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On the Generality of the Unfolded Chua's Circuit

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Abstract

In this paper, we study the generality of Chua's oscillator by deriving a class of vector fields that Chua's oscillator is equivalent to. For the class of vector fields with a *scalar* nonlinearity, we prove that under certain conditions, two such vector fields are *topologically conjugate* if the Jacobian matrices at each point have the same eigenvalues and the equilibrium points are matched up. We show how these conditions are related to the complete state observability of a corresponding linear system. These results are used to show that the n-dimensional Chua's oscillator is topologically conjugate to almost every vector field in this class. We comment on the special case when the vector field is piecewise-linear and in particular when the vector field is 2-segment piecewise-linear. These results are illustrated by transforming several systems studied in the literature into equivalent Chua's oscillators.

We also extend some of these results to the case of several scalar nonlinearities. As a corollary we prove that almost all piecewise-linear vector fields with parallel boundary planes are topologically conjugate if the boundary planes and equilibrium points are the same and the eigenvalues in corresponding regions are the same.

1 Introduction

It is known that the unfolded Chua's circuit, also known as Chua's oscillator [R. Madan (Guest Editor), 1993], with an odd-symmetric 3-segment piecewise-linear nonlinearity is *topologically conjugate* to almost all 3-dimensional systems with a 3-segment odd-symmetric piecewise-linear continuous vector field [Chua, 1993; Shil'nikov, 1994]. The purpose of this paper is to extend this result to the case where the *nonlinearity* is no longer odd-symmetric or piecewise-linear but may be *any arbitrary continuous* function and the dimension of the system may also be arbitrary. We will do this in two steps. In the first step, we show that under certain conditions, nonlinear vector fields are topologically conjugate if the Jacobian matrices at each point have the same eigenvalues (Sec. 2). In the second step, we show that Chua's oscillator can synthesize almost any eigenvalue patterns of vector fields in our class (Sec. 3). These steps are combined in Sec. 4 to give the main result of this paper. Secs. 5 and 6 are devoted to the special case of piecewise-linear vector fields. In Sec. 7 the results in Sec. 2 are extended to systems with several scalar nonlinearities. We show that almost all piecewise-linear vector fields with parallel boundary planes are determined, up to topological conjugacy, by the boundary planes, equilibrium points and the eigenvalues in each region. This generalizes previously known results on linearly conjugacy of piecewise-linear vector fields.

We use lowercase, bold uppercase and bold lowercase letters for scalars (or scalar-valued functions), matrices and vectors respectively. The transpose of a matrix $\mathbf{A}$ is denoted $\mathbf{A}^T$. The vector $\mathbf{0}$ denotes the zero vector and $\mathbf{e}_i$ denotes the i-th unit vector, i.e., $\mathbf{e}_1 = (1, 0, \ldots, 0)^T$. Let $\chi(\mathbf{A}) = \det(\lambda\mathbf{I} - \mathbf{A})$ denote the characteristic polynomial of the matrix $\mathbf{A}$. The integer n is usually used to denote the size of matrices and vectors.

2 Topological Conjugacy under Equivalence of Eigenvalues

We consider the following class of vector fields in *Lur'e* form:

Definition 1 *The class $\mathcal{C}(\mathbf{w})$ consists of vector fields of the form:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + f\left(\mathbf{w}^T\mathbf{x}\right)\mathbf{b} \tag{1}$$

where $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a real-valued continuous function. The class $\mathcal{C}$ is defined as $\mathcal{C} = \bigcup_{\mathbf{w}} \mathcal{C}(\mathbf{w})$.

If $\mathbf{w} = \mathbf{0}$, $\mathcal{C}(\mathbf{w})$ reduces to the class of affine systems. When $\mathbf{w} \neq \mathbf{0}$, there exists a nonsingular matrix $\mathbf{M}$ such that $\mathbf{M}^T\mathbf{w} = \mathbf{e}_1$. Using the transformation $\mathbf{x} \to \mathbf{M}\mathbf{x}$, system (1) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + f\left(\mathbf{e}_1^T\mathbf{x}\right)\mathbf{b} \tag{2}$$

where $\mathbf{A} \to \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ and $\mathbf{b} \to \mathbf{M}^{-1}\mathbf{b}$. Thus for $\mathbf{w} \neq 0$, vector fields in $\mathcal{C}(\mathbf{w})$ are topologically conjugate to vector fields in $\mathcal{C}(\mathbf{e}_1)$.

Vector fields in class $\mathcal{C}(\mathbf{w})$ can be considered to be nonlinear vector fields where the nonlinearity occurs only in the direction $\mathbf{w}$ and the changes in the Jacobian matrix are of a fixed form. In this paper, we will mainly be working with vector fields in class $\mathcal{C}$. By translating an equilibrium point to the origin, a seemingly larger class of vector fields can be shown to be reducible to class $\mathcal{C}$.

Definition 2 *A point* $\mathbf{x}^*$ *is a virtual equilibrium point of the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *if*

- $\mathbf{x}^*$ *is not an equilibrium point of the system; i.e.,* $\mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$.
- *There exists a point* $\hat{\mathbf{x}}$ *where the Jacobian matrix exists and such that* $\mathbf{x}^*$ *is an equilibrium point of the system linearized at point* $\hat{\mathbf{x}}$*; i.e.,* $\hat{\mathbf{A}}\mathbf{x} + \hat{\mathbf{b}}$ *is the linearized vector field at* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{A}}\mathbf{x}^* + \hat{\mathbf{b}} = \mathbf{0}$*, where* $\hat{\mathbf{A}} = D\mathbf{f}(\mathbf{x})|_{\mathbf{x}=\hat{\mathbf{x}}}$ *and* $\hat{\mathbf{b}} = \mathbf{f}(\hat{\mathbf{x}}) - \hat{\mathbf{A}}\hat{\mathbf{x}}$.

The property of a system possessing a (real or virtual) equilibrium point is generic; a real or virtual equilibrium point exists whenever the Jacobian matrix at some point is nonsingular. In general, there are uncountably many equilibrium points in a system. For example, consider the simple first order circuit shown in Fig. 1a. The virtual equilibrium points are found by linearizing the nonlinear resistor around some point and finding the intersection with the v-axis which corresponds to the equilibrium point of the linearized circuit. Some of the equilibrium points of this circuit is shown in Fig. 1b, where the blue curve is the v-i characteristic of the nonlinear resistor. The red tangent lines correspond to the affine v-i characteristics of the resistor in the linearized circuit.

Definition 3 *The class* $C'(\mathbf{w})$ *consists of vector fields of the form:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + h\left(\mathbf{w}^T\mathbf{x}\right)\mathbf{b} + \mathbf{c} \tag{3}$$

where $h(\cdot)$ *is a continuous real-valued function such that the system has at least one equilibrium point which can be real or virtual. We define* $C' = \bigcup_{\mathbf{w}} C'(\mathbf{w})$.

