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ABSTRACT

Most optimal design problems can only be solved through discretization. One solution strategy is to expand the original problem into an infinite sequence of finite dimensional, approximating nonlinear programming problems, which can be solved using standard algorithms. In this paper, an expansion strategy based on the concept of consistent approximations is proposed for certain optimal beam design problems, where the beam is modelled using Euler-Bernoulli beam theory. It is shown that any accumulation point of the sequence of the stationary points of the family of approximating problems is a stationary point of the original, infinite-dimensional problem. Numerical results are presented for problems of optimal design of fixed beams.

Key Words: optimal design, discretization theory, epiconvergence, consistent approximations, algorithm convergence theory.

AMS(MOS) subject classification. primary 49Q10; secondary 65K10

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1. INTRODUCTION

In the last 15 years, we have witnessed great activity in the development of computational procedures for the solution of optimal design problems (see, e.g., [4, 5, 8, 9, 10, 14] and references therein). In general, the use of such computational procedures involves the replacement of the set of admissible designs, the laws describing the behavior of the system under study, the cost function, the constraints, and the optimality conditions by appropriately discretized counterparts. Clearly, to be of any value, these discretizations must satisfy some consistency conditions. The consistency conditions for approximating problems that we find in the optimal design literature deal only with convergence of global minimizers of approximating problems to a global minimizer of the original problem (see, e.g., [6, 8, 11, 14]). As we will see in Section 2, this is related to the concept of epiconvergence of the approximating problems to the original problem. However, in the absence of convexity, nonlinear programming algorithms can only be shown to compute stationary points that are, hopefully, local minimizers of the approximating problems. As we will show by example in Section 2, this fact can lead to serious pathologies, such as the convergence of stationary points of the approximating problems to a nonstationary point of the original problem.

In [16] we find a theory of consistent approximations dealing with the expansion of an infinite dimensional problem into an infinite sequence of finite dimensional approximating problems, each with a finite number of constraints. In [16], because of the abstract problem formulation, as well as for algorithmic reasons, optimality conditions are expressed in terms of zeros of *optimality functions*. In [16] consistency of approximating problems is characterized in terms of the Kuratowski convergence of the constrained epigraphs of their cost functions and of the hypographs of their optimality functions to those of the original problem.

In addition, we find in [16] a set of diagonalization strategies designed to make efficient use of well-polished finite dimensional optimization codes and finite dimensional consistent approximations in computing approximate solutions to infinite dimensional problems. These diagonalization strategies take the form of a *master algorithm* that chooses a level of discretization and calls a finite minimax or nonlinear programming algorithm to iterate on the current approximating problem until some discretization refinement test is satisfied. At that point the master algorithm increases the discretization and uses the last point computed to initialize a finite minimax or nonlinear programming algorithm to iterate on the next approximating problem, until the discretization refinement test is again satisfied, and so on, until a final termination test is satisfied.

In this paper, we consider the optimal design of Euler-Bernoulli beams, subject to continuum constraints, such as constraints on vertical deflection, shear stress, and normal stress at the extreme fiber. Although beams with non-uniform cross sections are more difficult to manufacture, in some areas where weight is at a premium, such as in aerospace applications, the construction of minimum weight beams may be quite realistic. Moreover, the problem of determining the optimal dimensions of a uniform beam subject to continuum constraints is a particular case of the problems we will deal with.

First we deal with cantilever beams. We propose an expansion of the original problem into an infinite family of approximating problems, construct corresponding optimality functions, and show that the approximating problems are consistent. Second, we extend our results to the problem of optimal design of a fixed beam. Finally, we make use of a diagonalization strategy presented in [16] and of a method of centers algorithm [17], to solve these optimal design problems numerically.

For ease of exposition we will restrict ourselves to beams with rectangular cross section, fixed width, and distributed loads. It is straightforward to generalize our results to beams whose cross sections are not necessarily rectangular, provided the cross sections have a horizontal and a vertical axis of symmetry, and the plane containing the vertical axis of symmetry also contains the loads. For instance, we can extend our results to the design of rectangular beams with varying depth and width, or the design of a cylindrical beam with varying radius.

The paper is organized as follows. We recall the basic definitions and results related to consistent approximations of optimization problems introduced in [16] in Section 2. In Section 3 we state the optimal design problem for a cantilever beam and propose an expansion into a sequence of approximating problems which we show to be consistent under appropriate conditions. The results in Section 3 are extended to fixed beams in Section 4. In Section 5 we discuss a diagonalization strategy for numerical solution of the optimal design problems under consideration. In Section 6, we present the results of a numerical experiment. Finally, in Section 7 we present our conclusions.

2. CONSISTENT APPROXIMATIONS

We begin by presenting a summary of the main definitions and results related to the concept of consistent approximations introduced in [16].

Let $\mathcal{B}$ be a topological vector space and consider the problem

$$\textbf{P} \qquad \min_{z \in Z} f(z) \tag{2.1a}$$

where $f : \mathcal{B} \rightarrow \mathbb{R}$ is continuous and $Z \subset \mathcal{B}$ is the constraint set. Let $\{ \mathcal{B}_N \}_{N=1}^{\infty}$ be a family of finite dimensional subspaces of $\mathcal{B}$ such that $\mathcal{B}_N = \mathcal{B}$ if $\mathcal{B}$ is finite dimensional ($\mathbb{R}^n$) and $\mathcal{B}_N \subset \mathcal{B}_{N+1}$, for all N, otherwise. Consider the family of approximating problems

$$\textbf{P}_N \qquad \min_{z \in Z_N} f_N(z), \quad N \in \mathbb{N}, \tag{2.1b}$$

where $f_N : \mathcal{B}_N \rightarrow \mathbb{R}$ is continuous, and $Z_N \subset \mathcal{B}_N$. To be of any use to us at all, the problems $\textbf{P}_N$

must, at least, converge epigraphically to $\mathbf{P}$, i.e., the epigraphs $E_N \triangleq \{ (z^0, z) \in \mathbb{R} \times Z_N \mid z^0 \geq f(z) \}$, of the problems $\mathbf{P}_N$, must converge, in the sense of Kuratowski, to the epigraph $E \triangleq \{ (z^0, z) \in \mathbb{R} \times Z \mid z^0 \geq f(z) \}$, of the problem $\mathbf{P}$. Equivalently:

Definition 2.1. [1, 7] The problems in the family $\{ \mathbf{P}_N \}_{N=1}^{\infty}$ *converge epigraphically* to $\mathbf{P}$, ($\mathbf{P}_N \rightarrow^{Epi} \mathbf{P}$) if : *(a)* for every $z \in Z$, there exists a sequence $\{ z_N \}_{N=1}^{\infty}$, with $z_N \in Z_N$, such that $z_N \rightarrow z$ and $\overline{\lim} f_N(z_N) \leq f(z)$; and *(b)* for every sequence $\{ z_{N_k} \}_{k=1}^{\infty}$, with $z_{N_k} \in Z_{N_k}$, such that $z_{N_k} \rightarrow z$ as $k \rightarrow \infty$, $z \in Z$ and $\underline{\lim} f_{N_k}(z_{N_k}) \geq f(z)$. □

Epigraphic convergence, or epiconvergence for short, can be viewed as a "zeroth order" consistency property. In particular, it ensures the following result.

Theorem 2.2. Suppose that $\mathbf{P}_N \rightarrow^{Epi} \mathbf{P}$, and that $\{ \hat{z}_N \}_{N=1}^{\infty}$ is a sequence such that $\hat{z}_N \in Z_N$ for all N and $\hat{z}_N \rightarrow \hat{z}$. *(a)* If the $\hat{z}_N$ are global minimizers for the $\mathbf{P}_N$, then $\hat{z}$ is a global minimizer of $\mathbf{P}$; *(b)* If $\hat{z}_N$ are strict local minimizers for the $\mathbf{P}_N$ whose radii of attraction do not converge to zero as $N \rightarrow \infty$, then $\hat{z}$ is a local minimizer of $\mathbf{P}$. □

The reader is referred to [1, 7] for the proof of Theorem 2.2 *(a)*, and to [16] for the proof of Theorem 2.2 *(b)*.

Optimization algorithms, applied to the finite dimensional problems $\mathbf{P}_N$, are only known to compute stationary points. As the following example shows, epiconvergence alone does not rule out the possibility that stationary points of the $\mathbf{P}_N$ converge to a nonstationary point of $\mathbf{P}$: Let $\mathcal{B} = \mathbb{R}^2$, so that $z = (x, y)$, and let $f(z) = f_N(z) = (x-2)^2$, $N \in \mathbb{N}$. Let

$$Z \triangleq \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2 \}, \tag{2.2a}$$

and, for all $N \in \mathbb{N}$, let

$$Z_N \triangleq \{ (x, y) \in \mathbb{R}^2 \mid (x-y)^2(x^2+y^2-2) \leq 0,\ x^2+y^2 \leq 2 + \frac{1}{N} \}. \tag{2.2b}$$

Then we see that $\mathbf{P}_N \rightarrow^{Epi} \mathbf{P}$. Nevertheless, the point $(1, 1)$ is feasible and satisfies the F. John optimality condition for all $\mathbf{P}_N$, but it is not a stationary point for the problem $\mathbf{P}$. The reason for this is an incompatibility of the constraint sets Z_N with the constraint set Z which shows up only at the level of optimality conditions.

To eliminate the possibility of pathologies such as in the above example, as well as some others, e.g, failure of derivatives to converge, [16] imposed a second condition on the approximating problems in terms of optimality conditions, which can be viewed as a "first order" consistency requirement. For the purpose of this condition, it is convenient to characterize stationary points as the zeros of optimality functions: $\theta : \mathcal{D} \rightarrow \mathbb{R}$ for $\mathbf{P}$ and $\theta_N : \mathcal{D}_N \rightarrow \mathbb{R}$ for $\mathbf{P}_N$, $N \in \mathbb{N}$, where $\mathcal{D} \subset \mathcal{B}$ and

$\mathcal{D}_N \subset \mathcal{B}_N$, i.e., the optimality functions may not be defined on the entire space. We will assume that $\mathcal{D}_N \subset \mathcal{D} \cap \mathcal{B}_N$, for all $N \in \mathbb{N}$.

Definition 2.3. A function $\theta : \mathcal{D} \to \mathbb{R}$ is an *optimality function* for **P** if, *(i)* $Z \subset \mathcal{D}$, *(ii)* $\theta(\cdot)$ is sequentially upper semicontinuous, *(iii)* $\theta(z) \le 0$ for all $z \in \mathcal{D}$, and *(iv)* $\theta(\hat{z}) = 0$ for any $\hat{z} \in Z$ that is a local minimizer for **P**. Similarly, a function $\theta_N : \mathcal{D}_N \to \mathbb{R}$ is an *optimality function* for $\mathbf{P}_N$ if, *(i)* $Z_N \subset \mathcal{D}_N$, *(ii)* $\theta_N(\cdot)$ is sequentially upper semicontinuous, *(iii)* $\theta_N(z) \le 0$ for all $z \in \mathcal{D}_N$, and *(iv)* $\theta_N(\hat{z}_N) = 0$ for any $\hat{z}_N \in Z_N$ that is a local minimizer for $\mathbf{P}_N$. □

Definition 2.4. Let $\theta(\cdot)$, $\theta_N(\cdot)$, $N \in \mathbb{N}$, be optimality functions for **P**, $\mathbf{P}_N$, respectively. The pairs $(\mathbf{P}_N, \theta_N)$, in the sequence $\{ (\mathbf{P}_N, \theta_N) \}_{N=1}^{\infty}$ are *weakly consistent approximations* to the pair $(\mathbf{P}, \theta)$, *if (i)* $\mathbf{P}_N \to^{Epi} \mathbf{P}$, *and (ii)* for any sequence $\{ z_N \}_{N \in K}$, $K \subset \mathbb{N}$, with $z_N \in \mathcal{D}_N$ for all $N \in K$, such that $z_N \to z$, $\overline{\lim}\, \theta_N(z_N) \le \theta(z)$. □

As a result of this definition, we immediately get the following result, which subsumes Theorem 2.2:

Theorem 2.5. Suppose that the pairs $(\mathbf{P}_N, \theta_N)$ in the sequence $\{ (\mathbf{P}_N, \theta_N) \}_{N=1}^{\infty}$ are weakly consistent approximations to the pair $(\mathbf{P}, \theta)$, and that $\{ \hat{z}_N \}_{N=1}^{\infty}$ is a sequence such that $\hat{z}_N \in Z_N$ for all N and $\hat{z}_N \to \hat{z}$.

(a) If the $\hat{z}_N$ are global minimizers for the $\mathbf{P}_N$, then $\hat{z}$ is a global minimizer of **P**.

(b) If $\hat{z}_N$ are strict local minimizers whose radii of attraction do not converge to zero, as $N \to \infty$, then $\hat{z}$ is a local minimizer of **P**.

(c) If $\overline{\lim}\, \theta_N(\hat{z}_N) = 0$, then $\theta(\hat{z}) = 0$. □

If we define a point $\hat{z}$ to be stationary for **P** if $\theta(\hat{x}) = 0$, then we see that Definition 2.3 permits nonfeasible points to be stationary (e.g., they can be stationary points for a problem with relaxed or modified constraints). This phenomenon can be removed by imposing an additional condition, as is done below:

Definition 2.6. Let $\theta(\cdot)$, $\theta_N(\cdot)$, $N \in \mathbb{N}$, be optimality functions for **P**, $\mathbf{P}_N$, respectively. The pairs $(\mathbf{P}_N, \theta_N)$, in the sequence $\{ (\mathbf{P}_N, \theta_N) \}_{N=1}^{\infty}$ are *consistent approximations* to $(\mathbf{P}, \theta)$, if they are weakly consistent approximations, and in addition $\theta(z) < 0$ for all $z \notin Z$ and $\theta_N(z) < 0$ for all $z \notin Z_N$, $N \in \mathbb{N}$. □

3. OPTIMAL DESIGN OF A CANTILEVER BEAM

Consider a cantilever beam of length $L > 0$ and rectangular cross-section, with constant width $b > 0$ and variable depth defined by a function $h : [0, L] \rightarrow \mathbb{R}$. The material of the beam has modulus of elasticity $E > 0$. Let $0 < \alpha < \beta < \infty$ and $\gamma \geq 0$ be given constants, and let $C[0, L]$ denote the space continuous real-valued functions defined on $[0, L]$. We assume that the set of admissible depth functions $h(\cdot)$ is given by

$$H_{ad} \triangleq \{ h \in C[0,L] \mid 0 < \alpha \leq h(x) \leq \beta, \ |dh(x)/dx| \leq \gamma, \ \text{for a. e. } x \in [0,L] \}. \quad (3.1a)$$

We model the beam using Euler-Bernoulli beam theory, and assume that it is subjected to a vertical load with density $l : H_{ad} \times [0, L] \rightarrow \mathbb{R}$ of the form

$$l(h, x) = m(x) - Kh(x), \quad x \in [0, L], \quad (3.1b)$$

where $K \geq 0$ is a given constant and $m \in L_\infty[0, L]$ is the density of an external load applied to the beam.

