ESTIMATING THE DIMENSIONAL CHARACTERISTICS OF TWO-DIMENSIONAL PATTERNS

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75-B LeConte Hall

Memorandum No. UCB/ERL M94/54

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Estimating the Dimensional Characteristics of Two-Dimensional Patterns

A.L.Zheleznyak and L.O.Chua

Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA

Abstract

Dynamical properties of two-dimensional patterns generated by spatially extended systems can be described via the characteristics of attractors in the matrix phase space of the associated translation (or translational-evolution) dynamical systems. Questions regarding the possibility of estimating the fractal dimensions of two-dimensional patterns from the fractal dimensions of one-dimensional observables scanning the patterns along a chosen path are investigated. The presented proofs state that the generalized dimensions of the *scanning* observables are lower bounds for estimating the corresponding generalized dimensions of two-dimensional patterns. Spatial field distributions defined as superposition of planar waves and different spatiotemporal patterns produced by cellular neural networks made of Chua's circuits are studied numerically. The results of computer experiments confirm the theoretical predictions presented in this paper.

1 Introduction

The *dimension* of an attractor or invariant set is one of the key notions in the study of dynamical systems with chaotic behavior [Mandelbrot, 1982; Farmer *et al.*, 1983]. It is well known that the dimension of a set invariant under the action of the shift operator along the trajectories of a system characterizes the number of effective degrees of freedom involved in the system's dynamics. Even only a knowledge about the "finiteness" of the dimension is important in essence because it implies to that the properties of the system are determined by the interaction among only a limited number of eigenmodes, and not by the influence of random forces. If, in addition, the dimension is not an integer, i.e. the trajectories are confined on a fractal set, then the process under consideration represents deterministic chaos. In spatially extended systems possessing typically infinite-dimensional attractors the computation of dimension-like characteristics, and the extraction of different kinds of scaling, have also been found to be very useful for an understanding of the underlying dynamics (see, e.g., Grassberger [1989]; Torcini *et al.* [1991; Tsimring [1993]).

To date a number of effective methods for calculating various dimensional characteristics have been developed. Using as a rule *time-delay* techniques they allow us to infer properties of the system directly from an *observable*, i.e. from a realization of one (or a few) state vector components of the system (see, e.g., Theiler [1989], Grassberger *et al.* [1991]). The main idea of all these methods was clearly formulated by Takens [1980], (see, also, Packard *et al.* [1980]). Let us examine, for example, a scalar observable, i.e. a sequence of scalar measurements $\underline{u} = \{u(i)\}$. For each integer m, called the *embedding dimension*, we can reconstruct a new dynamical system with a phase space of $m-$ dimensional vectors, $U_i^{(m)} = (u(i), u(i+1), ..., u(i+m-1))$, and some one-parameter shift operator $S^k : U_i^{(m)} \rightarrow U_{i+k}^{(m)}$, acting on this space. As proven by Takens [1980] and Mañe [1980] (see, also, the refined formulation in Sauer & Yorke [1993]), under certain conditions many dynamical invariants (including the fractal dimension of invariant sets) of this reconstructed system coincide with those of the original system. If the fractal dimension computed from an observable is a finite one, then the observable and the process of interest are called *finitely-generated.*

The ideas of the study of dynamical systems directly from observables are now very popular and widespread not only in computational, but also in real experiments. However, remarkable progress has been achieved mainly for systems with complex *temporal* dynamics. Recently, a new approach was proposed that generalizes these ideas to the analysis of disordered spatial field distributions - *snapshots* and complex spatio-temporal *patterns* in spatially extended systems [Afraimovich *et al.*, 1992; 1993; Zheleznyak & Chua 1994]. It is natural to consider spatial field distributions and spatio-temporal patterns as observables. But they are no longer vectors; for a planar geometry the observables are matrices $\underline{u} = \{u_{ij}\}$, where $u_{ij} = u(x_i, y_j)$ for spatial, and $u_{ij} = u(t_i, x_j)$ for spatio-temporal distributions, respectively. In general, the observables $\underline{u}$ are $d-$ dimensional tensors $\{u_{\mathbf{i}}; \mathbf{i} \in \mathbf{Z}^d\}$ (or $\{u_{\mathbf{i}}; \mathbf{i} \in \mathbf{Z} \times \mathbf{Z}^{d-1}\}$), but for simplicity we will assume in this paper that the observables are two-dimensional. In this case we can reconstruct a new dynamical system with a phase space of $(m \times m)$ - matrices

$$U_{ij}^{(m)} = \{u_{kl}; k = i, ..., i+m-1; l = j, ..., j+m-1\}, \tag{1}$$

and define a *translation* (or *translational evolution*) operator T, which acts on this space as: $T^{(\rho_1,\rho_2)} : U_{ij} \to U_{i+\rho_1 j+\rho_2}$, where $\rho_1, \rho_2 \in \mathbf{Z}$ (or $\rho_1 \in \mathbf{Z}^+$, $\rho_2 \in \mathbf{Z}$) . Note that the operator T now depends on two parameters and the trajectory of the reconstructed system is a *discrete* surface, i.e. a set of points lying on a two-dimensional surface, but not on a curve.

The study of dimensional characterictics of these dynamical systems with a matrix phase space can reveal new important information about the properties of the underlying processes generating the distribution. t on a curve Unfortunately, such analysis is usually based on massive numerical computations. It is therefore very tempting to estimate the properties of the distributions by applying efficient techniques and tools which have been developed in recent years for studying time series. It is apparent, that given a two-dimensional pattern (snapshot) we can construct various one-dimensional observables by *scanning* the picture in different ways. Treating these new, *scanning observables* by standard tools we can reconstruct dynamical systems with, generally speaking, distinctive properties. However, it looks reasonable, that some characteristics of the two-dimensional observables and associated one-dimensional scanning observables should be correlated.

In this paper we present some theoretical and numerical results confirming that, at least in particular cases, such relations do exist between the fractal dimensions of two-dimensional observables and their one-dimensional scanning observables.

2 Dimensions of two-dimensional patterns

Let us first introduce the *dimensions* of two-dimensional patterns and their corresponding scanning observables. Suppose we study the dynamical properties of a spatially extended system and have at our disposal a two-dimensional pattern of a field distribution $\underline{u}$ defined at the nodes of an integer lattice, i.e. $\underline{u} = \{u_{ij}\}$. Let us fix a pattern of a finite size N (for simplicity we will take a square picture , i.e. $i, j = 1, ..., N$), and suppose that N may be taken as large as desired. Then, for each integer m, and a *measurements resolution* ε we can introduce the approximate pointwise dimension of the pattern at each node (i, j) as

$$\tilde{D}_p^{(2)}(i,j,m,N,\varepsilon) = \frac{\log B_{ij}^{(m \cdot m)}(N,\varepsilon)}{\log \varepsilon}. \tag{2}$$

Here

$$B_{ij}^{(m \cdot m)}(N,\varepsilon) = \frac{1}{(N-m)^2} \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|) \tag{3}$$

denotes the pointwise mass function; $U_{ij}^{(m)}$ denotes $(m \times m)$ – matrices defined by (1), so that m^2 is the dimension of the embedding space, $\|\cdot\|$ defines a matrix norm in the space $\mathbf{R}^{m^2}$, and Θ denotes the Heaviside function:

$$\Theta(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 1. \end{cases}$$

If the function $B_{ij}^{(m \cdot m)}(N,\varepsilon)$ exhibits a scaling property when the size of the pattern tends to infinity and the resolution tends to zero, i.e. the limit

$$D_p^{(2)}(i,j,m) = \lim_{\varepsilon \to 0} \overline{\lim_{N \to \infty}} \tilde{D}_p^{(2)}(i,j,m,N,\varepsilon) \tag{4}$$

exists and does not depend on the embedding dimension when $m > m^*$, then the value $D_p^{(2)}(i,j,m)$ defines the pointwise dimension of the two-dimensional pattern $\underline{u}$ at the node (i, j).

The pointwise dimension is a *local* characteristic of the observable. Averaging in different ways the pointwise mass function over the observable, we can define *global* dimensional characteristics. Following the ideas of Grassberger & Procaccia [1983] we can introduce a whole spectrum of two-dimensional generalized dimensions $D_q^{(2)}$, which quantify the multifractal properties of the underlying dynamics. A one-parameter family of generalized dimensions of the pattern can be defined as the limit

$$D_q^{(2)}(m) = \lim_{\varepsilon\to 0} \overline{\lim_{N\to\infty}} \tilde{D}_q^{(2)}(m, N, \varepsilon), \qquad -\infty < q < \infty. \tag{5}$$

The approximate generalized dimensions $\tilde{D}_q^{(2)}(m, N, \varepsilon)$ have the form

$$\tilde{D}_q^{(2)}(m, N, \varepsilon) = \frac{\log G_q^{(2)}(m, N, \varepsilon)}{\log \varepsilon}, \tag{6}$$

where the generalized average mass function

$$G_q^{(2)}(m, N, \varepsilon) \equiv [\langle B_{ij}^{(m\cdot m)}(N, \varepsilon)\rangle^{q-1}]^{1/q-1} \tag{7}$$

can be written in a form analogous to that proposed by Pawelzik & Shuster [1987]:

$$G_q^{(2)}(m, N, \varepsilon) = \left\{\frac{1}{(N-m)^2} \sum_{i,j=1}^{N-m} [B_{ij}^{(m\cdot m)}(N, \varepsilon)]^{q-1}\right\}^{1/q-1} \equiv \tag{8}$$

$$\left\{\frac{1}{(N-m)^2} \sum_{i,j=1}^{N-m} \left[\frac{1}{(N-m)^2 - 1} \sum_{\substack{i',j'=1 \\ (i',j')\neq(i,j)}}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|)\right]^{q-1}\right\}^{1/q-1}.$$

Note that $q = 2$ gives an estimate of the easiest to compute two-dimensional correlation dimension $\tilde{D}_2^{(2)}(m)$, and that $q = 1$ gives an approximate value of the two-dimensional information dimension $\tilde{D}_1^{(2)}(m)$.

