A UNIFIED FRAMEWORK FOR SYNCHRONIZATION AND CONTROL OF DYNAMICAL SYSTEMS

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Memorandum No. UCB/ERL M94/28

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A Unified Framework for Synchronization and Control of Dynamical Systems

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Abstract

In this paper, we give a framework for synchronization of dynamical systems which unifies many results in synchronization and control of dynamical systems, in particular chaotic systems. We define concepts such as asymptotical synchronization, partial synchronization and synchronization error bounds. We show how asymptotical synchronization is related to asymptotical stability. The main tool we use to prove asymptotical stability and synchronization is Lyapunov stability theory. We illustrate how many previous results on synchronization and control of chaotic systems can be derived from this framework. We will also give a characterization of robustness of synchronization and show that master-slave asymptotical synchronization in Chua's oscillator is robust.

1 Introduction

Recently, there has been much interest in the dynamics of coupled chaotic circuits and systems. Research has been carried out in areas such as controlling unstable periodic orbits [Chen and Dong, 1993b; Chen, 1993; Chen and Dong, 1993a; Kocarev *et al.*, 1993], stabilizing aperiodic orbits in chaotic systems [Pyragas, 1993], mutual coupling between two chaotic circuits [Chua *et al.*, 1993b; Rul'kov *et al.*, 1992], master-slave synchronization of chaotic systems [Pecora and Carroll, 1990; Pecora and Carroll, 1991; Carroll and Pecora, 1991; He and Vaidya, 1992], and implementations of novel communication systems [Oppenheim *et al.*, 1992; Kocarev *et al.*, 1992; Parlitz *et al.*, 1992; Cuomo and Oppenheim, 1993b; Cuomo and Oppenheim, 1993a; Halle *et al.*, 1993; Wu and Chua, 1993; Dedieu *et al.*, 1993]. In this paper we give a unified framework of synchronization which puts these results under one umbrella. We show how asymptotical synchronization is related to asymptotical stability. We show how asymptotical stable systems can be cascaded and connected while preserving asymptotical stability. We define concepts such as synchronization error and partial synchronization. The main tool we use to prove asymptotical stability and synchronization is Lyapunov stability theory. This is in contrast to other methods of proving synchronization which require the computation of numerical quantities such as conditional Lyapunov exponents. We will use Chua's oscillator as our prototypical chaotic system in several examples to illustrate ideas.

Finally we give conditions under which asymptotically synchronization is robust. We show that master-slave asymptotical synchronization in Chua's oscillator is robust.

The outline of this paper is as follows. In Sec. 2 asymptotical stability of systems is defined and Lyapunov's direct method is used to prove asymptotical stability. In Sec. 3 we show two ways in which asymptotically stable systems can be connected while preserving asymptotical stability. In Sec. 4 concepts about synchronization are defined and their relations to stability are given, along with several examples illustrating ideas. In Sec. 5 the question of robustness of synchronization is addressed. In Sec. 6 two more examples are discussed in detail to illustrate several ideas.

2 Asymptotic Stability of Dynamical Systems

In this paper we assume that the right hand side of our systems of ordinary differential equations (ODE) is continuous. Furthermore, we assume that all systems of ODE's under consideration have existence and uniqueness of solutions for all time (thus we can speak of a flow of a dynamical system). We use the Euclidean norm on vectors in $\mathbb{R}^n$, although most of the results can be stated for other norms in $\mathbb{R}^n$ as well. The norm on $n \times n$ matrices will be the one induced by the norm in $\mathbb{R}^n$.

We show that asymptotic synchronization of systems can be achieved if some related systems are asymptotically stable. We use the following definitions of asymptotically stability of a system [Yoshizawa, 1966]:

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

We denote by $\mathbf{x}(t, \mathbf{x}_0, t_0)$ the (unique) solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. Let S_α be the set of $\mathbf{x}$ such that $\|\mathbf{x}\| \leq \alpha$. In the following, H^* will be a positive real number. We denote $C(\mathbb{R}, D)$ as the set of continuous functions $f : \mathbb{R} \to D$ with values in D.

Definition 1 *The system (1) is* uniform-stable *with respect to H^* if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $t \geq t_0$*

$$\|\mathbf{x}(t, \mathbf{x}_0, t_0) - \mathbf{x}(t, \mathbf{x}_1, t_0)\| < \epsilon$$

whenever

$$\|\mathbf{x}_1 - \mathbf{x}_0\| < \delta(\epsilon)$$

and $\mathbf{x}_0 \in S_{H^*}$, $\mathbf{x}_1 \in S_{H^*}$.

Definition 2 *The system (1) is* quasi-uniform-asymptotically stable *with respect to H^*, if there exists $\delta > 0$ such that such that for every $\epsilon > 0$, there exists $T(\epsilon) \geq 0$ such that*

$$\|\mathbf{x}(t, \mathbf{x}_0, t_0) - \mathbf{x}(t, \mathbf{x}_1, t_0)\| \leq \epsilon$$

for all $t \geq t_0 + T(\epsilon)$ whenever

$$\|\mathbf{x}_1 - \mathbf{x}_0\| < \delta$$

and $\mathbf{x}_0 \in S_{H^*}$, $\mathbf{x}_1 \in S_{H^*}$.

Definition 3 *The system (1) is* uniform-asymptotically stable *with respect to H^*, if it is uniform-stable and quasi-uniform-asymptotically stable with respect to H^*.*

Definition 4 *Let $\mathcal{D} \subset C(\mathbb{R}, \mathbb{R}^n)$ be a set of continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$. The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \eta(t), t)$ is uniform-asymptotically stable with respect to H^*,* i-uniformly *with respect to all $\eta(t) \in \mathcal{D}$ if it is uniform-asymptotically stable with respect to H^* for all $\eta(t) \in \mathcal{D}$ and the constants δ and $T(\epsilon)$ in definitions 1 and 2 can be chosen to be independent of $\eta(t)$.*

Note that these definitions do not specify whether $\mathbf{x}(t, \mathbf{x}_0, t_0)$ is bounded or not. We say the system (1) is uniform-(asymptotically) stable if it is uniform-(asymptotically) stable with respect to all $H^* > 0$. We will next consider how to show that a system is asymptotically stable.

Definition 5 ([Vidyasagar, 1978]) *A function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to belong to class K if*

1. *$\alpha(\cdot)$ is continuous and nondecreasing,*
2. *$\alpha(0) = 0$,*
3. *$\alpha(p) > 0$ whenever $p > 0$.*

A basic technique for proving asymptotical stability is by Lyapunov's direct method. We assume that all Lyapunov functions we consider are continuous. For a Lyapunov function $V(t, \mathbf{x}, \mathbf{y})$, the generalized derivative along the trajectories of system (2) is defined as:

$$\dot{V}(t, \mathbf{x}, \mathbf{y}) = \lim \sup_{h \to 0^+} \frac{1}{h} \left[V(t + h, \mathbf{x} + h\mathbf{f}(\mathbf{x}, t), \mathbf{y} + h\tilde{\mathbf{f}}(\mathbf{y}, t)) - V(t, \mathbf{x}, \mathbf{y}) \right]$$

Theorem 1 *Consider the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \dot{\mathbf{y}} = \tilde{\mathbf{f}}(\mathbf{y}, t) \tag{2}$$

Suppose that D_1 and D_2 are open sets such that if $\mathbf{x}_0 \in S_{H^*}$, $\mathbf{y}_0 \in S_{H^*}$ then $\mathbf{x}(t, \mathbf{x}_0, t_0) \in D_1$, $\mathbf{y}(t, \mathbf{y}_0, t_0) \in D_2$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t, \mathbf{x}, \mathbf{y})$, locally Lipschitzian in $\mathbf{x}$ and $\mathbf{y}$, exists on $\mathbb{R} \times D_1 \times D_2$ such that for all $t \geq t_0$, $\mathbf{x} \in D_1$, $\mathbf{y} \in D_2$,

$$a(\|\mathbf{x} - \mathbf{y}\|) \leq V(t, \mathbf{x}, \mathbf{y}) \leq b(\|\mathbf{x} - \mathbf{y}\|)$$

where $a(\cdot)$ and $b(\cdot)$ are functions in class K. Suppose that there exists $\mu > 0$ such that for all $t \geq t_0$ and $\|\mathbf{x} - \mathbf{y}\| \geq \mu$,

$$\dot{V}(t, \mathbf{x}, \mathbf{y}) \leq -c$$

for some constant $c > 0$ where $\dot{V}(t, \mathbf{x}, \mathbf{y})$ is the generalized derivative of V along the trajectories of (2).

If there exists $\delta > 0$ such that $a(\delta) > b(\mu)$, then for each $\mathbf{x}_0 \in S_{H^*}$ and $\mathbf{y}_0 \in S_{H^*}$ there exists $t_1 \geq t_0$ such that for all $t \geq t_1$,

$$\|\mathbf{x}(t, \mathbf{x}_0, t_0) - \mathbf{y}(t, \mathbf{y}_0, t_0)\| \leq \delta$$

Furthermore, if $\|\mathbf{x}_0 - \mathbf{y}_0\| \leq \mu$ then

$$\|\mathbf{x}(t, \mathbf{x}_0, t_0) - \mathbf{y}(t, \mathbf{y}_0, t_0)\| \leq \delta$$

for all $t \geq t_0$.

Proof If $V(t,\mathbf{x}(t),\mathbf{y}(t)) > b(\mu)$ for all $t \geq t_0$ then $b(\|\mathbf{x}(t)-\mathbf{y}(t)\|) > b(\mu)$ and thus $\|\mathbf{x}(t)-\mathbf{y}(t)\| > \mu$ for all $t \geq t_0$. This implies that $\dot{V}(t,\mathbf{x},\mathbf{y}) \leq -c < 0$ for all $t \geq t_0$ which contradicts the fact that $V(t,\mathbf{x}(t),\mathbf{y}(t)) \geq 0$. Thus there exists $t_1 \geq t_0$ such that $V(t_1,\mathbf{x}(t_1),\mathbf{y}(t_1)) \leq b(\mu)$. Now we show that $V(t,\mathbf{x}(t),\mathbf{y}(t)) \leq b(\mu)$ for all $t \geq t_1$. By way of contradiction, suppose that there exists $t_2 > t_1$ such that $V(t_2,\mathbf{x}(t_2),\mathbf{y}(t_2)) > b(\mu)$. Then there exists $\epsilon > 0$ such that $V(t_2,\mathbf{x}(t_2),\mathbf{y}(t_2)) > b(\mu)+\epsilon$. By continuity of $V(t,\mathbf{x}(t),\mathbf{y}(t))$ with respect to t, there exists $t \in [t_1,t_2)$ such that $V(t,\mathbf{x}(t),\mathbf{y}(t)) = b(\mu)+\epsilon$. Let

$$t_3 = \sup\{t \in [t_1,t_2) : V(t,\mathbf{x}(t),\mathbf{y}(t)) = b(\mu)+\epsilon\}$$

Then $V(t_3,\mathbf{x}(t_3),\mathbf{y}(t_3)) = b(\mu)+\epsilon \leq b(\|\mathbf{x}(t_3)-\mathbf{y}(t_3)\|)$. Therefore $\|\mathbf{x}(t_3)-\mathbf{y}(t_3)\| > \mu$ and $\dot{V} \leq -c < 0$ at t_3. So there exists t_4 such that $t_3 < t_4 < t_2$ and

$$V(t_4,\mathbf{x}(t_4),\mathbf{y}(t_4)) < b(\mu)+\epsilon$$

Therefore there exists $t_5 \in (t_4,t_2)$ such that $V(t_5,\mathbf{x}(t_5),\mathbf{y}(t_5)) = b(\mu)+\epsilon$ contradicting the fact that t_3 is the largest such t.

Thus we have $a(\|\mathbf{x}(t)-\mathbf{y}(t)\|) \leq V(t,\mathbf{x}(t),\mathbf{y}(t)) \leq b(\mu) < a(\delta)$ for all $t \geq t_1$. Therefore $\|\mathbf{x}(t)-\mathbf{y}(t)\| \leq \delta$ for all $t \geq t_1$. Furthermore, if $\|\mathbf{x}_0-\mathbf{y}_0\| \leq \mu$ then $V(t_0,\mathbf{x}(t_0),\mathbf{y}(t_0)) \leq b(\mu)$ and thus for all $t \geq t_0, V(t,\mathbf{x}(t),\mathbf{y}(t)) \leq b(\mu)$ which implies that $\|\mathbf{x}(t)-\mathbf{y}(t)\| \leq \delta$ for all $t \geq t_0$. ■

Theorem 2 *Suppose that D_1 and D_2 are open sets such that if $\mathbf{x}_0 \in S_{H^*}$, $\mathbf{y}_0 \in S_{H^*}$ then $\mathbf{x}(t,\mathbf{x}_0,t_0) \in D_1$, $\mathbf{y}(t,\mathbf{y}_0,t_0) \in D_2$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t,\mathbf{x},\mathbf{y})$, locally Lipschitzian in $\mathbf{x}$ and $\mathbf{y}$, exists on $\mathbb{R} \times D_1 \times D_2$ such that for all $t \geq t_0$, $\mathbf{x} \in D_1$, $\mathbf{y} \in D_2$,*

$$a(\|\mathbf{x}-\mathbf{y}\|) \leq V(t,\mathbf{x},\mathbf{y}) \leq b(\|\mathbf{x}-\mathbf{y}\|)$$

where $a(\cdot)$ and $b(\cdot)$ are in class K, and for all $t \geq t_0$,

$$\dot{V}(t,\mathbf{x},\mathbf{y}) \leq -c(\|\mathbf{x}-\mathbf{y}\|)$$

for some function $c(\cdot)$ in class K where $\dot{V}(t,\mathbf{x},\mathbf{y})$ is the generalized derivative of V along the trajectories of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t), \quad \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y},t)$$

Then the system (1) is uniform-asymptotically stable *with respect to H^*.*

Proof Note that for each $H^* > \delta > 0$, $0 < a(\delta) \leq b(\delta)$, so there exists $\mu > 0$ such that $a(\delta) > b(\mu)$. Therefore $\dot{V}(t,\mathbf{x},\mathbf{y}) \leq -c(\mu) < 0$ for all $\|\mathbf{x}-\mathbf{y}\| \geq \mu$, so given $\mathbf{x}_0 \in S_{H^*}$ and $\mathbf{y}_0 \in S_{H^*}$, for each $\delta > 0$ there exists by theorem 1 (for the case $\mathbf{f} = \tilde{\mathbf{f}}$) a time $t_1 > t_0$ such that

$$\|\mathbf{x}(t,\mathbf{x}_0,t_0) - \mathbf{y}(t,\mathbf{y}_0,t_0)\| \leq \delta$$

for all $t \geq t_1$. Furthermore, if $\|\mathbf{x}_0-\mathbf{y}_0\| \leq \mu$ then

$$\|\mathbf{x}(t,\mathbf{x}_0,t_0) - \mathbf{y}(t,\mathbf{y}_0,t_0)\| \leq \delta$$

for all $t \geq t_0$. ■

Example: In [Cuomo and Oppenheim, 1993b; Cuomo and Oppenheim, 1993a] a similar Lyapunov function is used to show that a system derived from the Lorenz system is uniform-asymptotically stable.