We will consider optimal cantilever beam design problems that can be stated in the form

$$\mathbf{P}_c \qquad \min_{h \in \mathbf{C}_c} f_c(h), \quad (3.1c)$$

where $\mathbf{C}_c \subset H_{ad}$ is the set of all admissible depth functions $h(\cdot)$ such that

$$\frac{d}{dx} V_c(h, x) = -l(h, x), \quad x \in [0, L], \quad V_c(h, L) = 0, \quad (3.1d)$$

$$\frac{d}{dx} M_c(h, x) = -V_c(h, x), \quad x \in [0, L], \quad M_c(h, L) = 0, \quad (3.1e)$$

$$\frac{d^2}{dx^2} y_c(h, x) = \frac{12 M_c(h, x)}{Ebh(x)^3}, \quad x \in [0, L], \quad y_c(h, 0) = \frac{d}{dx} y_c(h, 0) = 0, \quad (3.1f)$$

$$\psi_c(h) \triangleq \max_{j \in \mathbf{q}} \max_{x \in [0, L]} \phi_c^j(h, x) - r^j(x) \leq 0, \quad (3.1g)$$

where (3.1d-f) are the Euler-Bernoulli beam equations for a cantilever beam whose depth is determined by the function $h(\cdot)$, and relate the deflection $y_c(h, \cdot)$, the bending moment $M_c(h, \cdot)$, the shear force $V_c(h, \cdot)$, and the load density $l(h, \cdot)$. We use the subscript c to differentiate the quantities associated with the cantilever beam from those of the fixed beam, to be considered in Section 4.

For any integer $q > 0$, let $\mathbf{q} \triangleq \{1, 2, \ldots, q\}$. The functions $\phi_c^j(\cdot, \cdot)$, $j \in \mathbf{q}$, are used to express continuum constraints. We assume that the functions $f_c(\cdot)$ and $\phi_c^j(\cdot, \cdot)$, $j \in \mathbf{q}$, are of the form

$$f_c(h) = \int_0^L \phi_c^0(h, x)\, dx, \quad (3.1h)$$

$$\phi_c^j(h,x) = \tilde{\phi}^j(h(x), M_c(h,x), V_c(h,x), y_c(h,x), x), \; j \in \bar{\mathbf{q}}, \tag{3.1i}$$

where, for $j \in \bar{\mathbf{q}} \triangleq \{0,1,\ldots,q\}$, $\tilde{\phi}^j : [\alpha,\beta] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,L] \to \mathbb{R}$.

Assumption 3.1. We will assume that

(a) The function $m(\cdot)$ is piecewise Lipschitz continuous, with finitely many points of discontinuity in $[0,L]$.

(b) The functions $r^j(\cdot)$, $j \in \mathbf{q}$, are Lipschitz continuously differentiable on $[0,L]$, and satisfy

$$\min_{j \in \mathbf{q}} \; \min_{x \in [0,L]} r^j(x) = \hat{r} > 0. \tag{3.1j}$$

(c) The functions $\tilde{\phi}^j(\cdot,\cdot,\cdot,\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$ are Lipschitz continuously differentiable.

(d) The feasible set $\mathbf{C}_c$ for $\mathbf{P}_c$ is not empty, that is, there exists an $h \in H_{ad}$ satisfying (3.1g). □

A variety of design problems can be expressed in the form (3.1a-j). For example, suppose that we wish to minimize the weight or volume of a cantilever beam of constant width, subject to constraints on the maximum normal stress at the extreme fiber $\sigma_{c,max}(h,\cdot)$, constraints on the maximum shear stress $\tau_{c,max}(h,\cdot)$, and constraints on the deflection $y_c(h,\cdot)$. Then we set

$$f_c(h) \triangleq \int_0^L h(\tau)d\tau, \tag{3.2a}$$

$$\phi_c^1(h,x) \triangleq \sigma_{c,max}(h,x) = \frac{6}{b}\frac{M_c(h,x)}{h(x)^2}, \quad \phi_c^2(h,x) \triangleq -\phi_c^1(h,x), \tag{3.2b}$$

$$\phi_c^3(h,x) \triangleq \tau_{c,max}(h,x) = \frac{3}{2}\frac{V_c(h,x)}{bh(x)}, \quad \phi_c^4(h,x) \triangleq -\phi_c^3(h,x), \tag{3.2c}$$

$$\phi_c^5(h,x) \triangleq y_c(h,x), \quad \phi_c^6(h,x) \triangleq -\phi_c^5(h,x), \tag{3.2d}$$

In the simplest case, the r^j are positive constants, with $r^1 = r^2$, $r^3 = r^4$, and $r^5 = r^6$.

The "natural" norm on $C[0,L]$ for establishing continuity and differentiability of solutions of (3.1d-f) with respect to depth functions is the sup-norm, $\|\cdot\|_\infty$. However, when we define optimality functions for our design problems, by extension of optimality functions for problems defined on $\mathbb{R}^n$, which is a Hilbert space, it is much more natural to use the $L_2[0,L]$ norm, $\|\cdot\|_2$. Since there is no inconvenience in also using the $L_2[0,L]$ norm for establishing continuity and differentiability of solutions of (3.1d-f) with respect to depth functions (see [3]), we adopt this norm for $C[0,L]$ in our analysis. Hence, we will work in the inner-product space $(C[0,L], \|\cdot\|_2, \langle\cdot,\cdot\rangle_2)$, where $\langle\cdot,\cdot\rangle_2$ denotes the usual inner-product on $L_2[0,L]$.

Existence of a solution to $\mathbf{P}_c$ follows from the Ascoli-Arzela theorem, which implies that the set H_{ad} is compact in $(C[0,L], \|\cdot\|_2)$. Proofs of existence of solutions to similar problems can be found in

[6, 8, 13].

We will obtain a sequence of consistent approximations in three steps. First, we will choose a dense family of finite dimensional subsets of H_{ad}, second, we will discretize the boundary value problem (3.1d-f), the cost function (3.1h), and the constraints (3.1g), and define approximating problems $\mathbf{P}_{c,N}$. Third, we will define appropriate optimality functions $\theta_c(\cdot)$ and $\theta_{c,N}(\cdot)$ for the original problem $\mathbf{P}_c$, and for the approximating problems $\mathbf{P}_{c,N}$ respectively.

For every positive integer N, we let $\Delta(N) \triangleq L/N$, and define the mesh $T_N \triangleq \{0, \Delta(N), 2\Delta(N), \ldots, L\}$, with nodes $x_{N,k} = (k-1)\Delta(N)$, $k \in \mathbf{N+1}$.

Let H_N denote the subset of $C[0,L]$ whose elements are piecewise affine on the mesh T_N. Let $H_{ad,N} \triangleq H_N \cap H_{ad}$. Any $h \in H_{ad,N}$ can be expressed as a unique linear combination of the functions

$$P_{N,k}(x) \triangleq \begin{cases} \dfrac{x - x_{N,k-1}}{\Delta(N)}, & \text{for all } x \in [x_{N,k-1}, x_{N,k}], \quad k \in \{2, \cdots, N+1\}, \\ \dfrac{x_{N,k+1} - x}{\Delta(N)}, & \text{for all } x \in [x_{N,k}, x_{N,k+1}], \quad k \in \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3a}$$

That is, for each $h \in H_{ad,N}$, there exists a unique $(\eta_1, \eta_2, \ldots, \eta_{N+1})^T \in \mathbb{R}^{N+1}$ such that

$$h(x) = \sum_{k=1}^{N+1} \eta_k P_{N,k}(x), \quad x \in [0, L]. \tag{3.3b}$$

The boundary value problem (3.1d-f) can be solved numerically by several different methods. The two most frequently used are finite difference and finite element methods. Although finite element methods have advantages for more complicated geometries in 2 or 3 dimensions and can be used with a variety of boundary conditions, their convergence analysis is somewhat more complex than that of finite difference methods. Hence, to keep the exposition as simple as possible, we discretize (3.1d-f) using finite differences. However, our results remain valid if one uses other discretization methods, provided Lemma 3.2, Lemma 3.9 and Theorem 3.10 below remain valid for the resulting approximations.

Before we integrate the second order differential equation (3.1f) numerically, we transform it to the first order form:

$$\frac{d}{dx}\begin{bmatrix} y_c(h,x) \\ y'_c(h,x) \end{bmatrix} = \begin{bmatrix} y'_c(h,x) \\ 12M_c(h,x)/Ebh(x)^3 \end{bmatrix}, \quad x \in [0,L], \quad y_c(h,0) = y'_c(h,0) = 0, \tag{3.4}$$

where $y'_c(h,x) \triangleq d/dx\, y_c(h,x)$.

Using Euler's forward method to discretize the ordinary differential equations (3.1d,e) and (3.4), we define the family $\mathbf{P}_{c,N}$ of approximating problems as follows:

$$\mathbf{P}_{c,N} \qquad \min_{h \in \mathbf{C}_{c,N}} f_{c,N}(h), \tag{3.5a}$$

where $\mathbf{C}_{c,N} \subset H_{ad,N}$ is the set of all depth functions $h(\cdot)$ in $H_{ad,N}$ such that for $k \in \mathbf{N}$,

$$V_{c,N}(h, x_{N,k}) = V_{c,N}(h, x_{N,k+1}) + \Delta(N) l(h, x_{N,k+1}), \quad V_{c,N}(h, x_{N,N+1}) = 0, \tag{3.5b}$$

$$M_{c,N}(h, x_{N,k}) = M_{c,N}(h, x_{N,k+1}) + \Delta(N) V_{c,N}(h, x_{N,k+1}), \quad M_{c,N}(h, x_{N,N+1}) = 0, \tag{3.5c}$$

$$\begin{bmatrix} y_{c,N}(h, x_{N,k+1}) \\ y'_{c,N}(h, x_{N,k+1}) \end{bmatrix} = \begin{bmatrix} y_{c,N}(h, x_{N,k}) + \Delta(N) y'_{c,N}(h, x_{N,k}) \\ y'_{c,N}(h, x_{N,k}) + \Delta(N) \dfrac{12 M_{c,N}(h, x_{N,k})}{E b h(x_{N,k})^3} \end{bmatrix}, \quad \begin{bmatrix} y_{c,N}(h, x_{N,1}) \\ y'_{c,N}(h, x_{N,1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.5d}$$

$$\psi_{c,N}(h) \triangleq \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \phi^j_{c,N}(h, x_{N,k}) - (1 + \Delta(N)^{1/2}) r^j(x_{N,k}) \le 0, \tag{3.5e}$$

$$f_{c,N}(h) \triangleq \sum_{k=1}^{N} \phi^0_{c,N}(h, x_{N,k}) \Delta(N). \tag{3.5f}$$

□

We define the functions $\phi^j_{c,N} : H_{ad,N} \times [0, L] \to \mathbb{R}$, $j \in \bar{\mathbf{q}}$, as the piecewise linear interpolation of the values $\tilde{\phi}^j(h(x_{N,k}), M_{c,N}(h, x_{N,k}), V_{c,N}(h, x_{N,k}), y_{c,N}(h, x_{N,k}), x_{N,k})$, on the mesh T_N, and hence

$$\phi^j_{c,N}(h, x_{N,k}) = \tilde{\phi}^j(h(x_{N,k}), M_{c,N}(h, x_{N,k}), V_{c,N}(h, x_{N,k}), y_{c,N}(h, x_{N,k}), x_{N,k}), \quad j \in \bar{\mathbf{q}}. \tag{3.6}$$

The term $\Delta(N)^{1/2}$ in the constraint (3.5e) is added to guarantee that for N large enough, the feasible set for $\mathbf{P}_{c,N}$ is not empty. This relaxation of the constraints will be needed in the proof of Theorem 3.3 *(a)* below.

The proof of the following lemma is given in Appendix I.

Lemma 3.2. *(a)* For every $h \in H_{ad}$, and positive integer N, there exists an $h_N \in H_{ad,N}$ such that

$$\max_{x \in [0, L]} | h(x) - h_N(x) | \le \gamma \Delta(N) ; \tag{3.7a}$$

(b) There exists a constant $C < \infty$ such that for any $h \in H_{ad}$, $N \ge 1$, and $h_N \in H_{ad,N}$

$$\max_{k \in \mathbf{N+1}} | V_c(h, x_{N,k}) - V_{c,N}(h_N, x_{N,k}) | \le C [\Delta(N) + \| h - h_N \|_2], \tag{3.7b}$$

$$\max_{k \in \mathbf{N+1}} | M_c(h, x_{N,k}) - M_{c,N}(h_N, x_{N,k}) | \le C [\Delta(N) + \| h - h_N \|_2], \tag{3.7c}$$

$$\max_{k \in \mathbf{N+1}} | y_c(h, x_{N,k}) - y_{c,N}(h_N, x_{N,k}) | \le C [\Delta(N) + \| h - h_N \|_2]; \tag{3.7d}$$

(c) There exists a constant $C < \infty$ such that for all $j \in \bar{\mathbf{q}}$, $N \geq 1$, $h \in H_{ad}$, and $h_N \in H_{ad,N}$,

$$\max_{x \in [0,L]} | \phi_c^j(h, x) - \phi_{c,N}^j(h_N, x) | \leq C [\Delta(N) + \| h - h_N \|_2], \tag{3.7e}$$

$$| \psi_c(h) - \psi_{c,N}(h_N) | \leq C [\Delta(N)^{1/2} + \| h - h_N \|_2], \tag{3.7f}$$

$$| f_c(h) - f_{c,N}(h_N) | \leq C [\Delta(N) + \| h - h_N \|_2]. \tag{3.7g}$$

□

In view of the definitions of $\psi_c(h)$ and $\psi_{c,N}(h)$, it is clear that the following holds:

$$\mathbf{C}_c = \{ h \in H_{ad} \mid \psi_c(h) \leq 0 \}, \quad \mathbf{C}_{c,N} = \{ h \in H_{ad,N} \mid \psi_{c,N}(h) \leq 0 \}. \tag{3.8}$$

Theorem 3.3. (Epiconvergence) *(a)* For every $h \in \mathbf{C}_c$, there exists a sequence $\{ h_N \}_{N=N_0}^{\infty}$, with $h_N \in \mathbf{C}_{c,N}$, such that $f_N(h_N) \to f(h)$ as $N \to \infty$. *(b)* Let $\{ h_N \}_{N=N_0}^{\infty}$ be a sequence such that $h_N \in \mathbf{C}_{c,N}$ and $h_N \to \hat{h}$ as $N \to \infty$, then $\hat{h} \in \mathbf{C}_c$, and $f_N(h_N) \to f(\hat{h})$.

Proof. Suppose $h \in \mathbf{C}_c$ is given. Then, by Lemma 3.2, for each integer N, there exists an $h_N \in H_{ad,N}$ such that (3.7a) holds. Clearly, $h_N \to h$ as $N \to \infty$. It now follows from (3.7g) that $f_{c,N}(h_N) \to f_c(h)$ as $N \to \infty$. To complete the proof of part *(a)*, it remains to show that there exists an N_0 such that $h_N \in \mathbf{C}_{c,N}$, for all $N \geq N_0$. Indeed, since $h \in \mathbf{C}_c$ by assumption, (3.8) and (3.7a,e) imply

$$\psi_{c,N}(h_N) \leq \psi_{c,N}(h_N) - \psi_c(h)$$

$$= \max_{j \in \mathbf{q}} \max_{x \in [0,L]} [\phi_{c,N}^j(h_N, x) - r^j(x)(1 + \Delta(N)^{1/2})] - \max_{j \in \mathbf{q}} \max_{x \in [0,L]} [\phi_c^j(h, x) - r^j(x)]$$

$$\leq \max_{j \in \mathbf{q}} \max_{x \in [0,L]} [| \phi_{c,N}^j(h_N, x) - \phi_c^j(h, x) | - r^j(x) \Delta(N)^{1/2}]$$

$$\leq C [\Delta(N) + \| h - h_N \|_2] - \hat{r}\, \Delta(N)^{1/2} \leq C [\Delta(N) + \| h - h_N \|_\infty] - \hat{r}\, \Delta(N)^{1/2}$$

$$\leq C (1 + \gamma) \Delta(N) - \hat{r}\, \Delta(N)^{1/2}, \tag{3.9}$$

where $\hat{r} > 0$ is as in (3.1j). It follows from (3.9) that there exist an N_0 such that for all $N \geq N_0$, $\psi_{c,N}(h_N) \leq 0$, which proves *(a)*.