Let us now construct a one-dimensional scanning observable $\underline{v}$ from the pattern of interest $\underline{u}$. Among the infinitely many different ways for scanning a two-dimensional pattern we will choose a simple path along the rows of the array $\underline{u}$, as shown in Fig.1. Thus, we have the following rule for relating the components of the two-dimensional observable $\underline{u} = \{u_{ij}; i,j = 1,...,N\}$ and the one-dimensional observable $\underline{v} = \{v_I; I = 1,...,N^2\}$:

$$v_I = u_{ij}, \quad I = (i-1)N + j; \; i,j = 1,...,N. \tag{9}$$

Having defined the scanning observable, we can further introduce in a standard fashion the one-dimensional pointwise dimension, $D_p^{(1)}$, and the generalized dimensions, $D_q^{(1)}$. Namely, we can define the approximate pointwise dimension at the node (i,j) for fixed m, N, and ε as

$$\tilde{D}_p^{(1)}(m,I,N,\varepsilon) = \frac{\log B_I^{(m)}(N,\varepsilon)}{\log \varepsilon}, \tag{10}$$

where the index I is related to (i,j) by the expression (9). Here $B_I^{(m)}(N,\varepsilon)$ denotes the pointwise mass function

$$B_I^{(m)}(N,\varepsilon) = \frac{1}{N^2 - m} \sum_{I'=1}^{N^2-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|), \tag{11}$$

$V_I^{(m)}$ denotes an m-dimensional vector

$$V_I^{(m)} = \{v_k; I \leq k \leq I + m - 1\},$$

and $\|\cdot\|$ denotes a vector norm in the space $\mathbf{R}^m$. If the limit

$$D_p^{(1)}(m,I) = \lim_{\varepsilon\to 0} \overline{\lim_{N\to\infty}} \tilde{D}_p^{(1)}(m,I,N,\varepsilon) \tag{12}$$

exists and does not depend on m (when $m > m_*$), then the quantity $D_p^{(1)}$ is the pointwise dimension of the scanning observable $\underline{v}$ at the node (i,j).

Analogously, we can define one-parameter families of the approximate generalized dimensions $\tilde{D}_q^{(1)}$ and the exact generalized dimensions $D_q^{(1)}$ of the scanning observable. The approximate generalized dimensions are defined by the expression

$$\tilde{D}_q^{(1)}(m,N,\varepsilon) = \frac{\log G_q^{(1)}(m,N,\varepsilon)}{\log \varepsilon}, \qquad -\infty < q < \infty, \tag{13}$$

where the generalized average mass function

$$G_q^{(1)}(m,N,\varepsilon) \equiv [\langle B_I^{(m)}(N,\varepsilon)\rangle^{q-1}]^{1/q-1} \tag{14}$$

has the form [Pawelzik & Shuster, 1987]:

$$G_q^{(1)}(m,N,\varepsilon) = \left\{ \frac{1}{N^2-m} \sum_{I=1}^{N^2-m} \left[\frac{1}{N^2-m-1} \sum_{\substack{I'=1 \\ I'\neq I}}^{N^2-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) \right]^{q-1} \right\}^{1/q-1}. \tag{15}$$

The exact generalized dimensions are defined as follow:

$$D_q^{(1)}(m) = \lim_{\varepsilon \to 0} \overline{\lim_{N\to\infty}} \tilde{D}_q^{(1)}(m,N,\varepsilon), \qquad -\infty < q < \infty. \tag{16}$$

3 Some relations between dimensions of two-dimensional field distributions

In this section we present some results which establish the relationships between the dimensions of two-dimensional and the corresponding scanning observables. In Sec. 2 we introduced definitions of these dimensions. The definitions admit an arbitrary choice of norms in the reconstructed matrix and vector phase spaces. Although all norms are equivalent in each finite-dimensional linear space, and the fractal dimensions are invariant with respect to the choice of the norm, the choice of a particular norm can influence the practical estimate of the dimension when the size "m" of the observable and the measurement resolution "ε" are finite.

In our proofs we will use the families of Hölder's vector norms

$$\|V_I^m\| = \{\sum_{L=0}^{m-1} |v_{I+L}|^s\}^{1/s}, \tag{17}$$

and Hölder's matrix norms

$$\|U_{ij}^m\| = \{\sum_{k,l=0}^{m-1} |u_{i+k,j+l}|^s\}^{1/s}. \tag{18}$$

For particular values of the parameter s these families of norms include some well-known norms: for $s = 1$ we get the *octaedric* norms, for $s = 2$ we get the *Euclidean* norms, and for $s = \infty$ we get the *cubic* norms (see Gantmacher [1960]). The main advantage of the last type of norms, also called *maximum* norms, is its ease of calculation, which enables us to write efficient computing codes. Note, however, that the question of the proper choice of a matrix norm which reflects optimally the structure of the patterns needs further study (see also the discussion in Abarbanel *et al.* [1993]).

Let us first formulate a theorem, relating the local pointwise dimensions of two-dimensional patterns.

Theorem 1 *For each integer m at any node (i,j) the pointwise dimension $D_p^{(2)}(i,j,m)$ of the two-dimensional pattern $\underline{u}$ and the pointwise dimension*

$D_p^{(1)}(I,m)$ *of the corresponding scanning observable* $\underline{v}$*, constructed along the path (9), are related by the inequality*

$$D_p^{(1)}(I,m) \le D_p^{(2)}(i,j,m). \tag{19}$$

Proof.

Consider the expression:

$$\sum_{I'=1}^{N^2-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) \equiv \sum_{\substack{i',j'=1 \\ I'=(i'-1)N+j'}}^{N-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) +$$

$$\sum_{i'=N-m+1}^{N} \sum_{\substack{j'=1 \\ I'=(i'-1)N+j'}}^{N-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) + \tag{20}$$

$$\sum_{i'=1}^{N-1} \sum_{\substack{j'=N-m+1 \\ I'=(i'-1)N+j'}}^{N} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|).$$

From the definitions of the vector and matrix norms (17),(18) it follows that

$$\|U_{ij}^{(m)} - U_{i'j'}^{(m)}\| \equiv \{\sum_{k,l=0}^{m-1} |u_{i+k,j+l} - u_{i'+k,j'+l}|^s\}^{1/s} \ge \{\sum_{l=0}^{m-1} |u_{i,j+l} - u_{i',j'+l}|^s\}^{1/s} \equiv \|V_I^{(m)} - V_{I'}^{(m)}\|$$

for all sets of indices $(i,j), I$, and $(i',j'), I'$ connected by the equality (9).

Taking into account that for any $\alpha \le \beta$

$$\Theta(\varepsilon - \alpha) \ge \Theta(\varepsilon - \beta),$$

and that the last two sums in (20) are always nonnegative we get

$$\sum_{I'=1}^{N^2-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) \geq \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|).$$

Further, after dividing both parts of this inequality by $1/(N^2 - m)$, we have

$$\frac{1}{N^2 - m} \sum_{I'=1}^{N^2-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) \geq$$

$$\left[\frac{1}{(N-m)^2} - \frac{m(2N-m-1)}{(N-m)^2(N^2-m)}\right] \cdot \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|) \geq$$

$$\frac{1}{(N-m)^2} \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|) - R(m, N), \tag{21}$$

where

$$R(m, N) = \frac{m(2N-m-1)}{N^2 - m}. \tag{22}$$

Recalling expressions (3) and (11), we can write

$$B_I^{(m)}(N, \varepsilon) + R(m, N) \geq B_{ij}^{(m \cdot m)}(N, \varepsilon). \tag{23}$$

Taking the logarithms of both parts of this inequality, dividing by $\log \varepsilon$, and taking into account that $\log \varepsilon < 0$ for small ε, we get the following inequality:

$$\frac{\log [B_I^{(m)}(N, \varepsilon) + R(m, N)]}{\log \varepsilon} \leq \frac{\log [B_{ij}^{(m \cdot m)}(N, \varepsilon)]}{\log \varepsilon}. \tag{24}$$

Finally, passing to the limits $N \to \infty$ and then $\varepsilon \to 0$, and taking into consideration that $R(N) \to 0$ as $N \to \infty$, we obtain:

$$D_p^{(1)}(I, m) \leq D_p^{(2)}(i, j, m). \tag{25}$$

Theorem 1 establishes the connection between the local dimensional characteristics of the two-dimensional observables and the associated one-dimensional scanning observables at each node of the given pattern. This relationship can be expanded to the whole spectrum of the generalized dimensions. Indeed, it follows from expressions (7) and (14) that the generalized dimensions may be introduced through different averages of the pointwise mass function over all points of the corresponding observable.

Let us consider the following expression

$$\left\{ \frac{1}{(N-m)^2} \sum_{\substack{i,j=1 \\ I=(i-1)N+j}}^{N-m} [B_I^{(m)}(N,\varepsilon) + R(m,N)]^{q-1} \right\}^{1/q-1} \equiv \mathcal{A}, \qquad (26)$$

where $R(m,N)$ is defined by (22), and $-\infty < q < \infty$.

