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D. TILBURY AND S. SASTRY

ELECTRONICS RESEARCH LABORATORY
DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCES
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720

dawnt@eecs.berkeley.edu

ABSTRACT. This paper presents the method of exterior differential systems for analyzing nonlinear systems. The Goursat normal form is presented, and conditions are given for converting Pfaffian systems into this normal form. Since the Brunovsky normal form is a special case of the Goursat normal form, we also show how the exact linearization conditions for control systems can be restated in the language of Pfaffian systems. In addition, we give new conditions for converting Pfaffian systems into Goursat form after prolongation, and for linearizing control systems using dynamic extension.

Several examples of mobile robots are examined, and it is shown that for some kinematic arrangements, a prolongation corresponding to a dynamic state feedback is needed to transform the corresponding Pfaffian system into Goursat form.

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1. Introduction

There has been a great deal of interest in the use of exterior differential systems for analyzing nonlinear control systems. We bring together here some of the results which have been recently published as well as add some of our own contributions to this area. We show that all of the main results in exact linearization of nonlinear systems can be restated in terms of exterior differential systems, and in addition, we present a new set of sufficient conditions for linearization by dynamic extension.

In this paper, we will only make use of a special type of exterior differential system called a Pfaffian system, and all of our definitions and results are specific to this case. Loosely speaking, a Pfaffian system is represented by a codistribution of one-forms defined on the state space, which we will assume to be a connected manifold. These one-forms may represent constraints on the system velocities, as in the case of mobile robots where the wheels roll without slipping. A control system with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ of the form $\dot{x} = f(x, u)$ can also be written as a Pfaffian system, with the constraints defined as the one-forms $\alpha^i = dx_i - f^i(x, u)dt$ on $\mathbb{R}^{n+m+1}$. Although special care must be taken for control systems to treat time differently from the state and input variables, we will show that the main results on transforming Pfaffian systems into normal forms will carry over into the special case of control systems.

For mechanical systems with linear velocity constraints, such as mobile robots with wheels that roll without slipping, exterior differential systems are the most appropriate method for analysis. Using methods from exterior differential systems, we can, under certain conditions, transform the Pfaffian system defined by the rolling constraints into a normal form and, in the new coordinates, easily find solution trajectories for the system.

The outline of this paper is as follows. First, we present some background material on exterior algebra and Pfaffian systems. We then give the definitions for the Goursat normal form and extended Goursat normal form. After stating necessary and sufficient conditions for converting Pfaffian systems into these normal forms, we show that several examples of mobile robot systems satisfy these conditions. We then consider systems which do not satisfy the conditions for this conversion, and present the concept of *prolongation* of a Pfaffian system. We give sufficient conditions for converting a Pfaffian system to extended Goursat normal form using a specific type of prolongation, and we show an example of a mobile robot system which satisfies these conditions. Finally, we turn our attention to control systems, expressed as Pfaffian systems. Since the Brunŏvsky form is a special case of Goursat normal form, all of the results for converting Pfaffian systems to Goursat form can be specialized to give conditions for converting control systems to Brunovsky linear form. We show that the special type of prolongation we proposed in this paper is the dual of dynamic extension, or adding integrators to some of the input channels. Thus our theorem for converting Pfaffian systems to extended Goursat normal form using prolongations can be specialized to give sufficient conditions for linearizing control systems by dynamic extension.

2. Exterior Algebra and Pfaffian systems

We give a brief overview of some of the definitions of exterior algebra and Pfaffian systems that we will use in this paper. The theory of exterior differential systems is powerful enough to analyze solutions of systems of partial differential equations; however, we will restrict ourselves in this paper to systems of first-order ordinary differential equations. We encourage the interested reader to consult the monograph by Bryant *et. al.* [1], from which most of this introductory material was taken, for more details.

A real vector space V or its dual (covector) space V^* generates an *exterior algebra* with the *exterior* or *wedge product* defined by

$$\begin{aligned} \alpha \wedge \beta &= -\beta \wedge \alpha \\ \alpha \wedge \alpha &= 0 \\ a\alpha \wedge (b\beta + c\gamma) &= (ab)\alpha \wedge \beta + (ac)\alpha \wedge \gamma \end{aligned}$$

for all $\alpha, \beta \in V(V^*)$, $a, b, c \in \Re$. The wedge product of two vectors is called a *two-vector*. We define $\Lambda^2(V)$ as the space of two-vectors. We can similarly build up higher vectors and define $\Lambda^k(V)$ as the space of all k-vectors. For completeness, we define $\Lambda^0(V) = \mathbb{R}$ and $\Lambda^1(V) = V$. The dimension of $\Lambda^k(V)$ is $\binom{n}{k}$. From the axioms, it follows that $\Lambda^k(V)$ is empty for $k > n$.

The exterior algebra over V is a graded algebra,

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V)$$

Any element $\lambda \in \Lambda(V)$ can be written uniquely as

$$\lambda = \lambda_0 + \lambda_1 + \cdots + \lambda_n$$

where $\lambda_i \in \Lambda^i(V)$ for $i = 1, \ldots, n$.

Now, consider a differentiable manifold M of dimension n and its cotangent bundle T^*M. We construct the bundle $\Lambda(T^*M)$ whose fibers are the exterior algebra of T_x^*M, that is:

$$\Lambda(T_x^*M) = \Lambda^0(T_x^*) \oplus \Lambda^1(T_x^*) \oplus \Lambda^2(T_x^*) \oplus \cdots \oplus \Lambda^n(T_x^*)$$

The bundle $\Lambda(T^*M)$ has $\Lambda^p(T^*M)$ as sub-bundles. A *section* of the bundle

$$\Lambda^p(T^*M) = \bigcup_{x \in M} \Lambda^p(T_x^*M)$$

over M is called an *exterior differential form* of degree p or simply a p-form.

For local coordinates on M denoted by $x = (x_1, \dots, x_n)$, a local basis for T_xM is:

$$\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}.$$

We denote its dual basis on T_x^*M as

$$\{dx_1, \dots, dx_n\}$$

defined by

$$\langle dx_j, \frac{\partial}{\partial x_i}\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In terms of these local coordinates, a p-form ω can be written as

$$\omega = \sum w_{i_1 \dots i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \quad i_1 < i_2 < \cdots < i_p$$

where the coefficient functions $w_{i_1 \dots i_p}(x)$ are smooth functions on M.

We will use the notation $\Omega^p(M)$ to mean the module (over the ring of smooth functions) of all smooth sections of $\Lambda^p(T^*M)$, and $\Omega(M) = \bigoplus \Omega^p(M)$ as the module of forms on M.

We begin by considering a codistribution I on M, spanned by s one-forms, that is

$$I = \mathrm{span}\{\alpha^1, \dots, \alpha^s\}$$

where α^i is in $\Omega^1(M)$ for $i = 1, \dots, s$.

Definition 1. ***Pfaffian Systems.***
On a manifold of dimension n, a *Pfaffian system* is the smallest ideal $\mathcal{I} \subset \Omega(M)$, generated by a codistribution I of one-forms spanned by $\{\alpha^1, \dots, \alpha^s\}$ which is closed under wedge products.

Any element of $\mathcal{I}$ can be written in the form:

$$\sigma = \sum_{i=1}^{s} \theta^i \wedge \alpha^i$$

where θ^i is any element in $\Omega(M)$. Throughout the course of this paper we will deliberately confuse the notation and refer to the codistribution I as the Pfaffian system.

The *dimension* of a Pfaffian system is defined to be s, the number of independent one-forms which generate it. Any $n - s$ linearly independent one-forms which are independent of I form a *complement* to I. The *codimension* of I is $n - s$.

An *integral curve* for a Pfaffian system is a curve $c(t) : (-\epsilon, \epsilon) \to M$ which satisfies the constraints, that is, $c^*(\alpha^i) = 0$ for all $\alpha^i \in I$. Here the notation $c^*(\alpha^i)$ is taken to mean $\langle \alpha^i, \frac{dc}{dt} \rangle$).

A (local) *independence condition* for a Pfaffian system is a one-form τ which does not vanish on integral curves, that is $c^*(\tau) \neq 0$. We add the additional condition that

τ be integrable, so that our Pfaffian systems will correspond to systems of first-order ordinary differential equations.

We will need the notion of *congruence* modulo a Pfaffian system. For $I = \{\alpha^1, \dots, \alpha^s\}$, we say that $\eta \equiv \xi \mod I$ if

$$\eta = \xi + \sum_{i=1}^{s} \theta^i \wedge \alpha^i$$

for some forms θ^i in $\Omega(M)$.

Definition 2. ***Exterior Derivative.***
The *exterior derivative* is defined as the unique map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ which satisfies the following properties:

(1) For $f \in \Omega^0(M)$,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n,$$

relative to a local coordinate chart, or the usual gradient.

(2) For $\alpha \in \Omega^r, \beta \in \Omega^s$,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta.$$

(3) $d^2 = 0$.

Definition 3. ***Derived Flag.***
Given a Pfaffian system $I = \{\alpha^1, \dots, \alpha^s\}$, the *derived flag* is defined to be the nested chain of codistributions given by $I^{(0)} = I$ and

$$I^{(k+1)} = \{\omega \in I^{(k)} : d\omega \equiv 0 \mod I^{(k)}\}$$

The construction is assumed to terminate at some N, when $I^{(N)} = I^{(N+1)}$. The *derived flag* is then defined to be the sequence of nested codistributions,

$$I = I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(N)}$$

We will assume that the dimension of $I^{(k)}$ is well-defined for all k.