Theorem 3 *For positive β and σ, $-3 < \mu < 1$ the following system derived from the Lorenz system is uniform-asymptotically stable i-uniformly with respect to all continuous $u(t)$ and $\eta_i(t)$.*

$$\begin{aligned}
\dot{x} &= \sigma(y-x) + \eta_1(t) \\
\dot{y} &= \mu x - y + u(t)(\rho - \mu - z) + \eta_2(t) \\
\dot{z} &= -\beta z + u(t)y + \eta_3(t)
\end{aligned}$$

Proof Define the Lyapunov function $V = \frac{1}{2\sigma}(x-x')^2 + \frac{1}{2}(y-y')^2 + \frac{1}{2}(z-z')^2$, where the primed variables are from an identical system. Then the derivative of V along the trajectories of the Lorenz systems is

$$\begin{aligned}
\dot{V} &= (x-x')((y-x)-(y'-x')) + (y-y')\left[-(y-y') + \mu(x-x') - u(t)(z-z')\right] \\
&\quad + (z-z')\left[-\beta(z-z') + u(t)(y-y')\right] \\
&= -(\tfrac{1+\mu}{2}(x-x') - (y-y'))^2 - \left(1 - \tfrac{(1+\mu)^2}{4}\right)(x-x')^2 - \beta(z-z')^2
\end{aligned}$$

Since $\left(1 - \frac{(1+\mu)^2}{4}\right) > 0$, the conditions of theorem 2 are satisfied. As V and $\dot{V}$ do not depend on $u(t)$ and $\eta_i(t)$, the asymptotical stability is i-uniform with respect to them. ■

It is well known that for the linear case $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \eta(t)$, uniform-asymptotical stability, which is equivalent to all the eigenvalues of $\mathbf{A}$ being in the open left half plane, implies the existence of a symmetric positive definite matrix $\mathbf{D}$ such that the Lyapunov function $V(\mathbf{x},\mathbf{y}) = (\mathbf{x}-\mathbf{y})^T\mathbf{D}(\mathbf{x}-\mathbf{y})$ satisfies the conditions of theorem 2. As we will see later, existence of Lyapunov functions of the form $(\mathbf{x}-\mathbf{y})^T\mathbf{D}(\mathbf{x}-\mathbf{y})$ allow us to show some results concerning the coupling of systems and robustness of synchronization.

Next we consider for what kind of systems a Lyapunov function can be found which allow us to prove asymptotical stability.

Definition 6 *A function $\mathbf{f} : D \to \mathbb{R}^n$ is increasing in some convex set $D \subset \mathbb{R}^n$ if for all $\mathbf{x}, \mathbf{x}' \in D$,*

$$(\mathbf{x}-\mathbf{x}')^T(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')) \geq 0 \tag{3}$$

A function $\mathbf{f} : D \to \mathbb{R}^n$ is strictly increasing in some convex set $D \subset \mathbb{R}^n$ if for all $\mathbf{x}, \mathbf{x}' \in D$, $\mathbf{x} \neq \mathbf{x}'$

$$(\mathbf{x}-\mathbf{x}')^T(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')) > 0 \tag{4}$$

A function $\mathbf{f} : D \to \mathbb{R}^n$ is uniformly increasing in some convex set $D \subset \mathbb{R}^n$ if there exists $\gamma > 0$ such that for all $\mathbf{x}, \mathbf{x}' \in D$

$$(\mathbf{x}-\mathbf{x}')^T(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')) \geq \gamma\|\mathbf{x}-\mathbf{x}'\|^2 \tag{5}$$

The following theorem characterizes these definitions for C^1 functions, where $\mathbf{Df}(\mathbf{x})$ denotes the Jacobian of $\mathbf{f}$ at $\mathbf{x}$.

Theorem 4 ([Chua and Green, 1976]) *A C^1 function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is*

(i) increasing *in $\mathbb{R}^n$ if and only if $\mathbf{Df}(\mathbf{x})$ is positive semi-definite for all $\mathbf{x} \in \mathbb{R}^n$.*

(ii) strictly increasing *in $\mathbb{R}^n$ if and only if $\mathbf{Df}(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathbb{R}^n$.*

(iii) uniformly increasing *in $\mathbb{R}^n$ if and only if for some $\lambda > 0$ $(\mathbf{Df}(\mathbf{x}) - \lambda\mathbf{I})$ is positive definite for all $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{I}$ is the $n \times n$ identity matrix.*

In [Chua and Green, 1976] a general class of systems were given for which an explicit Lyapunov function V can be found which satisfies the conditions of theorem 2. The system has the following general form:

$$\dot{\mathbf{z}} = -\mathbf{g}(\mathbf{h}(\mathbf{z}), \mathbf{u}(t)) \tag{6}$$

where $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{h}(\mathbf{z}) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^k$. The function $\mathbf{u}(t)$ can be considered as the input to the otherwise autonomous system.

Theorem 5 *Consider the system (6). Let B be some set in $\mathbb{R}^k$. If $\mathbf{g}(\mathbf{z}, \mathbf{u})$ is a uniformly increasing function from $\mathbb{R}^n \to \mathbb{R}^n$ for all fixed $\mathbf{u} \in B$ such that the constant γ in definition 6 does not depend on $\mathbf{u}$ and $\mathbf{h}$ is given by:*

$$\mathbf{h}(\mathbf{z}) = \mathbf{\Gamma}\mathbf{z} \tag{7}$$

where $\mathbf{\Gamma}$ is a symmetric and positive definite matrix, then the system is uniform-asymptotically stable i-uniformly with respect to all continuous input $\mathbf{u}(t)$ with values in B.

Proof Construct the Lyapunov function

$$V(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^T \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})$$

Note that $\|\mathbf{\Gamma}^{-1}\|^{-1}\|\mathbf{x} - \mathbf{y}\|^2 \leq V(\mathbf{x}, \mathbf{y}) \leq \|\mathbf{\Gamma}\| \quad \|\mathbf{x} - \mathbf{y}\|^2$. Its derivative along the trajectories of the system

$$\begin{aligned} \dot{\mathbf{x}} &= -\mathbf{g}(\mathbf{h}(\mathbf{x}), \mathbf{u}(t)) \\ \dot{\mathbf{y}} &= -\mathbf{g}(\mathbf{h}(\mathbf{y}), \mathbf{u}(t)) \end{aligned} \tag{8}$$

is

$$\begin{aligned} \dot{V}(\mathbf{x}, \mathbf{y}) &= 2(\mathbf{x} - \mathbf{y})^T \mathbf{\Gamma}(\dot{\mathbf{x}} - \dot{\mathbf{y}}) \\ &= -2(\mathbf{x} - \mathbf{y})^T \mathbf{\Gamma}(\mathbf{g}(\mathbf{\Gamma}\mathbf{x}, \mathbf{u}(t)) - \mathbf{g}(\mathbf{\Gamma}\mathbf{y}, \mathbf{u}(t))) \\ &\leq -2\gamma\|\mathbf{\Gamma}\mathbf{x} - \mathbf{\Gamma}\mathbf{y}\|^2 = -2\gamma\|\mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\|^2 \end{aligned} \tag{9}$$

So the conditions of theorem 2 are satisfied. As V and $\dot{V}$ do not depend on $\mathbf{u}(t)$, the asymptotical stability is i-uniform with respect to all $\mathbf{u}(t)$. ■

Theorem 6 *Consider the system (6). If $\mathbf{h}$ is a uniformly increasing function from $\mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{g}$ is given by:*

$$\mathbf{g}(\mathbf{z}, \mathbf{u}) = \mathbf{G}_z\mathbf{z} + \mathbf{G}_u(\mathbf{u}) \tag{10}$$

where $\mathbf{G}_z$ is a symmetric and positive definite matrix, and $\mathbf{G}_u(\cdot)$ is continuous, then the system is uniform-asymptotically stable i-uniformly with respect to all continuous input $\mathbf{u}(t)$.

Proof Construct the Lyapunov function

$$V(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^T \mathbf{G}_z^{-1}(\mathbf{x} - \mathbf{y})$$

Its derivative along the trajectories of system (8) is

$$\begin{aligned} \dot{V}(\mathbf{x}, \mathbf{y}) &= 2(\mathbf{x} - \mathbf{y})^T \mathbf{G}_z^{-1}(\dot{\mathbf{x}} - \dot{\mathbf{y}}) \\ &= -2(\mathbf{x} - \mathbf{y})^T \mathbf{G}_z^{-1}(\mathbf{G}_z\mathbf{h}(\mathbf{x}) - \mathbf{G}_z\mathbf{h}(\mathbf{y})) \\ &= -2(\mathbf{x} - \mathbf{y})^T(\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y})) \\ &\leq -2\gamma\|\mathbf{x} - \mathbf{y}\|^2 \end{aligned} \tag{11}$$

and the conditions of theorem 2 are satisfied. As V and $\dot{V}$ do not depend on $\mathbf{u}(t)$, i-uniformity follows. ∎

Chua's oscillator [Chua *et al.*, 1993a] is a simple electronic circuit which for certain values of the parameters becomes chaotic. The circuit has the following state equations:

$$\begin{array}{rcl} \frac{dv_1}{dt} & = & \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] \\ \frac{dv_2}{dt} & = & \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} & = & -\frac{1}{L}(v_2 + R_0 i_3) \end{array} \tag{12}$$

where $G = \frac{1}{R}$ and

$$f(v_1) \quad = \quad G_b v_1 + \frac{1}{2}(G_a - G_b)\{|v_1 + E| - |v_1 - E|\} \tag{13}$$

There exists a set of parameter values where the linear components are strictly passive (R, R_0, C_1, C_2, $L > 0$) and the nonlinear element is active ($G_a < G_b < 0$) such that the circuit becomes chaotic.

However, if the nonlinear element is strictly increasing, then the circuit is uniform-asymptotically stable:

Theorem 7 *The system*

$$\begin{array}{rcl} \frac{dv_1}{dt} & = & \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] + \eta_1(t) \\ \frac{dv_2}{dt} & = & \frac{1}{C_2}[G(v_1 - v_2) + i_3] + \eta_2(t) \\ \frac{di_3}{dt} & = & -\frac{1}{L}(v_2 + R_0 i_3) + \eta_3(t) \end{array} \tag{14}$$

is uniform-asymptotically stable i-uniformly with respect to all continuous $\eta_i(t)$*'s when* R, R_0, C_1, C_2, L, G_a, $G_b > 0$.

Proof We intend to use theorem 6. Equation (14) can be written as

$$\begin{pmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_3}{dt} \end{pmatrix} = \begin{pmatrix} \frac{1}{C_1} & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{1}{L} \end{pmatrix} \left[\begin{pmatrix} -\frac{1}{R} & \frac{1}{R} & 0 \\ \frac{1}{R} & -\frac{1}{R} & 1 \\ 0 & -1 & -R_0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ i_3 \end{pmatrix} + \begin{pmatrix} -f(v_1) \\ 0 \\ 0 \end{pmatrix} \right] + \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \eta_3(t) \end{pmatrix} \tag{15}$$

Thus we have

$$\mathbf{G}_z = \begin{pmatrix} \frac{1}{C_1} & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{1}{L} \end{pmatrix}$$

$$\mathbf{G}_u(\mathbf{u}) = \begin{pmatrix} -\eta_1(t) \\ -\eta_2(t) \\ -\eta_3(t) \end{pmatrix}$$

and

$$\mathbf{h}(\mathbf{z}) = \begin{pmatrix} \frac{1}{R} & -\frac{1}{R} & 0 \\ -\frac{1}{R} & \frac{1}{R} & -1 \\ 0 & 1 & R_0 \end{pmatrix} \mathbf{z} + \begin{pmatrix} f(z_1) \\ 0 \\ 0 \end{pmatrix}$$

Now we show that $\mathbf{h}$ is uniformly increasing. First note that the number $\lambda = \lambda(x_1, x_1') = \frac{f(x_1)-f(x_1')}{x_1-x_1'}$ which depends on x_1 and x_1' satisfies[1] $\min(G_a, G_b) \leq \lambda \leq \max(G_a, G_b)$.

$$\begin{aligned}(\mathbf{x}-\mathbf{x}')^T(\mathbf{h}(\mathbf{x})-\mathbf{h}(\mathbf{x}')) &= (\mathbf{x}-\mathbf{x}')^T\left[\begin{pmatrix} \frac{1}{R} & -\frac{1}{R} & 0 \\ -\frac{1}{R} & \frac{1}{R} & -1 \\ 0 & 1 & R_0 \end{pmatrix}(\mathbf{x}-\mathbf{x}') + \begin{pmatrix} (f(x_1)-f(x_1')) \\ 0 \\ 0 \end{pmatrix}\right] \\ &= \frac{1}{R}((x_2-x_2')-(x_1-x_1'))^2 + \lambda(x_1-x_1')^2 + R_0(x_3-x_3')^2 \end{aligned} \tag{16}$$

Since G_a,$G_b > 0$, $\lambda > 0$. Thus the conditions of theorem 6 are satisfied and we have uniform-asymptotical stability[2]. ∎

3 Connecting Asymptotically Stable Systems

In this section, we consider how two asymptotically stable systems can be connected into a bigger system without losing the property of asymptotical stability.