Let $\{ h_N \}_{N=N_0}^{\infty}$ be a sequence as in *(b)*. The Ascoli-Arzela Theorem implies that H_{ad} is compact in $(C[0,L], \| \cdot \|_2)$, and hence that it is closed. Since $\mathbf{C}_{c,N} \subset H_{ad,N} \subset H_{ad}$ for all $N \in \mathbb{N}$, it follows that $\hat{h} \in H_{ad}$. The facts that $\hat{h} \in \mathbf{C}_c$, that is, that $\psi_c(\hat{h}) \leq 0$, and that $f_{c,N}(h_N) \to f_c(\hat{h})$ follow directly from (3.7f) and (3.7g) respectively.

$\square$

Next we will develop optimality functions $\theta_c(\cdot)$, and $\theta_{c,\mathcal{N}}(\cdot)$, for the problems $\mathbf{P}_c$, and $\mathbf{P}_{c,\mathcal{N}}$, respectively.

The mappings $h \mapsto M_c(h,\cdot)$, $h \mapsto V_c(h,\cdot)$, and $h \mapsto y_c(h,\cdot)$, from H_{ad} into $(C[0,L], \|\cdot\|_2)$, and defined by (3.1d-f), have Lipschitz continuous Gateaux differentials on $h \in H_{ad}$. This is a direct consequence of the differentiability properties of solutions of ordinary differential equations (see [3]).

Let $D_1V_c(\cdot,\cdot;\cdot)$, $D_1M_c(\cdot,\cdot;\cdot)$, and $D_1y_c(\cdot,\cdot;\cdot)$ denote the Gateaux differentials of the mappings $h \mapsto V_c(h,\cdot)$, $h \mapsto M_c(h,\cdot)$, and $h \mapsto y_c(h,\cdot)$ respectively. It can be shown (see [3]) that for any $h, h' \in H_{ad}$, the following relations hold:

$$\frac{d}{dx}D_1V_c(h,x;h'-h) = K(h'(x)-h(x)), \quad x \in [0,L], \quad D_1V_c(h,L;h'-h) = 0, \tag{3.10a}$$

$$\frac{d}{dx}D_1M_c(h,x;h'-h) = -D_1V_c(h,x;h'-h), \quad x \in [0,L], \quad D_1M_c(h,L;h'-h) = 0, \tag{3.10b}$$

$$\frac{d^2}{dx^2}D_1y_c(h,x;h'-h) = \frac{12}{Ebh(x)^3}\Big(D_1M_c(h,x;h'-h) - 3M_c(h,x)\frac{h'(x)-h(x)}{h(x)}\Big), \quad x \in [0,L],$$

$$D_1y_c(h,0;h'-h) = \frac{d}{dx}D_1y_c(h,0;h'-h) = 0. \tag{3.10c}$$

Consider the functions defined in (3.1g) and (3.1h,i). It follows from the Lipschitz continuous differentiability of $\tilde{\phi}^j(\cdot,\cdot,\cdot,\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$, with respect to all their arguments, and of $M_c(h,\cdot)$, $V_c(h,\cdot)$, and $y_c(h,\cdot)$ with respect to h, that $\phi_c^j(\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$, and $f_c(\cdot)$, are Lipschitz continuously differentiable functions of h on H_{ad}. We will denote by $D_1\phi_c^j(\cdot,\cdot;\cdot)$, $j \in \bar{\mathbf{q}}$, and $Df_c(\cdot;\cdot)$ the Gateaux differentials of the functions $h \mapsto \phi_c^j(h,\cdot)$ and $h \mapsto f_c(h)$ respectively.

Lemma 3.4. There exists a constant $C < \infty$ such that for any $h, \tilde{h} \in H_{ad}$, $h', h'' \in H_{ad}$,

$$|Df_c(h;h'-h) - Df_c(\tilde{h};h''-\tilde{h})| \le C[\|h-\tilde{h}\|_2 + \|h'-h''\|_2], \tag{3.11a}$$

and for all $j \in \bar{\mathbf{q}}$,

$$\|D_1\phi_c^j(h,\cdot;h'-h) - D_1\phi_c^j(\tilde{h},\cdot;h''-\tilde{h})\|_\infty \le C[\|h-\tilde{h}\|_2 + \|h'-h''\|_2]. \tag{3.11b}$$

Proof. Both inequalities are a direct consequence of the Lipschitz continuity of the Gateaux differentials of $M_c(h,\cdot)$, $V_c(h,\cdot)$ and $y_c(h,\cdot)$, and the Lipschitz continuous differentiability of the functions $\tilde{\phi}^j(\cdot,\cdot,\cdot,\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$, in (3.1h,i). $\square$

Next, we define the function $F_c : H_{ad} \times H_{ad} \to \mathbb{R}$ by

$$F_c(h, h') \triangleq \max \{ f_c(h') - f_c(h) - \omega\psi_c(h)_+ \, , \max_{j \in \mathbf{q}} \max_{x \in [0,1]} \phi_c^j(h', x) - r^j(x) - \psi_c(h)_+ \} \, , \quad (3.12a)$$

where $\psi_c(h)_+ \triangleq \max \{ \psi_c(h), 0 \}$, and $\omega > 0$ is a parameter to be used in method of centers type algorithms. Note that *(i)* for all $h \in H_{ad}$, $F_c(h, h) = 0$, and *(ii)* if $\hat{h} \in H_{ad}$ is a local minimizer for $\mathbf{P}_c$ then, since $\psi_c(h) > 0$ when h is infeasible, and since $f_c(h) \geq f_c(\hat{h})$ for all feasible h in a ball about $\hat{h}$, $\hat{h}$ must also be a local minimizer for the problem

$$\min_{h \in H_{ad}} F_c(\hat{h}, h) \, . \tag{3.12b}$$

This fact is used in [2] to obtain the following first order optimality condition for $\mathbf{P}_c$:

Proposition 3.5. If $\hat{h}$ is a local minimizer for $\mathbf{P}_c$, then

$$\hat{h} \in H_{ad} \text{ and } d_2F_c(\hat{h}, \hat{h}; h' - \hat{h}) \geq 0, \quad \text{for all } h' \in H_{ad} \, , \tag{3.13}$$

where $d_2F_c(\hat{h}, \hat{h}; h' - \hat{h})$ denotes the (one-sided) directional derivative of $F_c(\cdot, \cdot)$ at $(\hat{h}, \hat{h})$, with respect to the second argument, in the direction $h' - \hat{h}$. □

Referring to [15], we see that for the purpose of constructing algorithms, it is useful to replace the first order linear approximation $d_2F_c(h, h; h' - h)$ of $F_c(h, h')$ in a neighborhood of h by the the following *convex* first order approximation:

$$\tilde{F}_c(h, h') \triangleq \max \{ Df_c(h; h' - h) - \omega\psi_c(h)_+ \, , \max_{j \in \mathbf{q}} \max_{x \in [0,L]} \phi_c^j(h, x) + D_1\phi_c^j(h, x; h' - h)$$

$$- \psi_c(h)_+ \} + \tfrac{1}{2}\|h' - h\|_2^2 . \tag{3.14}$$

In view of (3.1b-i) and Assumption 3.1 *(c)*, it should be clear that, for all $j \in \bar{\mathbf{q}}$ and $x \in [0.L]$, the mappings $h \mapsto \phi_c^j(h, x)$ and $h \mapsto \psi_c(h)$ are continuous on H_{ad}. Hence, as a consequence of the definition of $\tilde{F}_c(\cdot, \cdot)$ and of (3.11a-b), we obtain the following result:

Lemma 3.6. $\tilde{F}_c : H_{ad} \times H_{ad} \to \mathbb{R}$ is continuous. □

We now define the optimality function $\theta_c : H_{ad} \to \mathbb{R}$ as follows:

$$\theta_c(h) \triangleq \min_{h' \in H_{ad}} \tilde{F}_c(h, h'). \tag{3.15}$$

From Lemma 3.6 and the fact that $H_{ad} \subset (C[0, L], \|\cdot\|_2)$ is compact (by the Ascoli-Arzela Theorem) it follows that $\theta_c(\cdot)$ is well-defined.

Theorem 3.7. *(a)* The function $\theta_c(\cdot)$ takes values in $(-\infty, 0]$; *(b)* $\theta_c : H_{ad} \to \mathbb{R}$ is upper semicontinuous; *(c)* For any $\hat{h} \in H_{ad}$, $\theta_c(\hat{h}) = 0$ if and only if *either* $\psi_c(\hat{h}) \le 0$ and (3.13) holds *or* $\psi_c(\hat{h}) > 0$ and $0 \in \partial\psi_c(\hat{h})$, where $\psi_c(\hat{h})$ denotes the Clarke generalized gradient [2] of $\psi_c(\cdot)$ at $\hat{h}$ (i.e., $\hat{h}$ satisfies the first order optimality condition for the problem $\min_{h \in H_{ad}} \psi_c(h)$).

Proof. Parts *(a)* and *(c)* can be deduced from Proposition 5.4 and Proposition 5.5 in [15]. We will prove part *(b)*.

Suppose $\{ h_j \}_{j=0}^{\infty} \subset H_{ad}$ is such that $h_j \to h \in H_{ad}$ as $j \to \infty$. Let $h' \in H_{ad}$ be such that $\theta_c(h) = \tilde{F}_c(h, h')$. Then

$$\theta_c(h_j) \le \tilde{F}_c(h_j, h'), \quad \forall j \in \mathbb{N}. \tag{3.16a}$$

Hence, taking $\overline{\lim}$ on both sides, and using Lemma 3.6, we get

$$\overline{\lim_{j \to \infty}}\, \theta_c(h_j) \le \overline{\lim_{j \to \infty}}\, \tilde{F}_c(h_j, h') = \tilde{F}_c(h, h') = \theta_c(h). \tag{3.16b}$$

Corollary 3.8. $\theta_c(\cdot)$ is an optimality function for $\mathbf{P}_c$. □

It follows from the Implicit Function Theorem (see, e.g., [12]) that the functions $h \mapsto M_{c,N}(h, \cdot)$, $h \mapsto V_{c,N}(h, \cdot)$, and $h \mapsto y_{c,N}(h, \cdot)$, mapping $H_{ad,N}$ into H_N, defined by (3.5b-d), are Lipschitz continuously differentiable. In fact, given $h, h' \in H_{ad,N}$, one can show, by differentiating (3.5b-d), that $D_1 M_{c,N}(h, x_{N,k}; h' - h)$ and $D_1 V_N(h, x_{N,k}; h' - h)$ are given by

$$D_1 V_{c,N}(h, x_{N,k}; h' - h) = D_1 V_{c,N}(h, x_{N,k+1}; h' - h) - K\Delta(N)(h'(x_{N,k+1}) - h(x_{N,k+1})),\ k \in \mathbf{N},$$

$$D_1 V_{c,N}(h, x_{N,N+1}; h' - h) = 0; \tag{3.17a}$$

$$D_1 M_{c,N}(h, x_{N,k-1}; h' - h) = D_1 M_{c,N}(h, x_{N,k}; h' - h) + \Delta(N) D_1 V_{c,N}(h, x_{N,k}; h' - h),\ k \in \mathbf{N},$$

$$D_1 M_{c,N}(h, x_{N,N+1}; h' - h) = 0; \tag{3.17b}$$

and $D_1 y_{c,N}(h, x_{N,k}; h' - h) = \delta y_{c,N}(h, x_{N,k})$, $k \in \mathbf{N+1}$, where $\delta y_{c,N}(h, x_{N,k})$, is the solution of

$$\begin{bmatrix} \delta y_{k+1} \\ \delta y'_{k+1} \end{bmatrix} = \begin{bmatrix} \delta y_k + \Delta(N)\delta y'_k \\ \delta y'_k + \dfrac{12\Delta(N)}{Ebh(x_{N,k})^3}\left(D_1 M_{c,N}(h, x_{N,k}; h' - h) - \dfrac{3M_{c,N}(h, x_{N,k})}{h(x_{N,k})}(h'(x_{N,k}) - h(x_{N,k}))\right) \end{bmatrix},$$

$$k \in \mathbf{N}, \quad \delta y_1 = \delta y'_1 = 0. \tag{3.17c}$$

Hence, because all the functions on the right hand side of (3.10a-c) are Lipschitz continuous on $[0, L]$, and the equations (3.17a-c) correspond to the integration of (3.10a-c) by Euler's method, we have the following result.

Lemma 3.9. There exists a constant $C < \infty$ such that for all positive integers N, $h, h' \in H_{ad,N}$,

$$\max_{k \in \mathbf{N+1}} |D_1 V_c(h, x_{N,k}; h'-h) - D_1 V_{c,N}(h, x_{N,k}; h'-h)| \le C \|h'-h\|_2 \Delta(N), \quad (3.18a)$$

$$\max_{k \in \mathbf{N+1}} |D_1 M_c(h, x_{N,k}; h'-h) - D_1 M_{c,N}(h, x_{N,k}; h'-h)| \le C \|h'-h\|_2 \Delta(N), \quad (3.18b)$$

$$\max_{k \in \mathbf{N+1}} |D_1 y_c(h, x_{N,k}; h'-h) - D_1 y_{c,N}(h, x_{N,k}; h'-h)| \le C \|h'-h\|_2 \Delta(N). \quad (3.18c)$$

□

It follows from (3.6) and Assumption 3.1*(c)*, that the functions $\phi^j_{c,N}(\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$, are Lipschitz continuously differentiable on $H_{ad,N}$. The differentials with respect to h of $\phi^j_{c,N}(\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$, and $f_c(\cdot)$, which we denote by $D_1\phi^j_{c,N}(\cdot,\cdot;\cdot)$ and $Df_{c,N}(\cdot;\cdot)$ respectively, are easily obtained from (3.5f), (3.6), and (3.17a-c), by applying the Chain Rule.

As a consequence of (3.1h,i), (3.5f), (3.6), Assumption 3.1*(c)*, Lemma 3.9, and Lemma 3.2 we get the following result:

Theorem 3.10. *(a)* There exists a constant $C < \infty$ such that for any positive integer N, $h, h' \in H_{ad,N}$,

$$|Df_c(h; h'-h) - Df_{c,N}(h; h'-h)| \le C \Delta(N) \|h'-h\|_2, \quad (3.19a)$$

$$\max_{k \in \mathbf{N+1}} |D_1\phi^j_c(h, x_{N,k}; h'-h) - D_1\phi^j_{c,N}(h, x_{N,k}; h'-h)| \le C \Delta(N) \|h'-h\|_2. \quad (3.19b)$$

□

We now define the finite-dimensional counterparts of $F_c(\cdot,\cdot)$, $\tilde{F}_c(\cdot,\cdot)$ and $\theta_c(\cdot)$, which we will denote by $F_{c,N}(\cdot,\cdot)$, $\tilde{F}_{c,N}(\cdot,\cdot)$, and $\theta_{c,N}(\cdot)$ respectively, as follows. For $h, h' \in H_{ad,N}$,

$$F_{c,N}(h, h') \triangleq \max\{ f_{c,N}(h') - f_{c,N}(h) - \omega\psi_{c,N}(h)_+, \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \phi^j_{c,N}(h', x_{N,k}) - \psi_{c,N}(h)_+ \}, \quad (3.20a)$$

$$\tilde{F}_{c,N}(h, h') \triangleq \max\Big\{ Df_{c,N}(h; h'-h) - \omega\psi_{c,N}(h)_+, \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \phi^j_{c,N}(h, x_{N,k}) - \psi_{c,N}(h)_+ + D_1\phi^j_{c,N}(h, x_{N,k}; h'-h) \Big\} + \tfrac{1}{2}\|h'-h\|_2^2, \quad (3.20b)$$

$$\theta_{c,N}(h) \triangleq \min_{h' \in H_{ad,N}} \tilde{F}_{c,N}(h, h'). \quad (3.20c)$$

Lemma 3.11. There exists a constant $C < \infty$ such that for all positive integers N. and $h, h' \in H_{ad,N}$,

$$|\tilde{F}_c(h, h') - \tilde{F}_{c,N}(h, h')| \le C\Delta(N)^{1/2} . \tag{3.21}$$

□

Lemma 3.11 followsfrom the boundedness of $H_{ad,N}$, (3.7f,g), the definitions of $\tilde{F}_c(\cdot,\cdot)$ and $\tilde{F}_{c,N}(\cdot,\cdot)$ in (3.14) and (3.20b) respectively, and Theorem 3.10. Results analogous to Theorem 3.7 and Corollary 3.8 hold for $\theta_{c,N}(\cdot)$.

