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Transition in dynamical regime by driving: a method of control and synchronization of chaos

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Abstract: We propose a simple method for the synchronization and the control of chaos. A general qualitative description of the method is presented.

The study of chaos in the last 20 years had a tremendous impact on the foundations of the sciences and on engineering. Synchronization [Afraimovich, Verichev & Rabinovich, 1986; Pecora & Carroll, 1990] and control [Ott, Grebogy & Yorke, 1990] of chaos have recently aroused a great deal of interests in light of their potential applications in engineering. In this letter, we consider the following dynamical system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{K}(\mathbf{y} - \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}) \end{aligned} \tag{1}$$

where $\mathbf{x}$, $\mathbf{y} \in \Re^n$ and $\mathbf{K}$ is an n by n diagonal matrix. $\mathbf{y}$ is referred to as the *goal*. Without loss of generality, we will assume that $\mathbf{f}(0) = \mathbf{g}(0) = 0$.

The dynamical system (1) describes a simple method to entrain $\mathbf{x}(t)$ to a goal $\mathbf{y}(t)$. By considering this system with a specified goal $\mathbf{y}$, the desired dynamical regime can be obtained. Thus, if the goal is a chaotic or a periodic signal, synchronization or control of chaos is achieved. Moreover, (1) is a simple alternative to Hubler and Jackson's non-feedback method of entrainment and migration [Jackson, 1991].

Equation (1) also describes a transition or a transformation from one chaotic attractor,$\mathbf{x}(t)$ to a second, $\mathbf{y}(t)$ as K_{ii} is varied. When $K_{ii} = 0$, for $i = 1, 2..., n$, two uncoupled and independent chaotic attractors coexist. When $K_{ii} \to \infty\ \forall i$, chaos synchronization between them occurs. But what happens when $0 < K_{ii} < \infty$?

Case f=g

Theorem 1: If $\mathbf{f} = \mathbf{g}$ and $|\mathbf{x}(t = 0) - \mathbf{y}(t = 0)|$ is sufficiently small then there exists finite values of K_{ii}, $\tilde{k}_i$ with $i = 1, 2, ..., n$ such that for $K_{ii} > \tilde{k}_i$, $\mathbf{x}(t)$ approaches the goal $\mathbf{y}(t)$.

Proof: First note that the inequalities

$$|a_{jj}| \quad > \quad \textstyle\sum_{i \neq j}^{n} |a_{ij}| \tag{2}$$

where $j = 1, 2, ..., n$, are sufficient for the stability of a matrix with a negative diagonal $\mathbf{A} = (a_{ij})$. Now, denote **u=x-y**, so that from (1) we have

$$\dot{\mathbf{u}} = \{ \begin{pmatrix} -K_{11} & 0 & ... & 0 \\ 0 & -K_{22} & ... & 0 \\ ... & ... & ... & 0 \\ 0 & 0 & ... & -K_{nn} \end{pmatrix} + D\mathbf{f}\ |_{\mathbf{u}=0} \} \mathbf{u} + O(\mathbf{x}, \mathbf{y}) = \mathbf{A}\mathbf{u} + O(\mathbf{x}, \mathbf{y}) \tag{3}$$

where $D\mathbf{f}$ is the Jacobian matrix of $\mathbf{f}$ and $O(\mathbf{x}, \mathbf{y})$ represents the higher-order terms. It is obvious that one can find K_{ii} such that (2) is satisfied and such that the matrix $\mathbf{A}$ has negative diagonal elements. Therefore, $\mathbf{u} = 0$ is asymptotically stable and $\mathbf{x}(t)$ approaches $\mathbf{y}(t)$ as $t \to \infty$ □

A few remarks are in order. First, Theorem 1 states that the signal $\mathbf{x}(t)$ and $\mathbf{y}(t)$ can be synchronized. If they are chaotic, the phenomenon is referred to as *chaos synchronization*. We note that in Pecora and Carroll's approach stable commonly driven subsystems can be synchronized if their Lyapunov exponents are all negative. Instead of searching for a stable subsystem, the present method provides a way to synchronize *any* chaotic system. The variation of the driving strength K_{ii} allow us to turn the Lyapunov exponents of the linearized system $\mathbf{u}$ negative. The negativity of the Lyapunov exponents ensures that the null solution of the linear equation in $\mathbf{u}$ is asymptotically stable. Furthermore a bidirectional coupling leads to mutual synchronization [Anishenko, Vadivasova, Postnov & Safanova, 1992; Kowalski,Albert & Gross, 1990]. This can be proved in a similar fashion to theorem 1.