The first case we consider is a cascade of two systems, i.e. the coupling is in one direction.

A converse theorem to theorem 2 exists:

Theorem 8 (Converse Theorem) *Consider the system (1). If* $\mathbf{f}(\mathbf{x}, t)$ *is Lipschitz continuous in* $\mathbf{x}$ *uniformly in* t*, i.e.,*

$$\|\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)\| \leq L\|\mathbf{x} - \mathbf{y}\|$$

for some $L > 0$ *and system (1) is uniform-asymptotically stable with respect to* H^**, there exists a Lyapunov function* $V(t, \mathbf{x}, \mathbf{y})$ *such that*

$$a(\|\mathbf{x}-\mathbf{y}\|) \leq V(t, \mathbf{x}, \mathbf{y}) \leq b(\|\mathbf{x}-\mathbf{y}\|)$$

where $a(\cdot)$ *and* $b(\cdot)$ *are in class* K*,*

$$|V(t, \mathbf{x}_1, \mathbf{y}_1) - V(t, \mathbf{x}_2, \mathbf{y}_2)| \leq M(\|\mathbf{x}_1 - \mathbf{x}_2\| + \|\mathbf{y}_1 - \mathbf{y}_2\|)$$

for some constant $M > 0$ *and*

$$\dot{V}(t, \mathbf{x}, \mathbf{y}) \leq -V(t, \mathbf{x}, \mathbf{y})$$

where $\dot{V}(t, \mathbf{x}, \mathbf{y})$ *is the generalized derivative of* V *along the trajectories of*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$$

Proof See [Yoshizawa, 1966, page 109]. ∎

Consider two systems

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t), \quad \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{u}(t), t)$$

where $\mathbf{u}(\cdot)$ is considered the input to the second system. Suppose that the first system is uniform-asymptoticaly stable and the second system is uniform-asymptotically stable i-uniformly with respect to all continuous inputs $\mathbf{u}$. When we cascade the two systems by feeding the state of the first system as the input to the second system, i.e. by setting $\mathbf{u} = \mathbf{x}_1$, then the resulting system is still uniform-asymptotically stable.

[1] If $x_1 = x_1'$ then we set $\lambda = G_a$.

[2] A circuit-theoretic approach can be used to provide a simpler proof of the above theorem. It only uses the facts that the elements are strictly increasing, the capacitors and the inductor are linear and the circuit topology satisfies a fundamental topological hypothesis [Chua and Green, 1976].

Theorem 9 *Suppose that the system*

$$\dot{\mathbf{x}}_1 = \mathbf{f}_2(\mathbf{x}_1, t)$$

is uniform-asymptotically stable with respect to H^* and the system

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{u}(t), t)$$

is uniform-asymptotically stable with respect to H^* i-unformly for all continuous $\mathbf{u}(t)$. *Suppose that*

$$\|\mathbf{f}_2(\mathbf{x}_2, \mathbf{u}, t) - \mathbf{f}_2(\mathbf{y}_2, \tilde{\mathbf{u}}, t)\| \leq L(\|\mathbf{x}_2 - \mathbf{y}_2\| + \|\mathbf{u} - \tilde{\mathbf{u}}\|)$$

for some $L > 0$. We denote

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

Then the system

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1, t) \\ \mathbf{f}_2(\mathbf{x}_2, \mathbf{x}_1, t) \end{pmatrix}$$

is uniform-asymptotically stable with respect to H^*.

Proof Let $\mathbf{x}(t_0), \mathbf{y}(t_0) \in S_{H^*}$. Then the conditions on uniform-asymptotical stability are satisfied for $\mathbf{x}_1$ and $\mathbf{y}_1$. So it only remains to show them for $\mathbf{x}_2$ and $\mathbf{y}_2$. Let $\mathbf{u}(t) = \mathbf{x}_1(t)$ and $\tilde{\mathbf{u}}(t) = \mathbf{y}_1(t)$. By the converse theorem, there exists a Lyapunov function

$$a(\|\mathbf{x}_2 - \mathbf{y}_2\|) \leq V(t, \mathbf{x}_2, \mathbf{y}_2) \leq b(\|\mathbf{x}_2 - \mathbf{y}_2\|)$$

such that

$$|V(t, \mathbf{x}_2, \mathbf{y}_2) - V(t, \tilde{\mathbf{x}}_2, \tilde{\mathbf{y}}_2)| \leq M(\|\mathbf{x}_2 - \tilde{\mathbf{x}}_2\| + \|\mathbf{y}_2 - \tilde{\mathbf{y}}_2\|)$$

for some constant $M > 0$ and a and b functions in class K and

$$\dot{V}(t, \mathbf{x}_2, \mathbf{y}_2) \leq -V(t, \mathbf{x}_2, \mathbf{y}_2)$$

where $\dot{V}(t, \mathbf{x}_2, \mathbf{y}_2)$ is the generalized derivative of V along the trajectories of

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{u}(t), t), \quad \dot{\mathbf{y}}_2 = \mathbf{f}_2(\mathbf{y}_2, \mathbf{u}(t), t)$$

Then when we take the generalized derivative of V along the trajectories of

$$\dot{\mathbf{x}} = \mathbf{f}_2(\mathbf{x}, \mathbf{u}(t), t), \quad \dot{\mathbf{y}} = \mathbf{f}_2(\mathbf{y}, \tilde{\mathbf{u}}(t), t)$$

we obtain

$$\dot{V}(t, \mathbf{x}_2, \mathbf{y}_2) \leq -a(\|\mathbf{x}_2 - \mathbf{y}_2\|) + ML\|\tilde{\mathbf{u}}(t) - \mathbf{u}(t)\|$$

So by theorem 1 $\|\mathbf{x}_2(t) - \mathbf{y}_2(t)\|$ can be arbitrarily small as $t \to \infty$ if we make $\|\mathbf{x}_1(t) - \mathbf{y}_1(t)\| = \|\mathbf{u}(t) - \tilde{\mathbf{u}}(t)\|$ arbitrarily small. As the second system is uniform-asymptotically stable i-uniformly with respect to all inputs, the bounds on the Lyapunov function constructed in the converse theorem and its generalized derivative do not depend on $\mathbf{u}$ and so the proof is complete. ∎

In the second case we consider linear coupling. For asymptotically systems which are either linear or satisfy the conditions of theorems 3, 5 and 6, an appropriate Lyapunov function exists of the form $V(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^T \mathbf{D}(\mathbf{x} - \mathbf{y})$, where $\mathbf{D}$ is symmetric positive definite. Two such systems can be coupled via linear coupling with the resulting system still being asymptotically stable.

Theorem 10 *Consider the systems* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ *and* $\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, t)$ *where* $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbb{R}^n$. *Suppose that* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are two symmetric positive definite matrices such that* $V_1(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^T \mathbf{D}_1 (\mathbf{x} - \mathbf{y})$ *and* $V_2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = (\tilde{\mathbf{x}} - \tilde{\mathbf{y}})^T \mathbf{D}_2 (\tilde{\mathbf{x}} - \tilde{\mathbf{y}})$ *satisfy the conditions of theorem 2 for uniform-asymptotical stability, for the two systems respectively. Let* $\mathbf{D}$ *be a positive semi-definite matrix. Define*

$$\mathbf{D}_3 = \mathbf{D}_1^{-1}\mathbf{D}, \quad \mathbf{D}_4 = \mathbf{D}_2^{-1}\mathbf{D}$$

then the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, t) + \mathbf{D}_3(\tilde{\mathbf{x}} - \mathbf{x}) \\ \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, t) + \mathbf{D}_4(\mathbf{x} - \tilde{\mathbf{x}}) \end{aligned} \tag{17}$$

is uniform-asymptotically stable.

Proof Consider the Lyapunov function $V(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{y}, \tilde{\mathbf{y}}) = V_1(\mathbf{x}, \mathbf{y}) + V_2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. The derivative of V along the trajectories of

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, t) + \mathbf{D}_3(\tilde{\mathbf{x}} - \mathbf{x}) \\ \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, t) + \mathbf{D}_4(\mathbf{x} - \tilde{\mathbf{x}}) \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, t) + \mathbf{D}_3(\tilde{\mathbf{y}} - \mathbf{y}) \\ \dot{\tilde{\mathbf{y}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{y}}, t) + \mathbf{D}_4(\mathbf{y} - \tilde{\mathbf{y}}) \end{aligned} \tag{18}$$

is

$$\begin{aligned} \dot{V} = \; & 2(\mathbf{x} - \mathbf{y})^T \mathbf{D}_1 (\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)) + 2(\tilde{\mathbf{x}} - \tilde{\mathbf{y}})^T \mathbf{D}_2 (\tilde{\mathbf{f}}(\tilde{\mathbf{x}}, t) - \tilde{\mathbf{f}}(\tilde{\mathbf{y}}, t)) \\ & -2(\mathbf{x} - \mathbf{y})^T \mathbf{D}_1 \mathbf{D}_3 (\mathbf{x} - \mathbf{y} - (\tilde{\mathbf{x}} - \tilde{\mathbf{y}})) - 2(\tilde{\mathbf{x}} - \tilde{\mathbf{y}})^T \mathbf{D}_2 \mathbf{D}_4 (\tilde{\mathbf{x}} - \tilde{\mathbf{y}} - (\mathbf{x} - \mathbf{y})) \end{aligned} \tag{19}$$

The first two terms add up to a function that is the negative of a function in class K, while the last two terms add up to

$$-2(\mathbf{x} - \mathbf{y} - (\tilde{\mathbf{x}} - \tilde{\mathbf{y}}))^T \mathbf{D} (\mathbf{x} - \mathbf{y} - (\tilde{\mathbf{x}} - \tilde{\mathbf{y}})) \tag{20}$$

which is nonpositive. So the result follows from theorem 2. ∎

For the special case where the two systems are identical, ($\mathbf{f} = \tilde{\mathbf{f}}$), we can choose $\mathbf{D}_3 = \mathbf{D}_4 = \alpha \mathbf{I}$ for $\alpha \geq 0$, which corresponds to linear diffusive mutual coupling. Here α serves as a measure of the coupling strengh between the two systems.

4 Asymptotical Synchronization of Two Identical Dynamical Systems

In this section we show the relationship between asymptotical stability and asymptotical synchronization. First we introduce some notations. For $\mathbf{x} \in \mathbb{R}^n$, we denote $\mathbf{x}_i = (x_1, \cdots, x_i) \in \mathbb{R}^i$ and $\mathbf{x}_{i,j} = (x_i, \cdots, x_j) \in \mathbb{R}^{j-i+1}$.

Next we need definitions of synchronization of dynamical systems defined by ordinary differential equations which are analogous to the definitions of stability and asymptotical stability.

Definition 7 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{y}, t) \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{x}, \mathbf{y}, t) \end{aligned} \tag{21}$$

Let $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^m$. *System (21) is uniform-synchronized via* $\mathbf{x}_{i,j}$ *and* $\mathbf{y}_{k,l}$, *where* $l - k = j - i$, *if for each* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that if* $\|\mathbf{x}_{i,j}(t_0) - \mathbf{y}_{k,l}(t_0)\| \leq \delta(\epsilon)$, *then* $\|\mathbf{x}_{i,j}(t) - \mathbf{y}_{k,l}(t)\| \leq \epsilon$ *for all* $t \geq t_0$ *for* $\mathbf{x}(t_0)$ *and* $\mathbf{y}(t_0)$ *in some neighborhoods of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively.*

Definition 8 *Let* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^m$*. The system (21) is uniform-asymptotically synchronized via* $\mathbf{x}_{i,j}$ *and* $\mathbf{y}_{k,l}$ *if it is uniform-synchronized via* $\mathbf{x}_{i,j}$ *and* $\mathbf{y}_{k,l}$ *and there exists* $\delta > 0$ *such that for all* $\epsilon > 0$ *there exists* $T(\epsilon) \geq 0$ *such that if*

$$\|\mathbf{x}_{i,j}(t_0) - \mathbf{y}_{k,l}(t_0)\| \leq \delta$$

and $t \geq t_0 + T(\epsilon)$*, then*

$$\|\mathbf{x}_{i,j}(t) - \mathbf{y}_{k,l}(t)\| \leq \epsilon$$

for $\mathbf{x}(t_0)$ *and* $\mathbf{y}(t_0)$ *in some neighborhoods of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively.*

In the above definitions, the difference in the states between the two systems goes to zero as $t \to \infty$. In the next definition, we allow for some synchronization error which can occur, for example, when the two systems are not exactly identical.