Remark 1 (*Maximally Nonholonomic*). The last member of the chain of codistributions, $I^{(N)}$, is called the *bottom derived system.* Since $I^{(N)} = I^{(N+1)}$, we have that $d\omega \equiv 0 \mod I^{(N)}$ for all $\omega \in I^{(N)}$, and by the Frobenius theorem, the bottom derived system is integrable. That is, there there exist functions $h_1, \dots, h_q$ such that $I^{(N)} = \{dh_1, \dots, dh_q\}$. Solution trajectories of I are then constrained to lie on level surfaces of h.

A Pfaffian system is said to be *nonholonomic* if $I^{(1)}$ is a proper subset of I. We will only work with systems which are *maximally nonholonomic*, that is, with bottom derived system $I^{(N)} = \{0\}$.

For reference, we state here a variant of the familiar Frobenius theorem:

Theorem 1. Frobenius [1].
Let $\{\omega^1, \dots, \omega^p\}$ *be a set of linearly independent one-forms, and* $f_1, \dots, f_q$ *a set of functions whose differentials are linearly independent of each other and of the* ω^i*'s. If*

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^p \wedge df_1 \wedge \cdots \wedge df_q = 0$$

for $i = 1, \dots, p$, *then there exist coordinate functions* $z_1, \dots, z_p$ *and coefficient functions* a_{ij}, b_{ij} *such that the one-forms* ω^i *can be written as:*

$$\omega^i = \sum_{j=1}^{p} a_{ij} dz_j + \sum_{j=1}^{q} b_{ij} df_j$$

The proof follows from the standard Frobenius theorem and the fact that the codistribution $\{\omega^1, \dots, \omega^p, df_1, \dots, df_q\}$ is integrable.

3. Goursat Normal Forms

A great deal of work has been done on transforming Pfaffian systems into normal forms. Of special interest to us in this paper is the Goursat normal form, originally proposed by Goursat for Pfaffian systems of codimension two, which has the property that all its integral curves can be expressed in terms of two arbitrary functions. We will also examine an extended Goursat normal form for Pfaffian systems of codimension $k > 2$, for which solution trajectories can be expressed in terms of k arbitrary functions.

Definition 4. ***Goursat Normal Form.***
A codimension two Pfaffian system I on $\mathbb{R}^n$ with generators of the form

$$I = \text{span}\{dz_n - z_{n-1}dz_1, \dots, dz_3 - z_2 dz_1\}.$$

is said to be in *Goursat Normal Form.*

If we define $\omega^1(z) = dz_n - z_{n-1}dz_1, \dots, \omega^{n-2}(z) = dz_3 - z_2 dz_1$, then the derived flag of I is given by

$$\begin{aligned}
I^{(0)} &= \{\omega^1, \ \omega^2, \ \cdots, \ \omega^{n-3}, \ \omega^{n-2}\} \\
I^{(1)} &= \{\omega^1, \ \omega^2, \ \cdots, \ \omega^{n-3}\} \\
&\vdots \\
I^{(n-4)} &= \{\omega^1, \ \omega^2\} \\
I^{(n-3)} &= \{\omega^1\} \\
I^{(n-2)} &= \{0\}
\end{aligned}$$

From the form of the Pfaffian system in the z coordinates, it follows that integral curves of the system are unconstrained in their z_1, z_n coordinates alone. Once $z_1(t), z_n(t)$ are specified as functions of some parameter t, the other coordinates are determined as functions of $z_1(t), z_n(t)$ and finitely many of their derivatives. The following classical theorem gives necessary and sufficient conditions for converting a Pfaffian system into Goursat normal form:

Theorem 2. Goursat Normal Form [1].
A Pfaffian system I of codimension two on $\mathbb{R}^n$ has a set of generators which are in Goursat normal form if and only if there exists a basis set of forms $\{\alpha^1, \ldots, \alpha^{n-2}\}$ for I and a one-form π satisfying the congruences:

$$\begin{array}{rcll} d\alpha^i & \equiv & -\alpha^{i+1} \wedge \pi & \bmod \alpha^1, \ldots, \alpha^i \quad i = 1, \ldots, n-3 \\ d\alpha^{n-2} & \not\equiv & 0 & \bmod I \end{array} \tag{1}$$

In [10] we showed that the Pfaffian system associated with the system of a car towing n trailers, generated by the constraints that each axle of wheels roll without slipping, satisfied the conditions for conversion to Goursat normal form. This was a system of codimension two, corresponding to the fact that the linear and angular velocities of the front car are the inputs, and thus freely specifiable.

In order to consider Pfaffian systems associated with mobile robots such as the firetruck [3, 9] or the multi-steering multi-trailer system [11] we need to work with systems of codimension greater than two. We have the following definition:

Definition 5. ***Extended Goursat Normal Form.***
A Pfaffian system I on $\mathbb{R}^{n+m+1}$ of codimension $m+1$ is in *extended Goursat normal form* if it is generated by n constraints of the form:

$$I = \text{span}\{dz_i^j - z_{i+1}^j dz^0 : j = 1, \ldots, m; i = 1, \ldots, s_j\}, \tag{2}$$

We note that this is a direct extension of the Goursat normal form, and all solution trajectories of (2) are determined by the $m+1$ functions $z^0(t), z_1^1(t), \ldots, z_1^m(t)$ and their derivatives with respect to the parameter t.

There are conditions due to Murray [6] for converting a Pfaffian system to extended Goursat normal form. We restate and prove this with the additional condition (correction) that π needs to be integrable:

Theorem 3. Extended Goursat Normal Form [6].
Let I be a Pfaffian system of codimension $m+1$. If there exists a set of generators $\{\alpha_i^j : j = 1, \ldots, m; i = 1, \ldots, s_j\}$ for I and an integrable *one-form π such that for all j,*

$$\begin{array}{rcll} d\alpha_i^j & \equiv & -\alpha_{i+1}^j \wedge \pi & \bmod I^{(s_j - i)} \quad i = 1, \ldots, s_j - 1 \\ d\alpha_{s_j}^j & \not\equiv & 0 & \bmod I \end{array} \tag{3}$$

then there exists a set of coordinates z such that I is in Goursat normal form,

$$I = \{dz_i^j - z_{i+1}^j dz^0 : j = 1, \ldots, m; i = 1, \ldots, s_j\}.$$

Proof. If the Pfaffian system is already in extended Goursat normal form, the congruences are satisfied with $\pi = dz^0$ and the basis of constraints $\alpha_i^j = dz_i^j - z_{i+1}^j dz^0$.

Now assume that we have found a basis of constraints for I which satisfies the congruences (3). It is easily checked that this basis is adapted to the derived flag, that is:

$$I^{(k)} = \{\alpha_i^j : j = 1, \dots, m; \quad i = 1, \dots, s_j - k\}$$

We will now construct the coordinates z which comprise the Goursat normal form.

Since π is integrable, any first integral of π can be used for the coordinate z^0. If necessary, we can rescale the constraints α_i^j so that the congruences (3) are satisfied with dz^0:

$$d\alpha_i^j \equiv -\alpha_{i+1}^j \wedge dz^0 \mod I^{(s_j - i)} \quad i = 1, \dots, s_j - 1$$

and we can renumber the constraints so that $s_1 \geq s_2 \geq \cdots \geq s_m$.

Now consider the last nontrivial derived system, $I^{(s_1-1)}$. The one-forms $\{\alpha_1^1, \dots, \alpha_{r_1}^1\}$ form a basis for this codistribution, where $s_1 = s_2 = \cdots = s_{r_1}$. From the fact that

$$d\alpha_1^j \equiv -\alpha_2^j \wedge dz^0 \mod I^{(s_1-1)},$$

it follows that the one-forms $\alpha_1^1, \dots, \alpha_1^{r_1}$ satisfy the Frobenius condition:

$$d\alpha_1^j \wedge \alpha_1^1 \wedge \cdots \wedge \alpha_1^{r_1} \wedge dz^0 = 0$$

and thus, by the Frobenius theorem, we can find coordinates $z_1^1, \dots, z_1^{r_1}$ such that

$$\begin{bmatrix} \alpha_1^1 \\ \vdots \\ \alpha_1^{r_1} \end{bmatrix} = A \begin{bmatrix} dz_1^1 \\ \vdots \\ dz_1^{r_1} \end{bmatrix} + B dz^0$$

We note that the matrix A must be nonsingular, since the α_1^j's are a basis for $I^{(s_1-1)}$ and they are independent of dz^0. Therefore, we can define a new basis $\bar{\alpha}_1^j$ as:

$$\begin{bmatrix} \bar{\alpha}_1^1 \\ \vdots \\ \bar{\alpha}_1^{r_1} \end{bmatrix} := A^{-1} \begin{bmatrix} \alpha_1^1 \\ \vdots \\ \alpha_1^{r_1} \end{bmatrix} = \begin{bmatrix} dz_1^1 \\ \vdots \\ dz_1^{r_1} \end{bmatrix} + (A^{-1}B) dz^0$$

and we define the coordinates $z_2^j := (A^{-1}B)_j$, so that the one-forms $\bar{\alpha}_1^j$ have the form

$$\bar{\alpha}_1^j = dz_1^j - z_2^j dz^0$$

for $j = 1, \dots, r_1$.

By the proof of the standard Goursat theorem, all of the coordinates in the j^{th} tower can be found from z_1^j and z^0, thus by the above procedure, we have effectively found all the coordinates in the first r_1 towers.

