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NON-AUTONOMOUS CHAOTIC CIRCUIT**

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Memorandum No. UCB/ERL M93/93

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# THE SIMPLEST DISSIPATIVE NON-AUTONOMOUS CHAOTIC CIRCUIT

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## ABSTRACT

We show by computer simulation that chaos can occur in a sinusoidally driven second-order circuit made of 3 linear elements and a Chua's diode. Unlike other non-autonomous second-order chaotic circuits whose nonlinear element is a nonlinear capacitor, the Chua's diode is a nonlinear resistor, and is therefore simpler from a circuit-theoretic point of view.

## 1. INTRODUCTION

A non-autonomous system described by ordinary differential equations must be at least of second order in order for chaotic phenomena to be possible[1]. In [2], Linsay had demonstrated chaos experimentally by driving a series circuit made of a linear resistor, a linear inductor, and a varactor diode with a sinusoidal voltage source. To simulate the chaotic waveforms from this circuit, the varactor diode was modelled in [3] by a 4-element circuit model, which includes 2 nonlinear capacitors and a nonlinear resistor. In other words, even though the physical circuit contains only 3 circuit elements, the circuit used for computer simulation in [3] contains 6 circuit elements, and the associated nonlinearities are exponential functions, thereby making any mathematical analysis intractable.

In this letter, we present another second-order chaotic circuit whose only nonlinear element is a Chua's diode, modelled by a 3-segment piecewise-linear v-i characteristic<sup>1</sup>. There are several reasons why this nonlinear element is preferred over the varactor diode. First, Chua's diode can be fabricated using 2 op-amps and 6 linear resistors[5], or as an IC chip [6]. Second, its piecewise linear nature makes it mathematically tractable. Third, Chua's diode is a nonlinear resistor [7], conceptually a simpler circuit element than a capacitor. Computer-simulations of this circuit show, among other interesting dynamical phenomena, the typical period-doubling route to chaos. In a future paper, we will present the experimental results obtained from this circuit, as well as a more detailed analysis.

## II. Circuit realization and chaos

The circuit realization of the above simple non-autonomous circuit is shown in Fig.1. It contains a capacitor, an inductor,

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<sup>1</sup> Although Chua's diode has been defined as a 2-terminal nonlinear resistor characterized by an arbitrary v-i curve, e.g., a cubic  $i=av + bv^3$  [4], in this paper, we will choose the usual 3-segment characteristic.

a linear resistor, an external periodic forcing and a nonlinear element, namely, the Chua's diode. By applying Kirchoff's laws to this circuit, the governing equations for the voltage  $v$  across the capacitor  $C$  and the current  $i_L$  through the inductor  $L$  are represented by the following set of two first order nonautonomous differential equations:

$$C \frac{dv_C}{dt} = i_L - g(v_C), \quad L \frac{di_L}{dt} = -Ri_L - v_C + f \sin(\Omega t), \quad (1)$$

where  $f$  is the amplitude and  $\Omega$  is the angular frequency of the external periodic force. The term  $g(v_C) = G_b v_C + 0.5(G_a - G_b)[|v_C + B_p| - |v_C - B_p|]$  is the mathematical representation of the characteristic curve of the Chua's diode [5,6]. Rescaling Eq.(1) as  $v_C = xB_p$ ,  $i_L = GyB_p$ ,  $G=1/R$ ,  $\omega = \Omega C/G$  and  $t = \tau C/G$  and then redefining  $\tau$  as  $t$  the following set of normalized equations are obtained:

$$\dot{x} = y - g(x), \quad \dot{y} = -\beta y - \beta x + F \sin(\omega t), \quad (\cdot = d/dt) \quad (2)$$

where  $\beta = C/LG^2$ , and  $F = f\beta/B_p$ .

Obviously  $g(x) = bx + 0.5(a-b)[|x+1| - |x-1|]$ , where  $a = G_a/G$  and  $b = G_b/G$ .

The dynamics of Eq.(2) now depends on the parameters  $\beta, a, b, \omega$  and  $F$ . In our present study, we fix  $\beta=1.0$ ,  $a = -1.27$ ,  $b = -0.68$  and  $\omega = 0.6$ . We use the amplitude  $F$  of the external periodic forcing as the control parameter and carry out a numerical simulation of Eq.(2) for increasing values of  $F$ . To start with, or the autonomous case (i.e.  $F=0$ ), we observe the existence of a fixed point attractor. When sinusoidal forcing

is included, by varying the amplitude  $F$  upwards from zero ( $F > 0$ ), the system exhibits the familiar period doubling bifurcation route to chaos, followed by periodic windows, etc. Figures 2a and 2b represent the chaotic attractor and its corresponding Poincaré map in the  $(x-y)$  plane for  $F=0.25$ . The initial conditions are chosen as  $(x(0), y(0)) = (0.5, 0.1)$ . It may be noted that the chaotic attractor in figure 2a has some structural resemblance to that of the double scroll chaotic attractor of the Chua's autonomous circuit[5,6]. Figure 2c depicts the power spectrum of the  $x$  signal which clearly indicates its broad band nature, a typical characteristic feature of the chaotic signal. Figure 3a depicts the one parameter bifurcation diagram in the  $(F-x)$  plane which indicates the familiar period doubling bifurcation sequence, chaos, windows, etc. Also, in Fig.3b the maximal Lyapunov spectrum in  $(F-\lambda_{\max})$  is plotted.