Definition 9 *Let* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^m$*. The system (21) is uniform-synchronized with error bound* ϵ *via* $\mathbf{x}_{i,j}$ *and* $\mathbf{y}_{k,l}$ *if there exists* $\delta > 0$ *and* $T \geq 0$ *such that if*

$$\|\mathbf{x}_{i,j}(t_0) - \mathbf{y}_{k,l}(t_0)\| \leq \delta$$

then $\|\mathbf{x}_{i,j}(t) - \mathbf{y}_{k,l}(t)\| \leq \epsilon$ *for all* $t \geq t_0 + T$*, for* $\mathbf{x}(t_0)$ *and* $\mathbf{y}(t_0)$ *in some neighborhoods of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively.*

We will say that the system (21) for the case $n = m$ is uniform-(asymptotically) synchronized (with error bound ϵ) if it is uniform-(asymptotically) synchronized (with error bound ϵ) via $\mathbf{x}$ and $\mathbf{y}$. When system (21) is synchronized via $\mathbf{x}_{i,j}$ and $\mathbf{y}_{k,l}$ but it is not synchronized via $\mathbf{x}$ and $\mathbf{y}$, then we say that the system is *partially synchronized.*

Clearly these narrow definitions are quite restrictive and do not include other types of synchronization such as phase locking and frequency entrainment [Blekhman, 1988]. However, these definitions can be applied to systems exhibiting aperiodic and chaotic behavior. Furthermore, they are easy to verify and their relation to asymptotical stability can be exploited. To achieve synchronization of system (21), somehow $\mathbf{x}$ and $\mathbf{y}$ must be related. This can be achieved through external forcing, coupling, etc.

We are now in a position to state the main theorem of this paper, which concerns the case of two systems with the same "functional" form:

Theorem 11 (Main Theorem) *Consider the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, \mathbf{y}, t) \tag{22}$$

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, \mathbf{y}, t) \tag{23}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\mathbf{f}$ *is defined on* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$*. Suppose that for* $\mathbf{x}_0 \in S_{H^*}$*,* $\mathbf{y}_0 \in S_{H^*}$*, the states* $\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_1$*,* $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_2$ *for some sets* D_1 *and* D_2*. Suppose that for each* $\eta_1(t) \in C(\mathbb{R}, D_1)$ *and* $\eta_2(t) \in C(\mathbb{R}, D_2)$ *the system*

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, t) = \mathbf{f}(\mathbf{z}, \eta_1(t), \eta_2(t), t) \tag{24}$$

is uniform-asymptotically stable at t_0 *with respect to* H^**. Then* $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ *as* $t \to \infty$ *when the initial states at* t_0 *satisfy* $\mathbf{x}_0 \in S_{H^*}$*,* $\mathbf{y}_0 \in S_{H^*}$*. If in addition system (24) is uniform-asymptotically stable i-uniformly with respect to all* $\eta_1(t) \in C(\mathbb{R}, D_1)$ *and* $\eta_2(t) \in C(\mathbb{R}, D_2)$ *then system (22-23) is uniform-asymptotically synchronized.*

Proof Follows directly from the definitions of uniform-asymptotical stability, where we set $\eta_1(t) = \mathbf{x}(t)$ and $\eta_2(t) = \mathbf{y}(t)$. ■

The two systems as defined in Eqs.(22)-(23) are said to be in the same *functional* form which is a generalization of the definition given in [Wu and Chua, 1993]. At first glance the two systems (22) and (23) appear to be *different*. The argument of $\mathbf{f}(\cdot,\cdot,\cdot,\cdot)$ consists of 4 components. The first component is due to the state variables of the system, i.e. $\mathbf{x}$ in system (22) and $\mathbf{y}$ in system (23). The second component is due to the state variables in system (22). The third component is due to the state variables in system (23). The fourth component is due to the dependence of $\mathbf{f}$ on time t. If we regard the second and third components as time varying dependences and write them as η_1 and η_2 respectively, we obtain,

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \eta_1(t), \eta_2(t), t)\\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, \eta_1(t), \eta_2(t), t)\end{aligned} \tag{25}$$

Regarding the system in this way, the two systems are *identical*, so in essence we are really talking about synchronization of two *identical* systems.

Next we will consider applications of the above theorem. In the following the corollaries are given without proofs as they follow directly from the above theorem.

4.1 External synchronizing excitation

We consider two systems which are excited by the same excitation $\eta(t)$.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \eta(t))\\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, \eta(t))\end{aligned} \tag{26}$$

Corollary 1 *Consider system (26). Suppose that $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \eta(t))$ is uniform-asymptotically stable with respect to H^*. Then $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ as $t \to \infty$ and the system is uniform-asymptotically synchronized when the initial states at t_0 satisfy $\mathbf{x}_0 \in S_{H^*}$,$\mathbf{y}_0 \in S_{H^*}$.*

The state vector $\mathbf{x}(t)$ can be aperiodic if $\eta(t)$ is aperiodic. For example, in the homogeneous driving of Pecora and Carroll, $\mathbf{x}$ is a stable *subsystem* of a chaotic system and $\eta(t)$ is the state vector of the rest of the chaotic system [Pecora and Carroll, 1990; Pecora and Carroll, 1991; Carroll and Pecora, 1991; He and Vaidya, 1992]:

Corollary 2 (Homogeneous Driving) *Consider the system*

$$\left.\begin{aligned}\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{v}, \mathbf{u})\\ \dot{\mathbf{u}} &= \mathbf{g}(\mathbf{v}, \mathbf{u})\end{aligned}\right\}\ \textit{driving system} \tag{27}$$

$$\dot{\mathbf{w}} = \mathbf{g}(\mathbf{v}, \mathbf{w})\ \big\}\ \textit{driven system} \tag{28}$$

where $\mathbf{v} \in \mathbb{R}^m$, $\mathbf{u}, \mathbf{w} \in \mathbb{R}^k$. Suppose that for $\mathbf{v}_0, \mathbf{u}_0 \in S_{H^*}$, $\mathbf{v}(t, \mathbf{v}_0, \mathbf{u}_0, t_0) \in D$ for some set D. If $\dot{\mathbf{u}} = \mathbf{g}(\eta(t), \mathbf{u})$ is uniform-asymptotically stable with respect to H^* for all $\eta(t) \in C(\mathbb{R}, D)$, then $\|\mathbf{w}(t) - \mathbf{u}(t)\| \to 0$ as $t \to \infty$ when $\mathbf{v}_0, \mathbf{u}_0, \mathbf{w}_0 \in S_{H^*}$. If in addition the asymptotical stability of $\dot{\mathbf{u}} = \mathbf{g}(\eta(t), \mathbf{u})$ is i-uniform with respect to all $\eta(t) \in C(\mathbb{R}, D)$, then the driving system and the driven system are asymptotically synchronized at t_0 via $\mathbf{u}$ and $\mathbf{w}$.

Thus in corollary 2 the driven system $\dot{\mathbf{w}} = \mathbf{g}$ is an asymptotically stable subsystem of the driving system.

4.2 Asymmetric coupling

Corollary 3 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x}, \mathbf{y}, t)) \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, \mathbf{g}(\mathbf{x}, \mathbf{y}, t)) \end{aligned} \tag{29}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\mathbf{f}$ *is defined on* $\mathbb{R}^n \times \mathbb{R}^m$. *Suppose that for* $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$, *and* $t \geq t_0$, $\mathbf{g}(\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0, t_0), \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0, t_0), t) \in D$ *for some set* D. *Suppose that for each* $\eta(t) \in C(\mathbb{R}, D)$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \eta(t)) \tag{30}$$

is uniform-asymptotically stable with respect to H^*. *Then* $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ *as* $t \to \infty$ *when* $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$ *at* t_0. *If in addition the asymptotical stability of (30) is i-uniform with respect to all* $\eta(t) \in C(\mathbb{R}, D)$, *then the system is uniform-asymptotically synchronized.*

Here $\mathbf{g}(\mathbf{x}, \mathbf{y}, t)$ can be thought of as the coupling between the two systems, where each system receives the *exact same* coupling.

Several special cases of this type of coupling follow:

4.2.1 Master-slave synchronization

When $\mathbf{g}(\mathbf{x}, \mathbf{y}, t) = (\mathbf{x}, t)$, we have the following corollary:

Corollary 4 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{x}, t) \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, \mathbf{x}, t) \end{aligned} \tag{31}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\mathbf{f}$ *is defined on* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. *Suppose that for* $\mathbf{x}_0 \in S_{H^*}$, *and* $t \geq t_0$, $\mathbf{x}(t, \mathbf{x}_0, t_0) \in D$ *for some set* D. *Suppose that for each* $\eta(t) \in C(\mathbb{R}, D)$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \eta(t), t) \tag{32}$$

is uniform-asymptotically stable with respect to H^*. *Then* $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ *as* $t \to \infty$ *when* $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$ *at* t_0. *If in addition the asymptotical stability of (32) is i-uniform with respect to all* $\eta(t) \in C(\mathbb{R}, D)$, *then the system is uniform-asymptotically synchronized.*

In particular, we have synchronization of chaotic systems when $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$ is chaotic. In [Wu and Chua, 1993], two examples, namely the Lorenz system and Chua's circuit, are given where master-slave synchronization is achieved between chaotic systems. For the case of Chua's circuit, $\mathbf{f}$ is chosen such that $\mathbf{f}(\mathbf{x}, \eta(t), t) = \mathbf{A}\mathbf{x} + \eta(t)$ for $\mathbf{A}$ a matrix with all eigenvalues in the open left half plane and thus the conditions of corollary 4 are satisfied.

4.2.2 Control via linear feedback

The master-slave configuration was used by Chen and Dong [Chen and Dong, 1993a] in a linear feedback control scheme to control a chaotic system to an unstable periodic orbit. They consider the system (31) where $\mathbf{f}(\mathbf{y}, \mathbf{x}, t) = \mathbf{g}(\mathbf{y}, t) + \mathbf{K}(\mathbf{x} - \mathbf{y})$.

Corollary 5 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{g}(\mathbf{x}, t) \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}, t) + \mathbf{K}(\mathbf{x} - \mathbf{y}) \end{aligned} \tag{33}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\mathbf{g}$ *is defined on* $\mathbb{R}^n \times \mathbb{R}$. *Suppose that for* $\mathbf{x}_0 \in S_{H^*}$, *and* $t \geq t_0$, $\mathbf{x}(t, \mathbf{x}_0, t_0) \in D$ *for some set* D. *Suppose that for each* $\eta(t) \in C(\mathbb{R}, D)$

$$\dot{\mathbf{z}} = [\mathbf{g}(\mathbf{z}, t) - \mathbf{K}\mathbf{z}] + \mathbf{K}\eta(t) \tag{34}$$

is uniform-asymptotically stable with respect to H^*. *Then* $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ *as* $t \to \infty$ *when* $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$ *at* t_0. *If in addition the asymptotical stability of (34) is i-uniform with respect to all* $\eta(t) \in C(\mathbb{R}, D)$, *then the system is uniform-asymptotically synchronized.*

Thus $-\mathbf{K}\mathbf{z}$ serves as a stabilizing linear feedback. $\mathbf{x}(t)$ can be from another identical dynamical system which $\mathbf{y}$ synchronizes to or in the case of Chen and Dong [Chen and Dong, 1993b; Chen, 1993] $\mathbf{x}(t)$ is an unstable trajectory of the system which we want $\mathbf{y}(t)$ to approach. In [Chen and Dong, 1993b; Chen, 1993], $\mathbf{K}$ was chosen such that the Chua's circuit which is initially chaotic, becomes asymptotically stable after adding linear feedback by satisfying the conditions of theorem 7. This linear feedback scheme was also used by Pyragas to stabilize the Rössler, Lorenz and Duffing Systems [Pyragas, 1993].

4.2.3 Communication systems

In master-slave synchronization, the coupling is unidirectional; the master system is not influenced by the slave system. Exploiting this, many authors have since utilized this property of master-slave synchronization to construct communication systems where the transmission is also unidirectional [Oppenheim *et al.*, 1992; Kocarev *et al.*, 1992; Parlitz *et al.*, 1992; Cuomo and Oppenheim, 1993b; Cuomo and Oppenheim, 1993a; Halle *et al.*, 1993; Dedieu *et al.*, 1993; Wu and Chua, 1993]. When $\mathbf{g}(\mathbf{x}, \mathbf{y}, t) = c(\mathbf{x}, s(t))$, we have

Corollary 6 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, c(\mathbf{x}, s(t)), t) && (35) \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, c(\mathbf{x}, s(t)), t) && (36) \end{aligned}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\mathbf{f}$ *is defined on* $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. *Suppose that for* $\mathbf{x}_0 \in S_{H^*}$, *and* $t \geq t_0$, $c(\mathbf{x}(t, \mathbf{x}_0, t_0), s(t)) \in D$ *for some set* D *in* $\mathbb{R}^m$. *Suppose that for each* $\eta(t) \in C(\mathbb{R}, D)$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \eta(t), t) \tag{37}$$

is uniform-asymptotically stable with respect to H^*. *Then* $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ *as* $t \to \infty$ *when* $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$ *at* t_0. *If in addition the asymptotical stability of (37) is i-uniform with respect to all* $\eta(t) \in C(\mathbb{R}, D)$, *then the system is uniform-asymptotically synchronized.*

The transmitter corresponds to the system (35) and the receiver corresponds to the system (36). Here $s(t)$ is an information-bearing signal. The continuous function $c(\cdot, \cdot)$ is a coding function which codifies the information signal $s(t)$ with the chaotic signal $\mathbf{x}(t)$. We assume that there is a decoding function $d(\cdot, \cdot)$ such that $d(\mathbf{x}, c(\mathbf{x}, s(t))) = s(t)$ which is continuous in $\mathbf{x}$. The signal $c(\mathbf{x}(t), s(t))$ is then transmitted to the receiver. When the transmitter and the receiver are asymptotically synchronized, $\mathbf{y}(t) \to \mathbf{x}(t)$ and thus $d(\mathbf{y}, c(\mathbf{x}, s(t))) \to s(t)$ as $t \to \infty$ by continuity of d. For communication systems, a design criterion might be that $c(\mathbf{x}, s(t)) \in \mathbb{R}^m$ is a short vector, i.e., m should be small.