Secondly, Theorem 1 provides a method to stabilized unstable periodic orbits embedded in a chaotic attractor of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Chaotic attractors are closures of a dense set of unstable periodic orbits [Eckmann & Ruelle, 1985; Auerbach, Cvitanović, Eckmann & Gunaratne,1987;Cvitonović, 1988]. By supplying external oscillators, $\mathbf{y}(t)$ that mimic an unstable periodic orbit of the chaotic attractor of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, we can synchronize $\mathbf{x}(t)$ to the external oscillators [Pyrogas,1992], thus stabilize an unstable periodic orbit. Therefore (1) is a useful method of control especially when the parameters of the system *to be controlled* are not accessible or cannot be altered.

Finally, the current method is also suitable for migration control [Jackson, 1991]. Many systems described by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ possess multiple basins of attraction. The dynamical regime in each of them may be very different. By setting the goal attractor in a different basin from the current attractor's, (1) can be used to switch between attractors in a multiple attractors system. We can also achieve migration by choosing the goal to be a point in the desired basin and release the control (set $K_{ii} = 0$) when this basin is reached.

Case $\mathbf{f} \neq \mathbf{g}$

For simplicity, we will assume that $K_{ii} = k$, for all $i = 1, 2, ..., n$. In this case (1) is rewritten as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + k \ (\mathbf{y} - \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}) \end{aligned} \tag{4}$$

where k is a real nonnegative parameter.

Theorem 2: For sufficiently small $|\mathbf{x}(t=0)| + |\mathbf{y}(t=0)|$ and $\epsilon = k^{-1}$, there exists t_0 such that $\mathbf{x}(t)$ converges uniformily to $\mathbf{y}(t)$ as $\epsilon \to 0^+$ on all closed subsets of $t_0 < t < \infty$.

Proof: (1) can be rewritten as

$$\begin{aligned} &\epsilon\dot{\mathbf{x}} = -(\mathbf{x} - \mathbf{y}) + \epsilon\mathbf{f}(\mathbf{x}) \\ &\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) \end{aligned} \tag{5}$$

To examine (5) for small positive ϵ it is convenient to make use of two systems which are associated with (5). The first system called the *degenerate* system is obtained by formally setting $\epsilon = 0$ in (5). This gives

$$\begin{aligned} \mathbf{x} &= \mathbf{y} \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}) \end{aligned} \tag{6}$$

The second system is obtained by making the 'streching' transformation of independent variable $\tau = t/\epsilon$ in (5) and then setting $\epsilon = 0$. This yields

$$\frac{d\mathbf{x}}{d\tau} = -(\mathbf{x} - \mathbf{y}) \tag{7}$$

and,

$$\frac{d\mathbf{y}}{d\tau} = 0 \tag{8}$$

Since the only solution of (8) is $\mathbf{y} = C =$ constant, (7) can be written as

$$\frac{d\mathbf{x}}{d\tau} = -(\mathbf{x} - C) \tag{9}$$

The system (9) is called a *boundary-layer* system. Since it is an asymptotically stable system, the proof of the theorem follows from the theorem of Tihonov [Tihonov, 1952] and its generalization [Hoppensteadt,1966] □

Let us assume that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a chaotic attractor $\mathcal{A}_f$ and system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ a chaotic attractor $\mathcal{A}_g$. For typical dynamical system (4), a chaotic attractor denoted by $\tilde{\mathcal{A}}_k$ exists for every $k \geq 0$. This can be proved for $k \to 0$ and $k \to \infty$ (this latter case actually follows from Theorem 2). By denoting the projection of $\tilde{\mathcal{A}}_k$ on the subspace $\mathbf{x}$, $\Pi(\tilde{\mathcal{A}}_k)$, by $\mathcal{A}_k$, it is clear from (6) that $\mathcal{A}_\infty = \mathcal{A}_g$. Thus when k varies from 0 to ∞, $\mathcal{A}_k$ is transformed from $\mathcal{A}_0 = \mathcal{A}_f$ to $\mathcal{A}_\infty = \mathcal{A}_g$.

We now present a qualitative description of this transformation. Let us consider the corresponding undriven system of (4)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - k\ \mathbf{x} \tag{10}$$

A typical vector field (10) undergoes various bifurcations from chaos to a fixed point as k increases from 0 to ∞. On the other hand the driven system (4) also bifurcates but for typical vector fields, the dynamical regime remains chaotic as k is increased. In fact, we have found that a typical system (4) bifurcates from chaos to hyperchaos and vice versa. If the origin of (10) is asymptotically stable then $\mathcal{A}_k$ becomes more and more similar or *equivalent* (see the definition below) to $\mathcal{A}_g$ as k is increased.[1] To illustrate this concept we choose the vector field $\mathbf{f}$ to be that of Chua's circuit [Chua, Komuro & Matsumoto, 1986] and $\mathbf{g}$ to be that of the Lorenz system [Sparrow, 1982]. In Fig.(1) we show the projection of (4) onto the subspace $\mathbf{x}$, $\Pi(\tilde{\mathcal{A}}_k)$ for $k = 0$, $k = 10$ and $k = 100$. The latter two correspond to a stable origin for (10). From Fig.(1c), one can see that the chaotic attractor is similar to the Lorenz attractor in Fig.(1d). To characterize this transition or transformation, we have calculated the mutual information [Fraser & Swinney,1986] between a component of $\mathbf{x}$ and $\mathbf{y}$ (see Fig.(2)). We notice a monotonic rise of the mutual information as k is increased indicating stronger correlation between $\mathbf{x}(t)$ and $\mathbf{y}(t)$.