Theorem 3.12. Suppose that $\{ h_N \}_{N=N_0}^{\infty}$, with $h_N \in H_{ad,N}$, is such that $h_N \to h$ as $N \to \infty$. Then $h \in H_{ad}$, and $\overline{\lim}_{N\to\infty} \theta_{c,N}(h_N) \le \theta_c(h)$.

Proof. Let $h' \in H_{ad}$ be such that $\theta_c(h) = \tilde{F}_c(h, h')$. Let $\{ h'_N \}_{N=N_0}^{\infty}$ be such that , $h'_N \in H_{ad,N}$, and $h'_N \to h'$ as $N \to \infty$. Then we have

$$\theta_{c,N}(h_N) \le \tilde{F}_{c,N}(h_N, h'_N) \le \tilde{F}_c(h_N, h'_N) + C\Delta(N)^{1/2} , \tag{3.22a}$$

where we made use of Lemma 3.11 to obtain the last inequality. Hence, taking $\overline{\lim}$ on both sides and using Lemma 3.6, we obtain

$$\overline{\lim_{N\to\infty}}\, \theta_{c,N}(h_N) \le \overline{\lim_{N\to\infty}}\, \tilde{F}_c(h_N, h'_N) = \tilde{F}_c(h, h') = \theta_c(h). \tag{3.22b}$$

Corollary 3.13. The sequence $\{ (\mathbf{P}_{c,N}, \theta_{c,N}) \}_{N=1}^{\infty}$ is a family of weakly consistent approximations to the pair $(\mathbf{P}_c, \theta_c)$. Furthermore, if for all $h \in H_{ad}$ such that $\psi_c(h) > 0$, $0 \notin \partial\psi_c(h)$, then $\{ (\mathbf{P}_{c,N}, \theta_{c,N}) \}_{N=1}^{\infty}$ is a family of consistent approximations to $(\mathbf{P}_c, \theta_c)$. □

4. OPTIMAL DESIGN OF A FIXED BEAM

Consider the problem of designing an optimal fixed beam subject to a load of the form (3.1b), with cost and constraints as in (3.1c-i). Let $M_c(h, \cdot)$ be determined by (3.1e) and define $S \triangleq \{ M(\cdot) \in L_2[0, L] \mid M(x) = M_c(h, x) + ax + b\ ,\ a, b \in \mathbb{R} \}$. It follows from the dual formulation of the variational problem associated with the bending of the beam (see [11, 18]), that the bending moment, $M_f(h, \cdot)$, for a fixed beam of depth $h \in H_{ad}$, is the minimizer of the functional $V(h, \cdot) : S \to \mathbb{R}$, defined by

$$V(h, M) \triangleq \frac{12}{Eb} \int_0^L \frac{M(x)^2}{h(x)^3}\, dx . \tag{4.1a}$$

Hence $M_f(h, \cdot)$ differs from $M_c(h, \cdot)$ by a linear term only. This linear term accounts for the difference in the bending moment due to the change in the reactions at the supports. Suppose that

$$M_f(h, x) = M_c(h, x) + g_1(h)x + g_2(h),\quad x \in [0, L], \tag{4.1b}$$

where $g_1(\cdot)$ and $g_2(\cdot)$ are real valued functions on H_{ad}.

It follows from (4.1b) and the first-order necessary condition of optimality for (4.1a) that, given $h \in H_{ad}$, $g(h) \triangleq [g_1(h)\, g_2(h)]^T$ satisfies the equation

$$\begin{bmatrix} \int_0^L \frac{x^2}{h(x)^3}dx & \int_0^L \frac{x}{h(x)^3}dx \\ \int_0^L \frac{x}{h(x)^3}dx & \int_0^L \frac{1}{h(x)^3}dx \end{bmatrix} \begin{bmatrix} g_1(h) \\ g_2(h) \end{bmatrix} = \begin{bmatrix} -\int_0^L \frac{M_c(h,x)x}{h(x)^3}dx \\ -\int_0^L \frac{M_c(h,x)}{h(x)^3}dx \end{bmatrix}. \tag{4.1c}$$

If we denote by $A(h)$ the matrix on the left hand side of (4.1c), and by $b(h)$ the vector on the right, then we can write equation (4.1c) as $A(h)g(h) = b(h)$. It follows from the Euler-Bernoulli beam equations and (4.1b) that the shear force at x, $V_f(h,x)$, is given by

$$V_f(h,x) = -\frac{d}{dx}M_f(h,x) = V_c(h,x) - g_1(h), \quad x \in [0,L], \tag{4.1d}$$

where $V_c(h,x)$ is determined by (3.1d). The deflection $y_f(h,\cdot)$ of the beam is the solution of the differential equation

$$\frac{d}{dx}\begin{bmatrix} y_f(h,x) \\ y'_f(h,x) \end{bmatrix} = \begin{bmatrix} y'_f(h,x) \\ 12M_f(h,x)/Ebh(x)^3 \end{bmatrix}, \; x \in [0,L], \; y_f(h,0) = y'_f(h,0) = 0, \tag{4.1e}$$

where $y'_f(h,x) \triangleq d/dx\, y_f(h,x)$.

Hence, with $M_f(h,\cdot)$, $V_f(h,\cdot)$, and $y_f(h,\cdot)$ determined by (4.1b), (4.1d) and (4.1e) respectively, we consider the following optimal design problem:

$$\mathbf{P}_f \qquad \min_{h \in \mathbf{C}_f} f_f(h), \tag{4.2a}$$

where

$$\mathbf{C}_f \triangleq \{ h \in H_{ad} \mid \psi_f(h) \le 0 \}, \tag{4.2b}$$

with

$$\psi_f(h) \triangleq \max_{j \in \mathbf{q}} \max_{x \in [0,L]} \phi_f^j(h,x) - r^j(x). \tag{4.2c}$$

where the $r^j(\cdot)$, $j \in \mathbf{q}$, satisfy Assumption 3.1*(b)*. The functions $f_f(\cdot)$, and $\phi_f^j(\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$, are defined by

$$f_f(h) \triangleq \int_0^L \phi_f^0(h,x)\,dx, \tag{4.2d}$$

$$\phi_f^j(h,x) \triangleq \tilde{\phi}^j(h(x), M_f(h,x), V_f(h,x), y_f(h,x), x), \; j \in \bar{\mathbf{q}}, \tag{4.2e}$$

where the $\tilde{\phi}^j(\cdot,\cdot,\cdot,\cdot,\cdot)$, $j \in \bar{\mathbf{q}}$, satisfy Assumption 3.1*(c)*. □

Given $h \in H_{ad,N}$, we compute approximations to the integrals in (4.1c) using the rectangle rule on the mesh T_N, with $M_c(h,\cdot)$ replaced by $M_{c,N}(h,\cdot)$, determined by (3.5c). Hence, we obtain an approximation $g_N(h) \triangleq [g_{1,N}(h)\ g_{2,N}(h)]^T$ to $g(h)$, which satisfies the equation

$$\begin{bmatrix} \sum_{j=1}^{N} \frac{(j-1)^2 \Delta(N)^3}{h(x_{N,j})^3} & \sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{h(x_{N,j})^3} \\ \sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{h(x_{N,j})^3} & \sum_{j=1}^{N} \frac{\Delta(N)}{h(x_{N,j})^3} \end{bmatrix} \begin{bmatrix} g_{1,N}(h) \\ g_{2,N}(h) \end{bmatrix} = \begin{bmatrix} -\sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{h(x_{N,j})^3} M_{c,N}(h, x_{N,j}) \\ -\sum_{j=1}^{N} \frac{\Delta(N)}{h(x_{N,j})^3} M_{c,N}(h, x_{N,j}) \end{bmatrix}. \quad (4.2f)$$

We will write equation (4.2f) as $A_N(h) g_N(h) = b_N(h)$.

Theorem 4.1. *(a)* There are constants $m, M \in (0,\infty)$ such that for all integers N, $h \in H_{ad}$, $h_N \in H_{ad,N}$, and $w \in \mathbb{R}^2$, we have

$$m \| w \|^2 \le w^T A(h) w \le M \| w \|^2 \ , \text{ and } m \| w \|^2 \le w^T A_N(h_N) w \le M \| w \|^2 \ ; \quad (4.3a)$$

(b) There exists a constant $C \in (0,\infty)$ such that for all integers N, $h \in H_{ad}$, and $h_N \in H_{ad,N}$,

$$\| A(h) - A_N(h_N) \| \le C [\| h - h_N \|_2 + \Delta(N)], \quad (4.3b)$$

$$\| b(h) - b_N(h_N) \| \le C [\| h - h_N \|_2 + \Delta(N)], \quad (4.3c)$$

$$\| g(h) - g_N(h_N) \| \le C [\| h - h_N \|_2 + \Delta(N)]. \quad (4.3d)$$

Proof. We begin with part *(a)*. To show that for any $h \in H_{ad}$, $A(h)$ is positive definite, we only need to show that the determinants of its two principal minors are positive. Clearly, $\int_0^L x^2/h(x)^3\, dx \ge L^3/3\beta^3 > 0$. Hence, we only need to show that $\det A(h) > 0$. Because all $h \in H_{ad}$ take values in $[\alpha, \beta]$, we have that for all $p_1, p_2, \varepsilon \in (0,\infty)$,

$$\int_0^1 \frac{(p_1 x - p_2)^2}{h(x)^3} dx \ge \int_0^1 \frac{(p_1 x - p_2)^2}{\beta^3} dx = \frac{p_1^2}{3\beta^3} + \frac{p_2^2}{\beta^3} - \frac{p_1 p_2}{\beta^3} \ge \frac{p_1^2}{3\beta^3} + \frac{p_2^2}{\beta^3} - \frac{\varepsilon p_1^2}{\beta^3} - \frac{p_2^2}{4\varepsilon\beta^3}. \quad (4.4a)$$

If we set $\varepsilon = 1/4$, and choose

$$p_1^2 = \int_0^1 \frac{1}{h(x)^3} dx \quad \text{and} \quad p_2 = \frac{1}{p_1} \int_0^1 \frac{x}{h(x)^3} dx, \quad (4.4b)$$

we get from (4.4a) that

$$\det A(h) = \int_0^1 \frac{x^2}{h(x)^3} dx \int_0^1 \frac{1}{h(x)^3} dx - \left[\int_0^1 \frac{1}{h(x)^3} dx \right]^2 = \int_0^1 \frac{(p_1 x - p_2)^2}{h(x)^3} dx \ge \frac{p_1^2}{12} \ge \frac{1}{12\beta^6}, \quad (4.4c)$$

which implies that $A(h) \in \mathbb{R}^{2\times 2}$ is positive definite.

Next we observe that all entries of $A(h)$ are bounded by $(L^3+1)/\alpha^3$. Hence there exists an $M \in (0,\infty)$ such that for all $w \in \mathbb{R}^2$,

$$w^T A(h) w \le M \|w\|^2 . \tag{4.4d}$$

Since the strictly positive lower bounds on the principal determinants are independent of $h \in H_{ad}$, it follows that the smallest eigenvalue of $A(h)$ is bounded away from 0 for all $h \in H_{ad}$.

The proof of the inequalities (4.3a) for $A_N(h_N)$ is similar and hence omitted.

Next, we prove part *(b)*. Inequalities (4.3b) and (4.3c) follow from the fact that $h(x)$ and $h_N(x)$ take values in $[\alpha,\beta]$, Holder's inequality, the fact that the rectangle rule is $O(\Delta(N))$, and from the definitions of $A(h)$ and $b(h)$, and $A_N(h_N)$ and $b_N(h_N)$ (see (4.1c) and (4.2f)). To prove (4.3d), we first note that $A(h)g(h) = b(h)$ and $A_N(h_N)g_N(h_N) = b_N(h_N)$ imply

$$g(h) - g_N(h_N) = A(h)^{-1}[(A_N(h_N) - A(h))g_N(h_N) + (b(h) - b_N(h_N))] , \tag{4.4e}$$

which, in view of part *(a)* and (4.3b,c), implies (4.3d). □

For $N = 1, 2, \ldots$, we define the approximating problems:

$$\mathbf{P}_{f,N} \qquad \min_{h \in \mathbf{C}_{f,N}} f_{f,N}(h) , \tag{4.5a}$$

where $\mathbf{C}_{f,N} \subset H_{ad,N}$ is the set of all depth functions $h \in H_{ad,N}$ such that

$$M_{f,N}(h, x_{N,k}) = M_{c,N}(h, x_{N,k}) + g_{1,N}(h)x_{N,k} + g_{2,N}(h) , \; k \in \mathbf{N+1} , \tag{4.5b}$$

$$V_{f,N}(h, x_{N,k}) = V_{c,N}(h, x_{N,k}) - g_{1,N}(h) , k \in \mathbf{N+1} , \tag{4.5c}$$

$$\begin{bmatrix} y_{f,N}(h, x_{N,k}) \\ y'_{f,N}(h, x_{N,k}) \end{bmatrix} = \begin{bmatrix} y_{f,N}(h, x_{N,k-1}) + \Delta(N) y'_{f,N}(h, x_{N,k-1}) \\ y'_{f,N}(h, x_{N,k-1}) + \Delta(N) \dfrac{12 M_{f,N}(h, x_{N,k-1})}{Ebh(x_{N,k-1})^3} \end{bmatrix} , \begin{bmatrix} y_{f,N}(h, x_{N,1}) \\ y'_{f,N}(h, x_{N,1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \tag{4.5d}$$

for $k \in \mathbf{N+1}$, and

$$\psi_{f,N}(h) \triangleq \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \phi^j_{f,N}(h, x_{N,k}) - (1 + \Delta(N)^{1/2}) r^j(x_{N,k}) \le 0 . \tag{4.5e}$$

In (4.5a) $f_{f,N}(\cdot)$ is defined by

$$f_{f,N}(h) \triangleq \Delta(N) \sum_{k=1}^{N} \phi^0_{f,N}(h, x_{N,k}) , \tag{4.5f}$$

We define the functions $\phi^j_{f,N} : H_{ad,N} \times [0,L] \to \mathbb{R}$, $j \in \bar{\mathbf{q}}$, as the piecewise linear interpolations of the values $\tilde{\phi}^j(h(x_{N,k}), M_{f,N}(h, x_{N,k}), V_{f,N}(h, x_{N,k}), y_{f,N}(h, x_{N,k}), x_{N,k})$, $k \in \mathbf{N+1}$, on the mesh T_N. Hence

$$\phi^j_{f,N}(h,x_{N,k}) = \tilde{\phi}^j(h(x_{N,k}), M_{f,N}(h,x_{N,k}), V_{f,N}(h,x_{N,k}), y_{f,N}(h,x_{N,k}), x_{N,k}), \quad j \in \bar{\mathbf{q}}. \tag{4.5g}$$

Using (4.1b), (4.1d), (4.1e), and (4.3d) we can prove a result similar to Lemma 3.2, relating the functions defining $\mathbf{P}_f$ and $\mathbf{P}_{f,N}$. In particular, we can show that estimates similar to those in (3.7e-g) hold for $\phi^j_f(h,\cdot)$, $\phi^j_{f,N}(h,\cdot)$, $\psi_f(\cdot)$, $\psi_{f,N}(\cdot)$, $f_f(h)$, and $f_{f,N}(h)$. Hence, exactly as in Theorem 3.3, it follows that $\mathbf{P}_{f,N} \to^{Epi} \mathbf{P}_f$.