Example: Consider the following communication system based on Chua's oscillator:

$$\begin{aligned}
\frac{dv_1}{dt} &= \frac{1}{C_1}[G(v_2 - v_1) - f(v_1 + r(t))] \\
\frac{dv_2}{dt} &= \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\
\frac{di_3}{dt} &= -\frac{1}{L}(v_2 + R_0 i_3) \\
\frac{d\tilde{v}_1}{dt} &= \frac{1}{C_1}[G(\tilde{v}_2 - \tilde{v}_1) - f(v_1 + r(t))] \\
\frac{d\tilde{v}_2}{dt} &= \frac{1}{C_2}[G(\tilde{v}_1 - \tilde{v}_2) + \tilde{i}_3] \\
\frac{d\tilde{i}_3}{dt} &= -\frac{1}{L}(\tilde{v}_2 + R_0 \tilde{i}_3)
\end{aligned} \tag{38}$$

where $r(t) = d(a + v_1(t))(b + v_2(t))(c + i_3(t))s(t)$ and a, b, c are chosen such that $(a + v_1(t))(b + v_2(t))(c + i_3(t))$ is positive for all time t. The signal $v_1(t) + r(t)$ is transmitted. For C_1, C_2, G, $L > 0$ and $R_0 \geq 0$, the two systems synchronize [Wu and Chua, 1993] and so

$$\tilde{s}(t) = \frac{(v_1(t) + r(t)) - \tilde{v}_1}{d(a + \tilde{v}_1(t))(b + \tilde{v}_2(t))(c + \tilde{i}_3(t))}$$

will converge to $s(t)$.

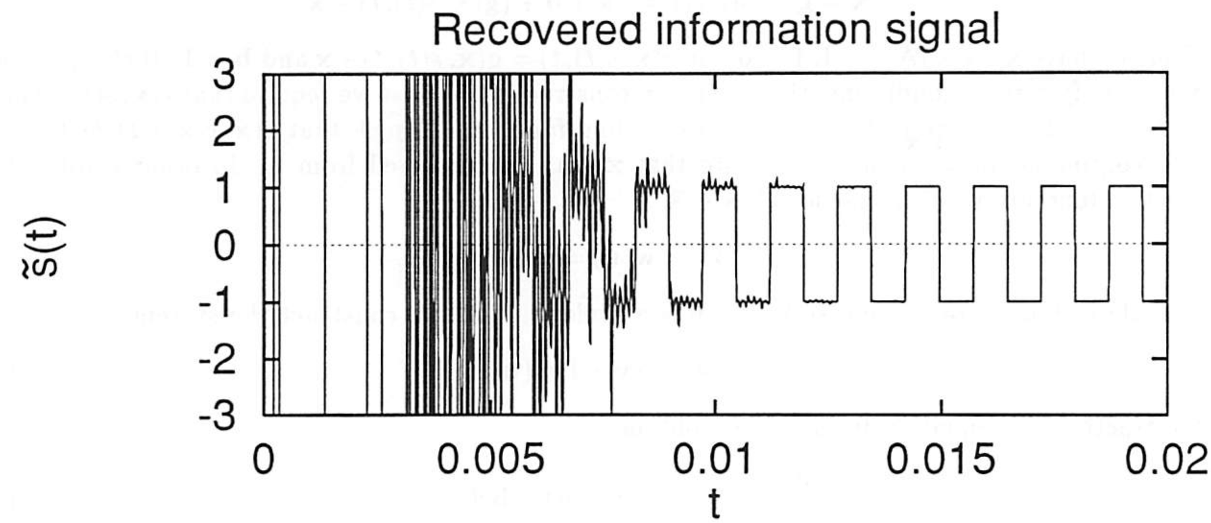


Figure 1: Information signal $\tilde{s}(t)$ recovered from $v_1(t) + r(t)$. The parameters are $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 0$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $G_b = -0.5mS$ and $a = 2.5$, $b = 0.51$, $c = 0.0025$, $d = 1$.

In Fig. 1 we show $\tilde{s}(t)$ when $s(t)$ is a square wave of amplitude 1 and frequency $\frac{2100}{\pi}Hz$ and the parameters are chosen as $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 0$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $G_b = -0.5mS$ and $a = 2.5$, $b = 0.51$, $c = 0.0025$, $d = 1$.

4.2.4 Inverse filtering

Consider the system

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, t, s(t)) \\
\mathbf{u} &= \mathbf{h}(\mathbf{x}, s(t))
\end{aligned} \tag{39}$$

We can think of this system as a nonlinear filter. The original signal $s(t)$ is being "filtered" by the system generating the output $\mathbf{u}(t)$, which is a filtered version of $s(t)$. The goal is now to perform an inverse filtering operation to recover the signal $s(t)$ from $\mathbf{u}(t)$. Unlike in linear systems, there is no straightforward way to perform the inverse filtering, even if the operation of the filter system (39) is completely specified. In general, there could be natural dynamics of the system which modifies the signal $s(t)$. The transmitter in the communication systems discussed above is a nonlinear filter which garbles the input information signal and the goal at the receiver is to recover the input signal given the output of the transmitter filter. What corollary 6 tells us is that if the filter is of the form (35) with $c(\cdot,\cdot)$ invertible (i.e., the function $d(\cdot,\cdot)$ exists) and $\mathbf{u} = c(\mathbf{x}, s(t))$, then we can recover $s(t)$ using a synchronized system.

However, in general the system is not given in such a form. In some cases, it is still possible to recover the signal $s(t)$ from $\mathbf{u}(t)$. Recall our notation that for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}_\rho = \{x_1, \ldots, x_\rho\}$ is a subvector of $\mathbf{x}$. Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}_\rho, t) + \mathbf{b}c(\mathbf{x}, s(t), t) && (40) \\ \mathbf{u} &= \mathbf{w}(\mathbf{x}) && (41) \end{aligned}$$

where $\mathbf{A}$ has all its eigenvalues in the open left half plane. Any system with input $s(t)$ can be put in the form (40) since

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, s(t), t) = -\mathbf{x} + \mathbf{0} + (\mathbf{g}(\mathbf{x}, s(t), t) + \mathbf{x})$$

Thus we have $\mathbf{x}_\rho = \mathbf{x}$, $\mathbf{A} = -\mathbf{I}$, $\mathbf{f} = \mathbf{0}$ and $c(\mathbf{x}, s(t), t) = \mathbf{g}(\mathbf{x}, s(t), t) + \mathbf{x}$ and $\mathbf{b} = \mathbf{I}$. If $c(\cdot,\cdot,\cdot)$, ρ and $\mathbf{w}(\cdot)$ satisfy certain conditions, then we can reconstruct $s(t)$. First we require that $c(\mathbf{x}, s(t), t)$ maps into $\mathbb{R}^\rho$. Next we require that there is a decoding function $d()$ such that $d(\mathbf{x}, c(\mathbf{x}, s(t), t), t) = s(t)$ and continuous in $\mathbf{x}$. Third we require that $\mathbf{x}_\rho$ can be recovered from $\mathbf{u}$. In other words, there exists a function $\tilde{\mathbf{w}}$ such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}_\rho = \tilde{\mathbf{w}}(\mathbf{u}) = \tilde{\mathbf{w}}(\mathbf{w}(\mathbf{x}))$$

The algorithm for recovering $s(t)$ from $\mathbf{u}$ is as follows. First we construct the system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{f}(\tilde{\mathbf{w}}(\mathbf{u}), t) \tag{42}$$

Subtracting this equation from (40) we obtain

$$\frac{d(\mathbf{x} - \mathbf{y})}{dt} = \mathbf{A}(\mathbf{x} - \mathbf{y}) + \mathbf{b}c(\mathbf{x}, s(t), t) \tag{43}$$

Thus $\mathbf{x}_\rho - \mathbf{y}_\rho$ at steady state is the output when $c(\mathbf{x}, s(t), t)$ is fed through a linear time-invariant filter. The transfer function is given by

$$\mathbf{h}(s) = \frac{(\mathbf{x}_\rho - \mathbf{y}_\rho)(s)}{c(s)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & 0 & \cdots \end{pmatrix} (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

where $c(s)$ is the Laplace transform of $c(\mathbf{x}(t), s(t), t)$. If $\mathbf{h}(s)$ is invertible for all s on the imaginary axis, then we can perform a linear inverse filtering operation to recover $c(\mathbf{x}, s(t), t)$ from $\mathbf{x}_\rho - \mathbf{y}_\rho$. Then we use Eq. (43) to obtain the steady state of $\mathbf{x} - \mathbf{y}$. Adding that to $\mathbf{y}$ gives us $\mathbf{x}$ from which we can recover $s(t)$ (asymptotically) from $c(\mathbf{x}(t), s(t), t)$ by using $d()$.

Example: Consider the state equations of Chua's oscillator with external forcing term $s(t)$:

$$\begin{aligned} \frac{dv_1}{dt} &= \frac{1}{C_1}[G(v_2 - v_1 + r(t)) - f(v_1)] \\ \frac{dv_2}{dt} &= \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} &= -\frac{1}{L}(v_2 + R_0 i_3) \end{aligned} \tag{44}$$

where $r(t) = d(a + v_1(t))(b + v_2(t))(c + i_3(t))s(t)$ and a, b, c are chosen such that $(a + v_1(t))(b + v_2(t))(c + i_3(t))$ is positive for all t. The goal is to recover $s(t)$ from $v_1(t)$. Equations (44) are in the form (40) where

$$\mathbf{A} = \begin{pmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & -\frac{R_0}{L} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} v_1 \\ v_2 \\ i_3 \end{pmatrix}$$

$\rho = 1$, $\mathbf{f}(v_1, t) = \begin{pmatrix} -\frac{f(v_1)}{C_1} & 0 & 0 \end{pmatrix}^T$, $\mathbf{b} = \begin{pmatrix} \frac{G}{C_1} & 0 & 0 \end{pmatrix}^T$, $c(\mathbf{x}, s(t), t) = d(a + v_1(t))(b + v_2(t))(c + i_3(t))s(t)$ and $\mathbf{u} = v_1$.

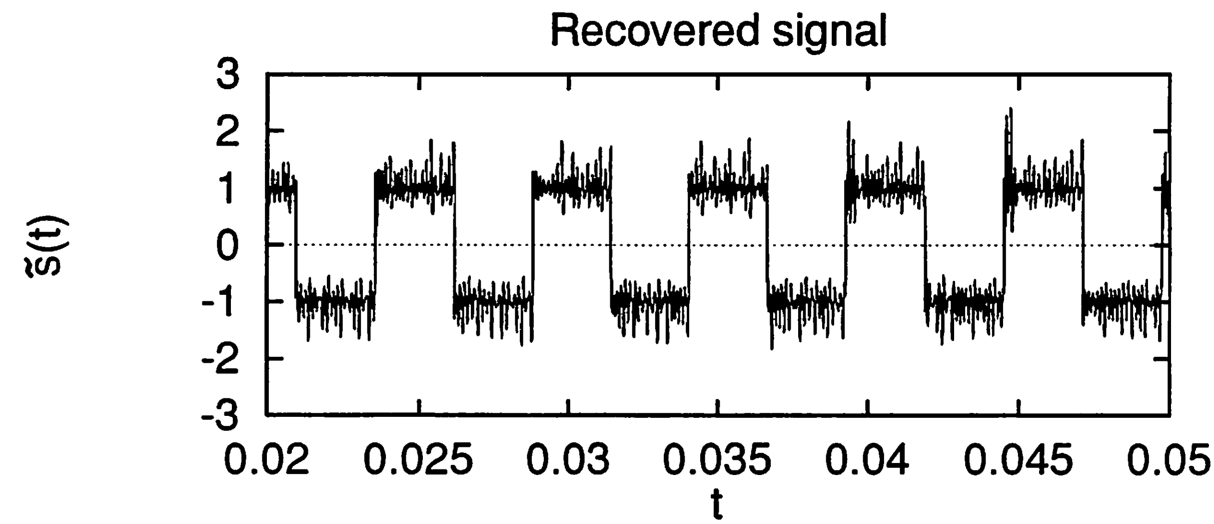


Figure 2: Signal $\tilde{s}(t)$ recovered from $v_1(t)$. The parameters are $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 0$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $G_b = -0.5mS$ and $a = 3$, $b = 0.7$, $c = 0.003$, $d = 1$.

In Fig. 2 we show $\tilde{s}(t)$ which was extracted from $v_1(t)$ by the above algorithm. The signal $s(t)$ is a square wave of amplitude 1 and frequency $\frac{600}{\pi}Hz$ and the parameters are chosen as $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 0$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $G_b = -0.5mS$ and $a = 3$, $b = 0.7$, $c = 0.003$, $d = 1$.