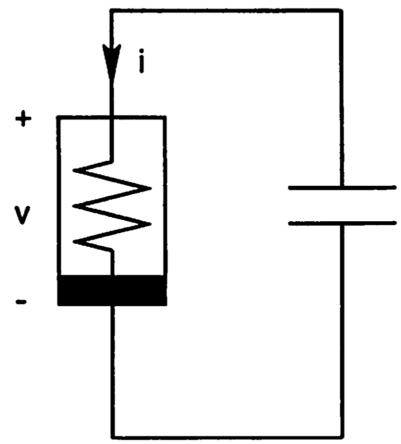
Lemma 1 *The vector fields in class* C' *is equivalent to a subset of the vector fields in class* C *in the sense that after a change of coordinates, Eq. (3) can be written as Eq. (1).*

Proof: Let $\mathbf{x}^*$ be an equilibrium point (which can be real or virtual) of system (3). $\mathbf{x}^*$ being an equilibrium point means that there exists a real number d such that $\mathbf{A}\mathbf{x}^* + d\mathbf{b} + \mathbf{c} = \mathbf{0}$. Let $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$. Then

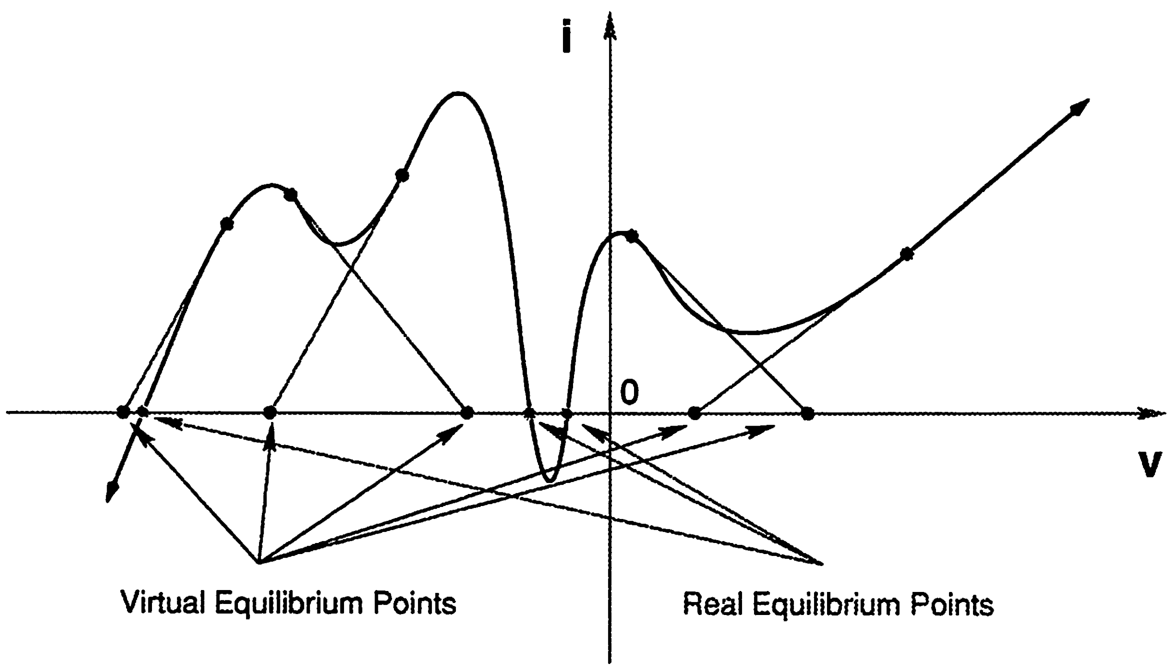
$$\dot{\mathbf{y}} = \dot{\mathbf{x}} = \mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{x}^* + h\left(\mathbf{w}^T(\mathbf{y} + \mathbf{x}^*)\right)\mathbf{b} + \mathbf{c} = \mathbf{A}\mathbf{y} + f\left(\mathbf{w}^T\mathbf{y}\right)\mathbf{b}$$

where $f\left(\mathbf{w}^T\mathbf{y}\right) = h\left(\mathbf{w}^T\mathbf{y} + \mathbf{w}^T\mathbf{x}^*\right) - d$. ■

Lemma 1 says that we can translate one of the equilibrium points to the origin, thereby obtaining a simpler form and reducing class C' to class C.



(a)



(b)

Figure 1: (a) First order nonlinear circuit consisting of a linear capacitor and a nonlinear resistor. (b) Some equilibrium points of the circuit in (a). The blue curve indicates the v-i characteristic of the nonlinear resistor. The red tangent lines correspond to the affine v-i characteristics of the resistor in the linearized circuit.

Definition 4 *The pair* $(\mathbf{A}, \mathbf{w})$ *satisfies condition* K *if the matrix*

$$\mathbf{K} = \mathbf{K}(\mathbf{A}, \mathbf{w}) = \begin{pmatrix} \mathbf{w}^T \\ \mathbf{w}^T\mathbf{A} \\ \mathbf{w}^T\mathbf{A}^2 \\ \vdots \\ \mathbf{w}^T\mathbf{A}^{n-1} \end{pmatrix} \tag{4}$$

is nonsingular, where $\mathbf{A}$ *is an* $n \times n$ *matrix and* $\mathbf{w}$ *is an* $n \times 1$ *vector.*

Note that $\mathbf{w}^T[\mathbf{K}(\mathbf{A}, \mathbf{w})]^{-1} = \mathbf{e}_1^T$ when $\mathbf{K}(\mathbf{A}, \mathbf{w})$ is invertible. Also note that if $(\mathbf{A}, \mathbf{w})$ satisfy condition K, then $\mathbf{w} \neq \mathbf{0}$. The set of $(\mathbf{A}, \mathbf{w})$ which satisfy condition K is of full measure. One of the consequences of $(\mathbf{A}, \mathbf{w})$ satisfying condition K is that $\mathbf{A}$ is similar to a matrix in companion form.

Lemma 2 *If* $(\mathbf{A}, \mathbf{w})$ *satisfies condition* K, *then*

$$\mathbf{K}\mathbf{A}\mathbf{K}^{-1} = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \ddots \\ & & \ddots & \ddots \\ -r_0 & -r_1 & \cdots & -r_{n-1} \end{pmatrix} \triangleq \hat{\mathbf{A}} \tag{5}$$

where $\chi(\mathbf{A}) = \lambda^n + r_{n-1}\lambda^{n-1} + \cdots + r_1\lambda + r_0$ *and* $\mathbf{K} = \mathbf{K}(\mathbf{A}, \mathbf{w})$ *is as defined in Eq. (4).*

Proof: By the Cayley-Hamilton theorem, we have

$$\mathbf{K}\mathbf{A} = \begin{pmatrix} \mathbf{w}^T\mathbf{A} \\ \mathbf{w}^T\mathbf{A}^2 \\ \mathbf{w}^T\mathbf{A}^3 \\ \vdots \\ \mathbf{w}^T\mathbf{A}^n \end{pmatrix} = \begin{pmatrix} \mathbf{w}^T\mathbf{A} \\ \mathbf{w}^T\mathbf{A}^2 \\ \mathbf{w}^T\mathbf{A}^3 \\ \vdots \\ \mathbf{w}^T(-r_{n-1}\mathbf{A}^{n-1} - \cdots - r_1\mathbf{A} - r_0\mathbf{I}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \ddots \\ & & \ddots & \ddots \\ -r_0 & -r_1 & \cdots & -r_{n-1} \end{pmatrix} \mathbf{K}$$

■

Note that any companion matrix in form (5) together with $\mathbf{e}_1$ satisfies condition K.

