VIA THE APPROXIMATE CHUA'S 1-D MAP

A. P. KUZNETSOV, S. P. KUZNETSOV, I. R. SATAEV
Institute of Radio-engineering and Electronics, Russian Academy of Sciences, Zelenaja 38, Saratov, 410019, Russia
and
L. O. CHUA
Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA

In this paper we investigate the features of the transition to chaos in a one-dimensional Chua's map which describes approximately the Chua's circuit. These features arise from the nonunimodality of this map. We show that there exists a variety of types of critical points, which are characterized by a universal self-similar topography in a neighborhood of each critical point in the parameter plane. Such universalities are associated with various cycles of the Feigenbaum's renormalization group equation.

On the cover of the first issue of the "International Journal of Bifurcation and Chaos" a beautiful color picture is presented, showing the topography of the dynamical behavior in the plane of two control parameters of the Chua's 1-D map which describes approximately the dynamics of the Chua's circuit.[1] From this picture, one can see that the borderline of chaos in the parameter plane of Chua's circuit possesses a complex fine structure. Its piecewise-smooth boundary curves are the critical lines, representing the loci of the accumulation points of period-doubling bifurcations.[2-5] Many cusps are located near the border of chaos, with two fold lines emanating from each cusp. Very narrow bands corresponding to regular (non-chaotic) regimes stretch far into regions representing chaotic regimes.

Transition to chaos via period-doubling cascades is a typically observed scenario while probing along an arbitrarily chosen path in the parameter plane from a region of regular motion to a chaotic regime. This scenario is characterized by the well known Feigenbaum's universality and scaling properties.[4,5] In particular, the period-doubling convergence to a critical line obeys a geometrical law with a universal exponent equal to 4.6692. It is natural to ask, whether there exists a generalization of such universality for the two-parameter situation? Such a generalization would allow us to uncover the fractal properties of the borderline of chaos in the parameter plane.

We shall show in this paper that the border of chaos in the Chua's 1-D map defined in references 6,7 contains an infinite set of critical points of codimension two. The vicinities of some of these points exhibit the property of self-similarity; namely, a universal two-parameter scaling property. These points may be found and classified with the help of an elegant procedure of constructing the binary tree of superstable orbits, first introduced in the analysis of circle maps by Shell et al.[8] The scaling properties observed in the vicinity of these points are determined by codes of itineraries on the binary tree.

Chua's circuit is an electronic system modeled by the following set of differential equations

$$\dot{x} = \alpha(y-h(x)),\ \dot{y} = x-y+z,\ \dot{z} = -\beta y, \tag{1}$$

where x, y, z are the dynamical variables, α and β are parameters, and $h(x)$ is a piecewise-linear function which is chosen in accordance with Chua et al.[6] as follows

$$\begin{aligned} h(x) &= (2x-3)/7, && x \geq 1 \\ &= -x/7, && |x| < 1 \\ &= (2x+3)/7, && x \leq -1 \end{aligned} \tag{2}$$

Using the Poincare section technique, the exact description of the system may be reduced to a two-dimensional map which, in turn, may be approximated by a one-dimensional map, henceforth called *the Chua's 1-D map.* The procedure for construction of this map

$$\pi^*: \qquad X \Rightarrow \pi^*(X)$$

is described in detail by Chua et al.[6,7] Unfortunately, it does not have a simple explicit representation.

It appears that the π^* map may have at least two extrema (a maximum and a minimum) and, thus, is *bimodal* in the region of states essential for the dynamics. This is precisely the circumstance responsible for the complex structure on the border of chaos in the (α, β) parameter plane.

In our following consideration, the double superstable period-2^n cycles will be of great significance. They are defined as cycles having two extremal points as their elements. In Fig. 1 the graph of the π^* map is shown corresponding to various points in the (α, β) parameter plane, where double superstable period-2^n cycles are realized. Taking the parameters of the period-2 superstable cycle as the initial point (see Fig. 1a), one can construct a *binary tree of superstable orbits* via the following procedure.