4.3 Symmetric or mutual coupling

Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x}, \mathbf{y}), t) \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, \mathbf{g}(\mathbf{y}, \mathbf{x}), t) \end{aligned} \tag{45}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{f}$ is defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. Again $\mathbf{g}(\mathbf{x}, \mathbf{y})$ can be thought of as the coupling between the two systems, but in this case each system received a coupling which is the same *relative to each other*. In other words, we have mutual coupling. Suppose that the coupling has the following form:

$$\begin{aligned} \mathbf{g}(\mathbf{x}, \mathbf{y}) &= \mathbf{h}(\mathbf{x}, \mathbf{i}(\mathbf{x}, \mathbf{y})) \\ \mathbf{g}(\mathbf{y}, \mathbf{x}) &= \mathbf{h}(\mathbf{y}, \mathbf{i}(\mathbf{x}, \mathbf{y})) \end{aligned} \tag{46}$$

The following corollary then follows from theorem 11:

Corollary 7 *Consider the system (45). Suppose that for* $\mathbf{x}_0, \mathbf{y}_0 \in S_{H^*}$*, and* $t \geq t_0$*,* $\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_1$ *and* $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_2$ *for some sets* D_1 *and* D_2*. Suppose that for each* $\eta_1(t) \in C(\mathbb{R}, D_1)$ *and* $\eta_2(t) \in C(\mathbb{R}, D_2)$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{h}(\mathbf{z}, \mathbf{i}(\eta_1(t), \eta_2(t)))) \tag{47}$$

is uniform-asymptotically stable with respect to H^**. Then* $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ *as* $t \to \infty$ *when* $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$ *at* t_0*. If in addition, the asymptotical stability of (47) is i-uniform with respect to all* $\eta_1(t) \in C(\mathbb{R}, D_1)$ *and* $\eta_2(t) \in C(\mathbb{R}, D_2)$ *then the system (45) is uniform-asymptotically synchronized.*

Condition (46) is satisfied if $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{h}(\mathbf{x}, \mathbf{i}(\mathbf{x}, \mathbf{y}))$ and $\mathbf{i}(\mathbf{x}, \mathbf{y}) = \mathbf{i}(\mathbf{y}, \mathbf{x})$. For example, consider the case $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{K}(\mathbf{y} - \mathbf{x})$. Since $\mathbf{g}(\mathbf{x}, \mathbf{y}) = -2\mathbf{K}\mathbf{x} + \mathbf{K}(\mathbf{x} + \mathbf{y})$, we have the following:

Corollary 8 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, t) + \mathbf{K}(\mathbf{y} - \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, t) + \mathbf{K}(\mathbf{x} - \mathbf{y}) \end{aligned} \tag{48}$$

Suppose that for $\mathbf{x}_0, \mathbf{y}_0 \in S_{H^*}$*, and* $t \geq t_0$*,* $\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_1$ *and* $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_2$ *for some sets* D_1 *and* D_2*. Suppose that for each* $\eta_1(t) \in C(\mathbb{R}, D_1)$ *and* $\eta_2(t) \in C(\mathbb{R}, D_2)$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t) - 2\mathbf{K}\mathbf{z} + \mathbf{K}\left[\eta_1(t) + \eta_2(t)\right] \tag{49}$$

is uniform-asymptotically stable with respect to H^**. Then* $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ *as* $t \to \infty$ *when* $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$ *at* t_0*. If in addition, the asymptotical stability of (49) is i-uniform with respect to all* $\eta_1(t) \in C(\mathbb{R}, D_1)$ *and* $\eta_2(t) \in C(\mathbb{R}, D_2)$ *then system (48) is uniform-asymptotically synchronized.*

In [Chua *et al.*, 1993b], a diagonal matrix $\mathbf{K}$ was chosen such that the linear feedback $-2\mathbf{K}\mathbf{x}$ causes the Chua's oscillator to become asymptotically stable. One such choice corresponds to resistive coupling between the two circuits.

4.4 Linear coupling

Notice the similarity of Eq. (49) with the case of linear feedback control (Sec. 4.2.2, Eq. (34)), where $-2\mathbf{K}$ can be considered a stabilizing feedback matrix. Thus we can conclude that *mutual* coupling in the form of (48) synchronizes the system better than *unidirectional* coupling in the form (33) in the sense that a smaller feedback matrix $\mathbf{K}$ is needed in mutual coupling than in unidirectional coupling. Also note that when the systems are synchronized; i.e. $\mathbf{x} = \mathbf{y}$, the two systems are decoupled, and we have:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$$

These two cases can be generalized as follows:

Corollary 9 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, t) + \mathbf{K}_1(\mathbf{y} - \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}, t) + \mathbf{K}_2(\mathbf{x} - \mathbf{y}) \end{aligned} \tag{50}$$

Suppose that for $\mathbf{x}_0, \mathbf{y}_0 \in S_{H^*}$*, and* $t \geq t_0$*,* $\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_1$ *and* $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0, t_0) \in D_2$ *for some sets* D_1 *and* D_2*. Suppose that for each* $\eta_1(t) \in C(\mathbb{R}, D_1)$ *and* $\eta_2(t) \in C(\mathbb{R}, D_2)$

$$\dot{\mathbf{z}} = [\mathbf{f}(\mathbf{z}, t) - (\mathbf{K}_1 + \mathbf{K}_2)\mathbf{z}] + \mathbf{K}_2\eta_1(t) + \mathbf{K}_1\eta_2(t) \tag{51}$$

is uniform-asymptotically stable with respect to H^*. Then $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ as $t \to \infty$ when $\mathbf{x}_0 \in S_{H^*}, \mathbf{y}_0 \in S_{H^*}$ at t_0. If in addition, the asymptotical stability of (51) is i-uniform with respect to all $\eta_1(t) \in C(\mathbb{R}, D_1)$ and $\eta_2(t) \in C(\mathbb{R}, D_2)$ then the system (50) is uniform-asymptotically synchronized.

In this case $\mathbf{K}_1 + \mathbf{K}_2$ is the stabilizing linear feedback matrix.

For example, using theorem 7, we can give conditions for which two linearly coupled Chua's oscillators will synchronize:

Corollary 10 *Consider the system:*

$$\begin{array}{rcl} \frac{dv_1}{dt} & = & \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] + k_{11}(\tilde{v}_1 - v_1) \\ \frac{dv_2}{dt} & = & \frac{1}{C_2}[G(v_1 - v_2) + i_3] + k_{12}(\tilde{v}_2 - v_2) \\ \frac{di_3}{dt} & = & -\frac{1}{L}(v_2 + R_0 i_3) + k_{13}(\tilde{i}_3 - i_3) \\ \frac{d\tilde{v}_1}{dt} & = & \frac{1}{C_1}[G(\tilde{v}_2 - \tilde{v}_1) - f(\tilde{v}_1)] + k_{21}(v_1 - \tilde{v}_1) \\ \frac{d\tilde{v}_2}{dt} & = & \frac{1}{C_2}[G(\tilde{v}_1 - \tilde{v}_2) + \tilde{i}_3] + k_{22}(v_2 - \tilde{v}_2) \\ \frac{d\tilde{i}_3}{dt} & = & -\frac{1}{L}(\tilde{v}_2 + R_0 \tilde{i}_3) + k_{23}(i_3 - \tilde{i}_3) \end{array} \qquad (52)$$

Suppose $C_1, C_2, G, L > 0$. If $k_{11} + k_{21} > \frac{\max(-G_a, -G_b)}{C_1}$, $k_{12} + k_{22} \geq 0$ and $k_{13} + k_{23} > -\frac{R_0}{L}$, then the two Chua's oscillators will asymptotically synchronize.

For the general case $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, under mild conditions it is possible to find a matrix $\mathbf{K}$ such that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) - \mathbf{K}\mathbf{x}$ is asymptotically stable.

Theorem 12 *If the Jacobian $\mathbf{D_x f}(\mathbf{x}, t)$ is uniformly bounded, i.e. $\|\mathbf{D_x f}(\mathbf{x}, t)\| < M$ for all $\mathbf{x}$ and t, then there exists $\mathbf{K}$ such that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) - \mathbf{K}\mathbf{x} + \eta(t)$ is uniform-asymptotically stable i-uniformly with respect to all continuous $\eta(t)$.*

Proof Because the Jacobian of $\mathbf{f}$ is uniformly bounded, there exists a matrix $\mathbf{K}$ such that $\mathbf{D_x f}(\mathbf{x}, t) - \mathbf{K} + \lambda \mathbf{I}$ is negative definite for all $\mathbf{x}$ and t and some $\lambda > 0$. For example, let $\mathbf{K}$ be a diagonal matrix with large entries. Then by theorem 4 the function $-\mathbf{f}(\mathbf{x}, t) + \mathbf{K}\mathbf{x}$ is uniformly increasing, and thus $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) - \mathbf{K}\mathbf{x} + \eta(t)$ is uniform-asymptotically stable i-uniformly with respect to all continuous $\eta(t)$ by theorem 5. ■

This theorem says that two identical systems with bounded Jacobian can by synchronized by appropriate linear coupling. In particular, in general two identical systems can be synchronized by large enough linear diffusive coupling between the corresponding state variables. Another way to synchronize *any* two identical systems can be found in [Wu and Chua, 1993]. In all these cases, the coupling is such that the two systems are decoupled when they are synchronized.

4.5 Discussion

From the above, we can draw the following conclusions. To synchronize two chaotic systems, we can follow the following algorithm: Two systems are needed; the original system $\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, \mathbf{x}, \mathbf{x}, t)$ which is chaotic, and $\dot{\mathbf{x}} = \mathbf{g}_2(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, \eta_1(t), \eta_2(t), t)$ which is asymptotically stable for all $\eta_1(t)$ and $\eta_2(t)$. The main idea is that the $\mathbf{g}_1(\mathbf{x}, t)$ depends on the state variable $\mathbf{x}$ in several ways and we want to find the dependencies on $\mathbf{x}$ such that when $\mathbf{x}$ in these dependencies is regarded as external time-varying input, the resulting system $\dot{\mathbf{x}} = \mathbf{g}_2(\mathbf{x}, t)$ is asymptotically stable. The difference between the two functions $\mathbf{g}_1(\mathbf{x}, t)$ and $\mathbf{g}_2(\mathbf{x}, t)$ will be used as the coupling between the two chaotic systems. The function $\mathbf{g}_2(\mathbf{x}, t)$ can be obtained from $\mathbf{g}_1(\mathbf{x}, t)$ through linear feedback

[Chen and Dong, 1993a], by identifying a stable subsystem [Pecora and Carroll, 1990; Pecora and Carroll, 1991; Carroll and Pecora, 1991; He and Vaidya, 1992] or in case of electronic circuits, by identifying the circuit elements which are active [Wu and Chua, 1993]. It would be preferable, for example in communication systems, if the difference between $\mathbf{g}_1(\mathbf{x},t)$ and $\mathbf{g}_2(\mathbf{x},t)$ is "small" in some sense, since that is what is being transmitted. For example, in Chua's oscillator $\mathbf{g}_2(\mathbf{x},t)$ can be found such that $\mathbf{g}_1(\mathbf{x},t)$ and $\mathbf{g}_2(\mathbf{x},t)$ differs only in one scalar variable. A possible source of such "small" change can be found in bifurcation diagrams. As a parameter is varied a chaotic system can become stable and vice versa. Thus the parameter change can be the source of the difference between $\mathbf{g}_1(\mathbf{x},t)$ and $\mathbf{g}_2(\mathbf{x},t)$. For example, in the x-coupled configuration of Chua's oscillators considered in [Chua *et al.*, 1993b] the systems $\mathbf{g}_1(\mathbf{x},t)$ and $\mathbf{g}_2(\mathbf{x},t)$ correspond to different values of the linear resistor that is placed in parallel with the nonlinear resistor.

5 Synchronization of Nearly Identical Systems and Robustness of Synchronization

So far, we have only considered systems which have the same functional form. Now we consider the question of robustness. This is related to the question of what happens when the two systems are "nearly" identical. When the two coupled systems are nearly identical, we expect (or at least hope) that the difference between their states is also small as $t \to \infty$. In other words, the system is synchronized with a small error bound (definition 9).

We say that the asymptotical synchronization of two systems is robust if for arbitrarily small $\epsilon > 0$, if we make the two systems close enough, then they will be synchronized with error bound ϵ. For this definition to make sense, we need to define what it means for two systems (or vector fields) to be "close" to each other. In practice, the system will depends on parameters.[3] So it seems reasonable to define closeness of systems as closeness of "natural" parameters.

