Linear Systems With Coefficient Matrices
Having Fields of Values
Not Containing The Origin *

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This is a continuation of [3] addressing a problem posed by Prof. Kahan. The problem is the following:

Given an $n \times n$ (complex) matrix $A$ whose field of values $F(A)$ does not contain the origin, is it necessary to pivot when solving the linear system $Ax=b$?

It is well known $F(A)$ is a compact convex set on the complex plane. Let’s draw two projecting lines $\ell_1$ and $\ell_2$ starting at the origin and “tangent”\(^1\) to the boundary of $F(A)$ such that $F(A)$ falls into the smaller section enclosed by $\ell_1$ and $\ell_2$ as shown in Figure 1. Let $\alpha$ be the angle of the section. Clearly $0 \leq \alpha < \pi$. Set $\theta = \pi - \alpha$. It is proved in [3] that if $\theta$ is reasonably

\(^1\) Here “tangent” may not be the right word since the boundary of $F(A)$ could not be smooth. So to be more rigorous, we could say that $\ell_1$ and $\ell_2$ are two support lines passing through the origin.

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large, no smaller than $0.1\pi$ (say), there is no danger of instability in solving $Ax = b$ without pivoting. However, if $\theta$ is rather small, $\theta = \epsilon$ (say), there is a potentiality that elements might grow by a factor $O\left(\frac{1}{\epsilon}\right)$. Recently, Ming Gu [2] improves this factor to $O\left(\frac{1}{\epsilon^2}\right)$, which is optimal as far as only order is concerned. This note adopts the idea developed in [1] where the case $A$ being real is studied. We will give a better bound which is asymptotically attainable.

It is easy to verify that scalar multiplications do not affect element growth in Gaussian elimination processes. Therefore, without loss of any generality, by rotating the matrix $A$ by an angle $\beta$ as $e^{i\beta}A$ we can assume that $\mathcal{F}(A)$ lies in the right half plane and the angles between the $y$-axis and $l_1$ and between the negative direction of the $y$-axis and $l_2$ are equal to $\frac{\beta}{2}$. Set $A = H + iS$, where

$$H = \frac{A + A^*}{2} = H^*, \quad S = \frac{A - A^*}{2i} = S^*$$

are both Hermitian, and moreover $H$ is positive definite.

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2When $A$ is real, $\mathcal{F}(A)$ is symmetric with respect to the $x$-axis, so either $A$ itself or $-A$ has the desired property.
Doing Gaussian elimination on $A$, we get a decomposition

$$A = H + iS = LD M^*,$$  \hspace{1cm} (1)

where $L$ and $M$ are unit lower triangular matrices, $D$ diagonal. Generally, they are all complex. The existence of the decomposition (1) is guaranteed by the assumption we made on the $F(A)$ (ref. [3]).

**Proposition 1** Write $D = \text{diag} \{d_1, d_2, \ldots, d_n\}$. Then

$$\Re d_j > 0, \quad j = 1, 2, \ldots, n,$$

where $\Re(\cdot)$ denotes the real part of a complex number.

Since $H = H^*$ is positive definite, it has a unique Cholesky decomposition $H = GG^*$, where $G$ is lower triangular. Now (1) gives

$$L^{-1}(H + iS)L^{-*} = DM^*L^{-*} \Rightarrow L^{-1}GG^*L^{-*} + iL^{-1}SL^{-*} = DM^*L^{-*},$$

which yields

$$(G^*L^{-*})^*G^*L^{-*} + iL^{-1}SL^{-*} = DM^*L^{-*}. \hspace{1cm} (2)$$

Let $e_j$ be the $j$th column of the $n \times n$ identity matrix. Comparing the $j$th diagonal entries of the two sides of (2) leads to

$$\Re d_j = \|G^*L^{-*}e_j\|_2^2$$

(3)

since $M^*L^{-*}$ is unit upper triangular. Therefore

$$\|G^*L^{-*}D^{-1/2}\|_F^2 = \sum_{j=1}^n \|G^*L^{-*}D^{-1/2}e_j\|_2^2 \leq \sum_{j=1}^n \frac{\|G^*L^{-*}e_j\|_2^2}{|d_j|} \leq n.$$  \hspace{1cm} (4)

On the other hand,

$$M^{-1}(H + iS)M^{-*} = M^{-1}LD \Rightarrow M^{-1}GG^*M^{-*} + iM^{-1}SM^{-*} = M^{-1}LD,$$

$D^{1/2}$ is not single-valued. But for our purpose it is good enough to pick any one of them and stick to it. $D^{-1/2} \overset{\text{def}}{=} (D^{1/2})^{-1}$. 