It should be clear that each entry of $A(h)$ and $b(h)$ in (4.1c) is a Lipschitz continuously differentiable function of $h \in H_{ad}$. Given $h \in H_{ad}$, and $h_* \in C[0,L]$, we denote by $DA(h;h_*)$, and $Db(h;h_*)$ the differentials of $A(\cdot)$, and $b(\cdot)$ at h, in the direction h_*. They are given by

$$DA(h;h_*) = -3\begin{bmatrix} \int_0^L \frac{x^2 h_*(x)}{h(x)^4}dx & \int_0^L \frac{x h_*(x)}{h(x)^4}dx \\ \int_0^L \frac{x h_*(x)}{h(x)^4}dx & \int_0^L \frac{h_*(x)}{h(x)^4}dx \end{bmatrix}, \tag{4.6a}$$

$$Db(h;h_*) = \begin{bmatrix} \int_0^L \frac{3M_c(h,x)h_*(x) - DM_c(h,x;h_*)h(x)}{h(x)^4}x\,dx \\ \int_0^L \frac{3M_c(h,x)h_*(x) - DM_c(h,x;h_*)h(x)}{h(x)^4}dx \end{bmatrix}. \tag{4.6b}$$

From (4.1c) and Theorem 4.1*(a)* it follows that $g: H_{ad} \to \mathbb{R}^2$ has Lipschitz continuous Gateaux differentials. In fact, for any $h \in H_{ad}$, $h_* \in C[0,L]$, the Gateaux differential of g at h in the direction h_*, $Dg(h;h_*)$, is the solution of

$$A(h)Dg(h;h_*) = Db(h;h_*) - DA(h;h_*)g(h). \tag{4.6c}$$

Therefore, from (4.1b-d) we obtain the following result:

Lemma 4.2. *(a)* The functions $h \mapsto V_f(h,\cdot)$, $h \mapsto M_f(h,\cdot)$, and $h \to y_f(h,\cdot)$, from H_{ad} into $(C[0,L], \|\cdot\|_2)$, have Lipschitz continuous Gateaux differentials at all $h \in H_{ad}$, in all directions $h_* \in C[0,L]$, which we denote by $D_1V_f(h,\cdot;h_*)$, $D_1M_f(h,\cdot;h_*)$, and $D_1y_f(h,\cdot;h_*)$ respectively. Moreover,

$$D_1V_f(h,x;h_*) = D_1V_c(h,x;h_*) - Dg_1(h;h_*), \quad x \in [0,L], \tag{4.7a}$$

$$D_1M_f(h,x;h_*) = D_1M_c(h,x;h_*) + Dg_1(h;h_*)x + Dg_2(h;h_*), \quad x \in [0,L], \tag{4.7b}$$

and $D_1y_f(h,\cdot;h_*)$ is the solution of the differential equation

$$\frac{d^2}{dx^2} D_1 y_f(h, x; h_*) = \frac{12}{Ebh(x)^3}\Big(D_1 M_f(h, x; h_*) - 3M_f(h, x)\frac{h_*(x)}{h(x)}\Big), \quad x \in [0, L],$$

$$D_1 y_f(h, 0; h_*) = \frac{d}{dx} D_1 y_f(h, 0; h_*) = 0. \tag{4.7c}$$

□

In view of theorem Lemma 4.2 and of (4.2e), the Chain Rule and Assumption 3.1*(c)* imply that $h \mapsto f_f(h)$, and $h \mapsto \phi_f^j(h, \cdot)$, $j \in \bar{\mathbf{q}}$, have Lipschitz continuous Gateaux differentials. We denote these differentials at $h \in H_{ad}$, in the direction $h_* \in C[0, L]$, by $Df_f(h; h_*)$, and $D_1\phi_f^j(h, \cdot; h_*)$, $j \in \bar{\mathbf{q}}$, respectively.

It follows directly from (4.2e), Lemma 4.2, and Assumption 3.1*(c)* that the following result holds:

Lemma 4.3. There exists a constant $C < \infty$ such that for any $h, \tilde{h} \in H_{ad}$, $h', h'' \in H_{ad}$,

$$|Df_f(h; h' - h) - Df_f(\tilde{h}; h'' - \tilde{h})| \le C[\|h - \tilde{h}\|_2 + \|h' - h''\|_2], \tag{4.8a}$$

and for all $j \in \bar{\mathbf{q}}$,

$$\|D_1\phi_f^j(h, \cdot; h' - h) - D_1\phi_f^j(\tilde{h}, \cdot; h'' - \tilde{h})\|_2 \le C[\|h - \tilde{h}\|_2 + \|h' - h''\|_2]. \tag{4.8b}$$

□

Proceeding as in Section 3, we define, for $h, h' \in H_{ad}$,

$$F_f(h, h') \triangleq \max\{f_f(h') - f_f(h) - \omega\psi_f(h)_+, \max_{j \in \mathbf{q}} \max_{x \in [0,1]} \phi_f^j(h', x) - r^j(x) - \psi_f(h)_+\}, \tag{4.9a}$$

$$\tilde{F}_f(h, h') \triangleq \max\{Df_f(h; h' - h) - w\psi_f(h)_+, \max_{j \in \mathbf{q}} \max_{x \in [0,L]} \phi_f^j(h, x) + D_1\phi_f^j(h, x; h' - h)$$

$$-\psi_f(h)_+\} + \tfrac{1}{2}\|h' - h\|_2^2, \tag{4.9b}$$

$$\theta_f(h) \triangleq \min_{h' \in H_{ad}} \tilde{F}_f(h, h'). \tag{4.9c}$$

One can show that results similar to Lemma 3.4, Proposition 3.5, and Lemma 3.6, and Theorem 3.7 also hold for $\tilde{F}_f(\cdot, \cdot)$, and $\theta_f(\cdot)$. Hence, by arguments similar to those used in Section 3, we obtain the following counterpart of Corollary 3.8:

Theorem 4.4. $\theta_f : H_{ad} \to \mathbb{R}$ is an optimality function for $\mathbf{P}_f$. □

The proof of Theorem 4.4 is identical to that of Theorem 3.8, and hence is omitted.

The mappings $h \in H_{ad,N} \mapsto A_N(h_N) \in \mathbb{R}^{2\times 2}$ and $h \in H_{ad,N} \mapsto b_N(h_N) \in \mathbb{R}^2$ are Lipschitz continuous differentiable. In fact, one can show that, for any $h_N \in H_{ad,N}$ and $h_* \in H_N$, their differentials, $DA_N(h_N; h_*)$ and $Db_N(h_N; h_*)$, are the discrete counterparts of $DA(h_N; h_*)$ and

$Db(h_N\,;h_*)$, obtained by discretizing (4.6a-b) using the rectangle rule, that is,

$$DA_N(h_N\,;h_*) = -3\begin{bmatrix} \sum_{j=1}^{N}\frac{(j-1)^2\Delta(N)^3 h_*(x_{N,j})}{h_N(x_{N,j})^4} & \sum_{j=1}^{N}\frac{(j-1)\Delta(N)^2 h_*(x_{N,j})}{h_N(x_{N,j})^4} \\ \sum_{j=1}^{N}\frac{(j-1)\Delta(N)^2 h_*(x_{N,j})}{h_N(x_{N,j})^4} & \sum_{j=1}^{N}\frac{\Delta(N) h_*(x_{N,j})}{h_N(x_{N,j})^4} \end{bmatrix}, \tag{4.10a}$$

$$Db_N(h_N\,;h_*) = \begin{bmatrix} -\sum_{j=1}^{N}(j-1)\Delta(N)^2\left[\frac{DM_{c,N}(h_N\,,x_{N,j}\,;h_*)}{h_N(x_{N,j})^3} - \frac{3M_{c,N}(h_N\,,x_{N,j})}{h_N(x_{N,j})^4}h_*(x_{N,j})\right] \\ -\sum_{j=1}^{N}\Delta(N)\left[\frac{DM_{c,N}(h_N\,,x_{N,j}\,;h_*)}{h_N(x_{N,j})^3} - \frac{3M_{c,N}(h_N\,,x_{N,j})}{h_N(x_{N,j})^4}h_*(x_{N,j})\right] \end{bmatrix}. \tag{4.10b}$$

Making use of Theorem 4.1*(a)* and the Implicit Function Theorem, one can show that $Dg_N(h_N\,;h_*)$ satisfies the equation

$$A_N(h_N)Dg_N(h_N\,;h_*) = Db_N(h_N\,;h_*) - DA_N(h_N\,;h_*)g_N(h_N)\,. \tag{4.10c}$$

Moreover, from Theorem 4.1 and the fact that the rectangle rule is $O(\Delta(N))$ we obtain the following:

Lemma 4.5. There exists a constant $C \in (0,\infty)$ such that for all $h_N\,,h'_N \in H_{ad,N}$,

$$\| DA(h_N\,;h'_N - h_N) - DA_N(h_N\,;h'_N - h_N)\| \le C\,\Delta(N)\|h'_N - h_N\|_2\,, \tag{4.11a}$$

$$\| Db(h_N\,;h'_N - h_N) - Db_N(h_N\,;h'_N - h_N)\| \le C\,\Delta(N)\|h'_N - h_N\|_2\,, \tag{4.11b}$$

$$\| Dg(h_N\,;h'_N - h_N) - Dg_N(h_N\,;h'_N - h_N)\| \le C\,\Delta(N)\|h'_N - h_N\|_2\,. \tag{4.11c}$$

□

Next, we define the discrete counterparts of $F_f(\cdot,\cdot)$, $\tilde{F}_f(\cdot,\cdot)$ and $\theta_f(\cdot)$, denoted by $F_{f,N}(\cdot,\cdot)$, $\tilde{F}_{f,N}(\cdot,\cdot)$ and $\theta_{f,N}(\cdot)$, respectively. Given $h\,,h' \in H_{ad,N}$,

$$F_{f,N}(h\,,h') \triangleq \max\{\,f_{f,N}(h') - f_{c,N}(h) - \omega\psi_{f,N}(h)_+\,,\max_{j\,\in\,\mathbf{q}}\max_{k\,\in\,\mathbf{N+1}}\phi^j_{f,N}(h'\,,x_{N,k}) - \psi_{f,N}(h)_+\,\} \tag{4.12a}$$

$$\tilde{F}_{f,N}(h\,,h') \triangleq \max\{\,Df_{f,N}(h\,;h'-h) - \omega\psi_{f,N}(h)_+\,,\max_{j\,\in\,\mathbf{q}}\max_{k\,\in\,\mathbf{N+1}}\phi^j_{f,N}(h\,,x_{N,k}) - \psi_{f,N}(h)_+$$
$$+D_1\phi^j_{f,N}(h\,,x_{N,k}\,;h'-h)\,\} + \tfrac{1}{2}\|h'-h\|_2^2\,, \tag{4.12b}$$

$$\theta_{f,N}(h) \triangleq \min_{h' \in H_{ad,N}} \tilde{F}_{f,N}(h\,,h')\,. \tag{4.12c}$$

Using (4.1b-d), Theorem 4.1, Lemmas 4.2, 4.3 and 4.5, we can prove results similar to Lemma 3.4, Lemma 3.9, Theorem 3.10, and Lemma 3.11. Hence, with arguments identical to those used in

Section 3, we conclude that the following is true:

Theorem 4.6. Suppose that $\{ h_N \}_{N=N_0}^{\infty}$, with $h_N \in H_{ad,N}$, is such that $h_N \to h \in H_{ad}$ as $N \to \infty$. Then $h \in H_{ad}$, and $\overline{\lim}_{N \to \infty} \theta_{f,N}(h_N) \le \theta_f(h)$.

Corollary 4.7. The sequence $\{ (\mathbf{P}_{f,N}, \theta_{f,N}) \}_{N=1}^{\infty}$ is a family of weakly consistent approximations to the pair $(\mathbf{P}_f, \theta_f)$. Furthermore, if for all $h \in H_{ad}$ such that $\psi_f(h) > 0$, $0 \notin \partial\psi_f(h)$, then $\{ (\mathbf{P}_{f,N}, \theta_{f,N}) \}_{N=1}^{\infty}$ is a family of consistent approximations to the pair $(\mathbf{P}_f, \theta_f)$.

5. A DIAGONALIZED OPTIMIZATION ALGORITHM

In this section we will describe a diagonalized implementable algorithm that uses consistent approximations and standard nonlinear programming software in computing an approximate solution to either problem $\mathbf{P}_c$ or problem $\mathbf{P}_f$. For this purpose, we will obtain $\mathbb{R}^{N+1}$ equivalents of the problems $\mathbf{P}_{c,N}$ and $\mathbf{P}_{f,N}$, which were originally defined on the function spaces H_N.

Given any $h \in H_N$, there exists a unique vector $\eta = (\eta_1, \dots, \eta_{N+1})^T \in \mathbb{R}^{N+1}$ satisfying (3.3b). In fact, in view of (3.3a), we have that $\eta_k = h(x_{N,k})$, $k \in \mathbf{N+1}$. We define the mapping $W_N : H_N \to \mathbb{R}^{N+1}$ by

$$W_N(h) \triangleq (\eta_1, \eta_2, \dots, \eta_{N+1})^T. \tag{5.1a}$$

Clearly, W_N is a bijection and the components of $\eta \in \mathbb{R}^{N+1}$ are the coordinates of $h \in H_N$ with respect to the basis set $\{ P_{N,k}(x) \}_{k=1}^{N+1}$. For any $h \in H_N$ and $\eta = W_N(h)$,

$$\| h \|_2^2 = \int_0^L \sum_{j=1}^{N+1} \eta_j P_{N,j}(x) \sum_{i=1}^{N+1} \eta_i P_{N,i}(x) dx = \sum_{i,j=1}^{N+1} \eta^i \eta_j \int_0^L P_{N,i}(x) P_{N,j}(x) dx = \eta^T Q_N \eta, \tag{5.2a}$$

where $Q_N \in \mathbb{R}^{(N+1)\times(N+1)}$ is given by

$$Q_N \triangleq \frac{\Delta(N)}{6} \begin{bmatrix} 2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 4 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & 4 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 & 4 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 & 2 \end{bmatrix}. \tag{5.2b}$$

Since $\{ P_{N,k}(x) \}_{k=1}^{N+1}$ is not orthonormal, we let $T_N \triangleq Q_N^{-1/2}$, and consider the mapping

$$\xi = T_N^{-1} W_N(h). \tag{5.3a}$$

Then the components of $\xi \in \mathbb{R}^{N+1}$ are the coordinates of $h \in H_N$ with respect to basis set $\{ B_{N,k}(x) \}_{k=1}^{N+1}$, given by

$$B_{N,k}(x) = W_N^{-1}(T_N e_k) = \sum_{j=1}^{N+1} (T_N)_{i,j} P_{N,j}(x), \quad k \in \mathbf{N+1}, \tag{5.3b}$$

where e_k denotes the k-th canonical basis vector in $\mathbb{R}^{N+1}$, and $(T_N)_{j,k}$ denotes the j,k-th entry of the matrix T_N. It follows from (5.2a) and (5.3a,b) that for any $i, j \in \mathbf{N+1}$,

$$\langle B_{N,i}, B_{N,j} \rangle_2 = (T_N e_i)^T Q_N (T_N e_j) = e_i^T e_j, \tag{5.3c}$$

which implies that the basis set $\{ B_{N,k}(x) \}_{k=1}^{N+1}$ is orthonormal.