Theorem 13 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{x},\mathbf{y},t) \\ \dot{\mathbf{y}} &= \tilde{\mathbf{f}}(\mathbf{y},\mathbf{x},\mathbf{y},t) \end{aligned} \tag{53}$$

where $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. *Let* $\mathbf{f}$ *defined on* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ *be fixed. Suppose that there exists* bounded *sets* D_1, D_2 *with the following property: There exists* $\mu > 0$ *such that if*

$$\sup_{\mathbf{x}\in D_1,\mathbf{y}\in D_2,t\geq t_0} \|\mathbf{f}(\mathbf{y},\mathbf{x},\mathbf{y},t) - \tilde{\mathbf{f}}(\mathbf{y},\mathbf{x},\mathbf{y},t)\| \leq \mu$$

and $\mathbf{x}_0 \in S_{H^*}$, $\mathbf{y}_0 \in S_{H^*}$, *then* $\mathbf{x}(t,\mathbf{x}_0,\mathbf{y}_0,t_0) \in D_1$, $\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_0,t_0) \in D_2$. *Suppose that for* $\eta_1 \in D_1$, $\eta_2 \in D_2$,

$$\|\mathbf{f}(\mathbf{x},\eta_1,\eta_2,t) - \mathbf{f}(\mathbf{x}',\eta_1,\eta_2,t)\| \leq L\|\mathbf{x} - \mathbf{x}'\|$$

Suppose that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\eta_1(t),\eta_2(t),t)$$

is uniform-asymptotically stable i-uniformly with respect to all $\eta_1(t) \in C(\mathbb{R},D_1)$ *and* $\eta_2(t) \in C(\mathbb{R},D_2)$. *Then system (53) is robustly synchronized in the sense that if for each* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *and* $T(\epsilon) > 0$ *such that if*

$$\sup_{\mathbf{x}\in D_1,\mathbf{y}\in D_2,t\geq t_0} \|\mathbf{f}(\mathbf{y},\mathbf{x},\mathbf{y},t) - \tilde{\mathbf{f}}(\mathbf{y},\mathbf{x},\mathbf{y},t)\| \leq \delta \tag{54}$$

[3] For example, the natural parameters of an electronic circuit are the component values of resistors, capacitors, etc.

and $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *is in some neighborhood of* $\mathbb{R}^n$, *then* $\|\mathbf{x}(t,\mathbf{x}_0,\mathbf{y}_0,t_0)-\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_0,t_0)\| \leq \epsilon$ *for all* $t \geq t_0 + T$.

Proof Let $H^* > 0$ and $\mathbf{x}_0, \mathbf{y}_0 \in S_{H^*}$ be fixed. We denote $\mathbf{u}(t) = \mathbf{x}(t,\mathbf{x}_0,\mathbf{y}_0,t_0)$ and $\mathbf{u}'(t) = \mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_0,t_0)$. By the converse theorem, there exists a Lyapunov function

$$a(\|\mathbf{x}-\mathbf{y}\|) \leq V(t,\mathbf{x},\mathbf{y}) \leq b(\|\mathbf{x}-\mathbf{y}\|)$$

with $a(\cdot)$ and $b(\cdot)$ in class K such that

$$|V(t,\mathbf{x},\mathbf{y}) - V(t,\mathbf{x}',\mathbf{y}')| \leq M(\|\mathbf{x}-\mathbf{x}'\| + \|\mathbf{y}-\mathbf{y}'\|)$$

for some constant $M > 0$ and

$$\dot{V}(t,\mathbf{x},\mathbf{y}) \leq -V(t,\mathbf{x},\mathbf{y})$$

where $\dot{V}(t,\mathbf{x},\mathbf{y})$ is the generalized derivative of V along the trajectories of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{u}',t), \quad \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y},\mathbf{u},\mathbf{u}',t)$$

Taking the generalized derivative of V along the trajectories of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{u}',t), \quad \dot{\mathbf{y}} = \tilde{\mathbf{f}}(\mathbf{y},\mathbf{u},\mathbf{u}',t)$$

we obtain

$$\dot{V} \leq -V(t,\mathbf{x},\mathbf{y}) + M\delta$$

for $\delta \leq \mu$ satisfying Eq. (54). So by theorem 1, the result follows. ∎

The above theorem says that if small perturbations of the second system cause the trajectories of both systems to remain within a fixed bounded set which is independent of the perturbations, then the synchronization is robust. The reason why we choose D_1 and D_2 to be bounded is that in many systems, arbitrarily small changes in the parameters still imply that

$$\sup_{\mathbf{x}\in\mathbb{R}^n, t\geq t_0} \|\mathbf{f}(\mathbf{x},t) - \tilde{\mathbf{f}}(\mathbf{x},t)\| = \infty$$

so to have a correspondence between small changes in parameters and small changes in the vector fields, we restrict the vector field to a bounded set.

The existence of the bounded sets D_1 and D_2 can be difficult to verify and obtain. Because of the form of the Lyapunov function of systems such as Chua's oscillators and Lorenz systems, for a master-slave configuration we can prove a stricter result on the robustness of synchronization.

Theorem 14 *Consider the system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{x},t) \\ \dot{\mathbf{y}} &= \tilde{\mathbf{f}}(\mathbf{y},\mathbf{x},t) \end{aligned} \tag{55}$$

where $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ *and* $\mathbf{f}$, $\tilde{\mathbf{f}}$ *are defined on* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. *Suppose that there exists a bounded set* D *such that for all* $\mathbf{x}_0 \in S_{H^*}$, $\mathbf{x}(t,\mathbf{x}_0,t_0) \in D$.

Let $V(\mathbf{x},\mathbf{y}) = (\mathbf{x}-\mathbf{y})^T\mathbf{D}(\mathbf{x}-\mathbf{y})$ *where* $\mathbf{D}$ *is a positive definite symmetric matrix.*

Let $\epsilon > 0$ *be given. Suppose* $\delta > 0$ *is such that for all* $\tilde{\mathbf{f}}$ *which satisfy*

$$\sup_{\mathbf{x}\in D, t\geq t_0} \|\mathbf{f}(\mathbf{x},\mathbf{x},t) - \tilde{\mathbf{f}}(\mathbf{x},\mathbf{x},t)\| \leq \delta$$

the following inequality holds

$$\dot{V}(\mathbf{x},\mathbf{y}) \leq -c\|\mathbf{x}-\mathbf{y}\|^2$$

for some constant $c > 0$ for $t \geq t_0$ and for all $\eta_1(t) \in C(\mathbb{R}, D)$ where $\dot{V}(\mathbf{x},\mathbf{y})$ is now the generalized derivative of V along the trajectories of

$$\begin{aligned} \dot{\mathbf{x}} &= \tilde{\mathbf{f}}(\mathbf{x},\eta_1(t),t) \\ \dot{\mathbf{y}} &= \tilde{\mathbf{f}}(\mathbf{y},\eta_1(t),t) \end{aligned}$$

If $\delta < \frac{\epsilon c}{2\|\mathbf{D}\|^{\frac{3}{2}}\|\mathbf{D}^{-1}\|^{\frac{1}{2}}}$, then the system (55) is uniform-synchronized with error bound ϵ.

Proof We use the same Lyapunov function $V(\mathbf{x},\mathbf{y}) = (\mathbf{x}-\mathbf{y})^T\mathbf{D}(\mathbf{x}-\mathbf{y})$, but now applied to the system (55). The derivative of V along the trajectories of system (55) is

$$\begin{aligned} \dot{V}(\mathbf{x},\mathbf{y}) &= 2(\mathbf{x}-\mathbf{y})^T\mathbf{D}(\mathbf{f}(\mathbf{x},\mathbf{x},t)-\tilde{\mathbf{f}}(\mathbf{y},\mathbf{x},t)) \\ &= 2(\mathbf{x}-\mathbf{y})^T\mathbf{D}(\tilde{\mathbf{f}}(\mathbf{x},\mathbf{x},t)-\tilde{\mathbf{f}}(\mathbf{y},\mathbf{x},t)) + 2(\mathbf{x}-\mathbf{y})^T\mathbf{D}(\mathbf{f}(\mathbf{x},\mathbf{x},t)-\tilde{\mathbf{f}}(\mathbf{x},\mathbf{x},t)) \\ &\leq -c\|\mathbf{x}-\mathbf{y}\|^2 + 2\delta\|\mathbf{D}\|\|\mathbf{x}-\mathbf{y}\| \end{aligned}$$

So if we choose $\delta < \frac{\epsilon c}{2\|\mathbf{D}\|^{\frac{3}{2}}\|\mathbf{D}^{-1}\|^{\frac{1}{2}}}$, then the conditions of theorem 1 are satisfied which gives the correct bound for the synchronization error. ■

This theorem says that if small pertubations of the receiver system do not affect the constant c in $\dot{V}$, then if the perturbation of the receiver system is small, the synchronization error will also be small. Notice that a bounded D exists if the state vector $\mathbf{x}(t)$ in the master system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{x},t)$ is bounded for some initial conditions $\mathbf{x}_0$ in S_{H^*} since the coupling does not affect the master system. The theorem then implies that $\mathbf{y}(t)$ will also be bounded.

For example, consider master-slave synchronization of two identical Chua's oscillators through linear feedback or as in [Wu and Chua, 1993]. Let us consider the component values C_1, C_2, L, R, G_a, G_b, R_0 as the *only* parameters of the system. Suppose the parameters are chosen such that the system is asymptotically synchronized and that the trajectory of the master system is bounded, with $\dot{V}(\mathbf{x},\mathbf{y},t) \leq -c\|\mathbf{x}-\mathbf{y}\|^2$. For arbitrary small $\mu > 0$ there exists $\delta > 0$ such that for all perturbations of the parameters smaller than δ, we have $\dot{V}(\mathbf{x},\mathbf{y},t) \leq -(c-\mu)\|\mathbf{x}-\mathbf{y}\|^2$. Thus the constant $(c-\mu)$ is independent of the perturbations of parameters provided they are small enough. By applying theorem 14, it follows that the synchronization error can be made arbitrarily small if the perturbation of the parameters is small enough. In other words, the asymptotical synchronization is robust. The same can be said for the master-slave configuration of the Lorenz system if we regard β, σ, and ρ as the only parameters of the system.

6 Two More Examples

In this section we will give two more examples of synchronized systems to illustrate some of the ideas above.

In the first example, we will illustrate partial synchronization and synchronization with error by means of two linearly coupled Chua's oscillators. The coupling will be weaker than what corollary 10 dictates, so that we will not have asymptotical synchronization.

The system we consider is:

$$
\begin{array}{rcl}
\frac{dv_1}{dt} &=& \frac{1}{C_1}[G(v_2 - v_1) - f(v_1) + k_{11}(\tilde{v}_1 - v_1)] \\
\frac{dv_2}{dt} &=& \frac{1}{C_2}[G(v_1 - v_2) + i_3 + k_{12}(\tilde{v}_2 - v_2)] \\
\frac{di_3}{dt} &=& -\frac{1}{L}[v_2 + R_0 i_3 - k_{13}(\tilde{i}_3 - i_3)] \\
\frac{d\tilde{v}_1}{dt} &=& \frac{1}{C_1}[G(\tilde{v}_2 - \tilde{v}_1) - \tilde{f}(\tilde{v}_1) + k_{21}(v_1 - \tilde{v}_1)] \\
\frac{d\tilde{v}_2}{dt} &=& \frac{1}{C_2}[G(\tilde{v}_1 - \tilde{v}_2) + \tilde{i}_3 + k_{22}(v_2 - \tilde{v}_2)] \\
\frac{d\tilde{i}_3}{dt} &=& -\frac{1}{L}[\tilde{v}_2 + R_0 \tilde{i}_3 - k_{23}(i_3 - \tilde{i}_3)]
\end{array}
\tag{56}
$$

where

$$
\tilde{f}(\tilde{v}_1) = \tilde{G}_b \tilde{v}_1 + \frac{1}{2}(\tilde{G}_a - \tilde{G}_b)\{|\tilde{v}_1 + E| - |\tilde{v}_1 - E|\} \tag{57}
$$

In the widely studied double scroll Chua's attractor, the parameters are such that there exist three equilibrium points in the system. In particular, $C_1, C_2, L > 0$ and $G_a < -\frac{1}{R} < G_b < 0$. This is the case we will consider. We also assume that $\tilde{G}_b = G_b$. The coupling is chosen such that $G_a < -\frac{1}{R} - (k_{11} + k_{21}) < G_b < 0$ and $k_{13} + k_{23} > -R_0$. We will choose $k_{12} + k_{22}$ to be positive. One such choice of coupling coefficients with $R_0 > 0$ is $k_{11} = k_{21} = k_{13} = k_{23} = 0$; i.e. there is only coupling between the state variables v_2 and $\tilde{v}_2$. We denote

$$
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ i_3 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{i}_3 \end{pmatrix}
$$

Consider the Lyapunov function $V(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(\mathbf{x}-\mathbf{y})^T \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{pmatrix} (\mathbf{x}-\mathbf{y})$. Its derivative along the trajectories of system (56) is

$$
\begin{array}{rcl}
\dot{V} &=& -\frac{1}{R}((x_2 - y_2) - (x_1 - y_1))^2 - (\lambda + (k_{11} + k_{21}))(x_1 - y_1)^2 \\
&& -(R_0 + (k_{13} + k_{23}))(x_3 - y_3)^2 - (k_{12} + k_{22})(x_2 - y_2)^2
\end{array}
\tag{58}
$$

where the number $\lambda = \lambda(x_1, y_1) = \frac{f(x_1) - \tilde{f}(y_1)}{x_1 - y_1}$ depends on x_1 and y_1. We set $\lambda = G_b$ if $x_1 = y_1$.