It is easy to show that the function

$$f(a_1, a_2, ..., a_N) = \{\sum_{I=1}^{N} a_I^{q-1}\}^{1/q-1}, \qquad a_I \geq 0$$

is always a monotonically increasing function of its arguments for all $-\infty < q < \infty$, because

$$\frac{\partial}{\partial a_I}\{\sum_{I=1}^{N} a_I^{q-1}\}^{1/q-1} \equiv a_I^{q-2}\{\sum_{I=1}^{N} a_I^{q-1}\}^{\frac{2-q}{q-1}} \geq 0.$$

Therefore, since $B_I, R \geq 0$, and the inequality (23) holds for all (i,j) and I, we can write

$$\mathcal{A} \geq \left\{ \frac{1}{(N-m)^2} \sum_{i,j=1}^{N-m} [B_{ij}^{(m\cdot m)}(N,\varepsilon)]^{q-1} \right\}^{1/q-1} \qquad (27)$$

$$\sum_{I'=1}^{N^2-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) \geq \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|).$$

Further, after dividing both parts of this inequality by $1/(N^2 - m)$, we have

$$\frac{1}{N^2 - m} \sum_{I'=1}^{N^2-m} \Theta(\varepsilon - \|V_I^{(m)} - V_{I'}^{(m)}\|) \geq$$

$$\left[\frac{1}{(N-m)^2} - \frac{m(2N-m-1)}{(N-m)^2(N^2-m)}\right] \cdot \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|) \geq$$

$$\frac{1}{(N-m)^2} \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{ij}^{(m)} - U_{i'j'}^{(m)}\|) - R(m, N), \tag{21}$$

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$$R(m, N) = \frac{m(2N - m - 1)}{N^2 - m}. \tag{22}$$

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Taking the logarithms of both parts of this inequality, dividing by $\log \varepsilon$, and taking into account that $\log \varepsilon < 0$ for small ε, we get the following inequality:

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Finally, passing to the limits $N \to \infty$ and then $\varepsilon \to 0$, and taking into consideration that $R(N) \to 0$ as $N \to \infty$, we obtain:

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$$\left\{ \frac{1}{(N-m)^2} \sum_{\substack{i,j=1 \\ I=(i-1)N+j}}^{N-m} [B_I^{(m)}(N,\varepsilon) + R(m,N)]^{q-1} \right\}^{1/q-1} \equiv \mathcal{A}, \tag{26}$$

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is always a monotonically increasing function of its arguments for all $-\infty < q < \infty$, because

$$\frac{\partial}{\partial a_I}\{\sum_{I=1}^{N} a_I^{q-1}\}^{1/q-1} \equiv a_I^{q-2}\{\sum_{I=1}^{N} a_I^{q-1}\}^{\frac{2-q}{q-1}} \geq 0.$$

Therefore, since $B_I, R \geq 0$, and the inequality (23) holds for all (i,j) and I, we can write

$$\mathcal{A} \geq \left\{ \frac{1}{(N-m)^2} \sum_{i,j=1}^{N-m} [B_{ij}^{(m\cdot m)}(N,\varepsilon)]^{q-1} \right\}^{1/q-1} \tag{27}$$

for any N, m, and q.

On the other hand,

$$\mathcal{A} = \left\{ \frac{1}{N^2 - m} \left[\sum_{I=1}^{N^2-m} [B_I^{(m)}(N,\varepsilon) + R(m,N)]^{q-1} - \right. \right.$$

$$\left. \sum_{I=(N-m)^2+1}^{N^2-m} [B_I^{(m)}(N,\varepsilon) + R(m,N)]^{q-1} \right] + \tag{28}$$

$$\left. \frac{m(2N-m-1)}{(N-m)^2(N^2-m)} \sum_{\substack{i,j=1 \\ I=(i-1)N+j}}^{N-m} [B_I^{(m)}(N,\varepsilon) + R(m,N)]^{q-1} \right\}^{1/q-1} .$$

Suppose that the average mass function is uniformly bounded by N, when $N \gg 1$: $0 < b_1 \leq B_I^{(m)}(N,\varepsilon) \leq b_2$ (where b_1 and b_2 denote some constants). Then, using Taylor's expansions in powers of $1/N$, we can obtain following estimates

$$[B_I^{(m)}(N,\varepsilon) + R(m,N)]^{q-1} = [B_I^{(m)}(N,\varepsilon)]^{q-1} + (q-1)[B_I^{(m)}(N,\varepsilon)]^{q-2} \cdot \frac{1}{N} + \ldots \sim$$

$$[B_I^{(m)}(N,\varepsilon)]^{q-1} + O(\frac{1}{N}),\ N \gg 1, \tag{29}$$

and

$$\mathcal{A} \sim \left\{ \frac{1}{N^2 - m} \sum_{I=1}^{N^2-m} [B_I^{(m)}(N,\varepsilon)]^{q-1} + O(\frac{1}{N}) \right\}^{1/q-1} \sim$$

$$\left\{ \frac{1}{N^2 - m} \sum_{I=1}^{N^2-m} [B_I^{(m)}(N,\varepsilon)]^{q-1} \right\}^{1/q-1} + O(\frac{1}{N}),\ N \gg 1. \tag{30}$$

Combining expressions (27) and (30) we obtain the asymptotic relation ($N \gg$

1)

$$\left\{\frac{1}{N^2-m}\sum_{l=1}^{N^2-m}[B_l^{(m)}(N,\varepsilon)]^{q-1}\right\}^{1/q-1}+O(\frac{1}{N})\geq$$

$$\left\{\frac{1}{(N-m)^2}\sum_{i,j=1}^{N-m}[B_l^{(m)}(N,\varepsilon)+R(m,N)]^{q-1}\right\}^{1/q-1}, \tag{31}$$

or (see (9),(15))

$$G_q^{(1)}(m,N,\varepsilon)+O(\frac{1}{N})\geq G_q^{(2)}(m,N,\varepsilon), \qquad N\gg 1, -\infty<q<\infty. \tag{32}$$

Further, taking logarithms and dividing both parts of the last inequality by $\log\varepsilon$, we have

$$\frac{\log G_q^{(1)}(m,N,\varepsilon)+O(\frac{1}{N})}{\log\varepsilon}\leq\frac{\log G_q^{(2)}(m,N,\varepsilon)}{\log\varepsilon}. \tag{33}$$

Finally, letting $N\to\infty$ and $\varepsilon\to 0$ we obtain the formulation of the following general result:

Theorem 2 *For each integer m and any value of the parameter $q\in(-\infty,\infty)$ the generalized dimensions $D_q^{(2)}(m)$ of the two-dimensional pattern $\underline{u}$ and the generalized dimensions $D_q^{(1)}(m)$ of the corresponding scanning observable $\underline{v}$ constructed along the path (9) are related by the inequality*

$$D_q^{(1)}(m)\leq D_q^{(2)}(m). \tag{34}$$

A trivial, but important consequence of this Theorem is

Corollary 1 *If the two-dimensional pattern $\underline{u}$ is* **finitely-generated**, *then the corresponding one-dimensional observable $\underline{v}$ scanning $\underline{u}$ along the path (9) is also* **finitely- generated**. *Conversely, if the scanning observable $\underline{v}$ is* **infinitely- generated**, *then the pattern $\underline{u}$ is also* **infinitely- generated**.

4 Computer experiments

The theorems presented in the previous section state that the fractal dimensions of the scanning observable give a lower bound for the corresponding dimensions of the two-dimensional patterns. To confirm these theoretical results we carried out calculations of the fractal dimension for several two-dimensional field distributions. We chose the correlation dimensions $D_2^{(2)}$ and $D_2^{(1)}$ because they give the most reliable and efficiently calculated dimensional characteristics. To reduce the computing time, which is rather high when calculating two-dimensional characteristics, we fixed a set of reference elements. In this case the average mass functions $G_q^{(2)}(m, N, \varepsilon)$ and $G_q^{(1)}(m, N, \varepsilon)$, which for $q = 2$ are called *correlation integrals*, take on the forms (compare with (9), (15))

$$G_2^{(2)}(m, N, \varepsilon) = \frac{1}{p_{ref} \cdot s_{ref} \cdot (N-m)^2} \sum_{p=1}^{p_{ref}} \sum_{s=1}^{s_{ref}} \sum_{i',j'=1}^{N-m} \Theta(\varepsilon - \|U_{i_p,j_s}^{(m)} - U_{i'j'}^{(m)}\|) \tag{35}$$

and

$$G_2^{(1)}(m, N, \varepsilon) = \frac{1}{P_{ref} \cdot (N^2-m)} \sum_{P=1}^{P_{ref}} \sum_{I'=1}^{N^2-m} \Theta(\varepsilon - \|V_{I_P}^{(m)} - V_{I'}^{(m)}\|). \tag{36}$$

Here $\{U_{i_p,j_s}; p = 1, ..., p_{ref}, s = 1, ..., s_{ref}\}$ denotes the set of reference matrices, and $\{V_{I_P}; P = 1, ..., P_{ref}\}$ denotes the set of reference vectors. We chose an equidistant distribution of the reference elements along the observables, i.e. we put

$$(i_p, j_s) = \left(\left[\frac{N-m}{p_{ref}}\right] \cdot (p-1), \left[\frac{N-m}{s_{ref}}\right] \cdot (s-1)\right) ;\ p = 1, ..., p_{ref}, s = 1, ..., s_{ref},$$

and

$$I_P = \left[\frac{N^2 - m}{P_{ref}}\right] \cdot (P-1);\ P = 1, ..., P_{ref},$$

where the function $[\,\cdot\,]$ denotes the integer part of its argument. To preserve the quantitative relations obtained in the previous sections we put $P_{ref} = p_{ref} \cdot s_{ref}$.

First of all, as test examples, we computed the correlation dimensions $D_2^{(2)}$ and $D_2^{(1)}$ of several artificially syntesized two-dimensional field distributions, which can be defined by linear superpositions of P_n planar waves:

$$u_{ij} = \mathrm{Re}\left[\sum_n^{P_n} e^{\mathrm{i}\bar{k}_n \cdot \begin{pmatrix} i \\ j \end{pmatrix}}\right],\ i,j = 1, ..., N. \tag{37}$$

Here $\mathrm{i} = \sqrt{-1}$, $\bar{k}_n$ denotes the wave vectors, and $\begin{pmatrix} i \\ j \end{pmatrix}$ denotes the indices of spatial position.