We will adopt the following definition of equivalence between two chaotic attractors [Afraimovich, Verichev & Rabinovich, 1986].

[1] This phenomenon has also bee observed recently by [Mayer-Kress, 1993]

Definition: A attractor $\mathcal{A}_k$ is *equivalent to* $\mathcal{A}_g$ if
(1) there exists a homeomorphism $h_1 : \Re^n \to \Re^n$ such that $h_1(\mathcal{A}_k) = \mathcal{A}_g$.
(2) there exists a diffeomorphism $h_2 : \mathcal{A}_k \to \mathcal{A}_g$ such that for any trajectory $\varphi_t(\mathbf{x}, \mathbf{y}) \subset \tilde{\mathcal{A}}_k$, $h_2(\Pi(\varphi_t(\mathbf{x}, \mathbf{y}))) = \zeta_{t+\alpha(t)}(\mathbf{y})$ with $\lim_{t\to\infty} \frac{t+\alpha(t)}{t} = 1$ where φ_t and ζ_t are the flow of (4) and $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$, respectively.

The first condition in this definition ensures topological equivalence of the chaotic attractors. Whereas the second condition ensures the equality of the Lyapunov exponents and dimensions of $\mathcal{A}_k$ and $\mathcal{A}_g$. Numerically, we found that the attractors shown in Fig.(1c) and (1d) are equivalent in the sense of the definition given above. We have calculated the templates for these attractors using the approach of Mindlin et al [Mindlin, Hou, Solari, Gilmore & Tufillaro, 1990] and found that they are the same. Moreover, their Lyapunov exponents and their correlation dimension are also the same. Fig.(3) shows the synchronization for $k = 100$. As expected, the synchronization is not perfect. But the equivalence between $\mathcal{A}_k$ and $\mathcal{A}_g$ can be shown rigorously for Eq.(4) if $\epsilon = k^{-1}$ is sufficiently small. The details of this proof will be presented elsewhere.

Only when $\epsilon = k^{-1} = 0$ or $\mathbf{f} = \mathbf{g}$ with $k \geq \tilde{k} \geq \tilde{k}_i \ \forall i$ does *perfect* synchronization occur. These cases are ideal and never encountered experimentally. A special case of $\mathbf{f} \neq \mathbf{g}$ (discussed in the previous paragraph) has been investigated experimentally [Halle, Wu, Itoh & Chua, 1993]. Namely, the case when both vector fields $\mathbf{f}$ and $\mathbf{g}$ are in the same mathematical form but with a mismatch in their parameters. In that experiment, the synchronization is not perfect but the chaotic attractor is similar to the goal attractor. Thus we conjecture that they are *equivalent* in the sense of the above definition. Perfect synchronization is in reality not necessary when we can get an *equivalence* between the chaotic attractor $\mathbf{x}(t)$ and the goal $\mathbf{y}(t)$. Our method provides an easy way to achieve this *equivalence*.

In conclusion, a simple method for synchronization and control of chaotic systems has been presented. A general qualitative description of the behavior of (1) has been given for the cases $\mathbf{f} = \mathbf{g}$ and $\mathbf{f} \neq \mathbf{g}$. The method we introduce is a useful method for synchronization of *any* chaotic systems. Moreover, the method is particularily useful for the control and the synchronization of chaotic systems whose parameters cannot be changed easily.

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Figure Captions

Figure 1.

(a): Double scroll attractor in the Chua's circuit [4] with $\alpha = 10.0$; $\beta = 19.42722$; $m_0 = -0.76483$; $m_1 = -1.41372$.