Lemma 3 *If* $\mathbf{M}$ *is nonsingular, then* $\mathbf{K}(\mathbf{A}, \mathbf{w})\mathbf{M} = \mathbf{K}(\mathbf{M}^{-1}\mathbf{A}\mathbf{M}, \mathbf{M}^T\mathbf{w})$. *In particular, for all nonsingular* $\mathbf{M}$,

$$(\mathbf{A}, \mathbf{w}) \quad \textit{satisfies condition } K \Longleftrightarrow (\mathbf{M}^{-1}\mathbf{A}\mathbf{M}, \mathbf{M}^T\mathbf{w}) \quad \textit{satisfies condition } K$$

Proof:

$$\mathbf{K}(\mathbf{A},\mathbf{w})\mathbf{M} = \begin{pmatrix} \mathbf{w}^T \\ \mathbf{w}^T\mathbf{A} \\ \mathbf{w}^T\mathbf{A}^2 \\ \vdots \\ \mathbf{w}^T\mathbf{A}^{n-1} \end{pmatrix} \mathbf{M} = \begin{pmatrix} \mathbf{w}^T\mathbf{M} \\ \mathbf{w}^T\mathbf{M}\mathbf{M}^{-1}\mathbf{A}\mathbf{M} \\ \mathbf{w}^T\mathbf{M}\mathbf{M}^{-1}\mathbf{A}^2\mathbf{M} \\ \vdots \\ \mathbf{w}^T\mathbf{M}\mathbf{M}^{-1}\mathbf{A}^{n-1}\mathbf{M} \end{pmatrix} = \mathbf{K}(\mathbf{M}^{-1}\mathbf{A}\mathbf{M}, \mathbf{M}^T\mathbf{w})$$

Since $\mathbf{M}$ is nonsingular, the second part of the lemma follows. ■

Lemma 4 *For all matrices* $\mathbf{A}$ *and all vectors* $\mathbf{b}$, $\mathbf{w}$

$$(\mathbf{A},\mathbf{w}) \quad \textit{satisfies condition } K \Longleftrightarrow (\mathbf{A}+\mathbf{b}\mathbf{w}^T,\mathbf{w}) \quad \textit{satisfies condition } K$$

Proof: The case $\mathbf{w} = \mathbf{0}$ is trivially true. For $\mathbf{w} \neq \mathbf{0}$ using Lemma 3 we only need to prove the case $\mathbf{w} = \mathbf{e}_1$. Let $\mathbf{K} = \mathbf{K}(\mathbf{A},\mathbf{e}_1)$ be defined as in (4). First note that $\mathbf{e}_1^T = \mathbf{e}_1^T\mathbf{K}$. We claim that $\mathbf{e}_1^T\left(\mathbf{A}+\mathbf{b}\mathbf{e}_1^T\right)^i$ can be written as $\mathbf{c}_i^T\mathbf{K}$ for some vector $\mathbf{c}_i$. Thus $\mathbf{c}_0 = \mathbf{e}_1$.

$$\mathbf{e}_1^T\left(\mathbf{A}+\mathbf{b}\mathbf{e}_1^T\right)^{i+1} = \mathbf{c}_i^T\mathbf{K}\left(\mathbf{A}+\mathbf{b}\mathbf{e}_1^T\right) = \mathbf{c}_i^T\hat{\mathbf{A}}\mathbf{K} + \mathbf{c}_i^T\mathbf{K}\mathbf{b}\mathbf{e}_1^T\mathbf{K} = \left(\mathbf{c}_i^T\hat{\mathbf{A}} + \mathbf{c}_i^T\mathbf{K}\mathbf{b}\mathbf{e}_1^T\right)\mathbf{K}$$

Thus $\mathbf{c}_{i+1}^T = \mathbf{c}_i^T\hat{\mathbf{A}} + \mathbf{c}_i^T\mathbf{K}\mathbf{b}\mathbf{e}_1^T$. Since $\mathbf{c}_0 = (1,0,\cdots)^T$ and $\mathbf{c}_1 = (\mathbf{e}_1^T\mathbf{b},1,0,\cdots)^T$, etc, it is clear that the matrix

$$\begin{pmatrix} \mathbf{c}_0^T \\ \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_{n-1}^T \end{pmatrix}$$

is lower triangular with 1's on the diagonal, i.e. it is nonsingular.

Since

$$\begin{pmatrix} \mathbf{e}_1^T \\ \mathbf{e}_1^T\left(\mathbf{A}+\mathbf{b}\mathbf{e}_1^T\right) \\ \mathbf{e}_1^T\left(\mathbf{A}+\mathbf{b}\mathbf{e}_1^T\right)^2 \\ \vdots \\ \mathbf{e}_1^T\left(\mathbf{A}+\mathbf{b}\mathbf{e}_1^T\right)^{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{c}_0^T \\ \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_{n-1}^T \end{pmatrix}\mathbf{K}$$

the result follows. ■

Lemma 5 *Let* $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *be matrices, and* $\mathbf{b}$, $\tilde{\mathbf{b}}$ *and* $\mathbf{w}$ *be vectors. For each real number* δ, *define* $\mathbf{A}_\delta$ *and* $\tilde{\mathbf{A}}_\delta$ *as follows:*

$$\mathbf{A}_\delta = \mathbf{A} + \delta\mathbf{b}\mathbf{w}^T, \qquad \tilde{\mathbf{A}}_\delta = \tilde{\mathbf{A}} + \delta\tilde{\mathbf{b}}\mathbf{w}^T$$

Then $\mathbf{A}_\delta$ *has the same eigenvalues as* $\tilde{\mathbf{A}}_\delta$ *for all* δ *if and only if* $\mathbf{A}_{\delta_1}$ *and* $\tilde{\mathbf{A}}_{\delta_1}$ *have the same eigenvalues and* $\mathbf{A}_{\delta_2}$ *and* $\tilde{\mathbf{A}}_{\delta_2}$ *have the same eigenvalues for some* $\delta_1 \neq \delta_2$.

Proof: One direction is clear. Without loss of generality we can assume that $\mathbf{w} = \mathbf{e}_1$. Let b_i and $\tilde{b}_i$ denote the ith entry of $\mathbf{b}$ and $\tilde{\mathbf{b}}$ respectively and let a_{ij} and $\tilde{a}_{ij}$ denote the (i,j)th entry of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ respectively. Expanding along the first column, $\chi(\mathbf{A}), \chi(\tilde{\mathbf{A}}), \chi(\mathbf{A}_\delta), \chi(\tilde{\mathbf{A}}_\delta)$ can be written as:

$$\begin{aligned}
\chi(\mathbf{A}) &= (\lambda - a_{11})p_1(\lambda) - a_{21}p_2(\lambda) - \cdots - a_{n1}p_n(\lambda) \\
\chi(\tilde{\mathbf{A}}) &= (\lambda - \tilde{a}_{11})\tilde{p}_1(\lambda) - \tilde{a}_{21}\tilde{p}_2(\lambda) - \cdots - \tilde{a}_{n1}\tilde{p}_n(\lambda) \\
\chi(\mathbf{A}_\delta) &= \chi(\mathbf{A}) - \delta b_1 p_1(\lambda) - \delta b_2 p_2(\lambda) - \cdots - \delta b_n p_n(\lambda) \\
\chi(\tilde{\mathbf{A}}_\delta) &= \chi(\tilde{\mathbf{A}}) - \delta \tilde{b}_1 \tilde{p}_1(\lambda) - \delta \tilde{b}_2 \tilde{p}_2(\lambda) - \cdots - \delta \tilde{b}_n \tilde{p}_n(\lambda)
\end{aligned}$$

for some polynomials p_i and $\tilde{p}_i$.