To find the coordinates for the other towers, we need to look at the lowest derived systems in which they appear. The coordinates for the longest towers were found first, next we will find those for the next-longest tower(s).

Consider the smallest integer k such that $\dim I^{(s_1-k)} > k\,r_1$; more towers will appear at this level. We know that a basis for $I^{(s_1-k)}$ is

$$\{\bar\alpha_1^1,\dots,\bar\alpha_k^1,\dots,\bar\alpha_1^{r_1},\dots,\bar\alpha_k^{r_1},\alpha_1^{r_1+1},\dots,\alpha_1^{r_1+r_2}\}$$

where $\bar\alpha_i^j = dz_i^j - z_{i+1}^j dz^0$ for $j = 1,\dots,r_1$, as found in the first step, and α_1^j for $j = r_1+1,\dots,r_2$ are the one-forms we started with, which satisfy the congruences (3) and are adapted to the derived flag. The lengths of these towers are $s_{r_1+1} = \cdots s_{r_1+r_2} = s_1 - k + 1$. For notational convenience, we will define $z_{(k)}^j := (z_1^j,\dots,z_k^j)$ for $j = 1,\dots,r_1$.

By the Goursat congruences, we know that $d\alpha_1^j \equiv -\alpha_2^j \wedge dz^0 \mod I^{(s_1-k)}$ for $j = r_1+1,\dots,r_1+r_2$, thus the Frobenius condition

$$d\alpha_1^j \wedge \alpha_1^{r_1+1} \wedge \cdots \wedge \alpha_1^{r_1+r_2} \wedge dz_1^1 \wedge \cdots \wedge dz_k^1 \wedge \cdots \wedge dz_1^{r_1} \wedge \cdots \wedge dz_k^{r_1} \wedge dz^0 = 0$$

is satisfied for $j = r_1+1,\dots,r_1+r_2$. Using the Frobenius theorem, we can find new coordinates, $z_1^{r_1+1},\dots,z_1^{r_1+r_2}$, such that

$$\begin{bmatrix} \alpha_1^{r_1+1} \\ \vdots \\ \alpha_1^{r_1+r_2} \end{bmatrix} = A \begin{bmatrix} dz_1^{r_1+1} \\ \vdots \\ dz_1^{r_1+r_2} \end{bmatrix} + B dz^0 + C \begin{bmatrix} dz_{(k)}^1 \\ \vdots \\ dz_{(k)}^{r_1} \end{bmatrix}$$

Since the congruences are only defined up to mod $I^{(s_1-k)}$, we can eliminate the last group of terms (those multiplied by the matrix C) by adding in the appropriate multiples of $\bar\alpha_i^j = dz_i^j - z_{i+1}^j dz^0$ for $j = 1,\dots,r_1$ and $i = 1,\dots,k$. This will change the B matrix, and we will be left with

$$\begin{bmatrix} \tilde\alpha_1^{r_1+1} \\ \vdots \\ \tilde\alpha_1^{r_1+r_2} \end{bmatrix} = A \begin{bmatrix} dz_1^{r_1+1} \\ \vdots \\ dz_1^{r_1+r_2} \end{bmatrix} + \tilde B dz^0$$

Again, we note that A must be nonsingular because the α_1^j's are linearly independent mod $I^{(s_1-k)}$ and also independent of dz^0, and so we can define

$$\begin{bmatrix} \bar\alpha_1^{r_1+1} \\ \vdots \\ \bar\alpha_1^{r_1+r_2} \end{bmatrix} := A^{-1} \begin{bmatrix} \tilde\alpha_1^{r_1+1} \\ \vdots \\ \tilde\alpha_1^{r_1+r_2} \end{bmatrix} + (A^{-1}\tilde B) dz^0 = \begin{bmatrix} dz_1^{r_1+1} \\ \vdots \\ dz_1^{r_1+r_2} \end{bmatrix} + (A^{-1}\tilde B) dz^0$$

We then define the coordinates $z_2^j := (A^{-1}\tilde B)_j$ for $j = r_1+1,\dots,r_1+r_2$ so that $\bar\alpha_1^j = dz_1^j - z_2^j dz^0$. Again, by the standard Goursat theorem, all of the coordinates in the towers $r_1+1,\dots,r_1+r_2$ are now defined.

The coordinates for the rest of the towers are defined in a manner exactly analogous to those of the second-longest tower.

If π is not integrable, then we cannot use the Frobenius theorem to find the coordinates. In the special case where $s_1 > s_2$, that is, there is only one tower which is longest, it can be shown that if there exists ***any*** π which satisfies the congruences, then there also exists an ***integrable*** π' which also satisfies the congruences (with a rescaling of the basis forms), see [2, 6]. However, if $s_1 = s_2$, or there are at least two towers which are longest, this is no longer true. Thus, we need the assumption that π is integrable. □

If I can be converted to extended Goursat normal form, then the derived flag of I has the structure:

$$\begin{array}{rlllllllll}
I = & \{\alpha_1^1, & \cdots, & \cdots, & \alpha_{s_1-1}^1, & \alpha_{s_1}^1, & \cdots, & \alpha_1^m, & \cdots, & \alpha_{s_m-1}^m, \ \alpha_{s_m}^m\} \\
I^{(1)} = & \{\alpha_1^1, & \cdots, & \cdots, & \alpha_{s_1-1}^1, & & \cdots, & \alpha_1^m, & \cdots, & \alpha_{s_m-1}^m\} \\
& \ \vdots & & & & & & \vdots & & \\
I^{(s_m-1)} = & \{\alpha_1^1, & \cdots, & \alpha_{s_1-s_m+1}^1, & & & \cdots, & \alpha_1^m\} & & \\
& \ \vdots & & & & & \vdots & & & \\
I^{(s_1-2)} = & \{\alpha_1^1, & \alpha_2^1\} & & & & & & & \\
I^{(s_1-1)} = & \{\alpha_1^1\} & & & & & & & & \\
I^{(s_1)} = & \{0\} & & & & & & & &
\end{array}$$

where the forms in each level have been arranged to show the different "towers" which result. The superscripts j indicate the tower to which each form belongs, and the subscripts i index the position of the form within the j^{th} tower. There are s_j forms in the j^{th} tower. An algorithm for converting systems to extended Goursat normal form is given in [2].

We also give another version of this theorem, which is easier to check, since it does not require finding a basis which satisfies the congruences but only one which is adapted to the derived flag. We also give the proof for this theorem, since only a special case is proved in [7].

Theorem 4. Extended Goursat Normal Form [7].
A Pfaffian system I of codimension $m+1$ on $\mathbb{R}^{n+m+1}$ can be converted to Goursat normal form if and only if there exists a one-form π such that $\{I^{(k)}, \pi\}$ is integrable for $k = 0, \ldots, N-1$.

Proof. The only if part is easily shown by taking $\pi = dz^0$ and noting that

$$\begin{aligned}
I^{(k)} &= \{dz_i^j - z_{i-1}^j dz^0 : \ j = 1, \ldots, m; i = k+1, \ldots, s_j\} \\
\{I^{(k)}, \pi\} &= \{dz_i^j, dz^0 : \qquad j = 1, \ldots, m; i = k+1, \ldots, s_j\},
\end{aligned}$$

which is integrable.

For the if part, assume that such a π exists. First, we find the derived flag of the system, $I =: I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(s_1)} = \{0\}$. We will iteratively construct a basis which is adapted to the derived flag and which satisfies the Goursat congruences (3).

We claim that the lengths of each tower are determined from the dimensions of the derived flag. Indeed, the longest tower of forms has length s_1. If the dimension of $I^{(s_1-1)}$ is r_1, then there are r_1 towers which each have length s_1; and we have $s_1 = s_2 = \cdots = s_{r_1}$. Now, if the dimension of $I^{(s_1-2)}$ is $2r_1 + r_2$, then there are r_2 towers with length $s_1 - 1$, and we have $s_{r_1+1} = \cdots = s_{r_1+r_2} = s_1 - 1$. We find each s_j similarly.

We note here that a π which satisfies the conditions must be in the complement of I, for if π were in I, then $\{I, \pi\}$ integrable means that I is integrable, and this contradicts our assumption that I is maximally nonholonomic, that is $I^{(N)} = \{0\}$ for some N.

Consider the last nontrivial derived system, $I^{(s_1-1)}$. Let $\{\alpha_1^1, \ldots, \alpha_1^{r_1}\}$ be a basis for $I^{(s_1-1)}$. The definition of the derived flag, specifically $I^{(s_1)} = \{0\}$, implies that

$$d\alpha_1^j \not\equiv 0 \mod I^{(s_1-1)} \quad j = 1, \ldots, r_1 \tag{4}$$

Also, the assumption that $\{I^{(k)}, \pi\}$ is integrable gives us

$$d\alpha_1^j \equiv 0 \mod \{I^{(s_1-1)}, \pi\} \quad j = 1, \ldots, r_1 \tag{5}$$

combining equations (4) and (5), we have that

$$d\alpha_1^j \equiv \pi \wedge \beta^j \mod I^{(s_1-1)} \quad j = 1, \ldots, r_1 \tag{6}$$

for some $\beta^j \not\equiv 0 \mod I^{(s_1-1)}$.

Now, we also have from the definition of the derived flag that

$$d\alpha_1^j \equiv 0 \mod I^{(s_1-2)} \quad j = 1, \ldots, r_1$$

which combined with (6) gives us that β^j is in $I^{(s_1-2)}$.

Claim. $\beta^1, \ldots, \beta^{r_1}$ are linearly independent mod $I^{(s_1-1)}$.