Suppose that a point (α, β) is found at which a period-2^n double superstable cycle is realized, the maximum being mapped into the minimum after p iterations, and the minimum being mapped to the maximum after $q = 2^n - p$ iterations. This cycle will henceforth be referred as (p, q)-type cycle. It's location in the parameter plane may be found at the intersection point of two curves $U(p)$ and $D(q)$: the $U(p)$ curve is defined by the condition that the maximum is mapped to the minimum after p iterations, and the $D(q)$ curve is defined by the condition that the minimum is mapped into the maximum after q iterations. Moving along the $U(p)$ curve and tracing the attractive cycle we come across the period-doubling bifurcation point and then find the point at which the period-2^{n+1} cycle becomes double superstable. Apparently, its type is $(p, p+2q)$. In a similar way the double superstable cycle of the $(2p+q, q)$-type may be found while moving along the D(q) curve.

In Fig. 1 the initial steps of this process are shown. We shall restrict our following considerations to the upper half of the full binary tree, where the corresponding orbits visit only two of the three piecewise-linear regions of the vector field (1). Also a qualitative picture of the tree is sketched in Fig. 2.

We shall now code the itineraries on the tree using the symbols U and D to designate the path along the U and D curves after each branching, respectively. While moving along the tree in accordance to an arbitrary code *UUDDUDU*..., one may observe a period-doubling bifurcation cascade. The accumulation point (*critical point*) of this cascade belongs to the borderline of chaos,[9] and the properties of universality and scaling in the neighborhood of this point are defined by the structure of the code.

To uncover these properties we turn now to renormalization group (RG) analysis, which is a generalization of that suggested by Feigenbaum.[4,5] Taking the point at which the π^* map has a maximum as our reference point $\bar{X}$, we consider further the translated map $f(x)=\pi^*(x+\bar{X})-\bar{X}$. Let us apply this mapping twice and rescale the dynamical variable to normalize the resulting map at the origin, namely, $f(0)=1$. Multiple repetition of this procedure leads to a recurrent functional equation

$$f_{n+1}(x) = \alpha_n f_n(f_n(x/\alpha_n)), \tag{3}$$

where $\alpha_n=1/f_n(f_n(0))$.

Let us now consider a critical point corresponding to some definite coding sequence of the symbols U and D and apply the above procedure to the Chua's 1-D map at this point. The functions $f_n(x)$ may be calculated via the chain rule

$$f(x) = f^{2^n}(x \cdot f^{2^n}(0))/f^{2^n}(0), \tag{4}$$

where $f^{2^n}(x)$ designates the 2^n-fold functional composition of the map $f(x)$. Such calculations for different integer n reveal a simple correlation between the structure of the itinerary code and the behavior of the iterations of the functional map (3).

If the code contains an infinite number of repetitions of the same symbol U as its tail, then the sequence f_n converges to a limit function corresponding to a fixed point of the functional map (3) (Fig. 3a). This limit function is a universal function describing the dynamics of one-dimensional maps at the so called *tricritical point*. Tricritical points are the terminal points of the Feigenbaum's critical lines in the parameter plane. They were introduced by Chang et al. while studying the two-parameter quartic map [10]

$$x_{n+1} = 1 + Ax_n^2 + Bx_n^4 . \tag{5}$$

These authors have also found the corresponding scaling factor α (see the Table) and the polynomial approximation for the universal function $g_T(x) = 1 - 1.8341x^4 + 0.0130x^8 + 0.3119x^{16}+\ldots$ It may be verified that it coincides for large n with the functions f_n shown in Fig. 3a.

Codes with trailing symbols D also correspond to a tricritical behavior. In this case the f_n sequence appears to converge to the function which is conjugate to the $g_T(x)$ via the variable change: $g^*_T(x)=[g_T(x^{1/2})]^2$. It is *a conjugate tricritical point* in accordance with the terminology introduced by Chang et al.[10] The function $g_T(x)$ may be obtained in this case also, but the origin must be taken at the minimum point of the $\pi^*(X)$ map rather than at the maximum.