By the assumption $\chi(\mathbf{A}_{\delta_1}) - \chi(\mathbf{A}_{\delta_2}) = \chi(\tilde{\mathbf{A}}_{\delta_1}) - \chi(\tilde{\mathbf{A}}_{\delta_2})$, i.e.,

$$(\delta_2 - \delta_1)(b_1p_1(\lambda) + b_2p_2(\lambda) + \cdots + b_np_n(\lambda)) = (\delta_2 - \delta_1)(\tilde{b}_1\tilde{p}_1(\lambda) + \tilde{b}_2\tilde{p}_2(\lambda) + \cdots + \tilde{b}_n\tilde{p}_n(\lambda)) \quad (6)$$

Since $\delta_2 - \delta_1 \neq 0$, we can multiply both sides of Eq. (6) by $\frac{\delta - \delta_1}{\delta_1 - \delta_2}$, add $\chi(\mathbf{A}_{\delta_1})$ to the left side, and add $\chi(\tilde{\mathbf{A}}_{\delta_1})$ to the right side to get $\chi(\mathbf{A}_\delta) = \chi(\tilde{\mathbf{A}}_\delta)$. ■

Lemma 6 *Let* $\mathbf{b}$ *be a vector and let* $(\mathbf{A}, \mathbf{w})$ *satisfies condition* K. *Then* $\mathbf{Kb}$ *is uniquely determined by the eigenvalues of* $\mathbf{A}$ *and* $\mathbf{A} + \mathbf{bw}^T$, *where* $\mathbf{K} = \mathbf{K}(\mathbf{A}, \mathbf{w})$ *is defined in Eq. (4). Furthermore,* $\mathbf{b}$ *is uniquely determined by* $\mathbf{A}$, $\mathbf{w}$ *and the eigenvalues of* $\mathbf{A} + \mathbf{bw}^T$.

Proof: Write the characteristic polynomials of $\mathbf{A}$ and $\mathbf{A} + \mathbf{bw}^T$ as:

$$\begin{aligned}
\chi(\mathbf{A}) &= \lambda^n + r_{n-1}\lambda^{n-1} + \cdots + r_1\lambda + r_0 \\
\chi(\mathbf{A} + \mathbf{bw}^T) &= \lambda^n + s_{n-1}\lambda^{n-1} + \cdots + s_1\lambda + s_0
\end{aligned} \quad (7)$$

Let $\mathbf{Kb} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n)^T$. Then $\mathbf{KAK}^{-1}$ is in companion form by Lemma 2 and

$$\mathbf{K}(\mathbf{A} + \mathbf{bw}^T)\mathbf{K}^{-1} = \mathbf{KAK}^{-1} + \mathbf{Kbe}_1^T = \begin{pmatrix} \tilde{b}_1 & 1 & 0 & \\ \tilde{b}_2 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots \\ \tilde{b}_n - r_0 & -r_1 & \cdots & -r_{n-1} \end{pmatrix} \quad (8)$$

By expanding along the first column, we see that the characteristic polynomial of $\mathbf{K}(\mathbf{A}+\mathbf{bw}^T)\mathbf{K}^{-1}$ is:

$$\begin{aligned}
\chi(\mathbf{K}(\mathbf{A} + \mathbf{bw}^T)\mathbf{K}^{-1}) = \; & \chi(\mathbf{KAK}^{-1}) \\
& -\tilde{b}_1(\lambda^{n-1} + r_{n-1}\lambda^{n-2} + \cdots + r_1) \\
& -\tilde{b}_2(\lambda^{n-2} + r_{n-1}\lambda^{n-3} + \cdots + r_2) \\
& - \cdots - \tilde{b}_{n-1}(\lambda + r_{n-1}) - \tilde{b}_n
\end{aligned} \quad (9)$$

which is equal to $\chi(\mathbf{A}+\mathbf{b}\mathbf{w}^T)$. Comparing Eq. (7) with Eq. (9) the following set of equations is obtained:

$$\begin{pmatrix} 1 & 0 & & 0 \\ r_{n-1} & 1 & \ddots & \\ r_{n-2} & r_{n-1} & \ddots & 0 \\ r_1 & r_2 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{pmatrix} = \begin{pmatrix} r_{n-1}-s_{n-1} \\ r_{n-2}-s_{n-2} \\ \vdots \\ r_0-s_0 \end{pmatrix} \tag{10}$$

So $\mathbf{b}$ is uniquely determined by $\mathbf{Kb}$ which in turn is uniquely determined by r_i and s_i which is determined by the eigenvalues of $\mathbf{A}$ and $\mathbf{A}+\mathbf{b}\mathbf{w}^T$.

■

Corollary 1 *Let* $\mathbf{A}$ *be a matrix and* $\mathbf{w}$, $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *be vectors. Suppose that* $(\mathbf{A},\mathbf{w})$ *satisfies condition* K. *The matrices* $\mathbf{A}+\mathbf{b}_1\mathbf{w}^T$ *and* $\mathbf{A}+\mathbf{b}_2\mathbf{w}^T$ *have the same eigenvalues if and only if* $\mathbf{b}_1=\mathbf{b}_2$.

Proof: One direction is clear. Now let $\mathbf{A}'=\mathbf{A}+\mathbf{b}_1\mathbf{w}^T$. Then $\mathbf{A}+\mathbf{b}_2\mathbf{w}^T=\mathbf{A}'+(\mathbf{b}_2-\mathbf{b}_1)\mathbf{w}^T$. Lemma 4 implies that $(\mathbf{A}',\mathbf{w})$ satisfies condition K. Applying Eq. (10) in Lemma 6 to $\mathbf{A}'$ and $\mathbf{A}'+(\mathbf{b}_2-\mathbf{b}_1)\mathbf{w}^T$ for the case where $r_i=s_i$ we see that $\mathbf{b}_2-\mathbf{b}_1=\mathbf{0}$. ■

The following theorem gives some concepts which are equivalent to condition K.

Theorem 1 *Let* $\mathbf{A}$ *be a matrix and* $\mathbf{w}$, $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *be vectors. The following statements are equivalent:*

1. *The linear system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ y &= \mathbf{w}^T\mathbf{x} \end{aligned} \tag{11}$$

   is completely state observable[1].

2. *The pair* $(\mathbf{A},\mathbf{w})$ *satisfies condition* K.

3. *No nontrivial subspace, which is invariant under* $\mathbf{A}$, *is orthogonal to* $\mathbf{w}$.

4. *The matrices* $\mathbf{A}+\mathbf{b}_1\mathbf{w}^T$ *and* $\mathbf{A}+\mathbf{b}_2\mathbf{w}^T$ *have the same eigenvalues if and only if* $\mathbf{b}_1=\mathbf{b}_2$.