Proof of Claim. The proof is by contradiction. Suppose there exists some combination of the β^j's, say

$$\beta = b_1\beta^1 + \cdots + b_{r_1}\beta^{r_1} \equiv 0 \mod I^{(s_1-1)}$$

with not all of the b_j's equal to zero. Consider $\alpha = b_1\alpha_1^1 + \cdots + b_{r_1}\alpha_1^{r_1}$. We must have $\alpha \neq 0$ because the α_1^j are a basis for $I^{(s_1-1)}$. The exterior derivative of α can be found by the product rule,

$$\begin{aligned}
d\alpha &= \sum_{j=1}^{r_1} b_j d\alpha_1^j + \sum_{j=1}^{r_1} db_j \wedge \alpha_1^j \\
&\equiv \sum_{j=1}^{r_1} b_j(\pi \wedge \beta^j) \mod I^{(s_1-1)} \\
&\equiv \pi \wedge (\sum_{j=1}^{r_1} b_j\beta^j) \mod I^{(s_1-1)} \\
&\equiv 0 \mod I^{(s_1-1)}
\end{aligned}$$

which implies that α is in $I^{(s_1)}$. However, this contradicts our assumption that $I^{(s_1)} = \{0\}$. Thus the β^j's are linearly independent mod $I^{(s_1-1)}$, and the claim is proven.

Define $\alpha_2^j := \beta^j$ for $j = 1, \ldots, r_1$. Note that these basis elements satisfy the first level of Goursat congruences, that is:

$$d\alpha_1^j \equiv -\alpha_2^j \wedge \pi \mod I^{(s_1-1)} \quad j = 1, \ldots, r_1$$

If the dimension of $I^{(s_1-2)}$ is greater than $2r_1$, then choose one-forms $\alpha_1^{r_1+1}, \ldots, \alpha_1^{r_1+r_2}$ such that that

$$\{\alpha_1^1, \ldots, \alpha_1^{r_1}, \alpha_2^1, \ldots, \alpha_2^{r_1}, \alpha_1^{r_1+1}, \ldots, \alpha_1^{r_1+r_2}\}$$

is a basis for $I^{(s_1-2)}$.

For the induction step, we assume that we have a basis for $I^{(i)}$,

$$\{\alpha_1^1, \ldots, \alpha_{k_1}^1, \alpha_1^2, \ldots, \alpha_{k_2}^2, \ldots, \alpha_1^c, \ldots, \alpha_{k_c}^c\}$$

which satisfies the Goursat congruences up to this level:

$$d\alpha_k^j = -\alpha_{k+1}^j \wedge \pi \mod I^{(s_j-k)} \quad j = 1, \ldots, c; \quad k = 1, \ldots, k_j - 1$$

Note we have assumed that c towers of forms have appeared in $I^{(i)}$. Consider only the last form in each tower that appears in $I^{(i)}$, that is $\alpha_{k_j}^j, j = 1, \ldots, c$. By the construction of this basis (or from the Goursat congruences), we have that $\alpha_{k_j}^j$ is in $I^{(i)}$ but is not in $I^{(i+1)}$, thus

$$d\alpha_{k_j}^j \not\equiv 0 \mod I^{(i)} \quad j = 1, \ldots, c$$

The assumption that $\{I^{(i)}, \pi\}$ is integrable assures us that

$$d\alpha_{k_j}^j \equiv 0 \mod \{I^{(i)}, \pi\} \quad j = 1, \ldots, c$$

thus we have that $d\alpha_{k_j}^j$ must be a multiple of π mod $I^{(i)}$,

$$d\alpha_{k_j}^j \equiv \pi \wedge \beta^j \mod I^{(i)} \quad j = 1, \ldots, c$$

for some $\beta^j \not\equiv 0 \mod I^{(i)}$. We also have, from the fact that $\alpha_{k_j}^k$ is in $I^{(i)}$ and the definition of the derived flag, that

$$d\alpha_{k_j}^j \equiv 0 \mod I^{(i-1)} \quad j = 1, \ldots, c$$

which implies that $\beta^j \in I^{(i-1)}$. By a similar argument to the claim above, we can show that the β^j's are independent mod $I^{(i)}$. We define $\alpha_{k_j+1}^j = \beta^j$, and thus

$$\{\alpha_1^1, \ldots, \alpha_{k_1+1}^1, \alpha_1^2, \ldots, \alpha_{k_2+1}^2, \ldots, \alpha_1^c, \ldots, \alpha_{k_c+1}^c\}$$

forms part of a basis of $I^{(i-1)}$. If the dimension of $I^{(i-1)}$ is greater than $k_1 + k_2 + \cdots + k_c + c$, then we complete the basis of $I^{(i-1)}$ with any linearly independent one-forms $\alpha_1^{c+1}, \ldots, \alpha_1^{c+r_c}$ such that

$$\{\alpha_1^1, \ldots, \alpha_{k_1+1}^1, \alpha_1^2, \ldots, \alpha_{k_2+1}^2, \ldots, \alpha_1^c, \ldots, \alpha_{k_c+1}^c, \alpha_1^{c+1}, \ldots, \alpha_1^{c+r_c}\}$$

is a basis for $I^{(i-1)}$.

Repeated application of this procedure will construct a basis for I which is not only adapted to the derived flag, but also satisfies the Goursat congruences.

We note that by assumption, π is integrable mod the last nontrivial derived system, $I^{(s_1-1)}$. Looking at the congruences (3), we see that any integrable one-form π' which is congruent to π up to a scaling factor,

$$\pi' = dt \equiv f\pi \mod I^{(s_1-1)}$$

will satisfy the same set of congruences up to a rescaling of the constraint basis by multiples of this factor f. □

4. Pfaffian Systems generated by Mobile Robots

We now consider some multi-steering mobile robot systems and show that the Pfaffian systems generated by the constraints that the wheels roll without slipping satisfy the extended Goursat conditions.

Example 1. ***Firetruck.***
Consider the example of a firetruck [3]. There are two steering wheels in the system: one at the front, for the driver, and another at the rear, for the tiller. We model

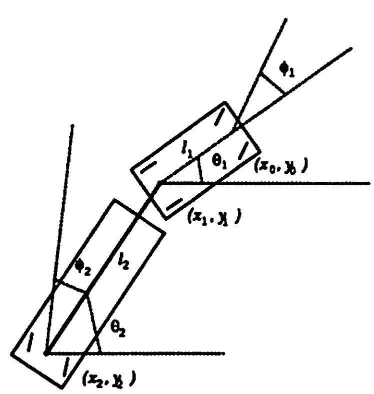

FIGURE 1. A sketch of the Firetruck, with steering wheels on the front and back axles.

the firetruck as a Pfaffian system generated by the constraints that the three pairs of wheels roll without slipping:

$$I = \{\alpha^0, \alpha^1, \alpha^2\}$$

In coordinates, the constraints can be written as:

$$\begin{aligned}\alpha^0 &= \sin(\theta_1+\phi_1)dx_0 - \cos(\theta_1+\phi_1)dy_0\\ \alpha^1 &= \sin\theta_1 dx_1 - \cos\theta_1 dy_1\\ \alpha^2 &= \sin(\theta_2+\phi_2)dx_2 - \cos(\theta_2+\phi_2)dy_2\end{aligned}$$

Since the coordinates x_0, y_0, x_2, y_2 are determined by $x_1, y_1, \theta_1, \theta_2$, we can parameterize the state space by $q = \{x_1, y_1, \theta_1, \theta_2, \phi_1, \phi_2\}$. A complement to the system is $\{dx_1, d\phi_1, d\phi_2\}$. It can be shown that the constraints are adapted to the derived flag,

$$\begin{aligned}I = &\ \{\alpha^0, \alpha^1, \alpha^2\}\\ I^{(1)} = &\ \{\alpha^1\}\\ I^{(2)} = &\ \{0\}\end{aligned}$$

and that the systems $\{I, dx_1\}$ and $\{I^{(1)}, dx_1\}$ are integrable. By Theorem 4, the system can be converted into extended Goursat normal form. We refer the reader to [3, 9] for the coordinate transformation (into chained form, which is the dual of Goursat form), and methods for steering the system.

A multi-steering trailer system was examined in [11], and it was shown how to transform such a system into chained form (which is the dual of Goursat normal form) using dynamic state feedback. That is, states were added to the system and this augmented system was transformed into the dual of Goursat normal form. In this section, we show that this augmentation is not always necessary; some configurations of the multi-steering trailer system can be transformed into Goursat form using only static state feedback.

We concentrate on the specific case of a 5-axle system with two steering wheels. Assuming that the first axle is steerable, there are four possible positions for the second steering wheel. We examine these four cases to show the variety of results that can be obtained from a relatively simple system configuration.

Example 2. ***5-axle, 1-2 steering trailer system.***
Consider the case of a 5-axle trailer system with the first two axles steerable. A sketch of this system is shown in Figure 2. The state space is parameterized by $q = \{x_5, y_5, \theta_5, \theta_4, \theta_3, \theta_2, \theta_1, \phi\}$, the x, y position of the last axle, the angle of each hitch θ_i, and the extra steering angle ϕ. We note that the x_i, y_i positions of the other axles can be written in terms of the state variables q.

The constraints are that each axle roll without slipping:

$$\begin{aligned}\alpha^i &= \sin\theta_i dx_i - \cos\theta_i dy_i \quad i = 1,3,4,5\\ \alpha^2 &= \sin\phi dx_2 - \cos\phi dy_2\end{aligned}$$

The Pfaffian system is $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$, a complement to this system is given by $\{d\phi, d\theta_1, dx_5\}$, and I has codimension three.