If an itinerary is k-periodic beginning from some step, then the functional sequence produced by Eq.(3) at the corresponding critical point becomes k-periodic too for large n. In Figs. 3b,c a period-two and a period-three examples are presented. Thus, the cycles of period k of the RG equation are responsible for the dynamics at the critical point in the case of period-k itineraries. To find the universal functions corresponding to the elements of these cycles is equivalent to finding the fixed points of the k-fold iterated RG equation

$$g(x) = \alpha g^{2^k}(x/\alpha), \tag{6}$$

where the scaling factor α is a product of factors α_n over all the k elements of the corresponding cycle. We have obtained the numerical solutions of Eq.(6) for k=2 and 3. The values of the scaling factors are presented in the Table. We have verified that the polynomial approximations calculated numerically are in agreement with the functions f_n obtained by iterating the Chua's 1-D map directly using Eq.(4)) at the corresponding critical points (Figs. 3b,c) for sufficiently large n.

Finally, if we choose a random sequence of symbols U and D as our itinerary, find the corresponding critical point in the (α, β) parameter plane, and choose the map at this point as the initial function for Eq.(3), then the resulting functional sequence f_n will be random too (Fig. 3d). In this case we must deal with chaotic dynamics of the RG map or, in other words, with *renormalization chaos*. The possibility of such phenomenon in the bimodal maps was discussed earlier by Gambaudo et al.[11]

The next step in our RG analysis consists of studying the behavior of the orbits of Eq.(3) in the vicinity of the solutions considered above. We shall limit our considerations to the case of periodic codes. This assumption gives us a possibility to reveal universal self-similar structures in the parameter plane near the borderline of chaos.

It was already mentioned that period-k cycles of Eq.(3) correspond to the fixed points of Eq.(6). Let us take a small perturbation of the fixed point, $f(x) = g(x) + h(x)$, and demand that after the RG transformation $f(x) \Rightarrow \alpha f^{2^k}(x/\alpha)$, this perturbation would preserve its form to within a constant multiplier. (Here α is a scaling factor found by solving the fixed point of Eq.(6).) As a result, we obtain the following eigenproblem:

$$\delta \cdot h(x) = \alpha[F_0^{N-1}(x)h(x/\alpha) + \sum_{m=1}^{N-2} F_m^{N-1}(x)h(g^m(x/\alpha)) + h(g^{N-1}(x/\alpha))], \tag{7}$$

where

$$F_m^{N-1}(x) = [\frac{d}{d\xi}(g^{N-1}(\xi))]_{\xi=g^{m+1}(x/\alpha)}, \quad N=2^k,\ k=1,2,3,...$$

For each of the cycles of the RG equation two related eigenvalues δ_1, δ_2 may be found, which exceed unity in modulus and are not connected to the infinitesimal change in variables. These eigenvalues are the factors which determine the scaling properties of the parameter plane topography in the vicinity of the corresponding critical points. Namely, if the coordinate axes in the parameter plane are chosen in a proper way (*scaling variables*), then the topography in a small vicinity of the critical point is reproduced after rescaling along these axes by the factors δ_1 and δ_2. It corresponds to a change of the temporal periods by a factor of 2^k. It should be noted that in small enough scales this topography is universal for all critical points having itineraries with different initial symbols but identical periodic tails. It is also universal for all other bimodal two-parameter 1-D maps.

As an example, Fig. 4a shows the universal parameter plane topography in the vicinity of the critical point related to the period-2 cycle of the RG equation. For the Chua's 1-D map the corresponding critical point has the coordinates α_c=3.3905335, β_c =4.0549327. The scaling variables ξ and η used in Fig. 3 are connected to the parameters of the initial map via the relation

$$\xi=0.86(\alpha-\alpha_c)-0.57(\beta-\beta_c), \quad \eta=0.41(\alpha-\alpha_c)-0.26(\beta-\beta_c). \qquad (8)$$

To illustrate the property of self-similarity, a fragment inside the small rectangle shown in Fig. 4a is reproduced in Fig. 4b after magnification. The scale change along the coordinate axes is given by the factors of δ_1 and δ_2 which have been found for this critical situation. Observe that Fig. 4a is well reproduced by Fig. 4b.

In conclusion we note that the problems discussed in this paper which are related to the renormalization group approach for explaining the complex fine structure on the border of chaos have recently attracted much attention among many theorists (see references [8-14]). Chua's circuit represents an ideal object for this research direction because it allows both a simple theoretical analysis and an electronic experimental investigation.

References

1. M. Komuro, R. Tokunaga, T. Matsumoto, L. O. Chua, and A. Hotta, "Global bifurcation analysis of the double scroll circuit", *Int. J. Bifurc. and Chaos* **1** (1991) 139-182.