Proof: The equivalence between the first two statements is a standard result in linear system theory [Chen, 1984]. The equivalence between statement 2 and 3 follows from the fact that $\mathbf{Kb}=\mathbf{0}$ if

---

[1] A (time-invariant) linear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

is said to be completely state observable if there exists a time $t>0$ such that for any initial state $\mathbf{x}_0$ at time 0, the knowledge of the output $\mathbf{y}$ over the time interval $[0,t]$ suffices to determine $\mathbf{x}_0$ [Chen, 1984].

and only if $\mathbf{w}^T\mathbf{A}^i\mathbf{b} = 0$ for all nonnegative integers i. From Corollary 1 statement 2 implies statement 4. Suppose that $(\mathbf{A}, \mathbf{w})$ does not satisfy condition K. Let $\mathbf{b} \neq \mathbf{0}$ be in the kernel of $\mathbf{K}(\mathbf{A}, \mathbf{w})$. This implies that $\mathbf{w}^T\mathbf{A}^i\mathbf{b} = 0$ for all nonnegative integers i. Now let $\lambda > |\mathbf{A}|$ be a real number and $\mathbf{M}^T\mathbf{w} = \mathbf{e}_1$ for $\mathbf{M}$ nonsingular. Since λ is not in the spectrum of $\mathbf{A}$, the matrix $\lambda\mathbf{I} - \mathbf{A}$ is invertible. Then $(\lambda\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{w}^T) = (\lambda\mathbf{I} - \mathbf{A})(\mathbf{I} - (\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{w}^T)$. By expanding $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ as a power series, we see that $\mathbf{w}^T(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \mathbf{e}_1^T\mathbf{M}^{-1}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = 0$, i.e., the first element of the vector $\mathbf{M}^{-1}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ is 0. Therefore the first row and all columns except the first column of $\mathbf{M}^{-1}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{e}_1^T$ consists of zero entries and thus $\det(\mathbf{I} - (\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{w}^T) = \det(\mathbf{I} - \mathbf{M}^{-1}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{e}_1^T) = 1$. Thus the characteristic polynomials of $\mathbf{A}$ and $\mathbf{A} + \mathbf{b}\mathbf{w}^T$ agree for all $\lambda > |\mathbf{A}|$. This implies that these polynomials are equal and thus $\mathbf{A}$ and $\mathbf{A} + \mathbf{b}\mathbf{w}^T$ have the same eigenvalues. This means that statement 4 is not satisfied. ■

Remark 1 *In studying the absolute stability problem (or Lur'e problem) of a system in form (1), it is sometimes assumed that the linear system (11) is observable [Vidyasagar, 1978], and thus* $(\mathbf{A}, \mathbf{w})$ *satisfies condition* K.

Remark 2 *Piecewise-linear systems where the Jacobian* $\mathbf{A}$ *and the normal vectors to the boundary planes* $\mathbf{w}$ *satisfy statement* 3 *in the above theorem are called* proper *in [Komuro, 1988]. This motivates us to give the following definition:*

Definition 5 *A vector field in* C *written in the form (1) is called* proper *if* $(\mathbf{A}, \mathbf{w})$ *satisfy condition* K.

Remark 3 *Proper vector fields form a set of full measure in* C. *Lemma 4 implies that Definition 5 is well-defined for a vector field in* C. *Furthermore, the matrix* $\mathbf{A}$ *in the definition can be replaced by the Jacobian matrix at any point.*

Remark 4 *Statement* 4 *in Theorem 1 provides a characterization for state observability of single-input single-output (SISO) systems. It says that an SISO system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{w}^T\mathbf{x} \end{aligned} \tag{12}$$

is state observable if and only if for all $\mathbf{b} \neq \mathbf{0}$, *output feedback moves the poles of the system.*

Consider a linear system of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Two such linear systems are *linearly conjugate*, i.e. topologically conjugate via a linear mapping, if the Jacobian matrices are similar, i.e. have the same Jordan form matrix. If we restrict ourselves to matrices which are similar to a matrix in companion form, then all Jordan blocks must be of maximum order, and the eigenvalues uniquely determine the Jordan form matrix up to permutation. Thus if we assume that the Jacobian matrices are similar to some matrix in companion form, then two linear systems are linearly conjugate if the eigenvalues are the same. Lemma 2 says that matrices $\mathbf{A}$ which satisfy condition K with some $\mathbf{w}$ are examples of such matrices. This result of topological conjugacy by matching of eigenvalues is generalized in the next theorem which is the main theorem in this section. It is a generalization of the global unfolding theorem in [Chua, 1993] and gives conditions under which vector fields in C are topologically conjugate whenever the eigenvalues of the Jacobian matrices are matched up at every point.

Theorem 2 *Consider the systems*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + f\left(\mathbf{w}^T\mathbf{x}\right)\mathbf{b} \tag{13}$$

and

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + f\left(\tilde{\mathbf{w}}^T\tilde{\mathbf{x}}\right)\tilde{\mathbf{b}} \tag{14}$$

Assume that both systems are proper. Then system (13) and system (14) are topologically conjugate if the eigenvalues of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are the same and the eigenvalues of $\mathbf{A}+\mathbf{b}\mathbf{w}^T$ and $\tilde{\mathbf{A}}+\tilde{\mathbf{b}}\tilde{\mathbf{w}}^T$ are the same.

Proof: The proof follows the same steps as in [Chua, 1993]. By hypothesis, the matrices $\mathbf{K} = \mathbf{K}(\mathbf{A},\mathbf{w})$ and $\tilde{\mathbf{K}} = \mathbf{K}(\tilde{\mathbf{A}},\tilde{\mathbf{w}})$ are nonsingular. Note that $\mathbf{w}^T\mathbf{K}^{-1} = \tilde{\mathbf{w}}^T\tilde{\mathbf{K}}^{-1} = \mathbf{e}_1^T$. Let us denote

$$\begin{array}{lllll} \chi(\mathbf{A}) & = & \chi(\tilde{\mathbf{A}}) & = & \lambda^n + r_{n-1}\lambda^{n-1} + \cdots + r_1\lambda + r_0 \\ \chi(\mathbf{A}+\mathbf{b}\mathbf{w}^T) & = & \chi(\tilde{\mathbf{A}}+\tilde{\mathbf{b}}\tilde{\mathbf{w}}^T) & = & \lambda^n + s_{n-1}\lambda^{n-1} + \cdots + s_1\lambda + s_0 \end{array} \tag{15}$$

Using the transformation $\mathbf{y} = \mathbf{K}\mathbf{x}$, we obtain a system of the form:

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{K}\mathbf{A}\mathbf{K}^{-1}\mathbf{y} + f(\mathbf{w}^T\mathbf{K}^{-1}\mathbf{y})\mathbf{K}\mathbf{b} \\ &= \hat{\mathbf{A}}\mathbf{y} + f(\mathbf{e}_1^T\mathbf{y})\mathbf{K}\mathbf{b} \end{aligned}$$

where $\hat{\mathbf{A}}$ is defined in Eq. (5).