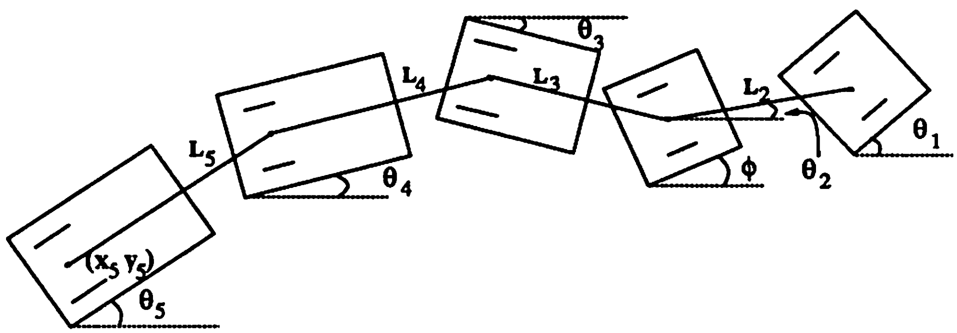

FIGURE 2. A 5-axle trailer system with the first two axles steerable.

This basis is adapted to the derived flag,

$$\begin{aligned} I &= \{\alpha^1,\ \alpha^2,\ \alpha^3,\ \alpha^4,\ \alpha^5\} \\ I^{(1)} &= \{\alpha^3,\ \alpha^4,\ \alpha^5\} \\ I^{(2)} &= \{\alpha^4,\ \alpha^5\} \\ I^{(3)} &= \{\alpha^5\} \\ I^{(4)} &= \{0\} \end{aligned}$$

and the congruences are satisfied with this basis as well:

$$\begin{aligned} d\alpha^5 &= c_5(q)\,\alpha^4\wedge dx_5 && \text{mod } I^{(3)} \\ d\alpha^4 &= c_4(q)\,\alpha^3\wedge dx_5 && \text{mod } I^{(2)} \\ d\alpha^3 &= c_3(q)\,\alpha^2\wedge dx_5 && \text{mod } I^{(1)} \\ d\alpha^2 &= c_2(q)\,d\phi\wedge dx_5 && \text{mod } I \\ d\alpha^1 &= c_1(q)\,d\theta_1\wedge dx_5 && \text{mod } I \end{aligned}$$

We note here that by a simple rescaling of the basis, the functions $c_i(q)$ can be eliminated to get the Goursat congruences (3) exactly. This is done as follows. First define $\bar{\alpha}^4 := -c_5(q)\alpha^4$ to get $d\alpha^5 \equiv -\bar{\alpha}^4\wedge dx_5 \mod I^{(3)}$. Then, taking the derivative of $\bar{\alpha}^4$, we see that

$$\begin{aligned} d\bar{\alpha}^4 &= -c_5(q)d\alpha^4 - dc_5(q)\wedge\alpha^4 \\ &\equiv -c_5(q)c_4(q)\alpha^3\wedge dx_5 \mod I^{(2)} \end{aligned}$$

since α^4 is in $I^{(2)}$. Defining $\bar{\alpha}^3 := c_5(q)c_4(q)\alpha^3$, we have $d\bar{\alpha}^4 = -\bar{\alpha}^3\wedge dx_5 \mod I^{(2)}$. The other constraints are scaled similarly. For the rest of this paper, we will assert that the Goursat congruences are satisfied if we have the modified congruences

$$\begin{aligned} d\alpha^j_i &\equiv c^j_{i+1}(q)\,\alpha^j_{i+1}\wedge\pi && \text{mod } I^{(s_j-i)} \quad i=1,\ldots,s_j-1 \\ d\alpha^j_{s_j} &\not\equiv 0 && \text{mod } I \end{aligned} \tag{7}$$

instead of the original congruences (3).

Example 3. ***5-axle, 1-5 steering trailer system.***
Now consider the same 5-axle system as in the previous example with the first and fifth (last) axles steerable, as sketched in Figure 3. The configuration space can be

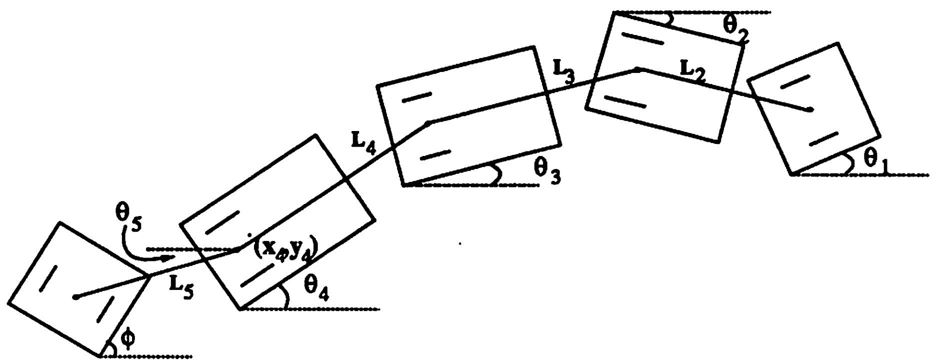

FIGURE 3. A 5-axle trailer system with steering wheels on the first and last axles.

parameterized by: $q = \{x_4, y_4, \theta_5, \theta_4, \theta_3, \theta_2, \theta_1, \phi\}$, the x, y position of the second-to-last axle, the angle of each hitch, and the angles of the steering wheels. As before, the x_i, y_i positions of the other axles can be written in terms of the coordinates q.

The constraints are that each axle roll without slipping,

$$\begin{aligned} \alpha^i &= \sin\theta_i dx_i - \cos\theta_i dy_i \quad i = 1,2,3,4 \\ \alpha^5 &= \sin\phi dx_5 - \cos\phi dy_5 \end{aligned}$$

The Pfaffian system is $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$. A complement to this system is given by $\{d\phi, d\theta_1, dx_4\}$; I has codimension three. This basis is adapted to the derived flag,

$$\begin{aligned} I &= \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5\} \\ I^{(1)} &= \{\alpha^2, \alpha^3, \alpha^4\} \\ I^{(2)} &= \{\alpha^3, \alpha^4\} \\ I^{(3)} &= \{\alpha^4\} \\ I^{(4)} &= \{0\} \end{aligned}$$

The congruences are satisfied with this basis as well,

$$\begin{aligned} d\alpha^4 &= c_4(q)\, \alpha^3 \wedge dx_4 \mod I^{(3)} \\ d\alpha^3 &= c_3(q)\, \alpha^2 \wedge dx_4 \mod I^{(2)} \\ d\alpha^2 &= c_2(q)\, \alpha^1 \wedge dx_4 \mod I^{(1)} \\ d\alpha^1 &= c_1(q)\, d\theta_1 \wedge dx \mod I \\ d\alpha^5 &= c_5(q)\, d\phi \wedge dx_4 \mod I \end{aligned}$$

and the Pfaffian system can be converted to extended Goursat normal form.

Example 4. ***5-axle, 1-4 steering trailer system.***
We now consider the 5-axle system with the first and fourth axles steerable, as sketched in Figure 4. The configuration space can be parameterized by the x, y

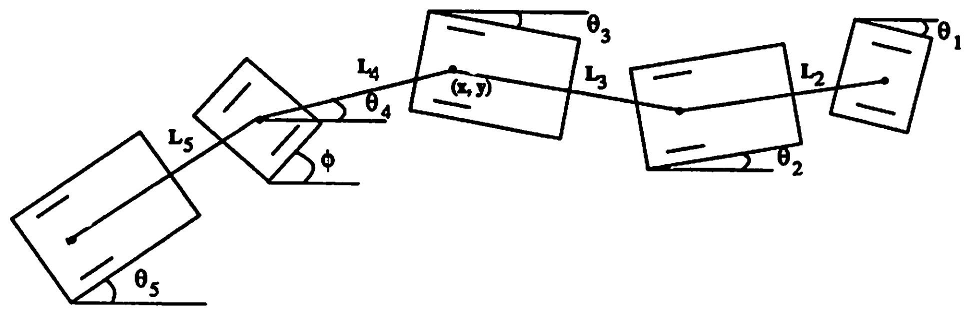

FIGURE 4. A 5-axle trailer system with the first and fourth axles steerable.

position of the third axle, the hitch angles θ_i, and the steering angle of the third axle ϕ. We let q represent the state, $q = \{x_3, y_3, \theta_5, \theta_4, \theta_3, \theta_2, \theta_1, \phi\}$. The other x_i, y_i positions can be written in terms of q.

The constraints are that each axle rolls without slipping:

$$\begin{aligned} \alpha^i &= \sin\theta_i dx_i - \cos\theta_i dy_i \quad i = 1,2,3,5 \\ \alpha^4 &= \sin\phi dx_4 - \cos\phi dy_4 \end{aligned}$$

The Pfaffian system is thus $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$ and a complement to this system is: $\{d\phi, d\theta_1, dx_3\}$. This basis is adapted to the the derived flag,

$$\begin{aligned} I &= \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5\} \\ I^{(1)} &= \{\alpha^2, \alpha^3, \alpha^5\} \\ I^{(2)} &= \{\alpha^3\} \\ I^{(3)} &= \{0\} \end{aligned}$$

and also satisfies the Goursat congruences:

$$\begin{aligned} d\alpha^3 &\equiv c_3(q)\, \alpha^2 \wedge dx_3 && \mod I^{(2)} \\ d\alpha^2 &\equiv c_2(q)\, \alpha^1 \wedge dx_3 && \mod I^{(1)} \\ d\alpha^1 &\equiv c_1(q)\, d\theta_1 \wedge dx_3 && \mod I \\ d\alpha^5 &\equiv c_5(q)\, \alpha^4 \wedge dx_3 && \mod I^{(1)} \\ d\alpha^4 &\equiv c_4(q)\, d\phi \wedge dx_3 && \mod I \end{aligned}$$

and thus the Pfaffian system can be converted to extended Goursat normal form.