### III. Conclusion

We have discussed the chaotic dynamics admitted by a very simple second-order non-autonomous circuit with a single nonlinear element namely the Chua's diode numerically. The circuit due to its simplicity naturally deserves further study including experimental confirmation, theoretical analysis as well as controlling and synchronization of chaos. We have already made progress along these lines and the details of these studies will be given elsewhere.



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**Figure captions**

**Fig. 1.** Circuit realization of the simple nonautonomous circuit.

**Fig. 2.** (a) Chaotic attractor in the (x-y) plane of Eq.(2) for  $F=0.25$ .

(b) Poincare map of (a).

(c) Power spectrum of the x signal of (a).

**Fig. 3.** (a) One parameter bifurcation diagram in the (F-x) plane.

(b) Maximal Lyapunov spectrum.

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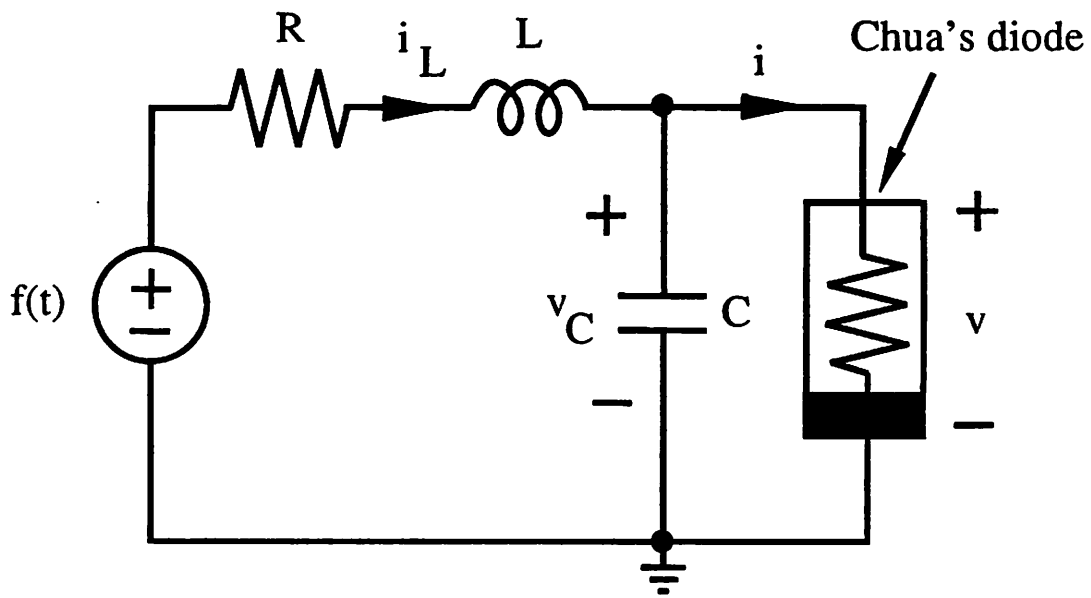