2. P. J. Myrberg, "Iteration der reelen polynome zweiten grades", *Ann. Acad. Sci. Fennica*, **A336** (1963) 1-18.

3. R. M. May, [1976] "Simple mathematical models with very complicated dynamics",

Nature **261** (1976) 459-467.

4. M. J. Feigenbaum, "Quantitative universality for a class of nonlinear transformations", *J. Stat. Phys.* **19** (1978) 25-52.

5. M. J. Feigenbaum, "The universal metric properties of nonlinear transformations", *J. Stat. Phys.* **21**, (1979) 669-706.

6. L. O. Chua, M. Komuro, and T. Matsumoto, "The double scroll family, Parts I and II", *IEEE Trans. Circuits and Systems* **CAS-33** (1986) 1072-1118.

7. M. Genot, "Application of 1-D map from Chua's circuit: A pictorial guide", *J. of Circuits, Systems and Computers* **3** (1993) No 2.

8. M. Shell, S. Fraser, and R. Kapral, "Subharmonic bifurcations in the sine map: An infinite hierarchy of cusp bistabilities", *Phys. Rev.* **A28** (1986) 373-379.

9. R. S. MacKay and C. Tresser, "Boundary of topological chaos for bimodal maps of the interval", *J. London Math. Soc.* **37** (1988) 164-181.

10. S. J. Chang, M. Wortis, and J. A. Wright, "Iterative properties of a one-dimensional quartic map. Critical lines and tricritical behavior", *Phys. Rev.* **A24** (1981) 2669-2684.

11. J. M. Gambaudo, J. E. Los, and C. Tresser, "A Horseshoe for the Doubling Operator Topological Dynamics for Metric Universality", *Phys. Lett.* **A123** (1987) 60-65.

12. S. Fraser and R. Kapral, "Universal vector scaling in one-dimensional maps", *Phys. Rev.* **A25** (1984) 3223-3233.

13. R. S. MacKay and C. Tresser, [1987] "Some flesh on the skeleton: The bifurcation structure of bimodal maps", *Physica* **D27** (1987) 412-422.

14. J. Carcasses, C. Mira, M. Bosch, C. Simo, and J. C. Tatjer, "Crossroad area - spring area transition (1) Parameter plane representation", *Int. J. Bifurc. and Chaos* **1** (1991) 183-196.

FIGURE CAPTIONS

Fig. 1. An illustration of the superstable orbit binary tree construction procedure. The itineraries are coded by a sequence of symbols U (up) and D (down). The double superstable cycles for the Chua's 1-D map corresponding to the branching points of the binary tree are presented. (a) α=2.42863139, β=2.66840554; (b) α=3.18121999, β=3.74564551; (c) α=3.00564702, β=3.41905639; (d) α=3.38557045, β=4.05622505; (e) α=3.30417266, β=3.92456471.

Fig. 2. Rough sketch of a binary tree location in the parameter plane of a bimodal 1D-map. The branching points correspond to double superstable cycles, their (p, q)-types are shown. Pieces of Feigenbaum's critical lines (shown dotted) are labeled by F. A codimension-two critical point is located at the end of each branch in the limit of infinite branchings, so there is an infinite number of such points. Some tricritical points are shown by "solid" circles, and conjugate tricritical points are shown by "open" circles. Also the critical points corresponding to period-2 (code *UDUDUD*...) and period-3 (code *UUDUUD*...) cycles of the RG equation are shown by a small square and a small triangle and labeled RG2 and RG3, respectively.

Fig. 3. Graphs of functions from the sequence $f_n(x)$ which were produced by iterating Eq.(3). Values of n are shown by numbers. The Chua's 1-D map was taken as the initial function f_0, the origin being chosen at the point of its maximum (see Fig.1), for the following parameter values:

(a) α=3.42646401, β=4.11924613, code *UUUUUU*... This correspond to a fixed point of the RG equation.

(b) α=3.39053349, β=4.05493271, code *UDUDUD*... This correspond to a period-2 cycle of the RG equation.

(c) α=3.47246137, β=4.18636722, code *UUDUUD*... This correspond to a period-3 cycle of the RG equation.