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which yields \((G^*M^{-*})^*G^*M^{-*} + iM^{-1}SM^{-*} = M^{-1}LD\), so
\[
\Re d_j = \|G^*M^{-*}c_j\|^2_2
\]
and
\[
\|G^*M^{-*}D^{-1/2}\|^2_F \leq n. \tag{5}
\]

It follows from (1) that
\[
LD^{1/2} = (GG^* + iS)M^{-*}D^{-1/2}
\]
\[
= (G + iSG^{-*})G^*M^{-*}D^{-1/2},
\]
\[
D^{1/2}M = D^{-1/2}L^{-1}(GG^* + iS)
\]
\[
= D^{-1/2}L^{-1}(G^* + iG^{-1}S).
\]

Thus
\[
\|LD^{1/2}\|_F \leq \sqrt{n}\|G + iSG^{-*}\|_2, \tag{6}
\]
\[
\|D^{1/2}M\|_F \leq \sqrt{n}\|G^* + iG^{-1}S\|_2. \tag{7}
\]

Notice that
\[
\|G + iSG^{-*}\|^2_2 = \|(G + iSG^{-*})(G^* - iG^{-1}S)\|_2
\]
\[
= \|GG^* - iS + iS + SG^{-*}G^{-1}S\|_2
\]
\[
= \|H + SH^{-1}S\|_2
\]
\[
\leq \|H\|_2 + \|SH^{-1}S\|_2,
\]
\[
\|G^* + iG^{-1}S\|^2_2 = \|(G - iSG^{-*})(G^* + iG^{-1}S)\|_2
\]
\[
= \|GG^* + iS - iS + SG^{-*}G^{-1}S\|_2
\]
\[
= \|H + SH^{-1}S\|_2
\]
\[
\leq \|H\|_2 + \|SH^{-1}S\|_2.
\]

Together with (6) and (7), we have\(^4\)
\[
\|L\|\|D\|\|M^*\|_F = \|LD^{1/2}\|\|D^{1/2}M\|_F
\]
\[
\leq n\|H + SH^{-1}S\|_2
\]
\[
\leq n(\|H\|_2 + \|SH^{-1}S\|_2). \tag{8}
\]

\(^4\)By \(|X|\), we mean its entrywise absolute value, i.e. \(|X| \overset{\text{def}}{=} (|x_{ij}|).\)
To relate this bound to the angle $\alpha$, we observe

$$
\|SH^{-1}S\|_2 = \|H^{-1/2}(H^{-1/2}SH^{-1/2})(H^{-1/2}SH^{-1/2})H^{1/2}\|_2 \\
\leq \|H^{1/2}\|_2 \|H^{-1/2}SH^{-1/2}\|_2^2 \|H^{1/2}\|_2 \\
= \|H\|_2 \|H^{-1/2}SH^{-1/2}\|_2^2.
$$

Lemma 1

$$
\|H^{-1/2}SH^{-1/2}\|_2 = \max_{x \neq 0} \frac{x^*Sx}{x^*Hx} = \tan \frac{\alpha}{2}.
$$

With those in mind, we get

Theorem 1

$$
\| |L||D||M^*\|_F \leq \|H\|_2 \left[ 1 + \left( \tan \frac{\alpha}{2} \right)^2 \right]. \quad (9)
$$

Roughly speaking, the bound in [2] is the one obtained by replacing the number inside [7] of (9) with $1 + \tan \frac{\alpha}{2} + \frac{3}{2} (\tan \frac{\alpha}{2})^2$.