Example 5. ***5-axle, 1-3 steering.***
The final instance of the 5-axle trailer system has the first and third axles steerable, as sketched in Figure 5. The state space is parameterized by: $q = \{x_5, y_5, \theta_5, \theta_4, \theta_3, \theta_2, \theta_1, \phi\}$,

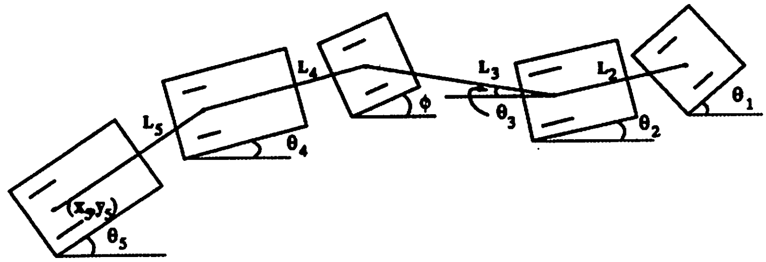

FIGURE 5. A 5-axle trailer system with the first and third axles steerable. This is the only configuration of the 5-axle system with two steering wheels which does not satisfy the conditions for converting to extended Goursat normal form.

and the other x_i, y_i can be written in terms of q.

The constraints are that each axle roll without slipping:

$$\begin{aligned} \alpha^i &= \sin\theta_i dx_i - \cos\theta_i dy_i \quad i = 1,2,4,5 \\ \alpha^3 &= \sin\phi dx_3 - \cos\phi dy_3 \end{aligned}$$

The Pfaffian system is $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$, and a complement to the system is given by $\{d\phi, d\theta_1, dx_5\}$.

This basis is adapted to the derived flag,

$$\begin{aligned} I = &\ \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5\} \\ I^{(1)} = &\ \{\alpha^2, \alpha^4, \alpha^5\} \\ I^{(2)} = &\ \{\alpha^5\} \\ I^{(3)} = &\ \{0\} \end{aligned}$$

however, the congruences are not satisfied:

$$\begin{aligned} d\alpha^5 &\equiv c_5(q)\, \alpha^4 \wedge dx_3 && \text{mod } I^{(2)} \\ d\alpha^4 &\equiv c_4(q)\, \alpha^3 \wedge dx_3 && \text{mod } I^{(1)} \\ d\alpha^2 &\equiv c_2(q)\, \alpha^1 \wedge dx_3 + k_2(q)\, \alpha^1 \wedge \alpha^3 && \text{mod } I^{(1)} \\ d\alpha^3 &\equiv c_3(q)\, d\phi \wedge dx_3 && \text{mod } I \\ d\alpha^1 &\equiv c_1(q)\, d\theta_1 \wedge dx_3 && \text{mod } I \end{aligned}$$

In order to have $\{I^{(2)}, \pi\}$ integrable, we must choose $\pi = dx_3$ (mod $\{\alpha^4, \alpha^5\}$). This will also give us $\{I^0, \pi\}$ integrable, but $\{I^1, \pi\}$ is *not* integrable. Thus, this system does not satisfy the conditions for conversion to extended Goursat normal form. We will return to this example in the next section.

5. PROLONGATIONS

If a Pfaffian system I of codimension k satisfies the necessary and sufficient conditions for converting into extended Goursat form, then its solution trajectories are determined by k arbitrary functions. However, even if a system cannot be transformed into Goursat form, its solution trajectories may still have this property. If so, then we say that I is ***absolutely equivalent*** in the sense of Cartan to the trivial system (the system with no constraints) on $\mathbb{R}^k$. Although we will not examine the concept of absolute equivalence in its full generality, we will give some sufficient conditions for a Pfaffian system to have a ***prolongation*** which can be converted to Goursat form, and thus the solution trajectories of I are determined by k independent functions.

A general type of prolongation which preserves a one to one correspondence between solution trajectories of the original and prolonged system is a Cartan prolongation.

Definition 6. *Cartan Prolongation.*
Let I be a Pfaffian system on a manifold M. A system J on $M \times \mathbb{R}^p$ is a *Cartan prolongation* of I if :

(1) $\pi^*(I) \subset J$
(2) For every solution curve $c : (-\epsilon, \epsilon) \to M$ of I there exists a unique solution curve $\tilde{c} : (-\epsilon, \epsilon) \to M \times \mathbb{R}^p$ of J with $\pi \circ \tilde{c} = c$.

If I is equipped with a given independence condition τ, then we also require that $\pi^*\tau$ be the independence condition for J.

A canonical way to prolong a system with independence condition dt is to take an integrable one-form $d\eta$ in the complement of I, and augment I with the additional form $d\eta - y\,dt$, where y is a new coordinate on $\mathbb{R}$. In effect, this adds the derivative of η (with respect to the independence condition) as a state variable. As long as all solution trajectories are "smooth enough" (we will assume C^∞), there will be a one-one correspondence between solution trajectories of the original and the prolonged system.

We consider a special type of Cartan prolongation which consists of many of these canonical prolongations.

Definition 7. *Prolongation by differentiation.*
Let I be a Pfaffian system of codimension $m+1$ on $\mathbb{R}^{n+m+1}$ with coordinates (z, v, t) for which dt is an independence condition and $\{dv_1, \dots, dv_m, dt\}$ forms a complement. Let $b_1, \dots, b_m$ be nonnegative integers and let b denote their sum. The system I augmented by

$$\begin{array}{llll} dv_1 - v_1^1 dt, & \dots, & dv_1^{b_1-1} - v_1^{b_1} dt, & \\ dv_2 - v_2^1 dt, & \dots, & & dv_2^{b_2-1} - v_2^{b_2} dt \\ & \vdots & & \ddots \\ dv_m - v_m^1 dt, & \dots, & \dots, & dv_m^{b_m-1} - v_m^{b_m} dt, \end{array}$$

is called a *prolongation by differentiation* of I. The augmented system is defined on $\mathbb{R}^{n+m+b+1}$.

We note here that since the original system and independence condition corresponded to a set of first order ordinary differential equations, the prolonged system has the same independence condition and also corresponds to a set of first order ordinary differential equations.

We can now give sufficient conditions for a Pfaffian system to have a prolongation which is equivalent to extended Goursat form.

Theorem 5. Conversion to Goursat form using prolongation by differentiation.

Consider a Pfaffian system $I = \{\alpha^1, \dots, \alpha^n\}$ on $\mathbb{R}^{n+m+1}$ with independence condition dz^0 and complement $\{dv_1, \dots, dv_m, dz^0\}$. If there exists a list of integers $b_1, \dots, b_m$ such that the prolonged system

$$\begin{aligned}\tilde{I} = \{ \quad & \alpha^1, \dots, \alpha^n, dv_1 - v_1^1 dz^0, \dots, dv_1^{b_1-1} - v_1^{b_1} dz^0, \\ & \dots, dv_m - v_m^1 dz^0, \dots, dv_m^{b_m-1} - v_m^{b_m} dz^0\}\end{aligned}$$

satisfies the condition that $\{\tilde{I}^{(k)}, dz^0\}$ is integrable for all k, then I can be transformed to extended Goursat normal form using a prolongation by differentiation.

Proof. The proof is by application of Theorem 4 to the prolonged system $\tilde{I}$. □

Although this is a very specific form of prolongation of a Pfaffian system, and the conditions of the theorem must be checked in a specific coordinate system with a given independence condition, there do exist practical systems which can be converted into extended Goursat form using this type of prolongation.

Example 6. ***5-axle, 1-3 steering, revisited.***

We return to the 5-axle trailer system with the first and third axles steerable, which did not satisfy the conditions for conversion to extended Goursat form.

Recall that the derived flag was of the form:

$$\begin{aligned} I &= \{\alpha^1,\ \alpha^2,\ \alpha^3,\ \alpha^4,\ \alpha^5\} \\ I^{(1)} &= \{\alpha^2,\ \alpha^4,\ \alpha^5\} \\ I^{(2)} &= \{\alpha^5\} \\ I^{(3)} &= \{0\} \end{aligned}$$

but that the congruences were not satisfied:

$$\begin{aligned} d\alpha^5 &\equiv c_5(q)\, \alpha^4 \wedge dx_3 && \mod I^{(2)} \\ d\alpha^4 &\equiv c_4(q)\, \alpha^3 \wedge dx_3 && \mod I^{(1)} \\ d\alpha^2 &\equiv c_2(q)\, \alpha^1 \wedge dx_3 + k_2(q)\, \alpha^1 \wedge \alpha^3 && \mod I^{(1)} \\ d\alpha^3 &\equiv c_3(q)\, d\phi \wedge dx_3 && \mod I \\ d\alpha^1 &\equiv c_1(q)\, d\theta_1 \wedge dx_3 && \mod I \end{aligned}$$

We can look at the equations for the exterior derivatives of the constraints to see if after prolongation, the augmented Pfaffian system will satisfy the conditions for conversion to extended Goursat form. We see that $\pi = dx_5$ will give us $\{I^{(2)}, \pi\}$ integrable. However, $\{I^{(1)}, dx_5\}$ is not integrable since $d\alpha^2$ has a term $k_2(q)\,\alpha^1 \wedge \alpha^3$. If either α^1 or α^3 could be added to $I^{(1)}$, that term would no longer cause a problem. We note that if α^1 were added to $I^{(1)}$, then $d\alpha^2 \equiv 0 \mod I^{(1)}$ and we will still have the same problem, except now with $I^{(2)}$. If we can somehow add α^3 to $I^{(1)}$, it appears that the conditions of Theorem 4 will be satisfied (the only thing remaining to be checked is that $d\alpha^3 \equiv 0 \mod \alpha^2, \alpha^3, \alpha^4, \alpha^5, dx_5$.)