Fig. 1

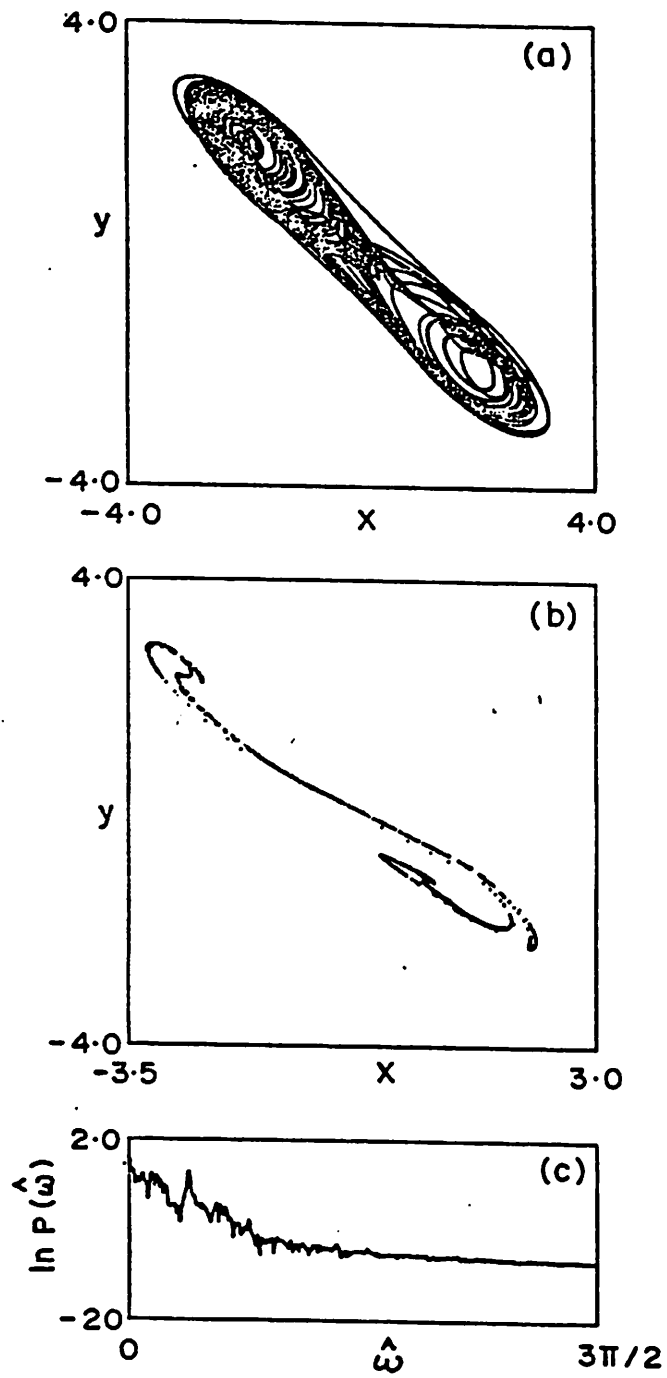


Fig.2

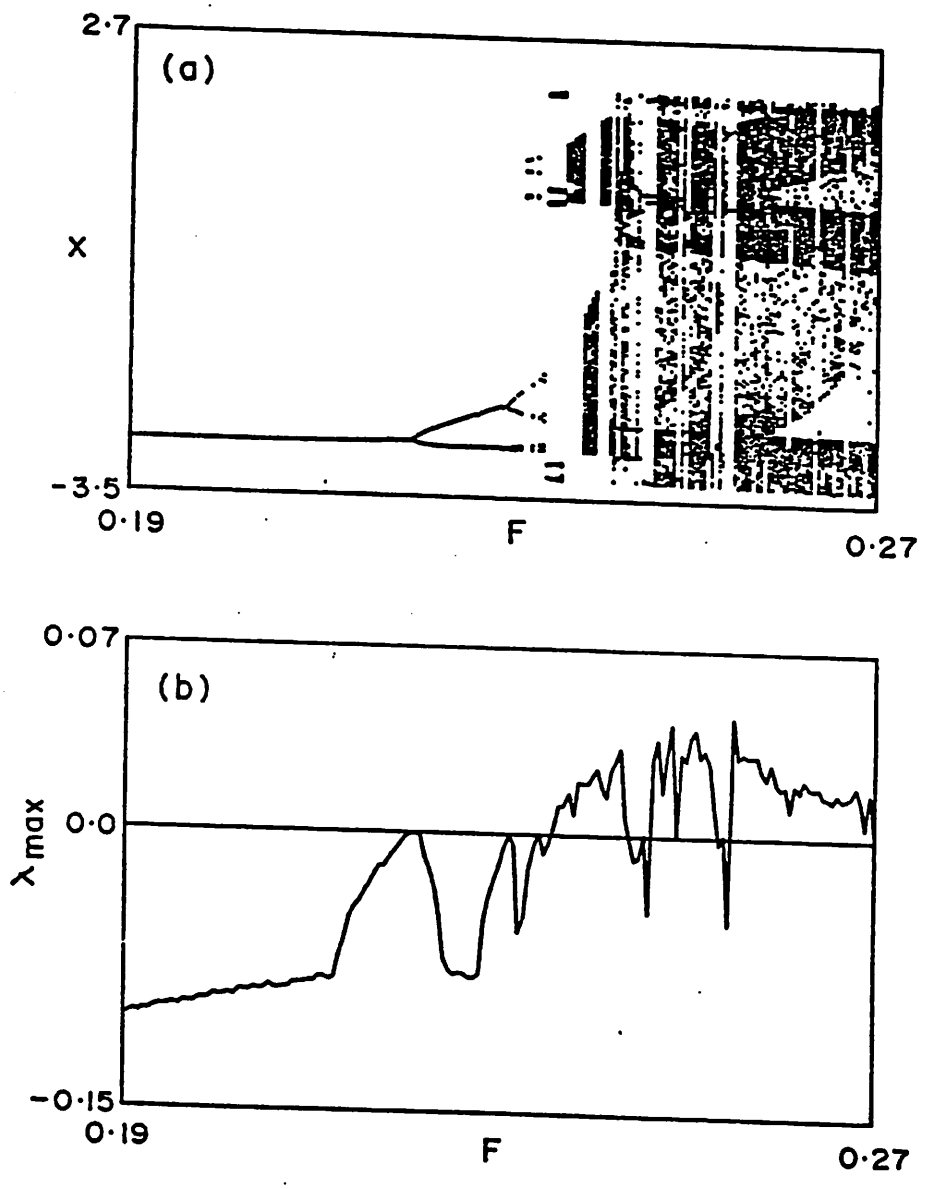


Fig.3