To each spatial distribution we can relate an invariant set in the matrix phase space of the translation dynamical system (see Afraimovich *et al.* [1992]; Zheleznyak & Chua [1994]). If the distribution is a superposition of n linearly independent planar waves (i.e. waves with linearly independent wave vectors), then the invariant set is an n-dimensional torus, i.e it has an integer dimension equal to n. In Fig.2, the spatial field distribution is shown which represents the superposition of two planar waves with wave vectors $\bar{k}_1 = (1,0)$ and $\bar{k}_2 = (0, \sqrt{3}/2)$. This distribution is periodic in both i and j directions, and the invariant set is a 2-dimensional torus in the associated matrix phase space. In Fig.3(a) the correlation integrals $G_2^{(2)}$ (solid lines) and $G_2^{(1)}$ (dotted lines) are presented, which are calculated directly from the two-dimensional pattern and from the scanning observable according to the formulas (35) and

(36), respectively. We put $N = 250$, $r_{ref} = s_{ref} = 16$, $R_{ref} = 256$. The odd values of the parameter m defining the dimension of the embedding space are taken from 1 (upper lines) to 9 (lowest lines). In Fig.3(b) the slopes of the coresponding correlation integrals are plotted, which give estimates for the correlation dimensions $D_2^{(2)}$ and $D_1^{(2)}$. We can see that over a rather wide interval of the resolution ε the approximate correlation dimension of the two-dimensional pattern, $D_2^{(2)}$, (solid lines) is, indeed, equal to 2 [1]; and the approximate correlation dimension of the scanning observable, $D_2^{(1)}$, (dotted lines) does not exceed $D_2^{(2)}$, thereby confirming the statement of Theorem 2.

In addition, we have calculated the correlation dimensions of the spatiotemoral patterns generated by a one-dimensional cellular neural network (CNN) made of Chua's circuits ([Madan, 1993],[Shil'nikov, 1994]), which mimics a spatially extended reaction-diffusion medium. The dynamics of the CNN is described by the following system of $3N$ ordinary differential equations:

$$\left\{ \begin{array}{lcl} \dot{x}_j &=& \alpha(y_j - x - h(x_j)) + D(x_{j+1} - 2x_j + x_{j-1}) \\ \dot{y}_j &=& x_j - y_j + z_j \\ \dot{z}_j &=& -\beta y_j \end{array} \right. \tag{38}$$

with periodic boundary conditions:

$$x_j(t) = x_{N+j}(t), \quad y_j(t) = y_{N+j}(t), \quad z_j(t) = z_{N+j}(t),\ j = 1, ..., N.$$

Here, $h(x) = m_1 x + 0.5(m_0 - m_1)[|x+1| - |x-1|]$ is a three-segment piecewise-linear function; D is a dissipative coupling coefficient; α, β, m_0, and m_1 are parameters of the uncoupled Chua's circuit ([Madan,1993]).

[1]The curves over the other regions of ε oscillate erratically due to the effects of discretization, the inevitable numerical noise, etc. This phenomena are typical in such computations (see [Theiler, 1989])

The behavior of the above system was studied by Zheleznyak & Chua [1994] for the set of parameters $(\alpha, \beta, m_0, m_1) = (9, 19, -8/7, -5/7)$. In this case two stable limit cycles, symmetrical with respect to the origin, exist in the phase space of the uncoupled Chua's circuit. It was demonstrated that the dynamics of the system (38) is very rich: from very simple spatially homogeneous and periodic in time to a complex fully developed spatiotemporal chaos. Also it was found that the chaotic patterns can have different, low or high, correlation dimensions depending on the initial conditions. To put it in other words, different strange attractors coexist in the associated matrix phase space of the system (38).

We have calculated the correlation dimensions for the spatiotemporal patterns generated by the system (38) with two distinct types of initial conditions:

$$x_j(0) = \sin(2\pi(j-1)/N), \quad y_j(0) = z_j(0) = 0.0, \tag{39}$$

and

$$x_j(0) = 1.1 + \sin(2\pi(j-1)/N), \quad y_j(0) = z_j(0) = 0.0. \tag{40}$$

These initial conditions were chosen to be different from those presented in [Zheleznyak & Chua,1994] to provide some diversity for comparison purposes. The spatiotemporal pattern for the initial conditions (39) with a dissipative coupling coefficient $D = 0.4$ is shown in Fig. 4. The length of the CNN was chosen to be $N = 256$, and we took $N_t = 500$ time units with a time step $\Delta t = 1$. Observe that for these initial conditions the spatiotemporal pattern splits into two sub-patterns due to the bistability of the cell's dynamics. This pattern has a *low* correlation dimension, because sub-patterns represent "small" systems (according to the terminology of Cross & Hohenberg [1993]), and the interaction between sub-patterns is weak (see Zheleznyak & Chua

[1994]). In Fig.5,a several plots of the correlation integrals $\log G_2^{(2)}$ (solid lines) and $\log G_2^{(1)}$ (dotted lines) *vs* $\log \varepsilon$ are presented. The corresponding plots of the slopes of the correlation integrals which give an approximation for the correlation dimensions $D_2^{(2)}$ (solid lines) and $D_2^{(1)}$ (dotted lines) are shown in Fig.5(b). Indeed, the correlation dimension $D_2^{(2)}$ calculated according to the expression (35) belongs to the range $[3.4, 3.9]$, and the correlation dimension $D_2^{(1)}$ of the corresponding scanning observable also falls within the same range. Thus, our numerical results agree well with the theory, inspite of the errors expected due to the finite size of the pattern and to nonoptimal choice of space and time lags, etc.

Spatiotemporal patterns generated by the system (38) with initial conditions (40) exhibit a different structure, as shown in Fig.6 for the dissipative coupling parameter $D = 0.4$. Such a choice of initial conditions does not cause the splitting of the pattern into sub-patterns, because all cells belong to the basin of attraction of the same limit cycle. That is why the system (38), (40) for D=0.4 can be considered as a "large" system, and generates a fully developed spatiotemporal pattern with a *high* fractal dimension of the associated attractor. Plots of the correlation integrals $\log G_2^{(2)}$ (solid lines) and $\log G_2^{(1)}$ (dotted lines), and their slopes $D_2^{(2)}$ (solid lines) and $D_1^{(2)}$ (dotted lines) *vs* $\log \varepsilon$ are presented in Fig.7(a) and 7(b) respectively. We can see that the inequality $D_2^{(1)} \leq D_2^{(2)}$ holds over a wide interval of the resolution ε and each m as predicted by Theorem 2.

The plots shown in Fig.7(b) have no intervals of ε where the correlation dimensions have a constant value. This phenomenon is typical for systems demonstrating a fully developed spatiotemporal chaos (with a high-dimensional attractors). However, both $D_2^{(2)}$ and $D_2^{(1)}$ reveal another type of scaling: a *linear growth* of the dimensions with the increase of the resolution ε. This scaling is manifested in the constant slopes of the graphs $D_2^{(2)}$ and $D_2^{(1)}$ *vs* $\log\varepsilon$ (see Fig.7(b)). The growth rate is proportional to the density of dimension [Tsimring, 1993], and, as we can see from the Fig.7(b), this rate is approximately the same for the correlation dimension of the two-dimensional pattern, $D_2^{(2)}$, and for the correlation dimension of the scanning observable, $D_2^{(1)}$. Thus, by analyzing the scanning observables we can, probably, estimate not only the dimensions of two-dimensional patterns, but also the densities of its dimensions.

5 Conclusion

In this paper we have demonstrated the possibility of estimating the dimensional characteristics of two-dimensional patterns generated by spatially extended systems, via its corresponding one-dimensional scanning observables. The relations found in this paper are sometimes not manifested so distinctively, because of the errors due to the imprecision in the numerical implementation (finite size of pattern, improper choice of the space and time lags, etc.), the inhomogeneous structure of attractors (lacunarity, multifractality), etc. But in all of the computations we have carried out, the behavior of dimensional characteristics of both two-dimensional patterns and scanning observables was qualitatively the same.

It looks reasonable that one can also estimate the entropy characteristics of two-dimensional patterns by analyzing the corresponding scanning observables in a similar way. And it is very important in general to find out what kinds of dynamical characteristics can be extracted and how these characteristics depend on the path of the scanning of the two-dimensional patterns by one-dimensional observables.

Acknowledgements

This work was supported in part by the Office of Naval Research under grant N 00014-89-J-1402, and by the National Science Foundation under grant MIP 86-14000.

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Figure Captions

Figure 1. The path of scanning the two-dimensional pattern $\underline{u} = \{u_{ij}; i, j = 1, ..., N\}$ by the one-dimensional observable $\underline{v} = \{v_I; I = 1, ..., N^2\}$.

Figure 2. Spatial pattern $\underline{u}$ defined as the linear superposition of two planar waves with wave vectors $\bar{k}_1 = (1, 0)$ and $\bar{k}_2 = (0, \sqrt{3}/2)$.