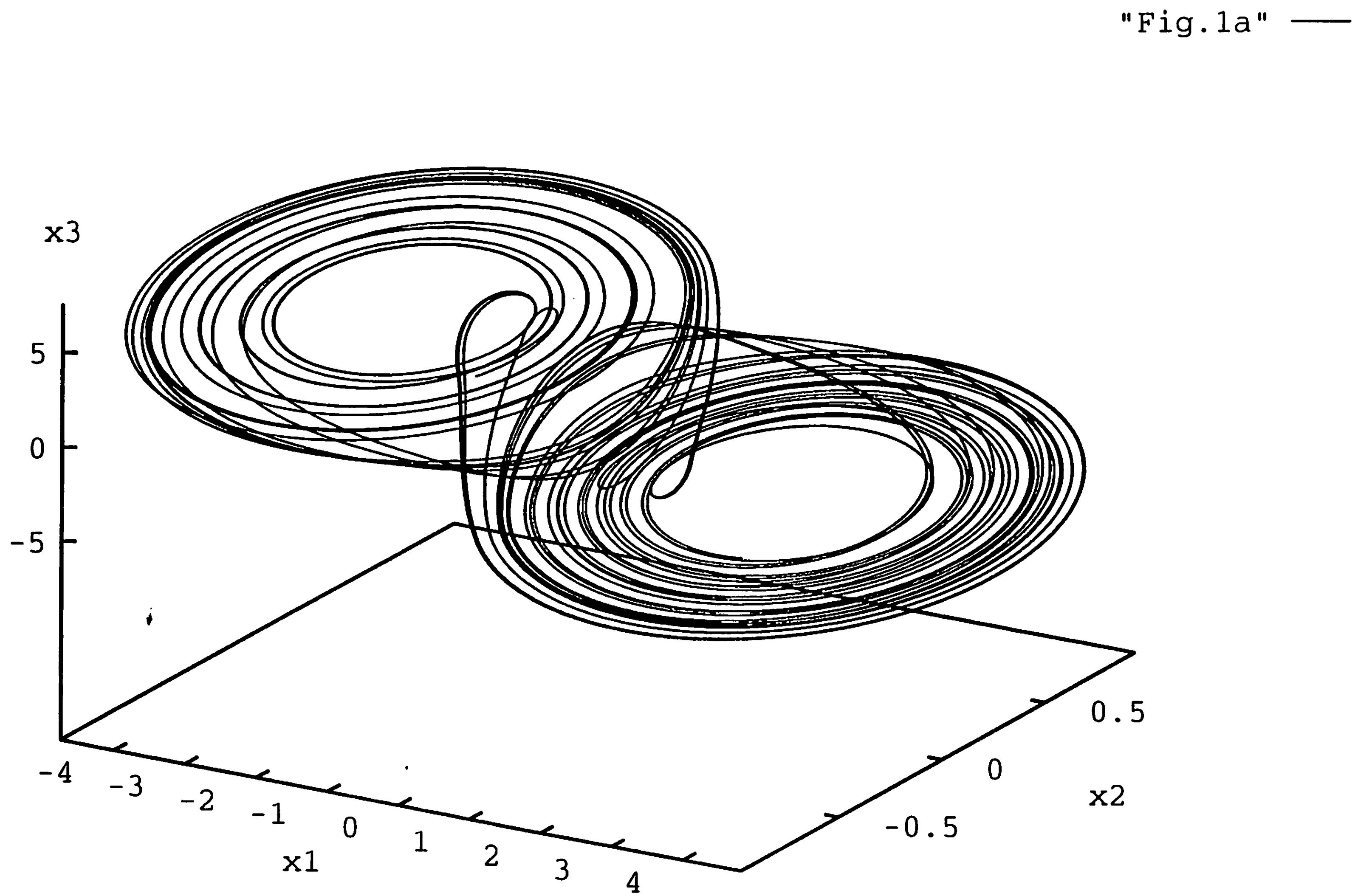
(b): Chua's circuit driven by a Lorenz attractor [15]; $k = 10$.