With α, β, and γ as in (3.1a), we let

$$\overline{H}_{ad,N} \triangleq \{ \xi \in \mathbb{R}^{N+1} \mid \alpha \le (T_N \xi)_k \le \beta, \; k \in \mathbf{N+1}, \text{ and } |(T_N \xi)_{k+1} - (T_N \xi)_k| \le \gamma \Delta(N), k \in \mathbf{N} \}, \tag{5.4a}$$

where $(T_N \xi)_k$ denotes the k-th entry of the vector $T_N \xi$. For any $\xi \in \overline{H}_{ad,N}$ and $k \in \mathbf{N+1}$, let

$$\bar{l}(\xi, x_{N,k}) \triangleq m(x_{N,k}) - K(T_N \xi)_k, \tag{5.4b}$$

with $m(\cdot)$ and $K \ge 0$ as in (3.1b). We define the problems $\overline{\mathbf{P}}_{c,N}$ as follows:

$$\overline{\mathbf{P}}_{c,N} \qquad \min_{\xi \in \overline{\mathbf{C}}_{c,N}} \bar{f}_{c,N}(\xi), \tag{5.4c}$$

where $\overline{\mathbf{C}}_{c,N} \subset \overline{H}_{ad,N}$ is the set of all ξ in $\overline{H}_{ad,N}$ such that for $k \in \mathbf{N}$,

$$\overline{V}_{c,N}(\xi, x_{N,k}) = \overline{V}_{c,N}(\xi, x_{N,k+1}) + \Delta(N)\bar{l}(\xi, x_{N,k+1}), \quad \overline{V}_{c,N}(\xi, x_{N,N+1}) = 0, \tag{5.4d}$$

$$\overline{M}_{c,N}(\xi, x_{N,k}) = \overline{M}_{c,N}(\xi, x_{N,k+1}) + \Delta(N)\overline{V}_{c,N}(\xi, x_{N,k+1}), \quad \overline{M}_{c,N}(\xi, x_{N,N+1}) = 0, \tag{5.4e}$$

$$\begin{bmatrix} \bar{y}_{c,N}(\xi, x_{N,k+1}) \\ \bar{y}'_{c,N}(\xi, x_{N,k+1}) \end{bmatrix} = \begin{bmatrix} \bar{y}_{c,N}(\xi, x_{N,k}) + \Delta(N)\bar{y}'_{c,N}(\xi, x_{N,k}) \\ \bar{y}'_{c,N}(\xi, x_{N,k}) + \Delta(N)\dfrac{12\overline{M}_{c,N}(\xi, x_{N,k})}{Eb(T_N\xi)_k^3} \end{bmatrix}, \quad \begin{bmatrix} \bar{y}_{c,N}(\xi, x_{N,1}) \\ \bar{y}'_{c,N}(\xi, x_{N,1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.4f}$$

and

$$\bar{\psi}_{c,N}(\xi) \triangleq \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \bar{\phi}^j_{c,N}(\xi, x_{N,k}) - (1 + \Delta(N)^{1/2}) r^j(x_{N,k}) \le 0, \tag{5.4g}$$

$$\bar{f}_{c,N}(\xi) \triangleq \sum_{k=1}^{N} \bar{\phi}^0_{c,N}(\xi, x_{N,k}) \Delta(N), \tag{5.4h}$$

$$\bar{\phi}^j_{c,N}(\xi, x_{N,k}) \triangleq \tilde{\phi}^j((T_N\xi)_k, \overline{M}_{c,N}(\xi, x_{N,k}), \overline{V}_{c,N}(\xi, x_{N,k}), \bar{y}_{c,N}(\xi, x_{N,k}), x_{N,k}), \quad j \in \bar{\mathbf{q}}. \tag{5.4i}$$

□

For any given N, the only fundamental difference between problems $\mathbf{P}_{c,N}$ and $\overline{\mathbf{P}}_{c,N}$ is that $\mathbf{P}_{c,N}$ is defined in a "coordinate-free" manner on the functional space H_N, while $\overline{\mathbf{P}}_{c,N}$ is defined on $\mathbb{R}^{N+1}$, in terms of the coordinates ξ corresponding to the basis set $\{ B_{N,k}(x) \}_{k=1}^{N+1}$. Hence, they are two equivalent statements of the same problem. Therefore, it should be clear that, given $h \in H_{ad,N}$ and $\xi = T_N^{-1} W_N(h)$, we have for all $k \in$ **N+1** and all $j \in \bar{\mathbf{q}}$,

$$\phi_{c,N}^j(h, x_{N,k}) = \bar{\phi}_{c,N}^j(\xi, x_{N,k}), \quad \psi_{c,N}(h) = \bar{\psi}_{c,N}(\xi), \quad f_{c,N}(h) = \bar{f}_{c,N}(\xi). \tag{5.5a}$$

Furthermore, from (5.2a) and (5.3a), if we let $\xi = T_N^{-1} W_N(h)$, we have

$$\| h \|_2^2 = W_N(h)^T Q_N W_N(h) = (T_N^{-1} W_N(h))^T (T_N^{-1} W_N(h)) = \| \xi \|^2, \tag{5.5b}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{N+1}$. Hence $h \mapsto T_N^{-1} W_N(h)$ is an isometry (it preserves norms) from $H_{ad,N} \subset (C[0,L], \|\cdot\|_2)$ onto $\bar{H}_{ad,N} \subset (\mathbb{R}^{N+1}, \|\cdot\|)$.

In view of (5.5a,b),(5.4a-i) and (3.5a-f), the following proposition should be obvious.

Proposition 5.1. Problems $\mathbf{P}_{c,N}$ and $\bar{\mathbf{P}}_{c,N}$ are equivalent in the following sense: *(a)* $h \in H_N$ is feasible for $\mathbf{P}_{c,N}$ if and only if $\xi = T_N^{-1} W_N(h)$ is feasible for $\bar{\mathbf{P}}_{c,N}$, and *(b)* $h \in \mathbf{C}_{c,N}$ is a global/local minimizer for $\mathbf{P}_{c,N}$ if and only if $\xi = T_N^{-1} W_N(h)$ is a global/local minimizer for $\bar{\mathbf{P}}_{c,N}$. □

Next, we compute the derivatives of the functions defining $\bar{\mathbf{P}}_{c,N}$ and define an optimality function for $\bar{\mathbf{P}}_{c,N}$.

Let the matrices $G_N^V, G_N^M \in \mathbb{R}^{N+1 \times N+1}$ be defined by

$$G_N^V \triangleq -K\Delta(N) \begin{bmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & 1 & 1 & \cdot & \cdot & 1 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}, \quad G_N^M \triangleq -\frac{1}{K} G_N^V G_N^V. \tag{5.6a}$$

The fact that the mappings $\xi \mapsto \bar{V}_{c,N}(\xi, x_{N,k})$ and $\xi \mapsto \bar{M}_{c,N}(\xi, x_{N,k})$, $k \in$ **N+1**, are Lipschitz continuously differentiable differentiable on $\bar{H}_{ad,N}$ follows from the Implicit Function Theorem and (5.4d-e). In view of (5.4b) and (5.6a), if we differentiate (5.4d,e) we obtain, for all $\xi, \xi' \in \bar{H}_{ad,N}$,

$$D_1 \bar{V}_{c,N}(\xi, x_{N,k}; \xi' - \xi) = G_N^V(k, :) T_N(\xi' - \xi), \quad k \in \mathbf{N+1}, \tag{5.6b}$$

$$D_1 \bar{M}_{c,N}(\xi, x_{N,k}; \xi' - \xi) = G_N^M(k, :) T_N(\xi' - \xi), \quad k \in \mathbf{N+1}, \tag{5.6c}$$

where for any matrix G, $G(k, :)$ denotes its k-th row.

We define the mapping $\tilde{G}_N^y : \bar{H}_{ad,N} \to \mathbb{R}^{(N+1)\times(N+1)}$ by

$$\tilde{G}_N^y(\xi)(k, :) \triangleq \frac{G_N^M(k, :)}{(T_N \xi)_k} - 3 M_{c,N}(\xi, x_{N,k}) e_k^T, \quad k \in \mathbf{N+1}. \tag{5.6d}$$

It can be shown, using (5.4f) and the Implicit Function Theorem, that the mappings $\xi \mapsto \bar{y}(\xi, x_{N,k})$, $k \in$ **N+1**, are Lipschitz continuously differentiable. For any $\xi, \xi' \in \bar{H}_{ad,N}$, $D_1 \bar{y}_{c,N}(\xi, x_{N,k}; \xi' - \xi) = \delta\bar{y}(\xi, x_{N,k})$, which is the solution of

$$\begin{bmatrix} \delta\bar{y}_{k+1} \\ \delta\bar{y}'_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta(N) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\bar{y}_k \\ \delta\bar{y}'_k \end{bmatrix} + \frac{12\Delta(N)}{Eb(T_N\xi)_k^3} \begin{bmatrix} \mathbf{0}^T \\ \tilde{G}_N^y(\xi)(k,:) \end{bmatrix} (\xi'-\xi),\ k \in \mathbf{N},\ \begin{bmatrix} \delta\bar{y}_1 \\ \cdot \\ \delta\bar{y}'_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5.6e)$$

for $k \in \mathbf{N}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{N+1}$. Equation (5.6e) is a linear difference equation of the form $v(k+1) = Av(k) + B(k)u$, $v(1) = 0$, whose solution is given by $v(k) = \sum_{j=1}^{k-1} A^{k-1-j} B(j)u$. Therefore, if we define

$$G_N^y(\xi)(k,:) \triangleq \Delta(N) \sum_{j=1}^{k-1} (k-1-j) \tilde{G}_N^y(\xi)(k,:)\ , \quad (5.6f)$$

it follows from (5.6d-f) that for all $k \in \mathbf{N+1}$,

$$D_1\bar{y}_N(\xi, x_{N,k}; \xi'-\xi) = \delta y_{c,N}(\xi, x_{N,k}) = G_N^y(\xi)(k,:)T_N(\xi'-\xi). \quad (5.6g)$$

To obtain derivatives of the mappings $\xi \mapsto \bar{\phi}_{c,N}^j(\xi, x_{N,k})$, $j \in \bar{\mathbf{q}}$, $k \in \mathbf{N+1}$, we apply the Chain Rule to (5.4i). First, for $\xi \in \bar{H}_{ad,N}$, $j \in \bar{\mathbf{q}}$, $k \in \mathbf{N+1}$, we define

$$G_{c,N}^{\phi,j}(\xi)(k,:) \triangleq \frac{\partial\tilde{\phi}^j((T_N\xi)_k, \bar{M}_{c,N}(\xi,x_{N,k}), \bar{V}_{c,N}(\xi,x_{N,k}), \bar{y}_{c,N}(\xi,x_{N,k}), x_{N,k})}{\partial h} e_k^T$$

$$+ \frac{\partial\tilde{\phi}^j((T_N\xi)_k, \bar{M}_{c,N}(\xi,x_{N,k}), \bar{V}_{c,N}(\xi,x_{N,k}), \bar{y}_{c,N}(\xi,x_{N,k}), x_{N,k})}{\partial M} G_N^M(k,:)$$

$$+ \frac{\partial\tilde{\phi}^j((T_N\xi)_k, \bar{M}_{c,N}(\xi,x_{N,k}), \bar{V}_{c,N}(\xi,x_{N,k}), \bar{y}_{c,N}(\xi,x_{N,k}), x_{N,k})}{\partial V} G_N^V(k,:)$$

$$+ \frac{\partial\tilde{\phi}^j((T_N\xi)_k, \bar{M}_{c,N}(\xi,x_{N,k}), \bar{V}_{c,N}(\xi,x_{N,k}), \bar{y}_{c,N}(\xi,x_{N,k}), x_{N,k})}{\partial y} G_N^y(\eta)(k,:)\ . \quad (5.7a)$$

Then it follows from the Chain Rule applied to (5.4i), (5.6b,c), and (5.6g) that for all $j \in \bar{\mathbf{q}}$ and all $k \in \mathbf{N+1}$,

$$D_1\bar{\phi}_{c,N}^j(\xi, x_{N,k}; \xi'-\xi) = G_{c,N}^{\phi,j}(\xi)(k,:)T_N(\xi'-\xi) = \langle T_N^T [G_{c,N}^{\phi,j}(\xi)(k,:)]^T, \xi-\xi' \rangle, \quad (5.7b)$$

where $\langle \cdot,\cdot \rangle$ denotes the Euclidean inner-product on $\mathbb{R}^{N+1}$. Hence, if we define

$$G_{c,N}(\xi) \triangleq \Delta(N) \sum_{k=1}^{N} G_{c,N}^{\phi,0}(\xi)(k,:)\ , \quad (5.7c)$$

we obtain from (5.4h) and (5.7b) that

$$D\bar{f}_{c,N}(\xi; \xi'-\xi) = \Delta(N) \sum_{k=1}^{N} D_1\bar{\phi}_{c,N}^0(\xi, x_{N,k}; \xi'-\xi) = \Delta(N) \sum_{k=1}^{N} G_{c,N}^{\phi,0}(\xi)(k,:)(\xi'-\xi)$$

$$= G_{c,N}(\xi)T_N(\xi' - \xi) = \langle T_N^T\, G_{c,N}(\xi)^T\, , \xi' - \xi \rangle\,. \tag{5.7d}$$

It is clear from (5.7b,d) that, for any $\xi \in \overline{H}_{ad,N}$, $j \in \mathbf{q}$, and $k \in \mathbf{N+1}$,

$$\nabla_1 \overline{\phi}_{c,N}^j(\xi, x_{N,k}) = T_N^T\, [G_{c,N}^{\phi j}(\xi)(k\,,:)]^T\,, \quad \nabla \overline{f}_{c,N}(\xi) = T_N^T\, G_{c,N}(\xi)^T\,.$$

Finally, we define the mappings $\overline{F}_{c,N} : \overline{H}_{ad,N} \times \overline{H}_{ad,N} \to \mathbb{R}$, $\tilde{\overline{F}}_{c,N} : \overline{H}_{ad,N} \times \overline{H}_{ad,N} \to \mathbb{R}$, and $\overline{\theta} : \overline{H}_{ad,N} \to \mathbb{R}$ by

$$\overline{F}_{c,N}(\xi, \xi') \triangleq \max \{ \overline{f}_{c,N}(\xi') - \overline{f}_{c,N}(\xi) - \omega\overline{\psi}_{c,N}(\xi)_+ , \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \overline{\phi}_{c,N}^j(\xi', x_{N,k}) - \overline{\psi}_{c,N}(\xi)_+ \}\,, \tag{5.8a}$$

$$\tilde{\overline{F}}_{c,N}(\xi, \xi') \triangleq \max \Big\{ \langle \nabla \overline{f}_{c,N}\, , \xi' - \xi \rangle - \omega\overline{\psi}_{c,N}(\xi)_+ , \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \overline{\phi}_{c,N}^j(\xi, x_k) - \overline{\psi}_{c,N}(\xi)_+$$

$$+ \langle \nabla_1 \overline{\phi}_{c,N}^j(\xi, x_{N,k}), \xi' - \xi \rangle \Big\} + \tfrac{1}{2}\|\xi' - \xi\|^2\,, \tag{5.8b}$$

$$\overline{\theta}_{c,N}(\xi) \triangleq \min_{\xi' \in \overline{H}_{ad,N}} \tilde{\overline{F}}_{c,N}(\xi, \xi'). \tag{5.8c}$$

Proposition 5.2. *(a)* $\overline{\theta}_{c,N} : \overline{H}_{ad,N} \to \mathbb{R}$ is an optimality function for $\overline{\mathbf{P}}_{c,N}$, *(b)* For any $h\,, h' \in H_{ad,N}$, $\xi = T_N^{-1}W_N(h)$, and $\xi' = T_N^{-1}W_N(h')$ we have

$$F_{c,N}(h\,, h') = \overline{F}_{c,N}(\xi, \xi'), \quad \theta_{c,N}(h) = \overline{\theta}_{c,N}(\xi). \tag{5.9a}$$

□

A proof of Proposition 5.2 *(a)* can be found in [15]. Part *(b)* is a direct consequence of (5.5), (3.20b,c), (5.8a-c), and of the fact that for any $h\,, h' \in H_{ad}$, $\xi = T_N^{-1}W_N(h)$, and $\xi' = T_N^{-1}W_N(h')$, we have

$$D_1\phi_{c,N}^j(h\,, x_{N,k}\,; h' - h) = D_1\overline{\phi}_{c,N}^j(\xi, x_{N,k}\,; \xi' - \xi), \quad Df_{c,N}(h\,; h' - h) = D\overline{f}_{c,N}(\xi\,; \xi' - \xi)\,, \tag{5.9b}$$

$$\tilde{F}_{c,N}(h\,, h') = \tilde{\overline{F}}_{c,N}(\xi, \xi')\,, \tag{5.9c}$$

which should be obvious, since $D_1\phi_{c,N}^j(h\,, x_{N,k}\,; h' - h)$ and $Df_{c,N}(h\,; h' - h)$ are just the "coordinate-free" counterparts of $D_1\overline{\phi}_{c,N}^j(\xi, x_{N,k}\,; \xi' - \xi)$ and $D\overline{f}_{c,N}(\xi\,; \xi' - \xi)$.