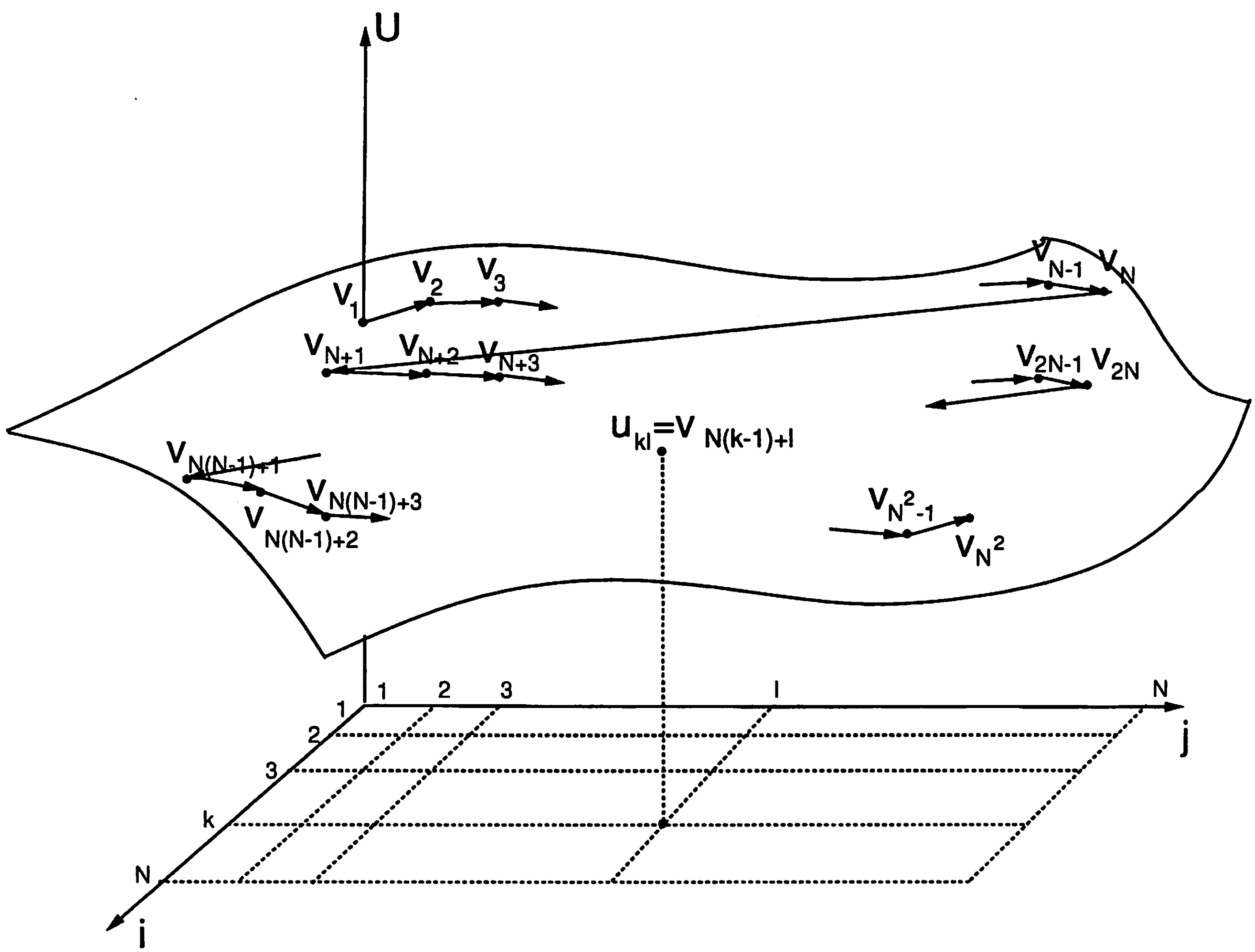
Figure 3. a) Plots of the correlation integrals of the two dimensional pattern shown in Fig.2, $G_2^{(2)}$, (solid lines) and for the sweepin observable, $G_2^{(1)}$, (dotted lines) *vs* ε in $\log - \log$ scale. Odd values of the parameter m are taken; upper lines correspond to $m = 1$, lowest lines correspond to $m = 9$. b) Plots of the correlation dimensions $D_2^{(2)}$ (solid lines) and $D_1^{(2)}$(dotted lines) calculated as the slopes of the corresponding correlation integrals presented in Fig.3(a).

Figure 4. Spatiotemporal pattern generated by a CNN made of dissipatively coupled Chua's circuits, described by the system (38) with initial conditions (39); dissipative coupling parameter $D = 0.4$.

Figure 5. The same as in Fig.3 for the spatiotemporal pattern shown in Fig.4.

Figure 6. Spatiotemporal pattern generated by a CNN made of dissipatively coupled Chua's circuits, described by the system (38) with initial conditions (40); dissipative coupling parameter $D = 0.4$.

Figure 7. The same as in Fig.3 for the spatiotemporal pattern shown in Fig.6.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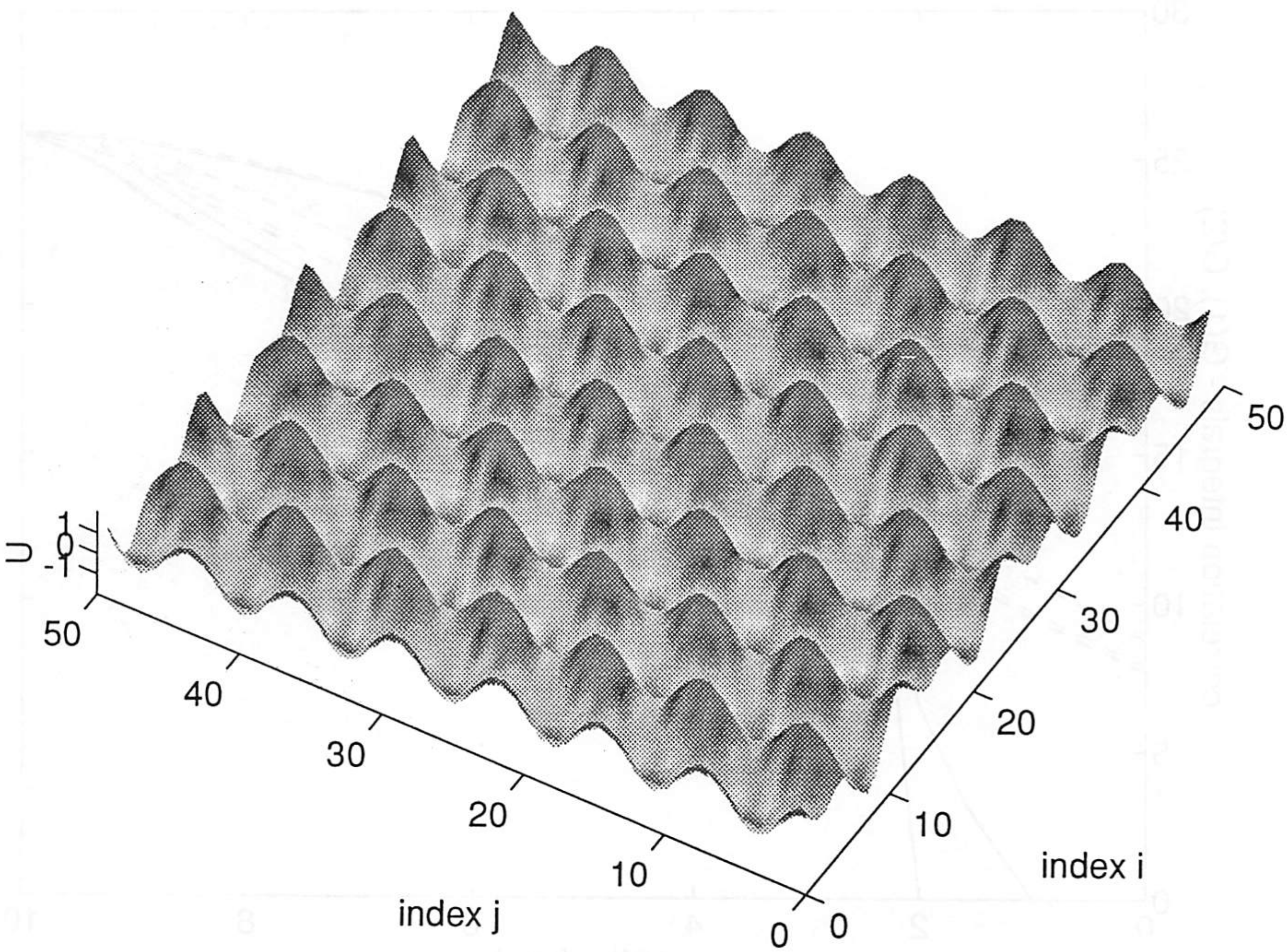
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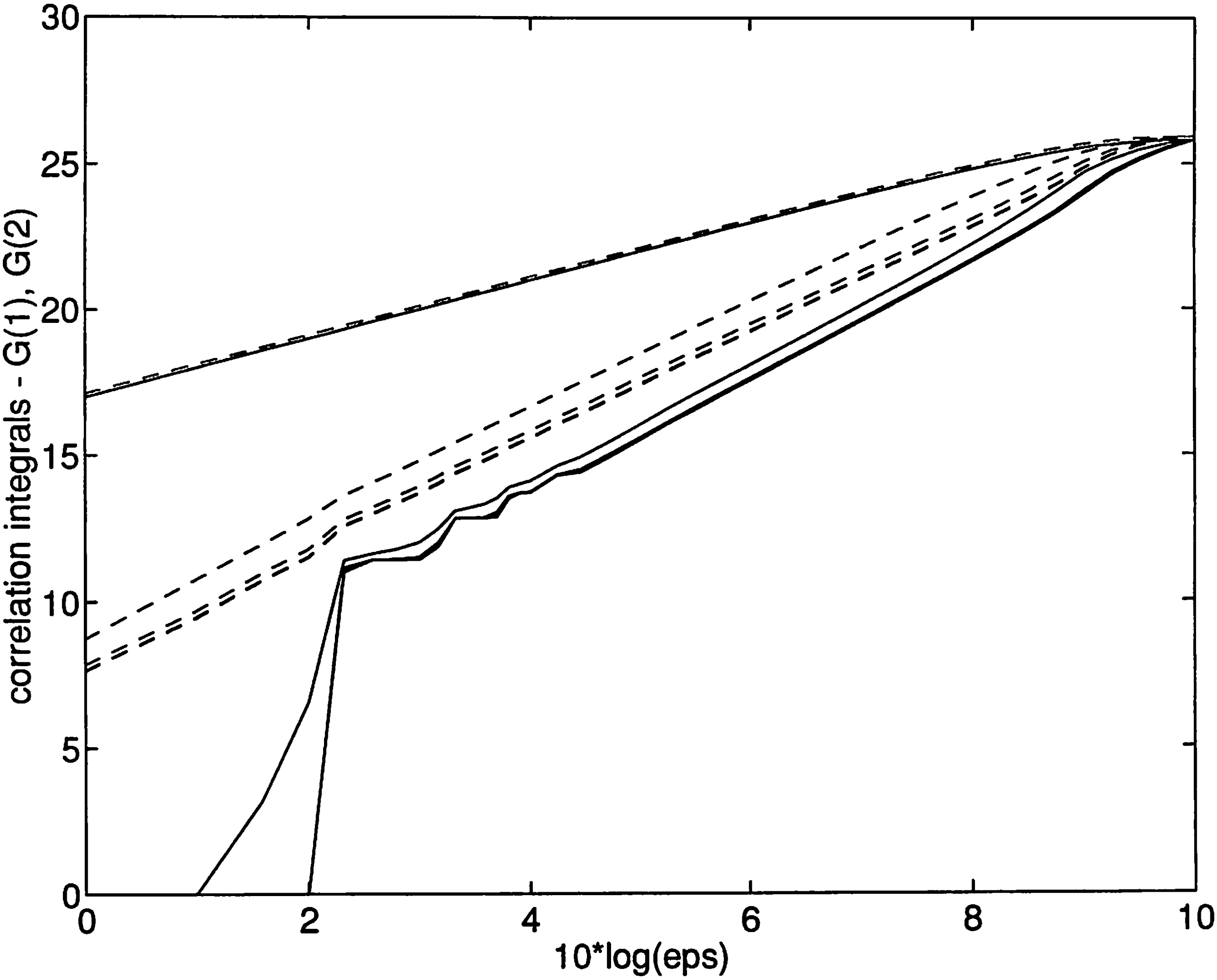