Similarly using $\tilde{\mathbf{y}} = \tilde{\mathbf{K}}\tilde{\mathbf{x}}$, we get

$$\begin{aligned} \dot{\tilde{\mathbf{y}}} &= \tilde{\mathbf{K}}\tilde{\mathbf{A}}\tilde{\mathbf{K}}^{-1}\tilde{\mathbf{y}} + f(\tilde{\mathbf{w}}^T\tilde{\mathbf{K}}^{-1}\tilde{\mathbf{y}})\tilde{\mathbf{K}}\tilde{\mathbf{b}} \\ &= \hat{\mathbf{A}}\tilde{\mathbf{y}} + f(\mathbf{e}_1^T\tilde{\mathbf{y}})\tilde{\mathbf{K}}\tilde{\mathbf{b}} \end{aligned}$$

By Lemma 6, the vectors $\mathbf{K}\mathbf{b}$ and $\tilde{\mathbf{K}}\tilde{\mathbf{b}}$ are uniquely determined by the coefficients r_i and s_i and are thus equal to each other. Thus the two systems above are identical, which means that systems (13) and (14) are topologically conjugate. ■

Remark 5 *By Lemma 5, for differentiable f, the condition that the eigenvalues of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are the same and the eigenvalues of $\mathbf{A}+\mathbf{b}\mathbf{w}^T$ and $\tilde{\mathbf{A}}+\tilde{\mathbf{b}}\tilde{\mathbf{w}}^T$ are the same is equivalent to the condition that the Jacobian matrices of the two systems (13) and (14) (with the same nonlinear function f) have the same eigenvalues at every point.*

Lemma 7 *Suppose* $(\mathbf{A}, \mathbf{w})$ *and* $(\tilde{\mathbf{A}}, \tilde{\mathbf{w}})$ *satisfy condition* K *and* $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *have the same eigenvalues. The transfer function* $g(s) = \mathbf{w}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ *is equal to the transfer function* $\tilde{g}(s) = \tilde{\mathbf{w}}^T(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{b}}$ *if and only if* $\mathbf{A} + \mathbf{b}\mathbf{w}^T$ *and* $\tilde{\mathbf{A}} + \tilde{\mathbf{b}}\tilde{\mathbf{w}}^T$ *have the same eigenvalues.*

Proof: By writing $(s\mathbf{I} - \mathbf{A})^{-1}$ and $(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}$ as a power series, we see that $g(s) = \tilde{g}(s)$ if and only $\mathbf{w}^T\mathbf{A}^i\mathbf{b} = \tilde{\mathbf{w}}^T\tilde{\mathbf{A}}^i\tilde{\mathbf{b}}$ for all nonnegative integers i. This is equivalent to $\mathbf{K}\mathbf{b} = \tilde{\mathbf{K}}\tilde{\mathbf{b}}$ where $\mathbf{K} = \mathbf{K}(\mathbf{A}, \mathbf{w})$ and $\tilde{\mathbf{K}} = \mathbf{K}(\tilde{\mathbf{A}}, \tilde{\mathbf{w}})$. The matrix $\mathbf{A} + \mathbf{b}\mathbf{w}^T$ is similar to $\mathbf{K}\mathbf{A}\mathbf{K}^{-1} + \mathbf{K}\mathbf{b}\mathbf{w}^T\mathbf{K}^{-1} = \hat{\mathbf{A}} + \mathbf{K}\mathbf{b}\mathbf{e}_1^T$. Similarly, $\tilde{\mathbf{A}} + \tilde{\mathbf{b}}\tilde{\mathbf{w}}^T$ is similar to $\hat{\mathbf{A}} + \tilde{\mathbf{K}}\tilde{\mathbf{b}}\mathbf{e}_1^T$. By Theorem 1, $\mathbf{K}\mathbf{b} = \tilde{\mathbf{K}}\tilde{\mathbf{b}}$ is equivalent to $\hat{\mathbf{A}} + \mathbf{K}\mathbf{b}\mathbf{e}_1^T$ and $\hat{\mathbf{A}} + \tilde{\mathbf{K}}\tilde{\mathbf{b}}\mathbf{e}_1^T$ having the same eigenvalues. ∎

Definition 6 ([Chen, 1984]) *A linear time-invariant dynamical system is* irreducible *if and only if there does not exist a linear time-invariant dynamical system of lesser dimension that has the same transfer-function matrix.*

The SISO system (12) is irreducible if and only if the pairs $(\mathbf{A}, \mathbf{w})$ and $(\mathbf{A}^T, \mathbf{b})$ both satisfy condition K [Chen, 1984].

Lemma 8 *Suppose the two systems*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{w}^T\mathbf{x} \end{aligned} \qquad\qquad \begin{aligned} \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{b}}\tilde{u} \\ \tilde{y} &= \tilde{\mathbf{w}}^T\tilde{\mathbf{x}} \end{aligned}$$

are irreducible. Then $\mathbf{w}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \tilde{\mathbf{w}}^T(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{b}}$ *if and only if* $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *have the same eigenvalues and* $\mathbf{A} + \mathbf{b}\mathbf{w}^T$ *and* $\tilde{\mathbf{A}} + \tilde{\mathbf{b}}\tilde{\mathbf{w}}^T$ *have the same eigenvalues.*

Proof: By [Chen, 1984, Theorem 5-20], if $\mathbf{w}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \tilde{\mathbf{w}}^T(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{b}}$, then $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are similar matrices. The rest follows from Lemma 7. ∎

From an input-output properties point of view, this has the following interpretation. The systems (13) and (14) can be decomposed into a linear and a nonlinear part as shown in Fig. 2a. Suppose both linear parts have the same transfer function $g(s)$. Then both systems can be depicted as Fig. 2b.

There is essentially only one irreducible state-space realization of $g(s)$, so if the linear parts are irreducible, then the two systems are linearly conjugate. This is what Lemma 8 and Theorem 2 say.

If the linear parts are observable but not controllable (i.e. $(\mathbf{A}, \mathbf{w})$ and $(\tilde{\mathbf{A}}, \tilde{\mathbf{w}})$ satisfy condition K, but $(\mathbf{A}^T, \mathbf{b})$ and $(\tilde{\mathbf{A}}^T, \tilde{\mathbf{b}})$ do not), then by Lemma 7 these 2 systems are still linearly conjugate if $\mathbf{A}$ and $\tilde{\mathbf{A}}$ have the same eigenvalues.

We say that a virtual equilibrium point $\mathbf{x}^*$ in system 1 and a virtual equilibrium point $\mathbf{y}^*$ in system 2 are *matched* if $\mathbf{x}^* = \mathbf{y}^*$ and the same point $\hat{\mathbf{x}}$ is used in both systems for the linearization (Definition 2). Two real equilibrium points are matched if they are the same.