We prolong I by differentiation, and augment it by the additional form $\omega = d\phi - v dx_3$. The derived flag of the augmented system is:

$$\begin{aligned}
\tilde{I} &= \{\alpha^1,\ \alpha^2,\ \omega,\ \alpha^3,\ \alpha^4,\ \alpha^5\} \\
\tilde{I}^{(1)} &= \{\alpha^2,\ \alpha^3,\ \alpha^4,\ \alpha^5\} \\
\tilde{I}^{(2)} &= \{\alpha^4,\ \alpha^5\} \\
\tilde{I}^{(3)} &= \{\alpha^5\} \\
\tilde{I}^{(4)} &= \{0\}
\end{aligned}$$

and the systems $\{\tilde{I}^{(k)}, dx_3\}$ are integrable for all k, as can be seen from looking at the Goursat congruences,

$$\begin{aligned}
d\alpha^5 &\equiv c_5(q)\,\alpha^4 \wedge dx_3 && \mod \tilde{I}^{(3)} \\
d\alpha^4 &\equiv c_4(q)\,\alpha^3 \wedge dx_3 && \mod \tilde{I}^{(2)} \\
d\alpha^3 &\equiv c_3(q)\,d\omega \wedge dx_3 && \mod \tilde{I}^{(1)} \\
d\omega &\equiv c_\omega(q)\,dv \wedge dx_3 && \mod \tilde{I} \\
d\alpha^2 &\equiv c_2(q)\,\alpha^1 \wedge dx_3 && \mod \tilde{I}^{(1)} \\
d\alpha^1 &\equiv c_1(q)\,d\theta_1 \wedge dx_3 && \mod \tilde{I}
\end{aligned}$$

Thus, the prolonged system $\tilde{I}$ can be converted into extended Goursat normal form. This extension is shown in [11], as are methods for steering this type of system.

6. Control Systems

As we mentioned in the introduction, control systems are a special type of Pfaffian system, and therefore all of the results presented thus far can be specialized to control systems. Most of the previous work analyzing nonlinear control systems has been from the vector field point of view, taking a system $\dot{x} = f(x) + g_1(x)u_1 + \cdots g_m(x)u_m$, and looking at properties of the vector fields f, g_i. The Pfaffian systems formulation is the dual of this.

Definition 8. ***Control System.***
A *control system* $\dot{x} = f(x, u)$ with the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, and the derivative of the state taken with respect to time $t \in \mathbb{R}$, generates a Pfaffian system I on $\mathbb{R}^{n+m+1}$

$$I = \{dx_i - f^i(x, u)dt : i = 1, \ldots, n\} \tag{8}$$

with complement $\{du_1, \ldots, du_m, dt\}$. The natural independence condition to choose is dt, since we want $dt \neq 0$ along all solution trajectories of the system.

Any Pfaffian system I of codimension $m+1$ on $\mathbb{R}^{n+m+1}$ with coordinates (x, u, t) can be called a *control system* if it has a set of generators of the form (8).

Brunovsky showed that any controllable linear system $\dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ can be converted to a "canonical" form given by

$$\begin{array}{llll} \dot{x}_1^1 = u_1 & \dot{x}_1^2 = u_2 & \cdots & \dot{x}_1^m = u_m \\ \dot{x}_2^1 = x_1^1 & \dot{x}_2^2 = x_1^2 & \cdots & \dot{x}_2^m = x_1^m \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \dot{x}_{k_m}^m = x_{k_m-1}^m \\ \vdots & \dot{x}_{k_2}^2 = x_{k_2-1}^2 & & \\ \dot{x}_{k_1}^1 = x_{k_1-1}^1 & & & \end{array} \tag{9}$$

with $n = k_1 + \cdots + k_m$. A control system is said to be *linearizable* if and only if it can be converted to Brunovsky form using a nonlinear coordinate change and state feedback. Since Brunovsky linear form for a control system is a special case of extended Goursat normal form (2) with $dz^0 = dt$ and $z_{s_1+1}^j = u_j$, the theorems for transforming to Goursat form can be specialized to give conditions for exact linearization.

Theorem 6. Exact Linearization [5].
If a control system I defined on $\mathbb{R}^{n+m+1}$ has a set of generators $\{\alpha_i^j : j = 1, \ldots, m; i = 1, \ldots, s_j\}$ such that for all j,

$$\begin{array}{lll} d\alpha_i^j \equiv -\alpha_{i+1}^j \wedge dt & \mod I^{(s_j-i)} & i = 1, \ldots, s_j - 1 \\ d\alpha_{s_j}^j \not\equiv 0 & \mod I & \end{array} \tag{10}$$

then there exists a set of coordinates z such that I is in Brunovsky normal form,

$$I = \{dz_i^j - z_{i+1}^j dt : j = 1, \ldots, m; i = 1, \ldots, s_j\}.$$

An algorithm for converting systems to Brunovsky normal form is also given in [5], and it is shown that if the control system is time-invariant and affine in the inputs, then the resulting feedback transformation is also autonomous and input-affine.

The control systems version of Theorem 4 is given by

Theorem 7. Exact Linearization [8].
A control system I can be converted to linear form if and only if $\{I^{(k)}, dt\}$ is integrable for every k.

By way of example, we examine a control system which is not linearizable but can be converted to Goursat normal form. The transformation scales time by a function of one of the states.

Example 7. ***Goursat normal form for a control system.***
Consider the single-input control system [4],

$$\begin{aligned}\dot{x}_1 &= x_2 + {x_3}^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\end{aligned}$$

This control system generates a Pfaffian system,

$$I = \{dx_1 - (x_2 + {x_3}^2)dt, dx_2 - x_3 dt, dx_3 - udt\} \tag{11}$$

I is of codimension two on $\mathbb{R}^5$ with coordinates (x, u, t). The derived flag of I is

$$\begin{aligned}I &= \{\alpha^1, \alpha^2, \alpha^3\} \\ I^{(1)} &= \{\alpha^1, \alpha^2\} \\ I^{(2)} &= \{\alpha^1\} \\ I^{(3)} &= \{0\}\end{aligned}$$

where the one-forms adapted to the derived flag are given by

$$\begin{aligned}\alpha^1 &= dx_1 - 2x_3 dx_2 + ({x_3}^2 - x_2)dt \\ \alpha^2 &= dx_2 - x_3 dt \\ \alpha^3 &= dx_3 - udt\end{aligned}$$

Note that this is not the basis of (11) which generated I. Since $\{I^{(2)}, dt\}$ is not integrable, the system is not feedback linearizable by Theorem 7.

We find however that the Goursat congruences (1) *are* satisfied, for $\pi = d\tau = dt - 2dx_3$:

$$\begin{aligned}d\alpha^1 &= \alpha^2 \wedge d\tau \\ d\alpha^2 &= c(u)\, \alpha^3 \wedge d\tau \\ d\alpha^3 &= c(u)\, du \wedge d\tau \qquad \text{mod } \alpha^3\end{aligned}$$

Thus, there does exist a transformation $\Phi(x, u, t) = (z, v, \tau)$ to Goursat normal form, which is given by

$$\begin{aligned}\tau &= t - 2x_3 \\ v &= \frac{u}{1 - 2u} \\ z_1 &= x_3 \\ z_2 &= x_2 - {x_3}^2 \\ z_3 &= x_1 - 2x_2 x_3 + \frac{2}{3}{x_3}^3\end{aligned}$$

and it is easily checked that

$$\begin{aligned}\frac{dz_1}{d\tau} &= v\\ \frac{dz_2}{d\tau} &= z_1\\ \frac{dz_3}{d\tau} &= z_2\end{aligned}$$

This formulation of the system makes it simple to analyze the possible trajectories of the system in state/time space. Also, a controller could be designed using linear techniques which would be valid in certain regions of the state space.

Converting control systems to Goursat normal form may not be the most useful thing to do. However, linearizing control system using dynamic extension is a problem that has been studied extensively. A *dynamic extension* of a control system is an augmented system with integrators added to the inputs; for example, a simple first-order dynamic extension is given by:

$$\begin{aligned}\dot{x} &= f(x,u)\\ \dot{u}_k &= v\end{aligned}$$

where an integrator is added to the k^{th} input channel.

The prolongation by differentiation which we defined in Section 5 is exactly the dual of dynamic extension in the language of forms. Thus, we have the control systems version of Theorem 5:

Theorem 8. Linearization by dynamic extension.
Consider a control system I on $\mathbb{R}^{n+m+1}$ with coordinates (x,u,t), independence condition dt, and complement $\{du_1,\dots,du_m,dt\}$. If there exists a prolongation by differentiation of dimension $b = b_1 + \cdots + b_m$ such that the augmented system

$$\begin{aligned}\tilde{I} = \{ \quad &\alpha^i = dx_i - f^i(x,u)dt: \quad i = 1,\dots,n;\\ &\beta_j^k = du_j^{k-1} - u_j^k dt: \qquad j = 1,\dots,m; k = 0,\dots,b_i\}\end{aligned}$$

on $\mathbb{R}^{n+m+b+1}$ satisfies the condition $\{\tilde{I}^{(k)}, dt\}$ is integrable for every k, then the original system I is linearizable by dynamic extension.