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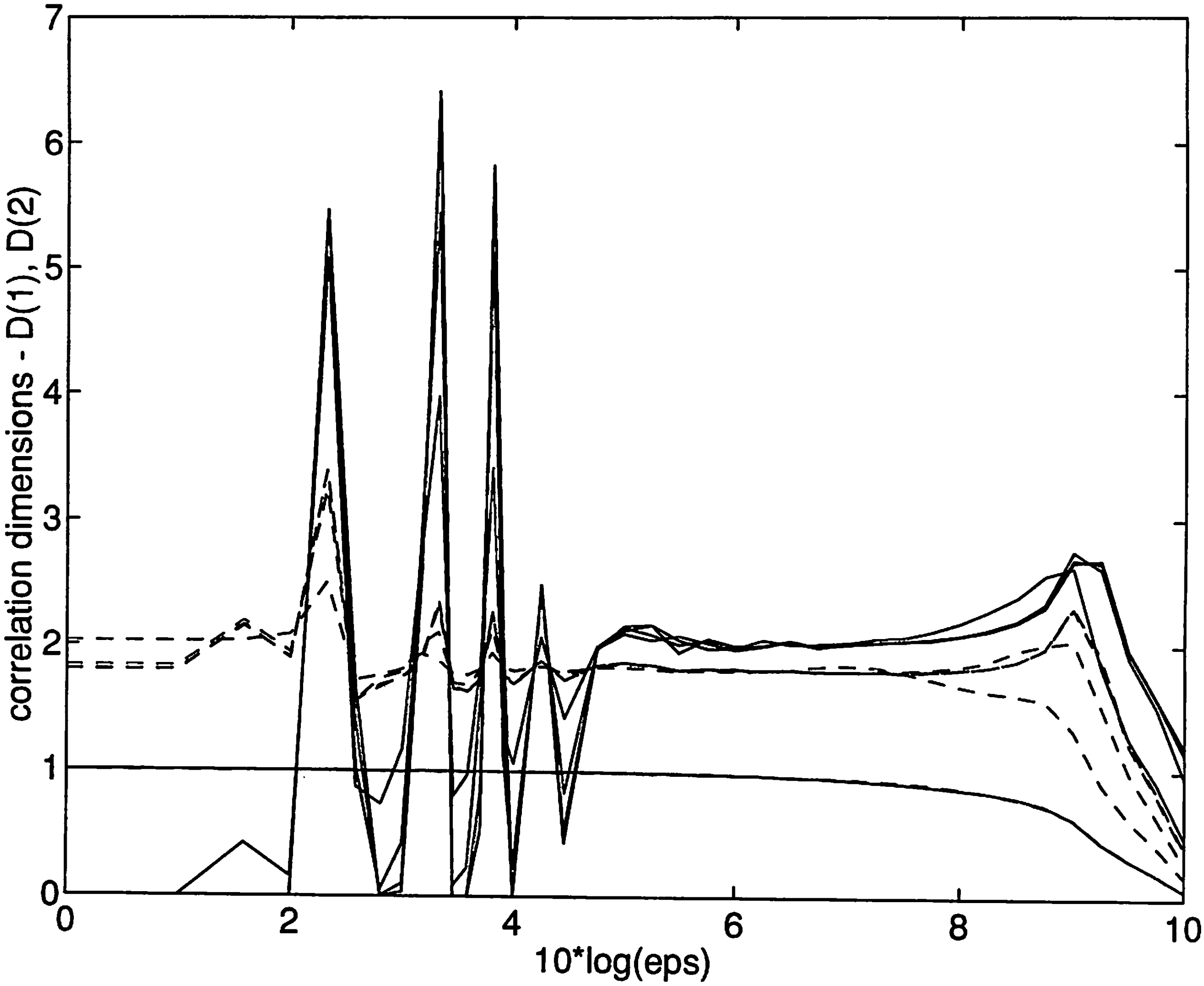

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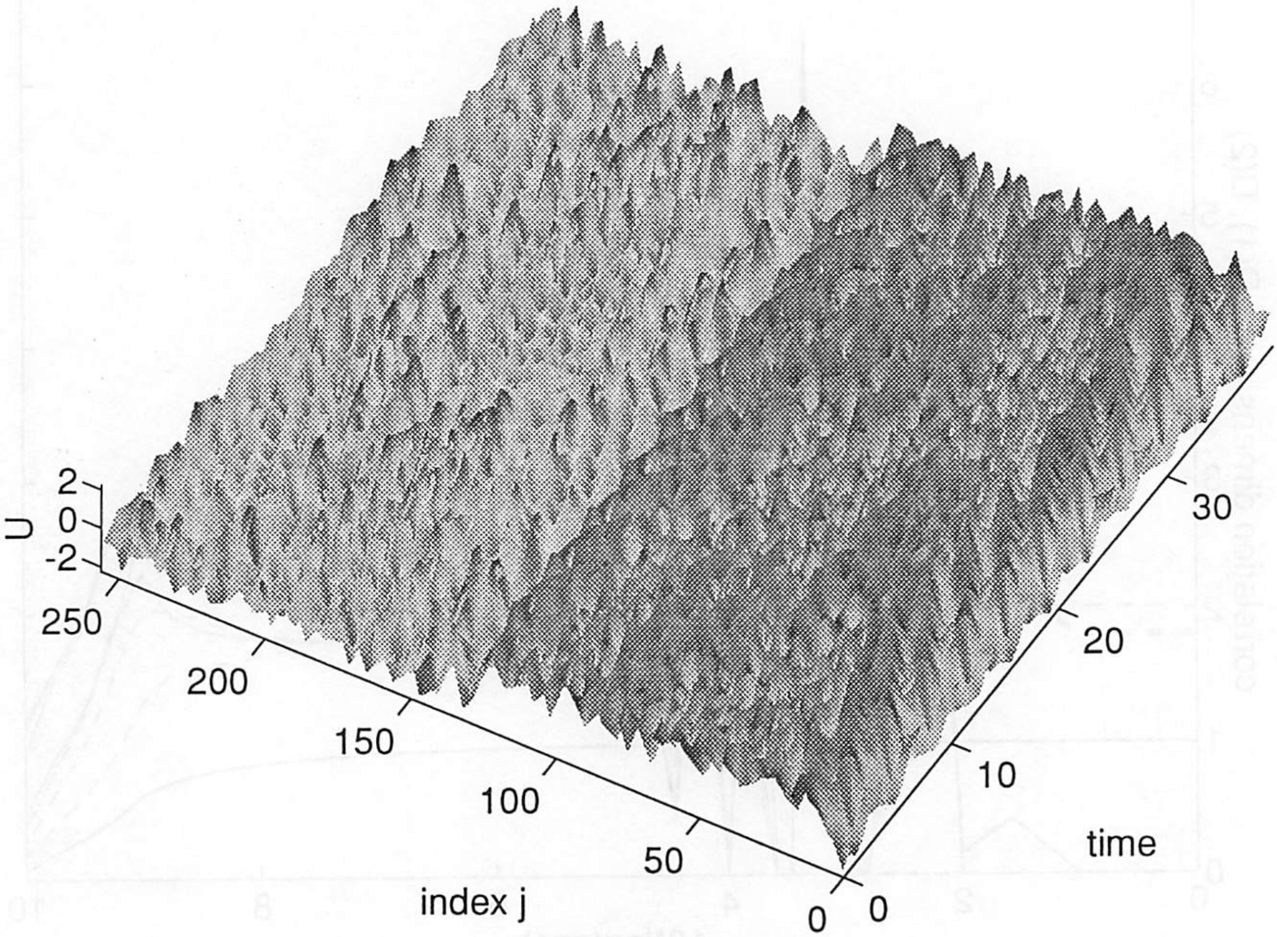