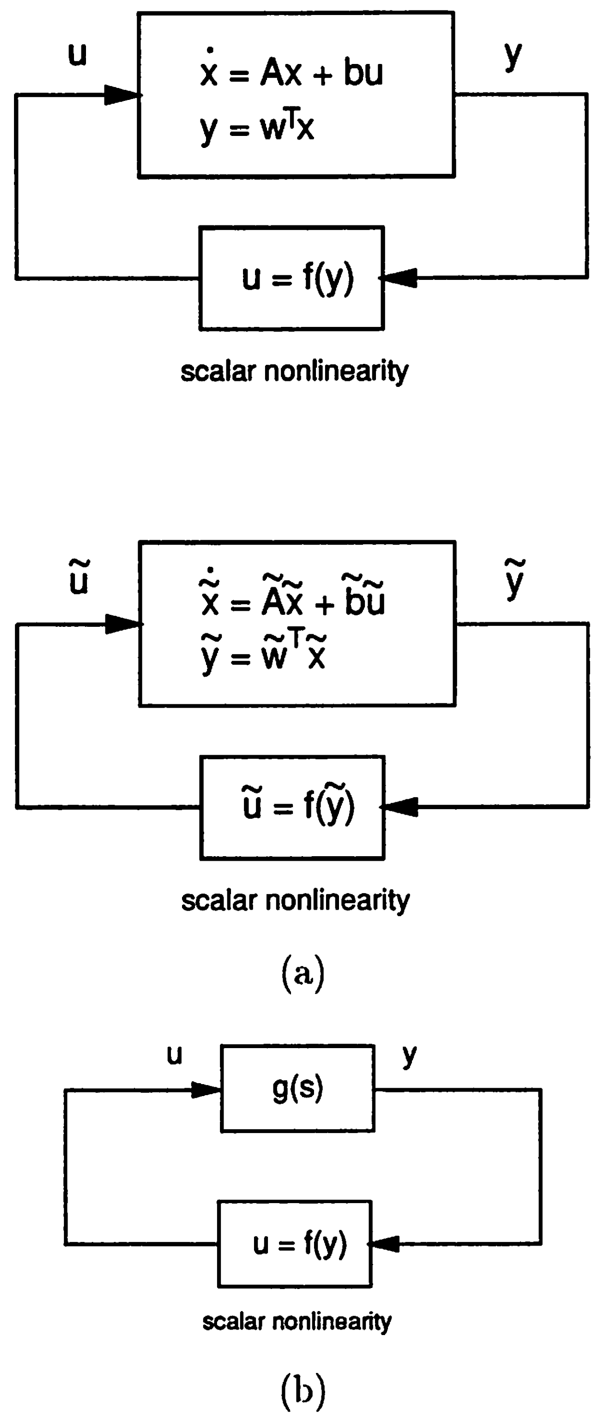


Figure 2: (a) System (13) and system (14) each decomposed as a linear system with (scalar) nonlinear feedback. (b) By expressing the transfer function of the linear part as $g(s)$, the systems in (a) can be depicted as shown.

The next theorem shows that matching the equilibrium points and the eigenvalues of the Jacobian matrices at every point are sufficient to guarantee topological conjugacy of vector fields in $\mathcal{C}(\mathbf{w})$.

Theorem 3 *Consider two systems in* $\mathcal{C}$*:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + f(\mathbf{w}^T\mathbf{x})\mathbf{b} \tag{16}$$

and

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + g(\tilde{\mathbf{w}}^T\mathbf{x})\tilde{\mathbf{b}} \tag{17}$$

where f and g are differentiable functions. Suppose that these two systems are proper and have at least one matched (real or virtual) equilibrium point and their Jacobian matrices at each point have the same eigenvalues, then these systems are topologically conjugate.

Proof: Fix $\mathbf{x}_1$ and define $\mathbf{A}' = \mathbf{A}+f'(\mathbf{w}^T\mathbf{x}_1)\mathbf{b}\mathbf{w}^T$ and $\tilde{\mathbf{A}}' = \tilde{\mathbf{A}}+g'(\tilde{\mathbf{w}}^T\mathbf{x}_1)\tilde{\mathbf{b}}\tilde{\mathbf{w}}^T$. Then Eqs. (16-17) can be written as follows:

$$\dot{\mathbf{x}} = \mathbf{A}'\mathbf{x} + \left(f(\mathbf{w}^T\mathbf{x}) - f'(\mathbf{w}^T\mathbf{x}_1)\mathbf{w}^T\mathbf{x}\right)\mathbf{b} \tag{18}$$

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}'\mathbf{x} + \left(g(\tilde{\mathbf{w}}^T\mathbf{x}) - g'(\tilde{\mathbf{w}}^T\mathbf{x}_1)\tilde{\mathbf{w}}^T\mathbf{x}\right)\tilde{\mathbf{b}} \tag{19}$$

Then $(\mathbf{A}', \mathbf{w})$ and $(\tilde{\mathbf{A}}', \tilde{\mathbf{w}})$ both satisfy condition K by Lemma 4 and $\mathbf{A}'$ and $\tilde{\mathbf{A}}'$ have the same eigenvalues.

Without loss of generality we can assume that $\mathbf{A}$ and $\tilde{\mathbf{A}}$ have the same eigenvalues, as otherwise we can always perform the above transformation. Assume that $\mathbf{b} \neq \mathbf{0}$ and $g'(\mathbf{e}_1^T\mathbf{y}) \neq 0$ for some $\mathbf{y}$. Similar to the proof of Theorem 2, by using the transformations $\mathbf{y} = \mathbf{K}\mathbf{x}$ and $\mathbf{y} = \tilde{\mathbf{K}}\mathbf{x}$ for the two systems respectively, we get

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + f(\mathbf{e}_1^T\mathbf{y})\mathbf{K}\mathbf{b}$$

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + g(\mathbf{e}_1^T\mathbf{y})\tilde{\mathbf{K}}\tilde{\mathbf{b}}$$

The Jacobian matrices of these systems at $\mathbf{y}$ are respectively $\hat{\mathbf{A}}+f'(\mathbf{e}_1^T\mathbf{y})\mathbf{K}\mathbf{b}\mathbf{e}_1^T$ and $\hat{\mathbf{A}}+g'(\mathbf{e}_1^T\mathbf{y})\tilde{\mathbf{K}}\tilde{\mathbf{b}}\mathbf{e}_1^T$. By Lemma 6, $f'(\mathbf{e}_1^T\mathbf{y})\mathbf{K}\mathbf{b} = g'(\mathbf{e}_1^T\mathbf{y})\tilde{\mathbf{K}}\tilde{\mathbf{b}}$ for all $\mathbf{y}$. Since $\mathbf{K}\mathbf{b} \neq \mathbf{0}$, this implies that $f'(\mathbf{e}_1^T\mathbf{y}) = cg'(\mathbf{e}_1^T\mathbf{y})$ for all $\mathbf{y}$ for some constant c. This means that there exists a constant e such that $f(\cdot) = cg(\cdot) + e$. Since $g'(\mathbf{e}_1^T\mathbf{y}) \neq 0$ for some $\mathbf{y}$, it follows that $c\mathbf{K}\mathbf{b} = \tilde{\mathbf{K}}\tilde{\mathbf{b}}$. Thus the two systems simplify to

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + g(\mathbf{e}_1^T\mathbf{y})\tilde{\mathbf{K}}\tilde{\mathbf{b}} + e\mathbf{K}\mathbf{b}$$

and

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + g(\mathbf{e}_1^T\mathbf{y})\tilde{\mathbf{K}}\tilde{\mathbf{b}}$$

Now if they share an equilibrium point $\mathbf{y}^*$, then $\hat{\mathbf{A}}\mathbf{y}^* + d\tilde{\mathbf{K}}\tilde{\mathbf{b}} + \epsilon\mathbf{K}\mathbf{b} = \mathbf{0}$ for the first system, and $\hat{\mathbf{A}}\mathbf{y}^* + d\tilde{\mathbf{K}}\tilde{\mathbf{b}} = \mathbf{0}$ for the second system, where $d = g(\mathbf{e}_1^T\mathbf{y}^*)$ if the equilibrium point is real, and $d = g'(\mathbf{e}_1^T\hat{\mathbf{y}})\mathbf{e}_1^T(\mathbf{y}^* - \hat{\mathbf{y}}) + g(\mathbf{e}_1^T\hat{\mathbf{y}})$ if the equilibrium point is virtual due to linearization around $\hat{\mathbf{y}}$. In either case, we see that $\epsilon\mathbf{K}\mathbf{b} = \mathbf{0}$, thus the two systems are topologically conjugate. The cases where $\mathbf{b} = \mathbf{0}$ or $g'(\mathbf{e}_1^T\mathbf{y}) = 0$ for all $\mathbf{y}$ are handled in a similar manner. ■

Since topological conjugacy is preserved under affine change of coordinates, Theorem 3 can be stated with more generality.