(d) α=3.46837499, β=4.17984652, code *UUDDUDU*...This correspond to renormalization chaos.

Fig. 4. Universal topography of the dynamical behavior in the parameter plane in the vicinity of the critical point corresponding to a period-2 cycle of the RG equation. The critical point is located exactly at the center of the picture; numbers designate the cycle periods. The coordinate axes correspond to the scaling variables defined by (8). To demonstrate the self-similarity phenomenon, a small fragment of the picture in (a) is reproduced with magnification along the vertical and horizontal axes by a factor equal to 14.5957 and 35.9286, respectively.

Table 1. Universal scaling constants for the critical behavior types associated with the period-1, -2 and -3 solutions of the RG equation

Type of criticality	Code	Phase space scaling factor α	Parameter space scaling factors $\delta_{1,2}$
Tricritical fixed point of the RG equation	...*UUUUUU*...	-1.69030297	7.28468622 2.85712414
Period-2 cycle of the RG equation	...*UDUDUD*...	-4.86264509	14.5957450 35.9286114
Period-3 cycle of the RG equation	...*UUDUUD*...	8.03026759	46.2910330 244.768707

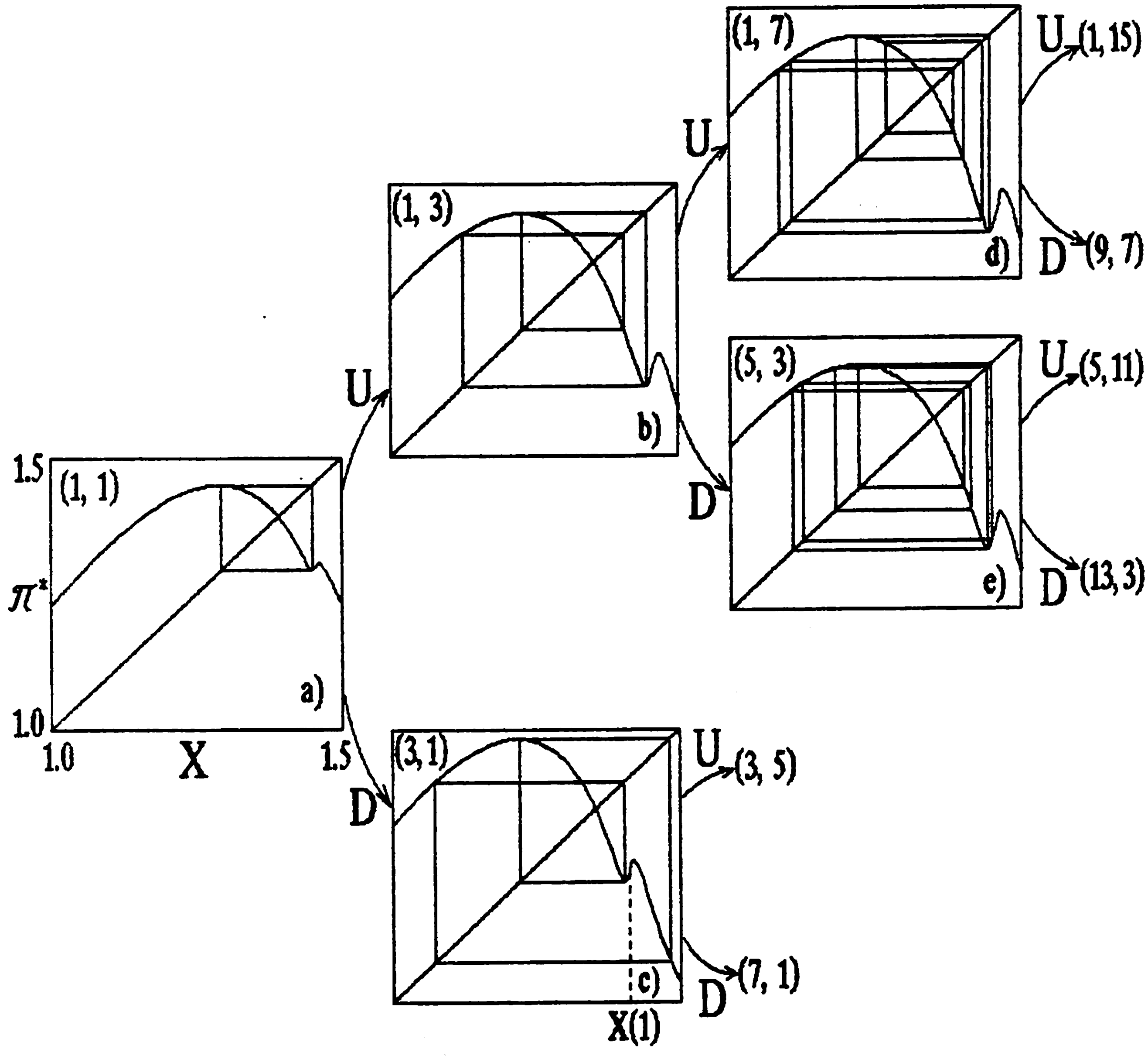

(1, 1)
1.5
π^*
1.0
1.0
X
1.5
a)
U
D
(1, 3)
b)
(3, 1)
c)
X(1)
U
(3, 5)
D
(7, 1)
U
D
(1, 7)
d)
U
(1, 15)
D
(9, 7)
(5, 3)
e)
U
(5, 11)
D
(13, 3)