The transcription of the approximating problems $\mathbf{P}_{f,N}$ is similar. If we consider (4.5a-g), where the $\mathbf{P}_{f,N}$ are defined on the finite dimensional function space H_N, we conclude that to obtain the transcriptions $\overline{\mathbf{P}}_{f,N}$ we need to define, in addition to the mappings defined in transcribing $\mathbf{P}_{c,N}$, the mapping $\overline{g}_N(\xi) \triangleq [\overline{g}_{1,N}(\xi)\;\; \overline{g}_{2,N}(\xi)]^T$, from $\overline{H}_{ad,N}$ into $\mathbb{R}^2$, which is the coordinate dependent counterpart of $g_N(\cdot)$ (see (4.2f)). To do that, consider the system of equations

$$\begin{bmatrix} \sum_{j=1}^{N} \frac{(j-1)^2\Delta(N)^3}{(T_N\xi)_j^3} & \sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{(T_N\xi)_j^3} \\ \sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{(T_N\xi)_j^3} & \sum_{j=1}^{N} \frac{\Delta(N)}{(T_N\xi)_j^3} \end{bmatrix} \begin{bmatrix} \bar{g}_{1,N}(\xi) \\ \bar{g}_{2,N}(\xi) \end{bmatrix} = \begin{bmatrix} -\sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{(T_N\xi)_j^3} \bar{M}_{c,N}(\xi,x_{N,j}) \\ -\sum_{j=1}^{N} \frac{\Delta(N)}{(T_N\xi)_j^3} \bar{M}_{c,N}(\xi,x_{N,j}) \end{bmatrix} \tag{5.10a}$$

which we write as $\bar{A}_N(\xi)\bar{g}_N(\xi) = \bar{b}_N(\xi)$. It should be obvious from (4.2f), (5.10a) and Theorem 4.1 that for any $h \in H_{ad,N}$ and $\xi = T_N^{-1}W_N(h)$, (5.10a) uniquely defines $\bar{g}_N(\xi)$, and

$$A_N(h) = \bar{A}_N(\xi), \; b_N(h) = \bar{b}_N(\xi), \; g_N(h) = \bar{g}_N(\xi). \tag{5.10b}$$

We define the problems $\bar{\mathbf{P}}_{f,N}$ as follows

$$\bar{\mathbf{P}}_{f,N} \qquad \min_{\xi \in \bar{\mathbf{C}}_{f,N}} \bar{f}_{f,N}(\xi), \tag{5.11a}$$

where $\bar{\mathbf{C}}_{f,N} \subset \bar{H}_{ad,N}$ is the set of all $\xi \in \bar{H}_{ad,N}$ such that

$$\bar{M}_{f,N}(\xi,x_{N,k}) \triangleq \bar{M}_{c,N}(\xi,x_{N,k}) + \bar{g}_{1,N}(\xi)x_{N,k} + \bar{g}_{2,N}(\xi)\;,\; k \in \mathbf{N+1}, \tag{5.11b}$$

$$\bar{V}_{f,N}(\xi,x_{N,k}) = \bar{V}_{c,N}(\xi,x_{N,k}) - \bar{g}_{1,N}(\xi), k \in \mathbf{N+1}, \tag{5.11c}$$

and, for $k \in \mathbf{N}$,

$$\begin{bmatrix} \bar{y}_{f,N}(\xi,x_{N,k+1}) \\ \bar{y}'_{f,N}(\xi,x_{N,k+1}) \end{bmatrix} = \begin{bmatrix} \bar{y}_{f,N}(\xi,x_{N,k}) + \Delta(N)\bar{y}'_{f,N}(\xi,x_{N,k}) \\ \bar{y}'_{f,N}(\xi,x_{N,k}) + \Delta(N)\frac{12\bar{M}_{f,N}(\xi,x_{N,k-1})}{Eb(T_N\xi)_k^3} \end{bmatrix}, \; \begin{bmatrix} \bar{y}_{f,N}(h,x_{N,1}) \\ \bar{y}'_{f,N}(h,x_{N,1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.11d}$$

$$\bar{\psi}_{f,N}(\xi) \triangleq \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \bar{\phi}^j_{f,N}(\xi,x_{N,k}) - (1+\Delta(N)^{1/2})r^j(x_{N,k}) \le 0. \tag{5.11e}$$

$$\bar{f}_{f,N}(\xi) \triangleq \Delta(N)\sum_{k=1}^{N} \bar{\phi}^0_{f,N}(\xi,x_{N,k}), \tag{5.11f}$$

$$\phi^j_{f,N}(\xi,x_{N,k}) \triangleq \tilde{\phi}^j((T_N\xi)_k, \bar{M}_{f,N}(h,x_{N,k}), \bar{V}_{f,N}(h,x_{N,k}), \bar{y}_{f,N}(h,x_{N,k}), x_{N,k}), \; j \in \bar{\mathbf{q}} \tag{5.11g}$$

□

It follows from (5.5a,b), (5.10b), and (5.11a-g) that a result analogous to Proposition 5.1 holds, establishing the equivalence of $\mathbf{P}_{f,N}$ and $\bar{\mathbf{P}}_{f,N}$.

It should be clear from (5.11b,c) that to obtain expressions for the gradients of the functions defining $\bar{\mathbf{P}}_{f,N}$ we need, in addition to the expressions computed in transcribing $\mathbf{P}_{c,N}$, expressions for the differentials of $\bar{g}_{1,N}(\xi)$ and $\bar{g}_{2,N}(\xi)$. First, given any $\xi \in \bar{H}_{ad,N}$, define

$$G_N^{DA}(\xi) \triangleq 3 \begin{bmatrix} \sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{(T_N\xi)_j^4}((j-1)\Delta(N)\bar{g}_{1,N}(\xi) + \bar{g}_{2,N}(\xi))e_j^T \\ \sum_{j=1}^{N} \frac{\Delta(N)}{(T_N\xi)_j^4}((j-1)\Delta(N)\bar{g}_{1,N}(\xi) + \bar{g}_{2,N}(\xi))e_j^T \end{bmatrix} . \tag{5.12a}$$

$$G_N^{Db}(\xi) \triangleq \begin{bmatrix} -\sum_{j=1}^{N} \frac{(j-1)\Delta(N)^2}{(T_N\xi)_j^3} \tilde{G}_N^{y}(\xi)(j,:) \\ -\sum_{j=1}^{N} \frac{\Delta(N)}{(T_N\xi)_j^3} \tilde{G}_N^{y}(\xi)(j,:) \end{bmatrix} , \tag{5.12b}$$

$$G_N^{Dg}(\xi) \triangleq \bar{A}_N(\xi)^{-1}[G_N^{Db}(\xi) + G_N^{DA}(\xi)] . \tag{5.12c}$$

Then it can be shown, using the Implicit Function Theorem and differentiating (5.10a) with respect to $\xi \in \bar{H}_{ad,N}$, that for any $\xi, \xi' \in \bar{H}_{ad}$,

$$D\bar{g}(\xi;\xi'-\xi) = [D\bar{g}_{1,N}(\xi,\xi'-\xi) \;\; D\bar{g}_{2,N}(\xi,\xi'-\xi)]^T = G_N^{Dg}(\xi)T_N(\xi'-\xi) . \tag{5.13a}$$

Hence, it follows directly from (5.11b,c) that

$$D_1\bar{V}_{f,N}(\xi, x_{N,k};\xi'-\xi) = [G_N^V(k,:) - G_N^{Dg}(\xi)(1,:)]T_N(\xi'-\xi) , \tag{5.13b}$$

$$D_1\bar{M}_{f,N}(\xi, x_{N,k};\xi'-\xi) = [G_N^M(k,:) + G_N^{Dg}(\xi)(1,:)x_{N,k} + G_N^{Dg}(\xi)(2,:)]T_N(\xi'-\xi) . \tag{5.13c}$$

We proceed exactly as we did above in the case of $\mathbf{P}_{c,N}$ to obtain expressions for $D_1\bar{y}_{f,N}(\xi, x_{N,k};\xi'-\xi)$, $k \in \mathbf{N+1}$, and consequently (using the Chain Rule) for the gradients corresponding to $D_1\bar{\phi}_{f,N}^j(\xi, x_{N,k};\cdot)$, $j \in \mathbf{q}$, $k \in \mathbf{N+1}$, $D\bar{f}_{f,N}(\xi;\cdot)$, which are denoted $\nabla\bar{\phi}_{f,N}^j(\xi, x_{N,k})$ and $\nabla\bar{f}_{f,N}(\xi)$ respectively.

Using these gradients we define for any $\xi, \xi' \in \bar{H}_{ad,N}$,

$$\bar{F}_{f,N}(\xi,\xi') \triangleq \max\{\bar{f}_{f,N}(\xi') - \bar{f}_{f,N}(\xi) - \omega\bar{\psi}_{f,N}(\xi)_+, \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \bar{\phi}_{f,N}^j(\xi', x_{N,k}) - \bar{\psi}_{f,N}(\xi)_+\} , \tag{5.14a}$$

$$\tilde{\bar{F}}_{f,N}(\xi,\xi') \triangleq \max\Big\{ \langle\nabla\bar{f}_{f,N}, \xi'-\xi\rangle - \omega\bar{\psi}_{f,N}(\xi)_+, \max_{j \in \mathbf{q}} \max_{k \in \mathbf{N+1}} \bar{\phi}_{f,N}^j(\xi, x_k) - \bar{\psi}_{f,N}(\xi)_+ + \langle\nabla_1\bar{\phi}_{f,N}^j(\xi, x_{N,k}), \xi'-\xi\rangle \Big\} + \tfrac{1}{2}\|\xi'-\xi\|^2 , \tag{5.14b}$$

$$\bar{\theta}_{f,N}(\xi) \triangleq \min_{\xi' \in \bar{H}_{ad,N}} \tilde{\bar{F}}_{f,N}(\xi,\xi'). \tag{5.14c}$$

Clearly, a result analogous to Proposition 5.2 holds, and hence $\bar{\theta}_{f,N}(\cdot)$ is an optimality func'tion for $\overline{\mathbf{P}}_{f,N}$, and for any $h, h' \in H_{ad,N}$, $\xi = T_N^{-1} W_N(h)$, and $\xi' = T_N^{-1} W_N(h')$, we have

$$F_{f,N}(h,h') = \bar{F}_{f,N}(\xi,\xi'), \quad \tilde{F}_{f,N}(h,h') = \tilde{\bar{F}}_{f,N}(\xi,\xi'), \quad \theta_{f,N}(h) = \bar{\theta}_{f,N}(\xi) \tag{5.15}$$

We will apply the algorithm described in [17] to solve problems $\mathbf{P}_c$ and $\mathbf{P}_f$ using the framework of consistent approximations, as suggested in [16]. When the algorithm is applied to solve $\mathbf{P}_c$, the functions $\bar{F}_N(\cdot,\cdot)$, $\tilde{\bar{F}}_N(\cdot,\cdot)$, and $\bar{\theta}_N(\cdot)$, in the statements below, are set equal to $\bar{F}_{c,N}(\cdot,\cdot)$, $\tilde{\bar{F}}_{c,N}(\cdot,\cdot)$, and $\bar{\theta}_{c,N}(\cdot)$ respectively. When the algorithm is applied to solve $\mathbf{P}_f$, $\bar{F}_N(\cdot,\cdot) \equiv \bar{F}_{f,N}(\cdot,\cdot)$, $\tilde{\bar{F}}_N(\cdot,\cdot) \equiv \tilde{\bar{F}}_{f,N}(\cdot,\cdot)$, and $\bar{\theta}_N(\cdot) \equiv \bar{\theta}_{f,N}(\cdot)$.

Algorithm 5.3.

Parameters: $a, b, s \in (0,1)$, $w, \varepsilon > 0$ and $N_0 \in \mathbf{N}$.

Data. $h_0 \in H_{ad,N_0}$.

Step 0. Set $i = 0$.

Step 1.

Inner-Step 0. Set $N = N_i$, $\xi_i = T_N^{-1} W_N(h_i)$.

Inner-Step 1. Compute

$$\bar{\theta}_N(\xi_i) = \min_{\xi' \in \bar{H}_{ad,N}} \tilde{\bar{F}}_N(\xi_i,\xi'), \tag{5.16a}$$

$$d_i = \arg \min_{\xi' \in \bar{H}_{ad,N}} \tilde{\bar{F}}_N(\xi_i,\xi'), \tag{5.16b}$$

Inner-Step 2. If $\bar{\theta}_N(\xi_i) = 0$, set $\xi_* = \xi_i$ and go to Step 3. Else, compute the *step size*

$$\lambda_i \triangleq \arg \max_{k \in \mathbf{N}} \{ b^k \mid \bar{F}_N(\xi_i + b^k d_i, \xi_i) \le b^k a \bar{\theta}_N(\xi_i) \}. \tag{5.16c}$$

Inner-Step 3. Set

$$\xi_* = \xi_i + \lambda_i d_i. \tag{5.16d}$$

Step 2. If

$$\bar{F}_N(\xi_i,\xi_*) \le -\varepsilon \Delta(N)^s, \tag{5.16e}$$

go to Step 3. Else, replace N_i by $2N_i$ and go to Inner-Step 0.

Step 3. Set $h_{i+1} = W_{N_i}^{-1} T_{N_i} \xi_*$, $N_{i+1} = N_i$, replace i by $i+1$, and go to Step 1. □

The following theorem on the convergence properties of Algorithm 5.1 can be deduced from Theorem 5.15 in [16].