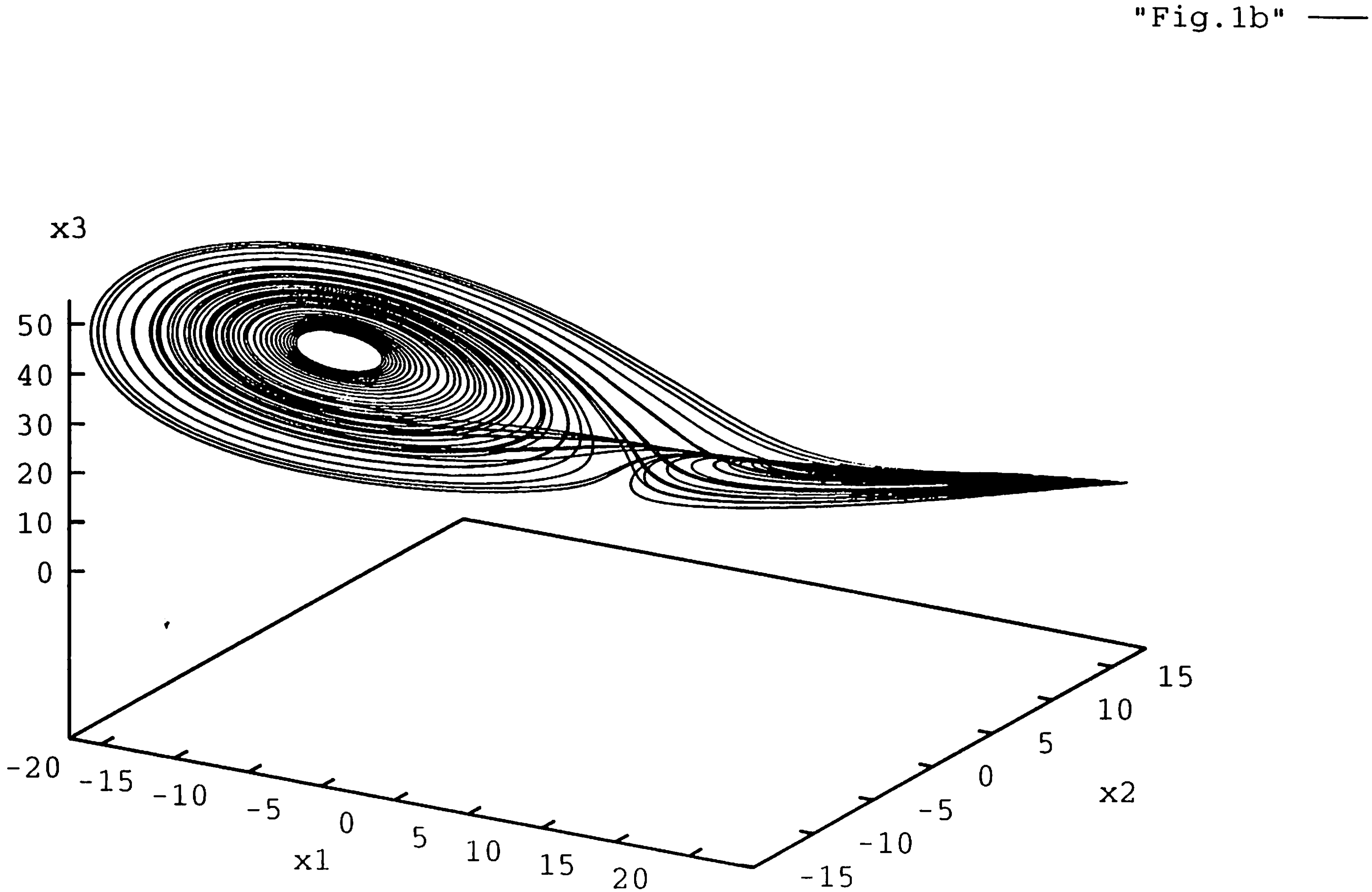
(c): Chua's circuit driven by a Lorenz attractor; $k = 100$.

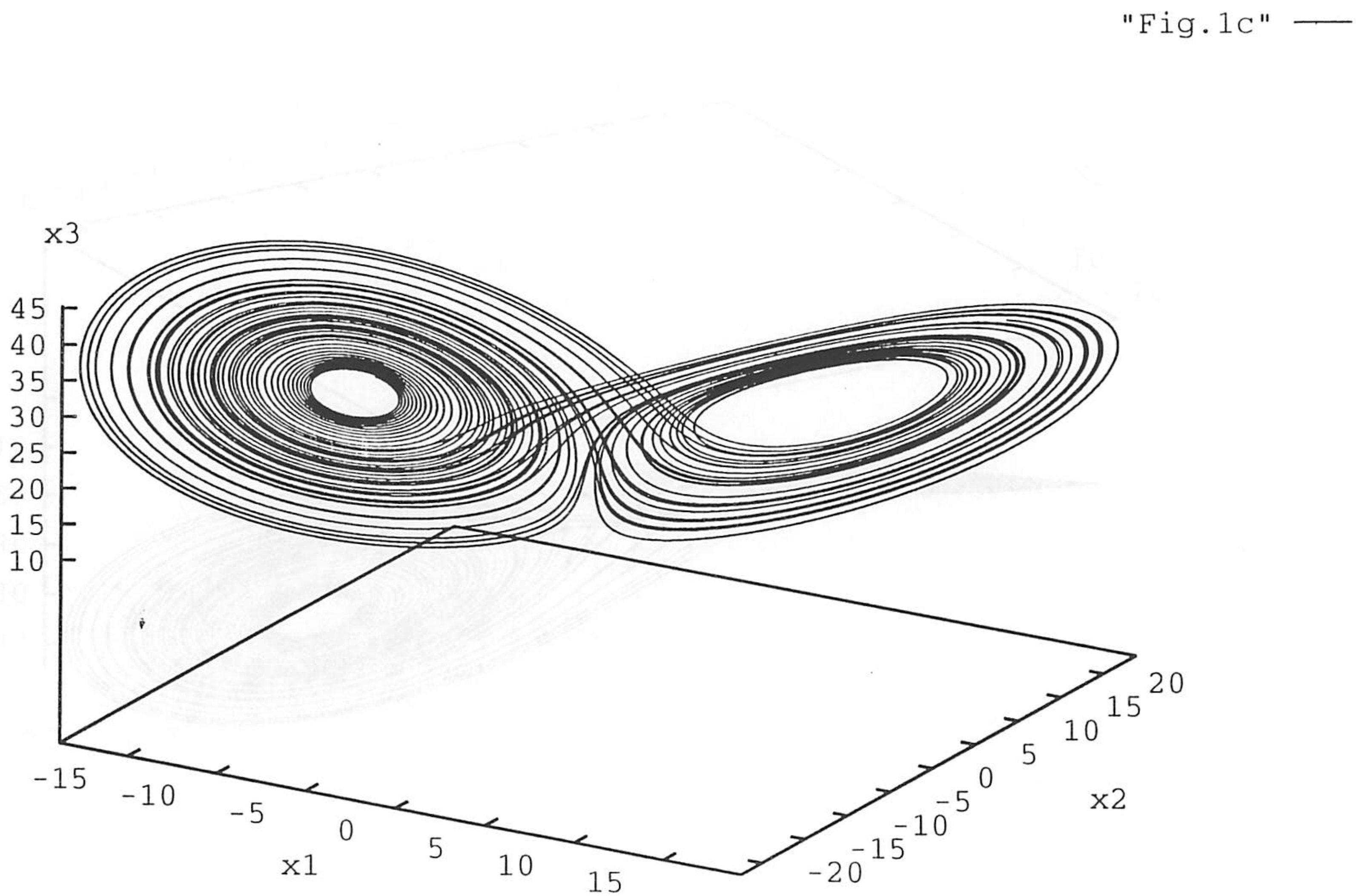
(d): Lorenz attractor with $\sigma = 10$, $b = 8/3$ and $r = 28$.