Fig. 1

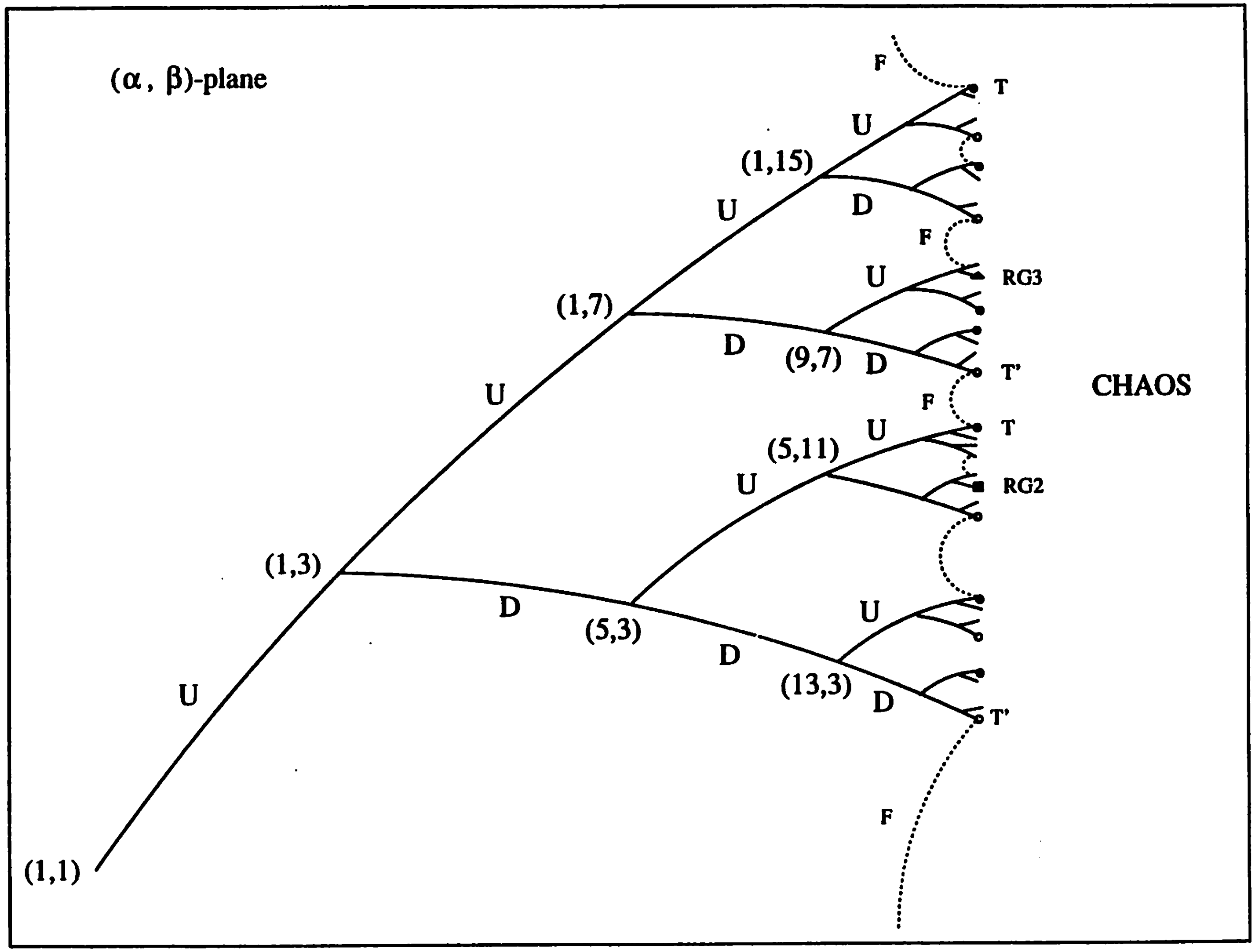

(α, β)-plane
F
T
U
(1,15)
D
U
F
RG3
U
(1,7)
D
(9,7)
D
T'
CHAOS
U
F
T
(5,11)
U
RG2
(1,3)
D
(5,3)
U
D
(13,3)
D
T'
U
F
(1,1)