Theorem 5.4. Suppose that Algorithm 5.1 has constructed an infinite sequence $\{ h_i \}_{i=0}^{\infty}$ that has an accumulation point $\hat{h}$. Then $\theta(\hat{h}) = 0$. □

6. NUMERICAL RESULTS

We will illustrate the use of consistent approximations and Algorithm 5.3 in solving a particular problem of the kind $\mathbf{P}_f$, that is, a fixed beam design. In our example, we assumed that $E = 10^7$ psi, $L = 50$ in, $b = 5$ in, $K = 0$ (we neglected the weight of the beam), $\alpha = 1.0$ in, $\beta = 5.0$ in, and $\gamma = 0.15$. We imposed continuum constraints on the maximum normal stress, on the maximum shear, and on the deflection, as follows

$$|\sigma_{f,max}(h, x)| \leq 30{,}000 \text{ psi} \quad , \quad \forall x \in [0, L] \; , \tag{6.1a}$$

$$|\tau_{f,max}(h, x)| \leq 15{,}000 \text{ psi} \quad , \quad \forall x \in [0, L] \; , \tag{6.1b}$$

$$|y_f(h, x)| \leq 0.1 \text{ in} \quad , \qquad \forall x \in [0, L] \; . \tag{6.1c}$$

The cost function was proportional to the total mass of the beam,

$$f_f(h) = \int_0^L h(x) dx \; . \tag{6.1d}$$

The load applied to the beam was

$$l(x) = \begin{cases} -1500 \;\; psi, & \text{if } x \in [20, 30], \\ 0, & \text{otherwise}, \end{cases} \tag{6.1e}$$

which clearly satisfies Assumption 3.1*(a)*. The initial discretization was set to $N = 8$ points, and the initial $h(\cdot)$ was constant, with value 2.85 in (see Figure 6.1(a)). This initial design, whose cost is 142.5, corresponds to the uniform beam of least mass which satisfies the constraints (for this $h(\cdot)$ the constraint on the displacement is active and the other two are slack).

In Figure 6.1(b), we find the beam obtained after 16 inner-steps of Algorithm 5.1. The discretization level at the end of the 16-th inner-step was $N = 128$. The corresponding cost was 124.05, about 87% of the initial cost. For the final design, the constraint on the deflection of the beam was active, and the constraints on the maximum normal stress and on the maximum shear stress were slack.

In Figure 6.2 we present the computed cost at each iteration as a percentage of the initial cost, 142.5, and the computed value of the optimality function θ_N, at each iteration. The number of discretization points used at each iteration is also shown in Figure 6.2. As our analysis indicates, for each given discretization the optimality function is driven to zero, but when the discretization is refined (at iterations 4, 8, 12 and 14), the value of the optimality function may decrease. However, as the algorithm progresses, the optimality function is eventually driven to zero, and therefore the computed depth functions $h_i(\cdot)$ approach a stationary point.

7 - CONCLUSION

We have shown that one can obtain consistent approximations, satisfying the axioms formulated in [16], for two classes of optimal beam design problems, involving Euler-Bernoulli cantilever and fixed beams, subject to continuum constraints, which include displacement, maximum shear stress, and maximum normal stress constraints. We have also demonstrated numerically how an algorithm first described in [17] and proposed for use with consistent approximations in [16], can be used to obtain an arbitrarily good approximation to a stationary point of these design problems.

We feel confident that consistent approximations can also be used to solve optimal design problems involving beams with one unilateral support, but the analysis involved is too extensive to include in the present paper. Finally, extensions to some design problems involving two dimensional beam models appear to be possible.

A1 - APPENDIX 1: PROOF OF LEMMA 3.2

We begin with part *(a)*. Given $h \in H_{ad}$, let h_N be the linear interpolate of h on the mesh T_N. Clearly, $h_N \in H_{ad,N}$. From (3.1a), we have that h is Lipschitz continuous with Lipschitz constant γ, and hence $\| h - h_N \|_\infty \le \gamma \Delta(N)$, which proves (3.7a).

Next we prove (3.7b). Let $h \in H_{ad}$, and $h_N \in H_{ad,N}$ be given. First, from (3.1d) it follows that

$$\frac{d}{dx}(V_c(h,x) - V_c(h_N,x)) = -K(h(x) - h_N(x)) \quad x \in [0,L], \quad V_c(h,L) - V_c(h_N,L) = 0. \quad \text{(A1.1a)}$$

Hence, integrating both sides of (A1.1a) and using Holder's inequality, we get that for all $x \in [0,L]$,

$$| V_c(h,x) - V_c(h_N,x) | \le K\sqrt{L}\, \| h - h_N \|_2 . \quad \text{(A1.1b)}$$

We will show that there exists a $C \in [K\sqrt{L}, \infty)$ such that for all $N \in \mathbb{N}$, and $h_N \in H_{ad,N}$,

$$\max_{k \in \mathbf{N+1}} | V_{c,N}(h_N, x_{N,k}) - V_c(h_N, x_{N,k}) | \le C \Delta(N) , \quad \text{(A1.1c)}$$

where $V_{c,N}(h_N, x_{N,k})$ is determined by (3.5b). Indeed, by Assumption 3.1*(a)*, $m(\cdot)$ is piecewise Lipschitz continuous. From (3.1a) and (3.1b), it follows that for any $h_N \in H_{ad,N} \subset H_{ad}$, $l(h_N, \cdot)$ is also piecewise Lipschitz continuous, and has finitely many points of discontinuity in $[0,L]$. Hence, there exists a constant C', independent of $N \in \mathbb{N}$ and of $h_N \in H_{ad,N}$, such that C' is a Lipschitz constant for $l(h_N, \cdot)$ on any subinterval of $[0,L]$ in which $l(h_N, \cdot)$ is continuous.

Consider the mesh T_N. In each mesh interval $[x_{N,k}, x_{N,k+1}]$, $k \in \mathbf{N}$, $l(h_N, \cdot)$ is either Lipschitz continuous or it has at least one point of discontinuity. There are at most finitely many mesh intervals, say $p \geq 0$, in which $l(h_N, \cdot)$ is discontinuous. Clearly, p is no larger than the number of discontinuities of $m(\cdot)$, and hence is independent of $N \in \mathbb{N}$. If we apply Euler's method to integrate (3.1d), obtaining (3.5b), the local truncation error, on each mesh interval where $l(h_N, \cdot)$ has at least one discontinuity, is bounded by $2\Delta(N)\max_{x \in [0,L]} | l(h_N, x) |$. In the intervals where $l(h_N, \cdot)$ is Lipschitz continuous, and there are at most $N - p$ of these, the local truncation error of Euler's Method is bounded by $C' \Delta(N)^2$. Therefore, there exists a constant $C \in [KL, \infty)$ such that for any $k \in \mathbf{N+1}$,

$$| V_{c,N}(h_N, x_{N,k}) - V_c(h_N, x_{N,k}) | \leq C' \Delta(N)^2 (N - p) + 2p \max_{x \in [0,L]} | l(h_N, x) | \Delta(N) \leq C \Delta(N) \; , \tag{A1.1d}$$

which proves (A1.1c). Inequality (3.7b) is a direct consequence of the triangle inequality, and (A1.1b,c). The proofs of inequalities (3.7c-d) are similar and hence omitted.

In view of Assumption 3.1*(c)*, (3.7e) is a direct consequence of part *(b)*, and the definitions of $\phi_c^j(\cdot, \cdot)$ and $\phi_{c,N}^j(\cdot, \cdot)$ in (3.1i) and (3.6b) respectively. Inequality (3.7f) follows from (3.7e), (3.1g), and (3.5e). Indeed, if we let $R \triangleq \max_{j \in \mathbf{q}} \max_{x \in [0,L]} r^j(x)$ and make use of (3.7e), it follows from (3.1g) and (3.5e) that

$$\psi_c(h) - \psi_{c,N}(h_N) \leq \max_{j \in \mathbf{q}} \max_{x \in [0,L]} \{ \phi_c^j(h, x) - r^j(x) - \phi_{c,N}^j(h_N, x) + (1 - \Delta(N)^{1/2}) r^j(x) \}$$

$$\leq C [\Delta(N) + \| h - h_N \|_2] + R \Delta(N)^{1/2} \leq C [\Delta(N)^{1/2} + \| h - h_N \|_2] \; , \tag{A1.2a}$$

where $\phi_{c,N}^j(h_N, \cdot) : [0, L] \to \mathbb{R}$ is the linear interpolate of $\{ \phi_{c,N}^j(h_N, x_{N,k}) \}_{k=0}^{N}$ on the mesh T_N.

In a similar way, an upper bound for $\psi_{c,N}(h_N) - \psi_c(h)$ can be obtained, namely

$$\psi_{c,N}(h_N) - \psi_c(h) \leq C [\Delta^{1/2} + \| h - h_N \|_2]. \tag{A1.2b}$$

which together with (A1.2a) implies (3.7f).

Finally, we prove (3.7g). First we note that because of Assumption 3.1 *(a)*, and because all $h \in H_{ad}$ take values between $[\alpha, \beta]$, the solutions of the differential equations (3.1d-f) are Lipschitz continuous functions on $[0, L]$. In fact, we can find a common Lipschitz constant for $V_c(h, \cdot)$, $M_c(h, \cdot)$ and $y_c(h, \cdot)$, for all $h \in H_{ad}$. In view of Assumption 3.1*(c)* we get that there exists a constant C such that for all $x, x' \in [0, L]$

$$| \phi_c^0(h, x) - \phi_c^0(h, x') | \leq C | x - x' | \; . \tag{A1.3a}$$

Hence,

$$|f_c(h) - f_{c,N}(h_N)| \leq \sum_{j=1}^{N} \int_{x_{N,j}}^{x_{N,j+1}} \{ |\phi_c^0(h, x) - \phi_c^0(h, x_{N,j})| + |\phi_c^0(h, x_{N,j}) - \phi_{c,N}^0(h_N, x_{N,j})| \} dx \ ,$$

(A1.3b)

which, in view of (A1.3a) and (3.7e), implies that there exists a constant C such that

$$|f_c(h) - f_{c,N}(h_N)| \leq C [\Delta(N) + \|h - h_N\|_2] \ . \tag{A1.3c}$$

□

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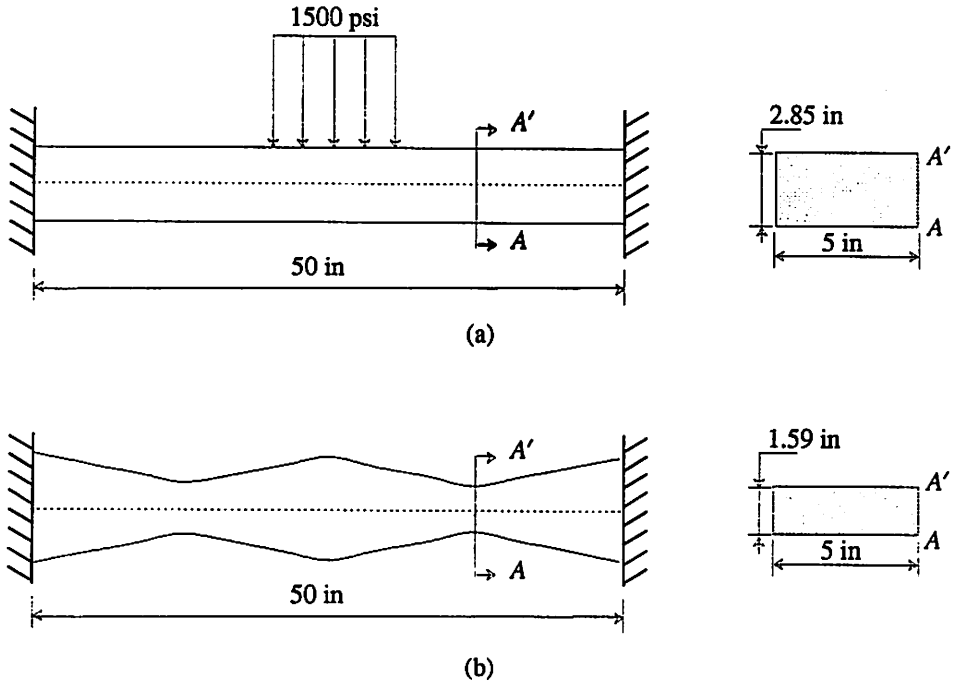

Fig. 6.1. *(a)* Initial Design; *(b)* Final Design.

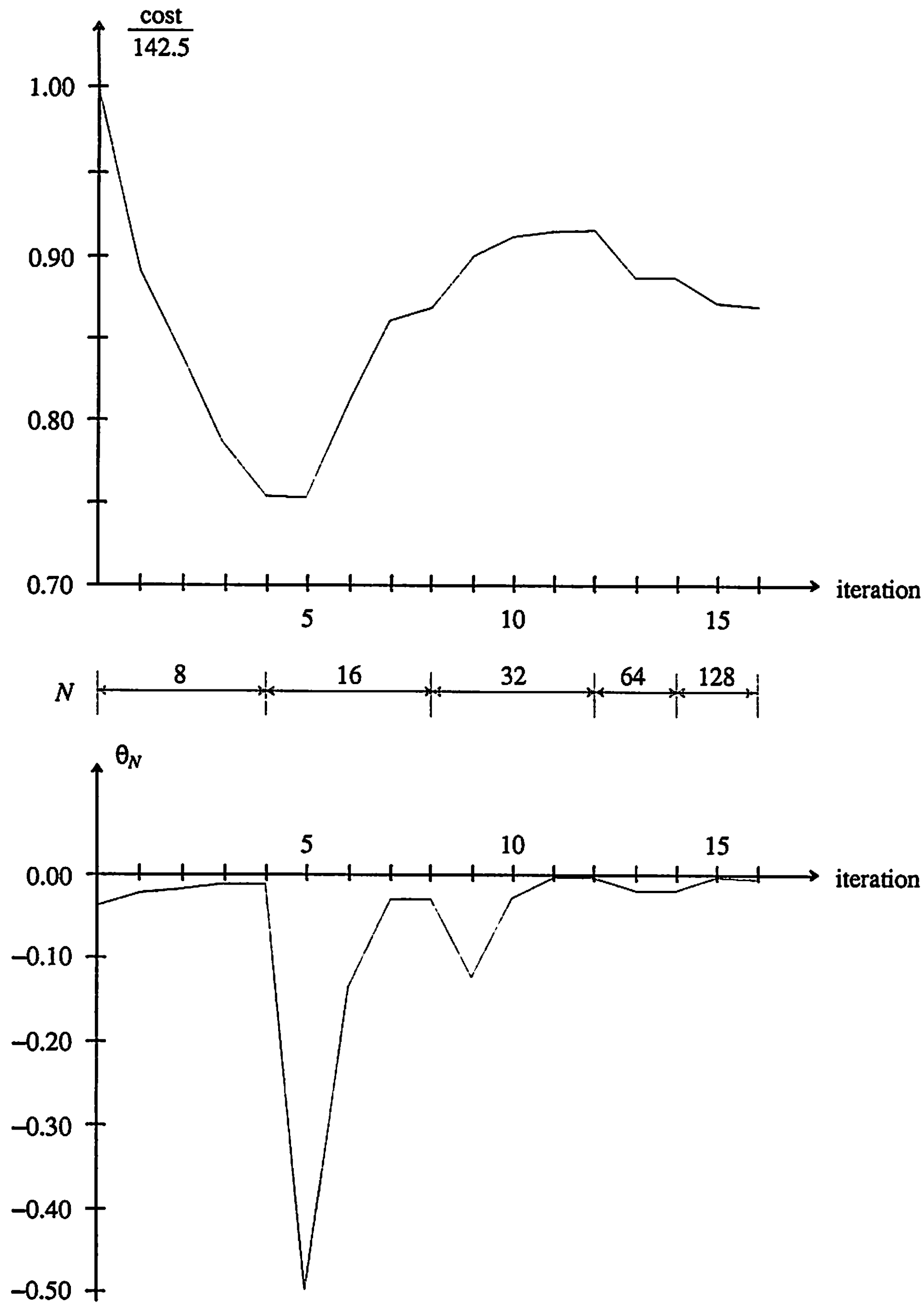

Fig. 6.2. Computed cost and computed optimality function