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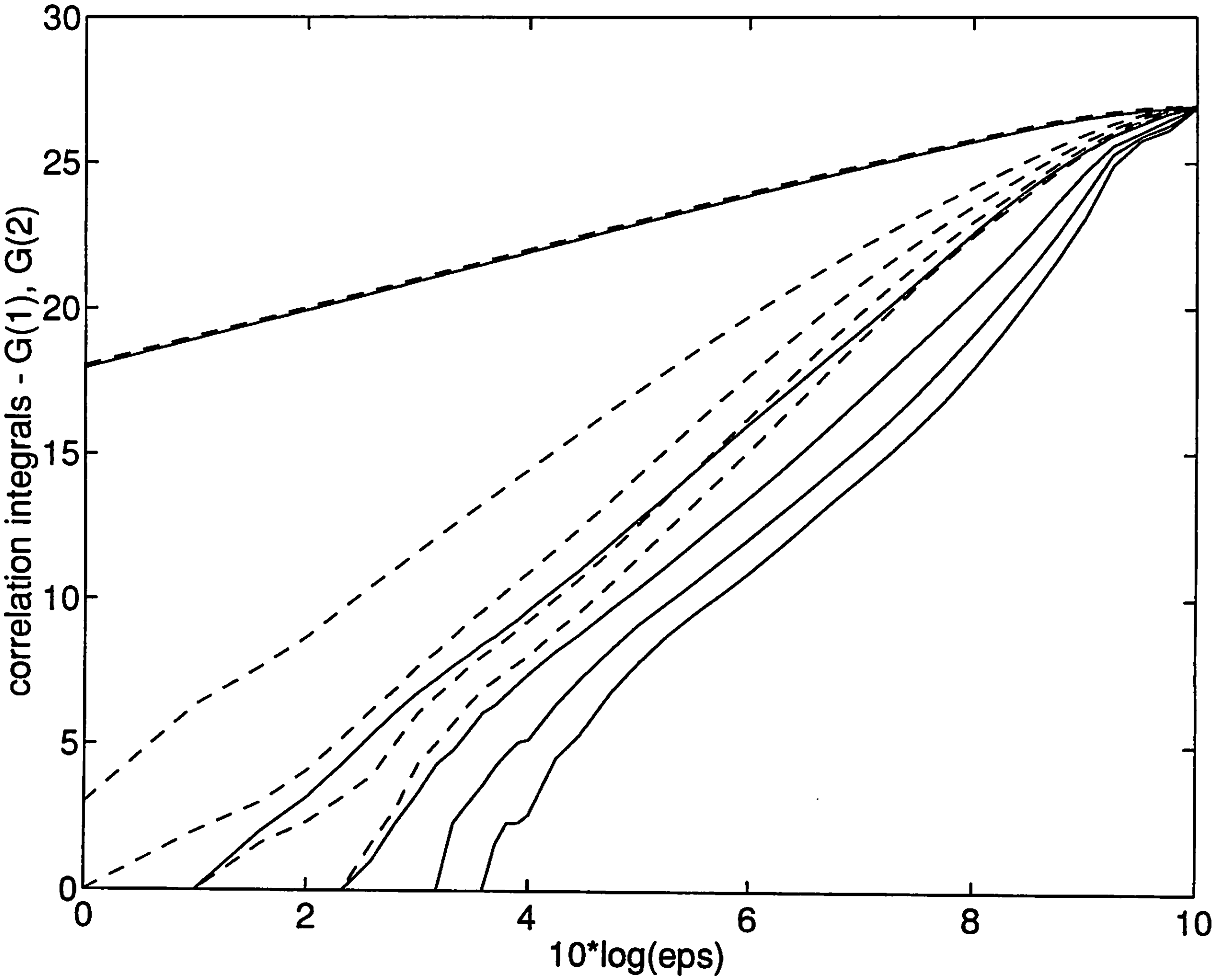

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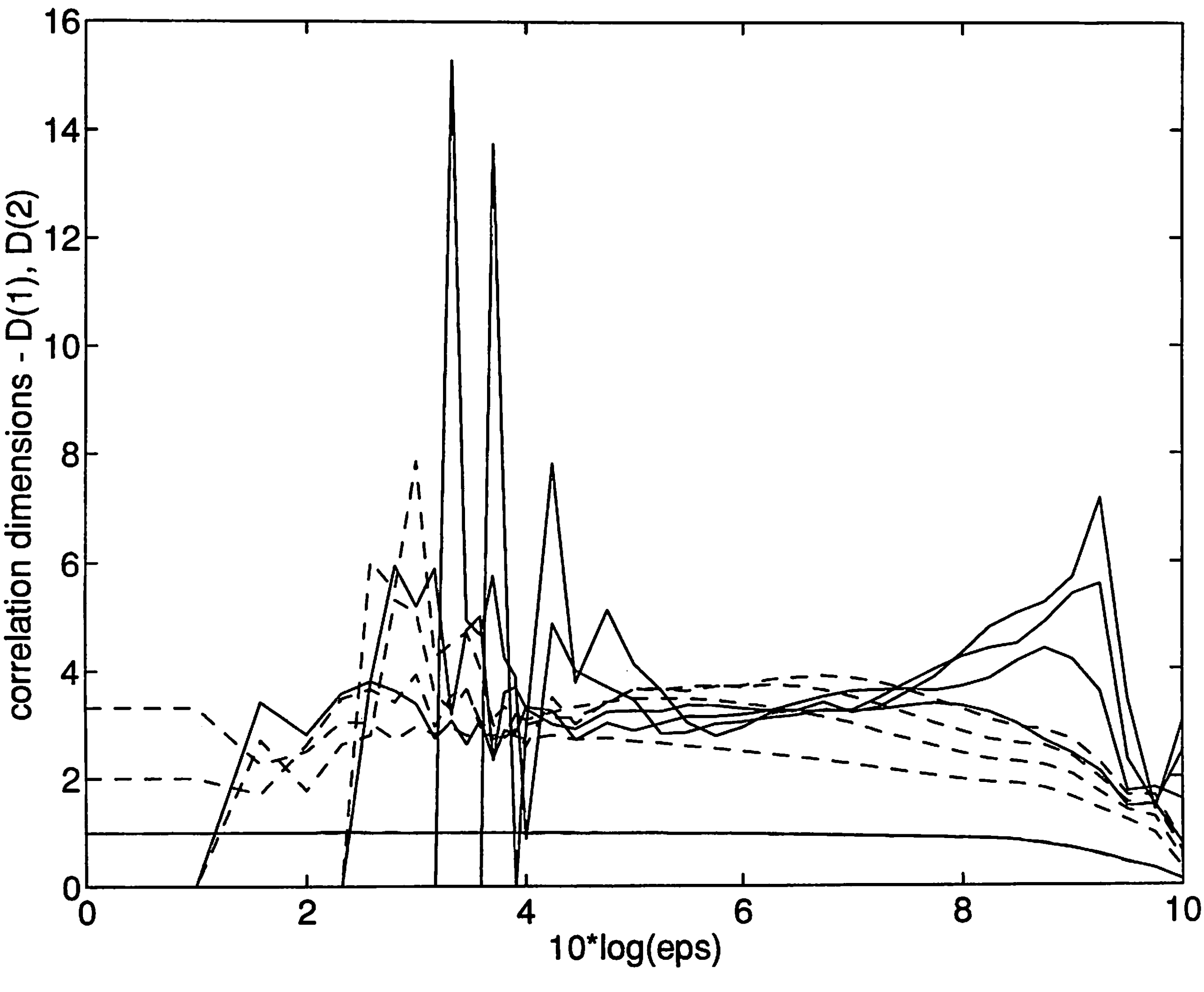