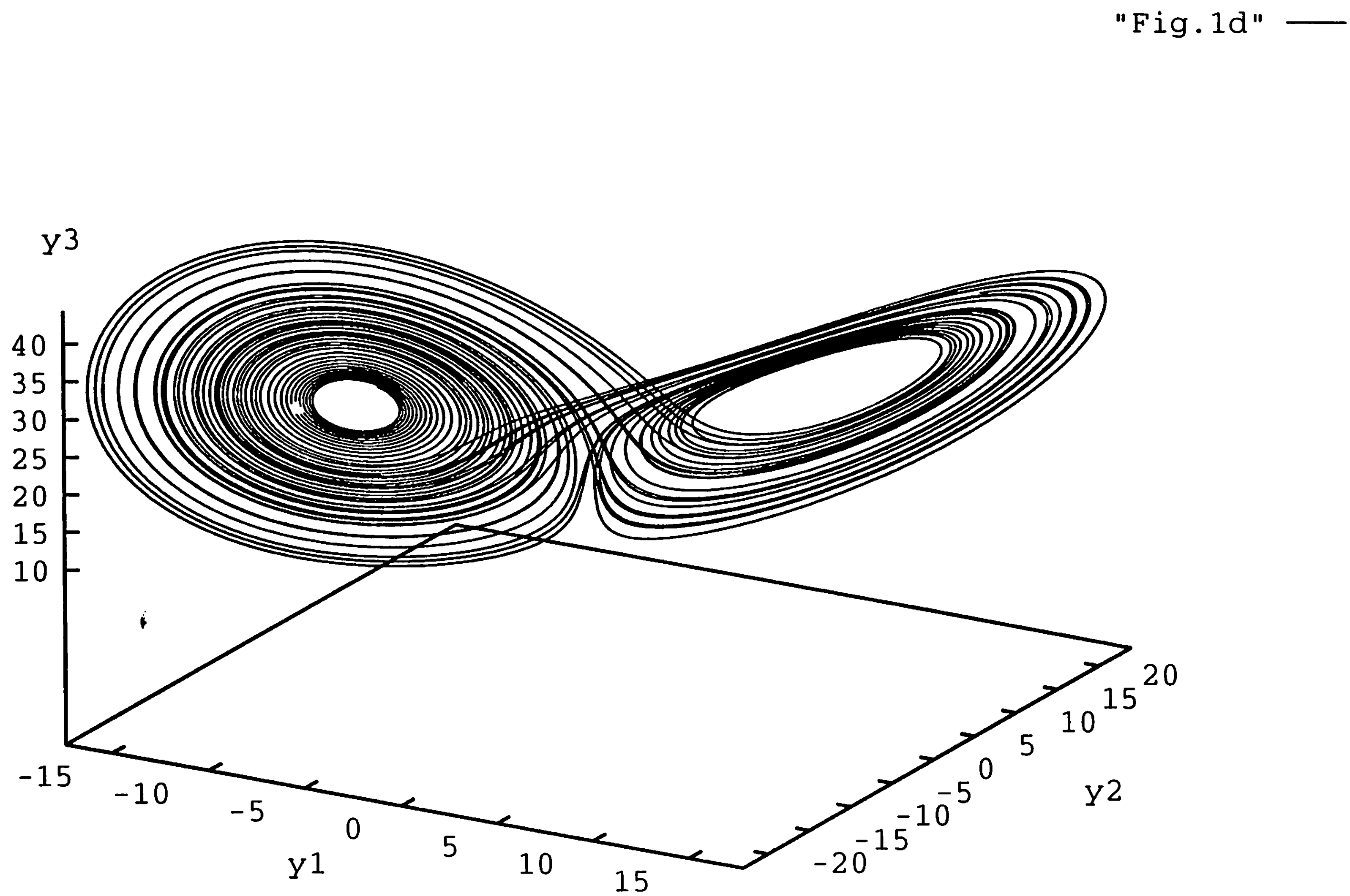
Figure 2. Mutual information (between x_2 and y_2) versus coupling coefficient k.