In what follows, we are going to present an example to shown that this inequality is at least asymptotically attainable in the sense that there are examples for which the two sides of (9) are arbitrarily close. Consider (ref. [3])

$$
A = \left( \begin{array}{cc}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{array} \right) \left( \begin{array}{cc}
1 & 2r \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{array} \right) = \left( \begin{array}{cc}
1-r & r \\
-r & 1+r
\end{array} \right),
$$

where $r$ is positive. It is known the field of values of $\mathcal{F}(A)$ is a disk with center 1 and radius $r$, i.e.

$$
\mathcal{F}(A) = \{ z \text{ complex : } |z-1| \leq r \}.
$$

So $0 \not\in \mathcal{F}(A)$ if $r < 1$ (which will be assumed hereafter). For this $A$, we have

$$
A = LD M^* = \left( \begin{array}{cc}
1 & 0 \\
-\frac{r}{1-r} & 1
\end{array} \right) \left( \begin{array}{cc}
1-r & 0 \\
0 & \frac{1}{1-r}
\end{array} \right) \left( \begin{array}{cc}
1 & \frac{r}{1-r} \\
0 & 1
\end{array} \right),
$$

$$
|L||D||M^*| = \left( \begin{array}{cc}
1 & 0 \\
\frac{r}{1-r} & 1
\end{array} \right) \left( \begin{array}{cc}
1-r & 0 \\
0 & \frac{1}{1-r}
\end{array} \right) \left( \begin{array}{cc}
1 & \frac{r}{1-r} \\
0 & \frac{1}{1-r}
\end{array} \right) = \left( \begin{array}{cc}
1-r & \frac{r}{1-r} \\
r & \frac{1+r}{1-r}
\end{array} \right),
$$

$$
H = \frac{A + A^*}{2} = \left( \begin{array}{cc}
1-r & 0 \\
0 & 1+r
\end{array} \right),
$$
\[ S = \frac{A - A^*}{2i} = \begin{pmatrix} 0 & -r \i \ 
 r & 0 \end{pmatrix}, \]

\[ SH^{-1} = \begin{pmatrix} 1 - r & 0 
 0 & 1 + r \end{pmatrix}, \]

\[ \tan \frac{\alpha}{2} = \frac{r}{\sqrt{1 - r^2}}. \]

Hence

\[ \| [L][D][M^*]_F \| = (1 - r)^2 + 2r^2 + \left( \frac{1 + r^2}{1 - r} \right)^2 \]

\[ = \frac{(1 - r)^4 + 2r^2(1 - r)^2 + (1 + r^2)^2}{(1 - r)^2}, \]

\[ 2 \| H \|_2 \left[ 1 + \left( \tan \frac{\alpha}{2} \right)^2 \right] = 2(1 + r) \left[ 1 + \frac{r^2}{1 - r^2} \right] = \frac{2}{1 - r}, \]

\[ 2(\| H \|_2 + \| SH^{-1} \|_2) = \frac{2}{1 - r}. \]

Define a function \( f(r) \) as follows:

\[ f(r) \stackrel{\text{def}}{=} \frac{\| [L][D][M^*]_F \|}{2 \| H \|_2 \left[ 1 + \left( \tan \frac{\alpha}{2} \right)^2 \right]} \]

\[ = \frac{1}{2} \sqrt{(1 - r)^4 + 2r^2(1 - r)^2 + (1 + r^2)^2} \]

\[ = 1 - (1 - r) + \frac{3}{4}(1 - r)^2 - \frac{1}{4}(1 - r)^3 - \frac{1}{32}(1 - r)^4 + O((1 - r)^5). \]

It is easy to see \( f(0) = 1/\sqrt{2} = 0.70710678118655, \lim_{r \to 1} f(r) = 1 = f(1) \) which shows that the inequalities (8) and (9) are asymptotically attainable! And \( \min_{0 \leq r \leq 1} f(r) \approx 0.614966762630915 \) at

\[ r = \frac{1}{2} - \frac{\sqrt{2/3}}{\sqrt{-9 + \sqrt{177}}} + \frac{\sqrt{-9 + \sqrt{177}}}{\sqrt{12^2}} \approx 0.273301174242. \]

To see how fast \( f(r) \) approaches 1 pictorially, we refer the reader to Figure 2, where the picture on the left is the graph of \( f(r) \) and the one on the right is that of \( 1 - f(r) \).
Figure 2: The functions $f(r)$ and $1 - f(r)$.

References