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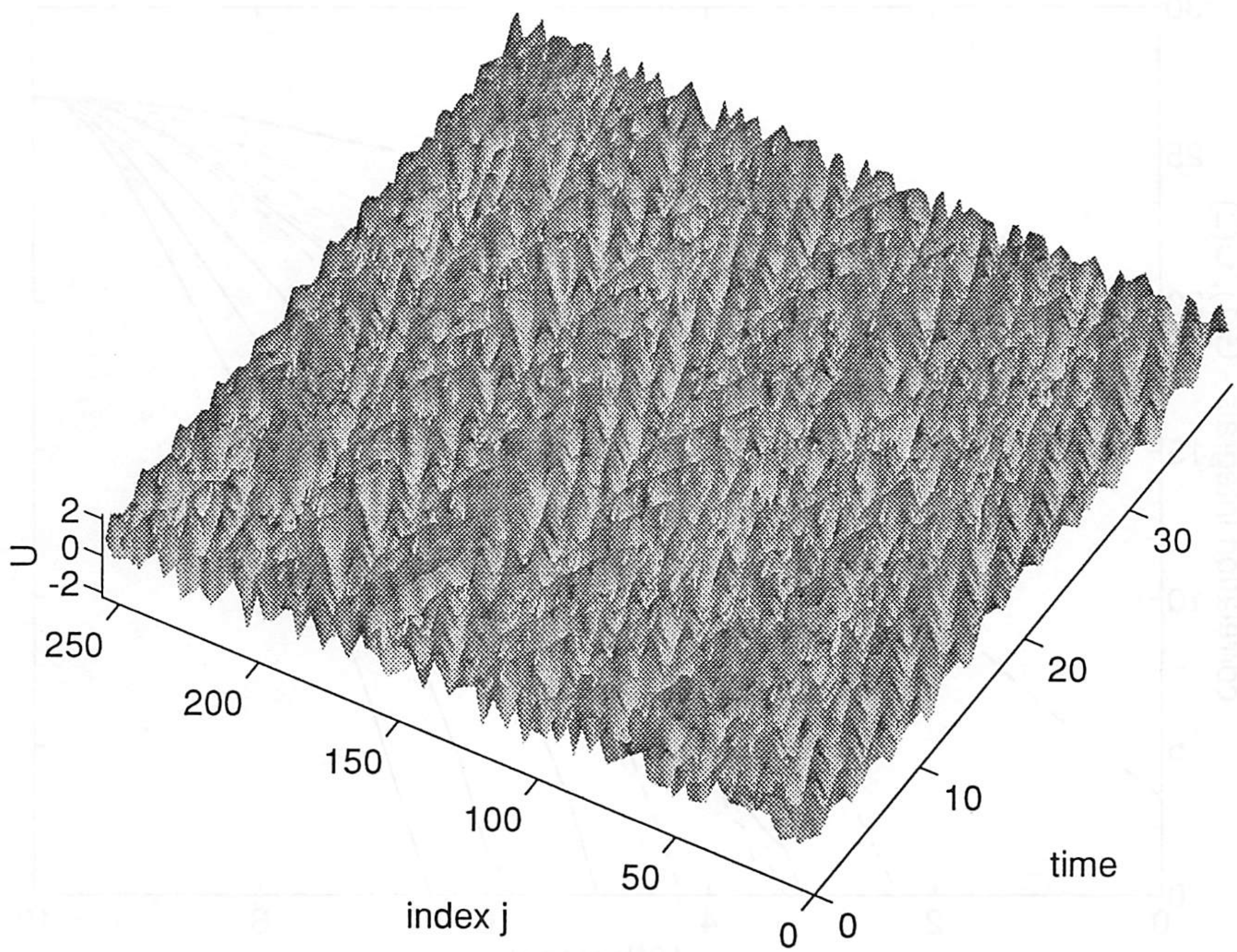

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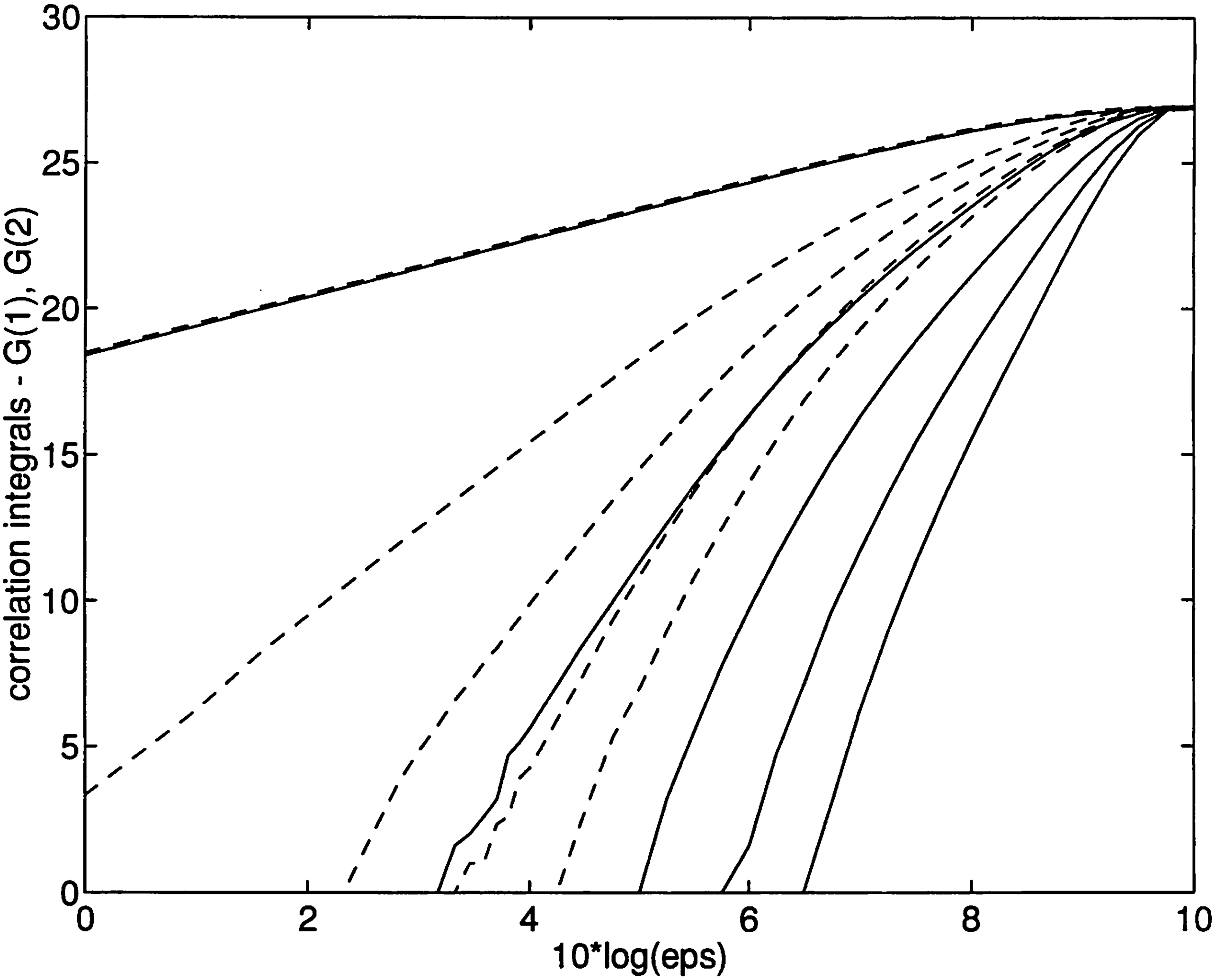

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correlation integrals - G(1), G(2)
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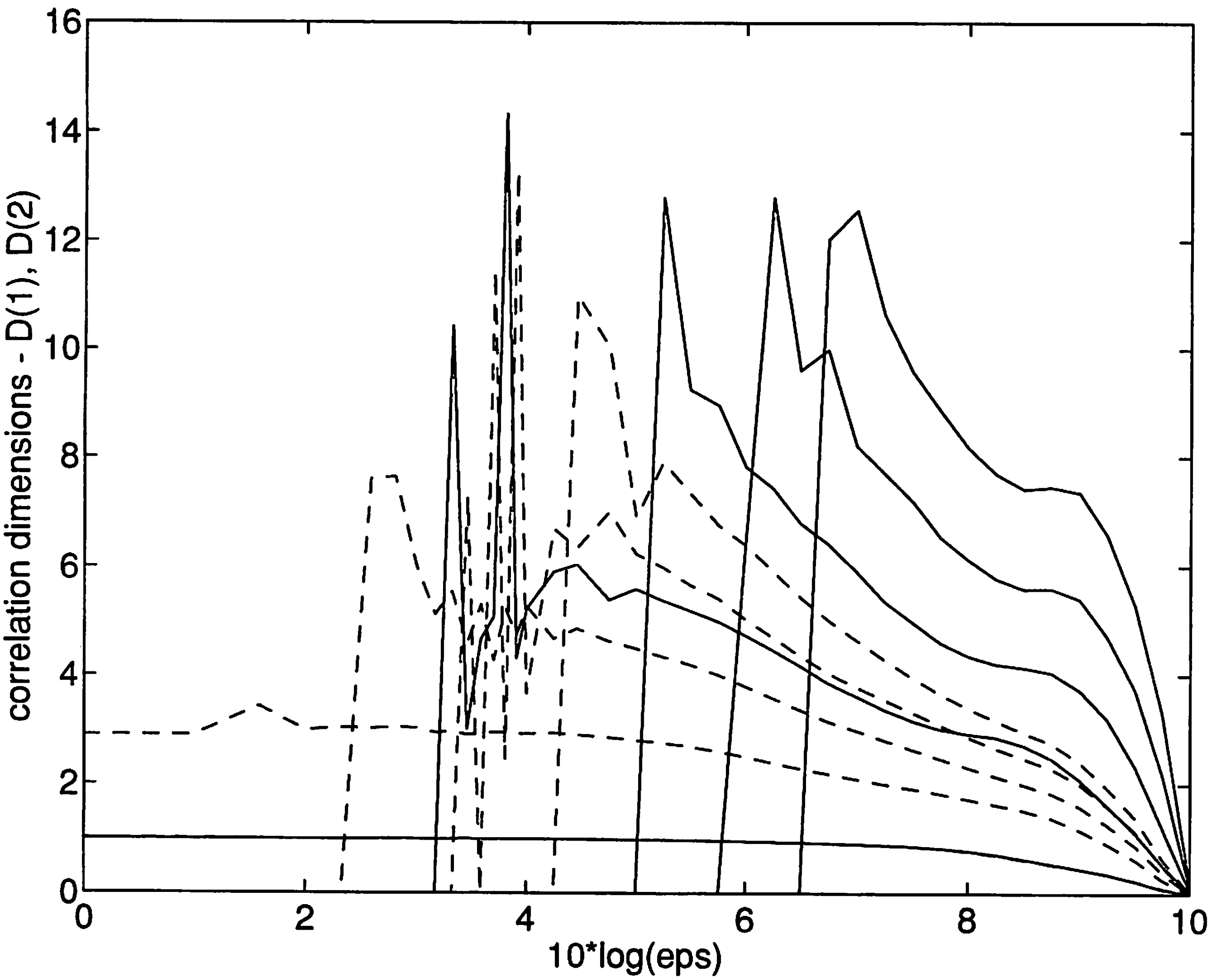
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