Figure 3. y_1 versus x_1


"Fig.1a"
x3
5
0
-5
-4
-3
-2
-1
0
1
2
3
4
x1
0.5
0
-0.5
x2


"Fig.1b"
x3
50
40
30
20
10
0
-20
-15
-10
-5
0
5
10
15
20
x1
15
10
5
0
-5
-10
-15
x2


"Fig.1c"
x3
45
40
35
30
25
20
15
10
-15
-10
-5
0
5
10
15
x1
-20
-15
-10
-5
0
5
10
15
20
x2



"Fig.1d"
y3
40
35
30
25
20
15
10
-15
-10
-5
0
5
10
15
y1
20
15
10
5
0
-5
-10
-15
-20
y2

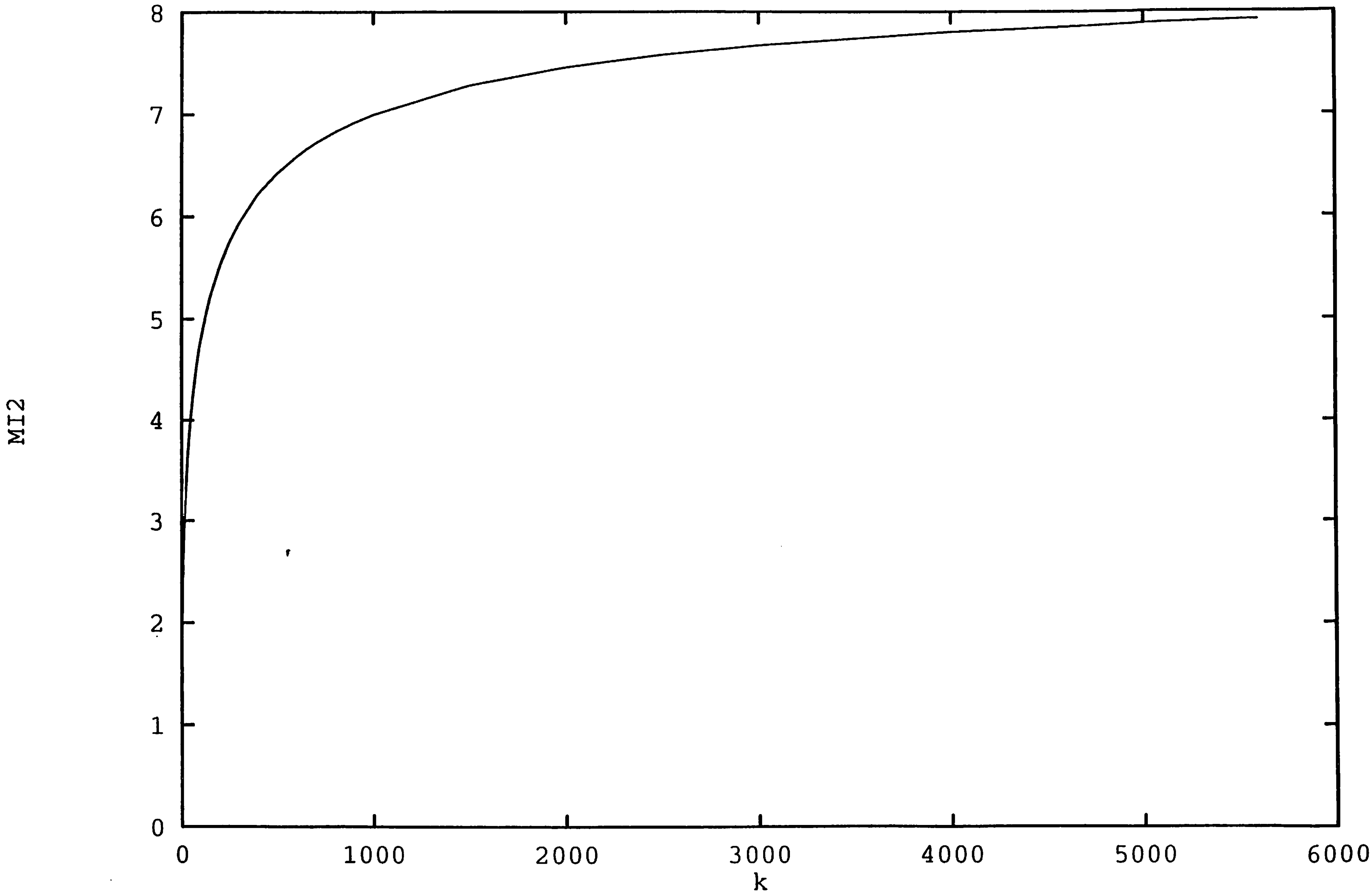


FIG. 9

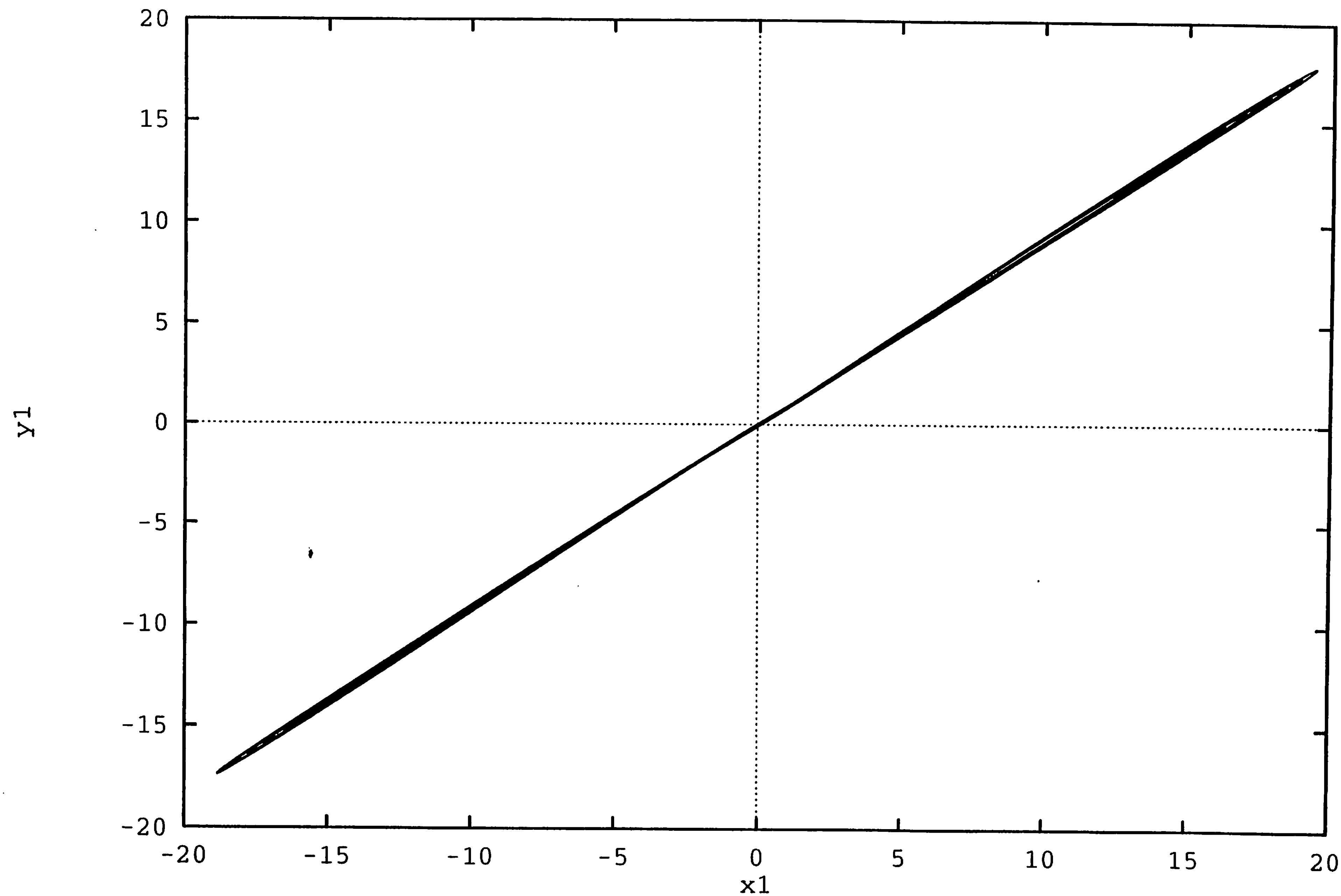


FIG. 3