Proof. Apply Theorem 7 to the extended system $\tilde{I}$. □

This theorem is similar to the one stated by Charlet, Lévine, and Marino [4] which also gave sufficient conditions for linearizing systems by dynamic extension. Their conditions also relied on the existence of some integers b_i which are the number of integrators added to the i^{th} input channel. However, the existence of a dynamic extension of order $b = (b_1,\dots,b_m)$ which is linearizable does not imply that the conditions of their theorem are satisfied for that b; whereas if there exists a dynamic extension of order $b = (b_1, \ldots, b_m)$ which can be linearized, our conditions will always be satisfied for that b.

We present a simple example to show how the theorem can be applied to linearize control systems using dynamic extension.

Example 8. ***A control system satisfying the conditions of Theorem 8 but not the conditions of*** **[4].**
Consider a 4-state, 2-input control system:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3 u_2 \\ \dot{x}_2 &= x_3 + x_1 u_2 \\ \dot{x}_3 &= u_1 + x_2 u_2 \\ \dot{x}_4 &= u_2 \end{aligned}$$

The corresponding Pfaffian system on $\mathbb{R}^7$ is

$$I = \{dx_1 - (x_2 + x_3u_2)dt, dx_2 - (x_3 + x_1u_2)dt, dx_3 - (u_1 + x_2u_2)dt, dx_4 - u_2dt\} \tag{12}$$

with independence condition dt and complement $\{du_1, du_2, dt\}$. The derived flag has the form:

$$\begin{aligned} I = {} & \{\alpha^1,\ \alpha^2,\ \alpha^3,\ \alpha^4\} \\ I^{(1)} = {} & \{\alpha^1,\ \alpha^4\} \\ I^{(2)} = {} & \{0\} \end{aligned}$$

The one-forms α^i which are adapted to the derived flag are not the same as those of (12) which generated I,

$$\begin{aligned} \alpha^1 &= dx_1 - x_2 dt \\ \alpha^2 &= dx_2 - (x_3 + x_1u_2)dt \\ \alpha^3 &= dx_3 - (u_1 + x_2u_2)dt \\ \alpha^4 &= dx_4 + x_3 dx_2 - x_3 dt \end{aligned}$$

The structure equations are fairly simple to find,

$$\begin{aligned} d\alpha^1 &= -\alpha^2 \wedge dt \\ d\alpha^2 &= -x_1 du_2 \wedge dt \\ d\alpha^3 &= -du_1 \wedge dt - x_2 du_2 \wedge dt \\ d\alpha^4 &= -\alpha^2 \wedge \alpha^3 + (x_3 + x_1u_2 - 1)\alpha^3 \wedge dt - (x_2u_2 + u_1)\alpha^2 \wedge dt \end{aligned}$$

and we note that $\{I^{(1)}, dt\}$ is not integrable, thus the system is not linearizable by static state feedback.

Now consider a prolongation by differentiation of I on $\mathbb{R}^{10}$,

$$\tilde{I} = \{I, \beta^1, \beta^2, \beta^3\}$$

with the additional one-forms

$$\begin{aligned} \beta^1 &= du_2 - v\,dt \\ \beta^2 &= dv - w\,dt \\ \beta^3 &= dw - z\,dt \end{aligned}$$

and additional coordinates v, w, z. A complement to $\tilde{I}$ is $\{du_1, dz, dt\}$. The derived flag of the extended system has the form

$$\begin{aligned} \tilde{I} &= \{\omega^1,\ \omega^2,\ \omega^3,\ \omega^4,\ \beta^1,\ \beta^2,\ \beta^3\} \\ \tilde{I}^{(1)} &= \{\omega^1,\ \omega^2,\ \omega^4,\ \beta^1,\ \beta^2\} \\ \tilde{I}^{(2)} &= \{\omega^1,\ \omega^4,\ \beta^1\} \\ \tilde{I}^{(3)} &= \{\omega^4\} \\ \tilde{I}^{(4)} &= \{0\} \end{aligned}$$

where the one-forms adapted to the derived flag are

$$\begin{aligned} \omega^1 &= dx_1 - u_2 dx_2 + ({u_2}^2 x_1 - x_2)dt \\ \omega^2 &= dx_2 - (u_2 x_1 + x_3)dt \\ \omega^3 &= dx_3 - (u_1 + u_2 x_2)dt \\ \omega^4 &= dx_4 - u_2 dt \end{aligned}$$

The structure equations are

$$\begin{aligned} d\omega^1 &\equiv (-1 + {u_2}^3 + v)\,\omega^2 \wedge dt \mod I^{(2)} \\ d\omega^2 &\equiv -\omega^3 \wedge dt \mod I^{(1)} \\ d\omega^3 &\equiv -du_1 \wedge dt \mod I \\ d\omega^4 &= -\beta^1 \wedge dt \\ d\beta^1 &= -\beta^2 \wedge dt \\ d\beta^2 &= -\beta^3 \wedge dt \\ d\beta^3 &= -dz \wedge dt \end{aligned}$$

from which it is easily seen that each $\{\tilde{I}^{(i)}, dt\}$ is integrable, hence the prolonged system $\tilde{I}$ can be feedback linearized.

We note that although this example can be linearized by dynamic extension, it is shown in [4] that it does not satisfy the sufficient conditions given in that paper.

Remark 2. ***Dynamic State Feedback.***
We have shown that a prolongation by differentiation of a control system corresponds to a dynamic extension. A general dynamic state feedback corresponds to adding

some states to the system and putting feedback around them, such as

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{z} &= g(x,z,v) \\ u &= \beta(x,z,v) \end{aligned}$$

This general form does not correspond to a Cartan prolongation, since there may not be a one-one correspondence between trajectories of the extended system and trajectories of the original system. This is especially obvious if the added z states have their own dynamics, independent of x, such as in the example due to [12]

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 \\ u &= \beta(z)v. \end{aligned}$$

where harmonics are added in the dynamic state feedback. This example corresponds to a prolongation of the original system, but not a Cartan prolongation, since there are many possible trajectories in (x,z) space for every trajectory in x, depending on the initial conditions of the z coordinates.

7. Conclusions

In this paper, we presented the method of exterior differential systems for analyzing nonlinear systems. We have given necessary and sufficient conditions for converting Pfaffian systems to Goursat normal form, and we contributed sufficient conditions for converting systems to Goursat form using prolongations. In addition, we showed how the techniques that we described for general Pfaffian systems could be specialized to control systems, and the conditions for exactly linearizing systems could be restated in the language of forms. Since dynamic extension is the dual of prolongation by differentiation, our theorem for converting Pfaffian systems to Goursat form using prolongation could be specialized to give conditions for converting control systems to Brunovsky form using dynamic extension and nonlinear feedback. We showed that these conditions are closer to necessary and sufficient than those which exist in the literature.

Future directions of research include investigating other types of prolongation besides prolongation by differentiation as well as algorithms for systematically prolonging Pfaffian systems to achieve equivalence to Goursat forms.

REFERENCES

1. R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior Differential Systems.* Springer-Verlag, 1991.
2. L. Bushnell, D. Tilbury, and S. S. Sastry. Extended Goursat normal forms with applications to nonholonomic motion planning. In *Proceedings of the IEEE Conference on Decision and Control*, pages 3447–3452, San Antonio, Texas, 1993.
3. L. Bushnell, D. Tilbury, and S. S. Sastry. Steering three-input chained form nonholonomic systems using sinusoids: The firetruck example. In *Proceedings of the European Control Conference*, pages 1432–1437, Groningen, The Netherlands, 1993.
4. B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM Journal of Control and Optimization*, 29(1):38–57, 1991.
5. R. B. Gardner and W. F. Shadwick. The GS algorithm for exact linearization to Brunovsky normal form. *IEEE Transactions on Automatic Control*, 37(2):224–230, 1992.
6. R. M. Murray. Applications and extensions of goursat normal form to control of nonlinear systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 3425–3430, 1993.
7. W. F. Shadwick and W. M. Sluis. Dynamic feedback for classical geometries. Technical Report FI93-CT23, The Fields Institute, Ontario, Canada, 1993.
8. W. M. Sluis. *Absolute Equivalence and its Applications to Control Theory.* PhD thesis, University of Waterloo, 1992.
9. D. Tilbury and A. Chelouah. Steering a three-input nonholonomic system using multirate controls. In *Proceedings of the European Control Conference*, pages 1428–1431, Groningen, The Netherlands, 1993.
10. D. Tilbury, R. Murray, and S. Sastry. Trajectory generation for the N-trailer problem using Goursat normal form. In *Proceedings of the IEEE Conference on Decision and Control*, pages 971–977, San Antonio, Texas, 1993.
11. D. Tilbury, O. Sørdalen, L. Bushnell, and S. Sastry. A multi-steering trailer system: Conversion into chained form using dynamic feedback. Technical Report UCB/ERL M93/55, Electronics Research Laboratory, University of California at Berkeley, 1993.
12. M. van Nieuwstadt, M. Rathinam, and R. M. Murray. Differential flatness and absolute equivalence. Technical report, California Institute of Technology, March 1994.