Fig. 2

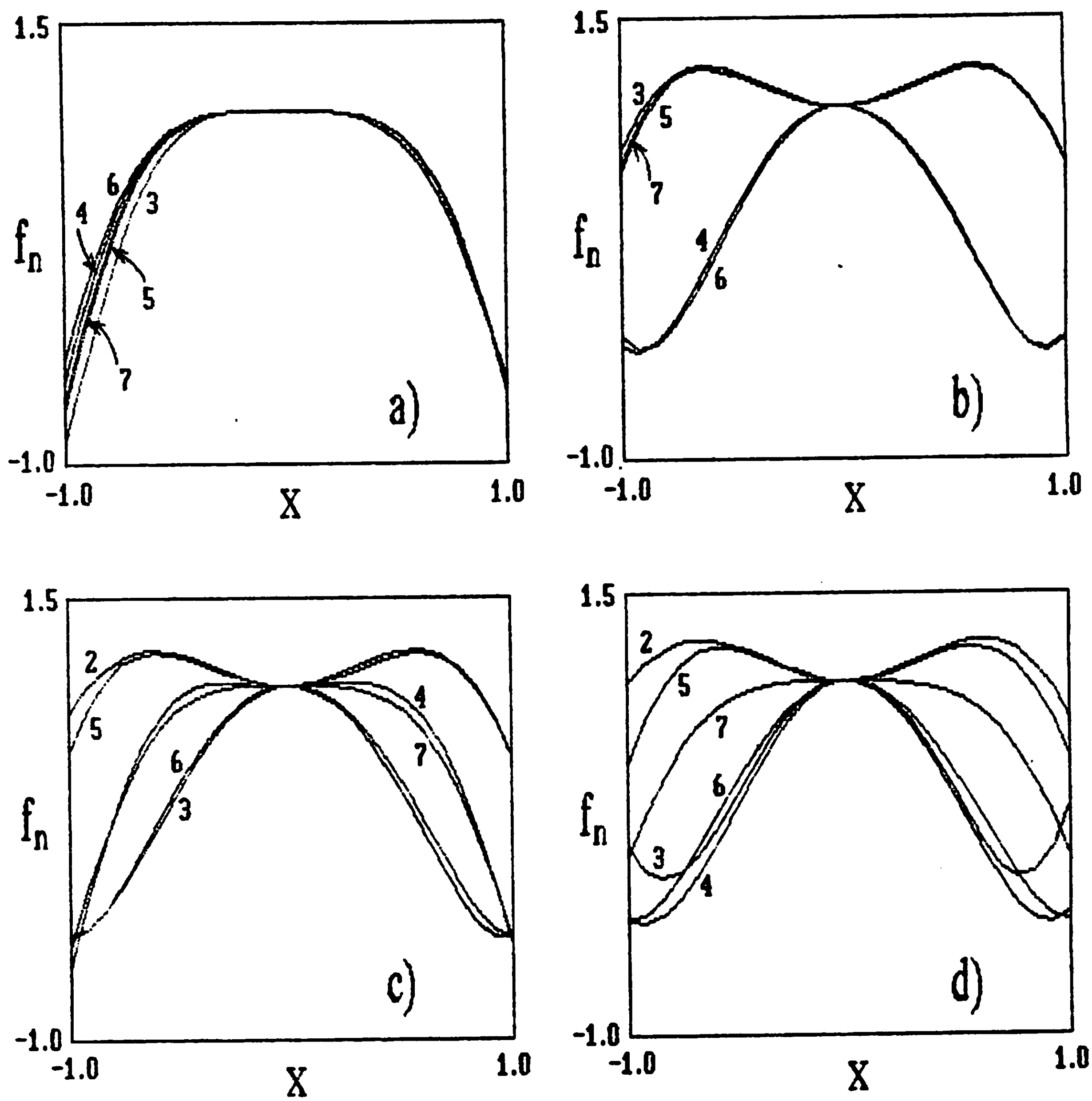
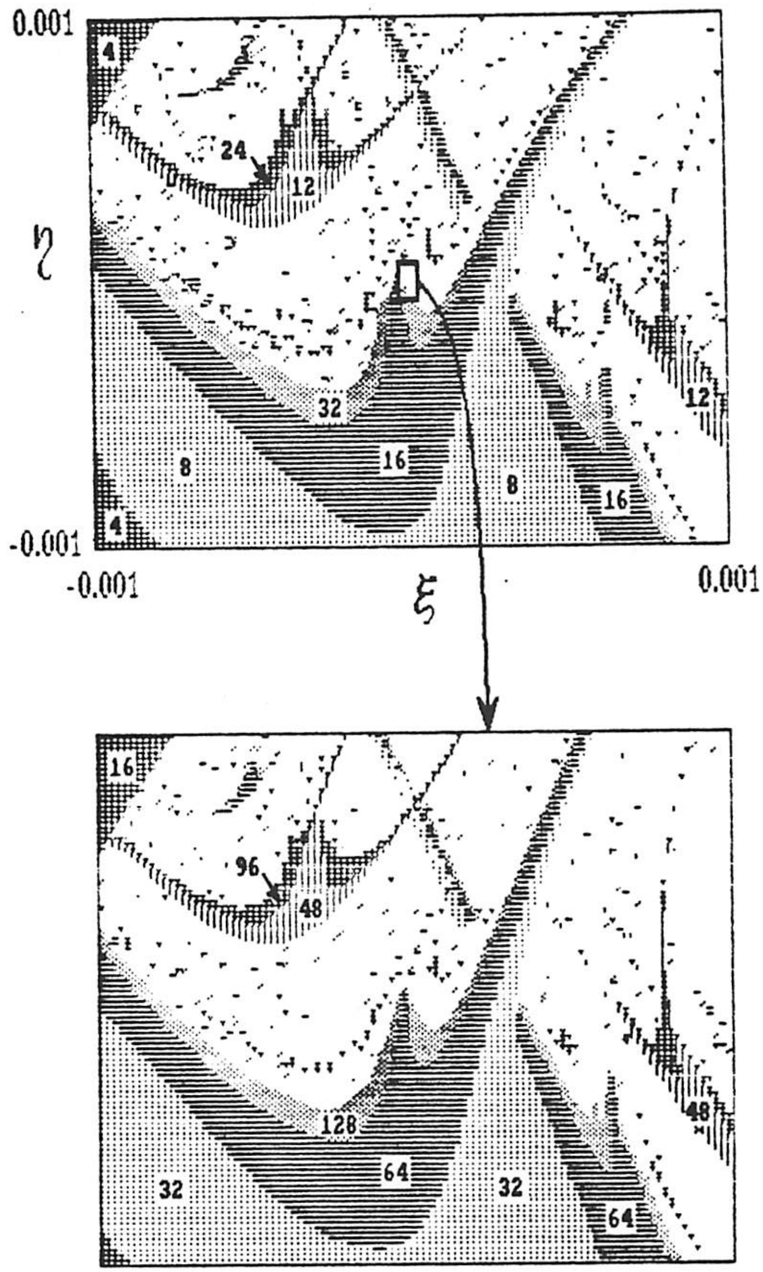


Fig.3


0.001
-0.001
-0.001
0.001
η
ξ
4
24
12
32
16
8
8
16
12
4
16
96
48
128
64
32
32
64
48

a)

b)

Fig.4