Now we are in a position to show that if $k_{12} + k_{22}$ is large enough, then $\mathbf{x} - \mathbf{y}$ will be eventually uniformly bounded; i.e. for all initial conditions, $\mathbf{x} - \mathbf{y}$ will eventually lie in a bounded set. For $\frac{1}{R} + (k_{11} + k_{21}) + G_b > \mu > 0$, Eq. (58) can be written as

$$
\begin{array}{rcl}
-\dot{V} &=& (-G_b + \mu + \lambda)(x_1 - y_1)^2 \\
&& + \left(\frac{1}{R} + G_b - \mu + (k_{11} + k_{21})\right)(\alpha(x_2 - y_2) - (x_1 - y_1))^2 \\
&& + \left(\frac{1}{R} + (k_{12} + k_{22}) - \frac{\alpha}{R}\right)(x_2 - y_2)^2 + (R_0 + (k_{13} + k_{23}))(x_3 - y_3)^2
\end{array}
\tag{59}
$$

where

$$
\alpha = \frac{1}{1 + R(G_b - \mu + (k_{11} + k_{21}))} > 0
$$

If we choose

$$
k_{12} + k_{22} > \frac{\alpha - 1}{R} \tag{60}
$$

then

$$
\left(\frac{1}{R} + (k_{12} + k_{22}) - \frac{\alpha}{R}\right) > 0
$$

It is not hard to show, by looking at the graphs of $f(\cdot)$ and $\tilde{f}(\cdot)$ that for each $\mu > 0$, there exists M such that if $|x_1 - y_1| > M$, then $\lambda = \frac{f(x_1)-\tilde{f}(y_1)}{x_1-y_1} > G_b - \mu$. Thus if $|x_1 - y_1| > M$, then $\dot{V}$ becomes negative. If $|x_1 - y_1| \leq M$, but $|x_2 - y_2|$ or $|x_3 - y_3|$ is large enough, then $\dot{V}$ is again negative. So $\dot{V}$ is negative if $\|\mathbf{x} - \mathbf{y}\|$ is large enough, and thus by theorem 1 $\|\mathbf{x} - \mathbf{y}\|$ is eventually uniformly bounded. This bound on $\|\mathbf{x} - \mathbf{y}\|$ still holds if $k_{12} + k_{22}$ is made larger. Note that this analysis cannot give us any indications whether $\mathbf{x}$ is bounded or not. In fact it is easy to find an example where $\mathbf{x}$ is unbounded.

Next we show that by making $k_{12}+k_{22}$ large enough, $|x_2-y_2|$ can be arbitrary small (as $t \to \infty$), using the fact that $\mathbf{x} - \mathbf{y}$ is eventually bounded. Subtracting equations, we get

$$\frac{d(v_2 - \tilde{v}_2)}{dt} = -\frac{G + k_{12} + k_{22}}{C_2}(v_2 - \tilde{v}_2) + \frac{1}{C_2}\left[G(v_1 - \tilde{v}_1) + i_3 - \tilde{i}_3\right] \tag{61}$$

Thus

$$\begin{aligned} v_2(t) - \tilde{v}_2(t) &= \exp(-\tfrac{G+k_{12}+k_{22}}{C_2}t)(v_2(t_0) - \tilde{v}_2(t_0)) \\ &\quad + \tfrac{1}{C_2}\int_{t_0}^{t} \exp(-\tfrac{G+k_{12}+k_{22}}{C_2}(t-\tau))\left[G(v_1(\tau) - \tilde{v}_1(\tau)) + i_3(\tau) - \tilde{i}_3(\tau)\right] d\tau \end{aligned} \tag{62}$$

So as $\mathbf{x} - \mathbf{y}$ is eventually bounded, $v_2 - \tilde{v}_2$ is eventually smaller than ϵ, where ϵ can be chosen to be arbitrarily small, as $k_{12} + k_{22}$ is chosen to be arbitrarily large.

Thus we have partial synchronization via v_2 and $\tilde{v}_2$ with error bound ϵ, where ϵ can be made arbitrarily small by making $k_{12} + k_{22}$ arbitrarily large.

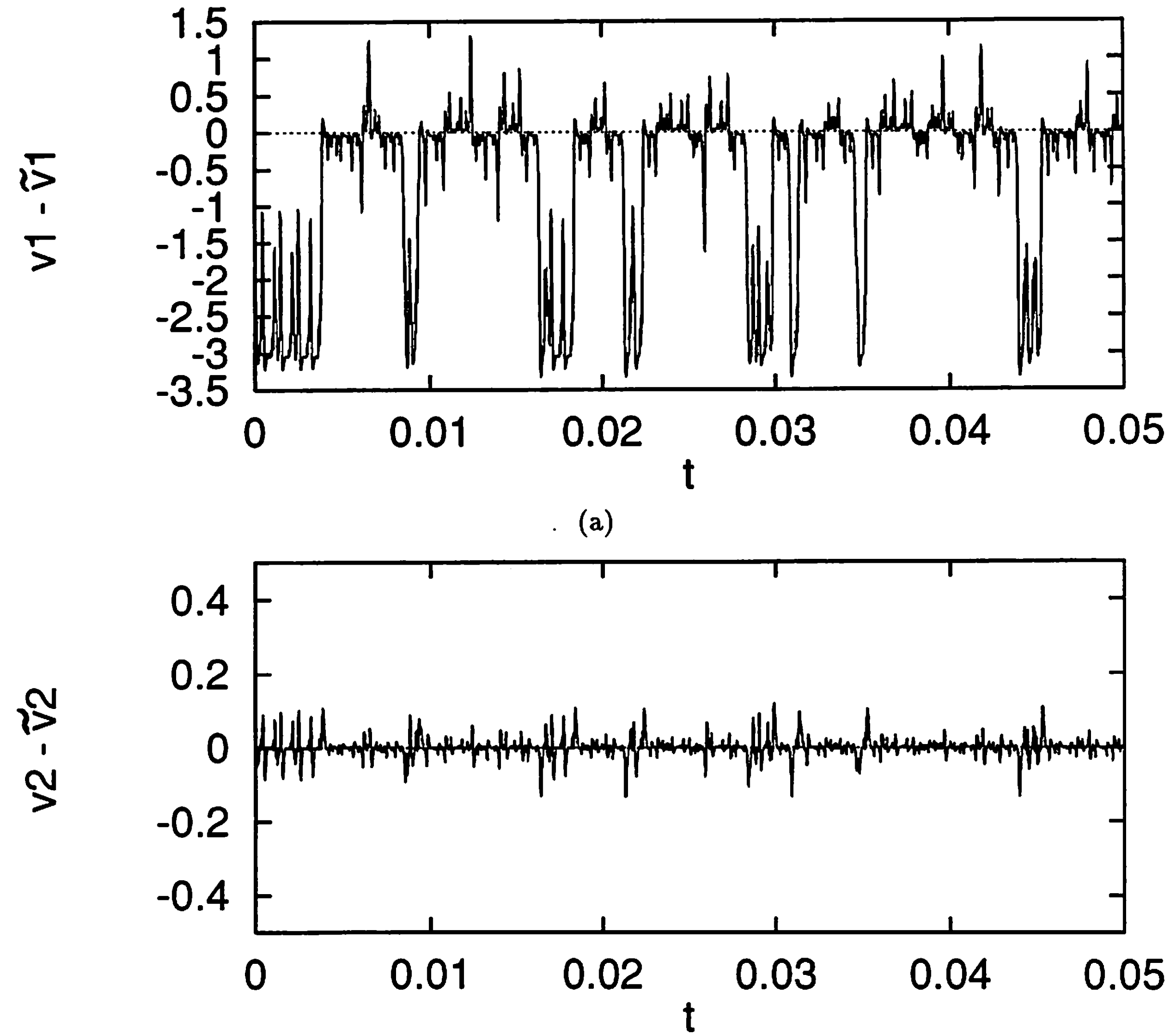
In Fig. 3 we show $v_1(t) - \tilde{v}_1(t)$ and $v_2(t) - \tilde{v}_2(t)$ when the parameters are chosen as $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 1$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $\tilde{G}_a = -0.808mS$, $G_b = \tilde{G}_b = -0.5mS$, $k_{11} = k_{21} = k_{12} = k_{13} = k_{23} = 0$ and $k_{22} = 0.007$. Note that only v_2 and $\tilde{v}_2$ are coupled. For these parameters, $\frac{\alpha-1}{R} \approx 1.75e^{-3}$ for small μ, so condition (60) is satisfied. As can be seen, the state variables $v_1(t)$ and $\tilde{v}_1(t)$ are not synchronized, while $v_2(t) - \tilde{v}_2(t)$ is relatively small.

In the computer simulations we have noticed that when $\tilde{G}_a = G_a$, the system appears to be asymptotically synchronized for large enough k_{22}. However, we were not able to prove this conjecture rigorously.

In the second example we will cascade a Chua's oscillator with a Lorenz system. The resulting system will be synchronized with an identical copy. In particular we consider the system:

$$\begin{aligned} \tfrac{dv_1}{dt} &= \tfrac{1}{C_1}[G(v_2 - v_1) - f(v_1)] \\ \tfrac{dv_2}{dt} &= \tfrac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \tfrac{di_3}{dt} &= -\tfrac{1}{L}(v_2 + R_0 i_3) \\ \tfrac{dx}{dt} &= \nu(\sigma(y - x) + k_1 v_1) \\ \tfrac{dy}{dt} &= \nu(-y + x(\rho - z) + k_2 v_2) \\ \tfrac{dz}{dt} &= \nu(-\beta z + xy + k_3 i_3) \end{aligned} \tag{63}$$

The constant ν serves to equalize the time scales of the two systems. A phase portrait in the x-y-z plane is shown in Fig. 4 for the parameter set $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 0$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $G_b = -0.5mS$ and $\sigma = 16$, $\rho = 45.6$, $\beta = 4$, $\nu = 1500$, $k_1 = k_2 = k_3 = 100$. From theorems 3, 9, and [Wu and Chua, 1993], it follows that the following system is uniform-asymptotically synchronized to system (63) when $x(t)$ is bounded,



(a)

(b)

Figure 3: Linear coupling of two Chua's oscillators. Only the state variables v_2 and $\tilde{v}_2$ are coupled. Plot of (a) $v_1(t) - \tilde{v}_1(t)$ and (b) $v_2(t) - \tilde{v}_2(t)$ versus time. The parameters are $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 1$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $\tilde{G}_a = -0.808mS$, $G_b = \tilde{G}_b = -0.5mS$, $k_{11} = k_{21} = k_{12} = k_{13} = k_{23} = 0$ and $k_{22} = 0.007$.

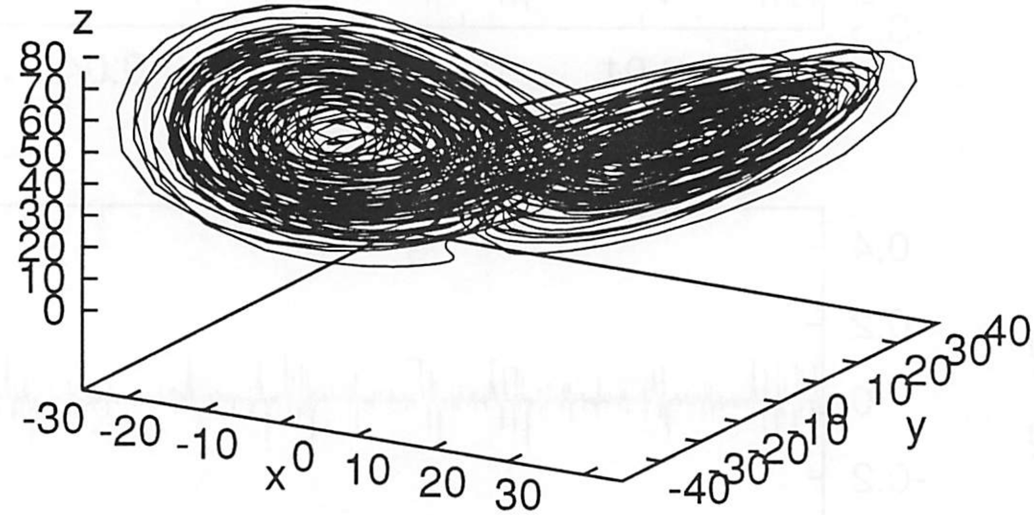


Figure 4: Cascading Chua's oscillator and Lorenz system. Phase portrait of the cascaded system in the x-y-z plane. The parameters are $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1428\Omega$, $R_0 = 0$, $L = 7.14mH$, $E = 1$, $G_a = -0.8mS$, $G_b = -0.5mS$ and $\sigma = 16$, $\rho = 45.6$, $\beta = 4$, $\nu = 1500$, $k_1 = k_2 = k_3 = 100$.

$R, R_0, C_1, C_2, L, \beta, \sigma > 0$ and $-3 < \mu < 1$.

$$\begin{array}{rcl} \frac{d\tilde{v}_1}{dt} & = & \frac{1}{C_1}[G(\tilde{v}_2 - \tilde{v}_1) - f(v_1)] \\ \frac{d\tilde{v}_2}{dt} & = & \frac{1}{C_2}[G(\tilde{v}_1 - \tilde{v}_2) + \tilde{i}_3] \\ \frac{d\tilde{i}_3}{dt} & = & -\frac{1}{L}(\tilde{v}_2 + R_0\tilde{i}_3) \\ \frac{d\tilde{x}}{dt} & = & \nu(\sigma(\tilde{y} - \tilde{x}) + k_1\tilde{v}_1) \\ \frac{d\tilde{y}}{dt} & = & \nu(\mu\tilde{x} - \tilde{y} + x(\rho - \mu - \tilde{z}) + k_2\tilde{v}_2) \\ \frac{d\tilde{z}}{dt} & = & \nu(-\beta\tilde{z} + x\tilde{y} + k_3\tilde{i}_3) \end{array} \quad (64)$$

This method can be used to cascade several chaotic systems which can be synchronized to identical copies. Similarly, theorem 10 can be used to connect several chaotic systems such as Chua's oscillator and Lorenz system which can be synchronized to identical copies.

7 Conclusions

In this paper, we give a unified framework to analyze synchronization and control between two dynamical systems. We show how asymptotical synchronization is related to asymptotical stability and give conditions which ensure synchronization. We illustrate how synchronizing systems can be cascaded and connected and give conditions for robust synchronization. The main tool used to prove asymptotical stability and asymptotical synchronization is Lyapunov's direct method.

Acknowledgements

This work is supported in part by the Office of Naval Research under grant N00014-89-J-1402, by the National Science Foundation under grant MIP 86-14000, and by the Josephine de Kármán Fellowship.

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