Theorem 4 *Consider two systems in C*

$$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + f(\mathbf{w}_1^T\mathbf{x})\mathbf{b}_1 = \mathbf{h}_1(\mathbf{x}) \tag{20}$$

and

$$\dot{\mathbf{y}} = \mathbf{A}_2\mathbf{y} + g(\mathbf{w}_2^T\mathbf{y})\mathbf{b}_2 = \mathbf{h}_2(\mathbf{y}) \tag{21}$$

where f and g are differentiable functions and $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times n$ matrices. Suppose that these two systems are proper. Suppose further that $\mathbf{T}$ is a nonsingular matrix and $\mathbf{t}$ is a vector such that $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}} = \mathbf{T}\tilde{\mathbf{x}} + \mathbf{t}$ are real equilibrium points of (20) and (21) respectively and $D\mathbf{h}_1(\mathbf{x})$ share the same eigenvalues as $D\mathbf{h}_2(\mathbf{T}\mathbf{x} + \mathbf{t})$ for all $\mathbf{x}$, then the two systems (20) and (21) are topologically conjugate.

Proof: Without loss of generality we can assume $\mathbf{b}_1$ and $\mathbf{b}_2$ are nonzero and f and g are *nonlinear* scalar functions. Using the transformation $\mathbf{y} = \mathbf{T}\mathbf{x} + \mathbf{t}$, system (21) can be written as

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}_2\mathbf{x} + \tilde{g}(\mathbf{w}_2^T\mathbf{T}\mathbf{x})\mathbf{T}^{-1}\mathbf{b}_2 + \mathbf{T}^{-1}\mathbf{A}_2\mathbf{t} \tag{22}$$

Then system (20) and (22) share an equilibrium point $\tilde{\mathbf{x}}$ and the Jacobian matrices at each point share the same eigenvalues. The result follows from Theorem 3 after using Lemma 1 to remove the term $\mathbf{T}^{-1}\mathbf{A}_2\mathbf{t}$. ■

3 Eigenvalue Patterns in the n-dimensional Chua's Oscillator

In this section we extend the 3-dimensional Chua's oscillator [Chua, 1993] to higher dimensions by adding additional linear inductors, capacitors and resistors. We will call this the *n-dimensional Chua's oscillator*. We show that the n-dimensional Chua's oscillator can synthesize almost every eigenvalue pattern of n-dimensional vector fields in C.

For even n the n-dimensional Chua's oscillator is shown in Fig. 3a. For odd n the n-dimensional Chua's oscillator is shown in Fig. 3b. The state equations of the n-dimensional Chua's oscillator

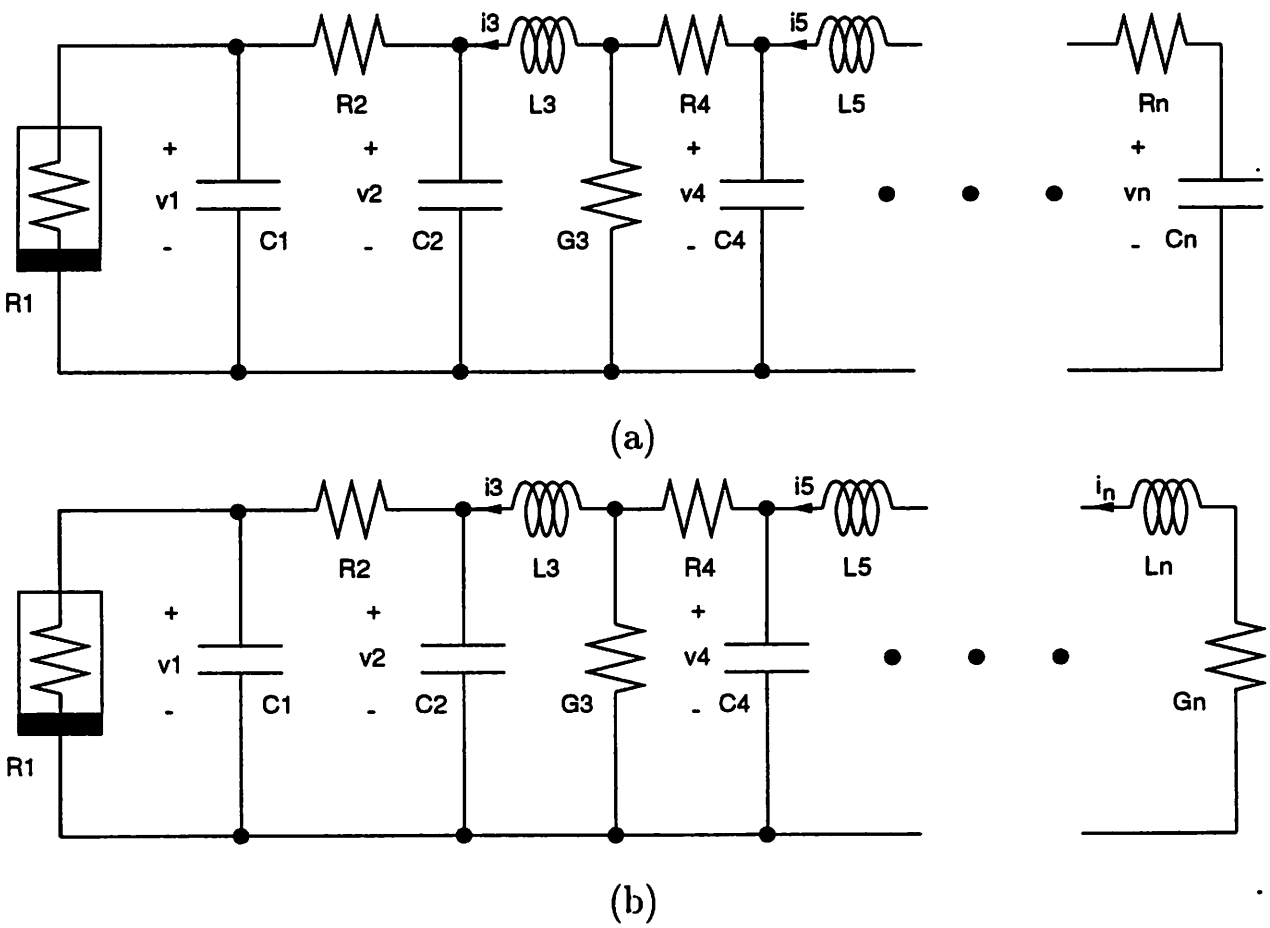


Figure 3: n-dimensional Chua's oscillator. (a) n is even. (b) n is odd.

with an *arbitrary* nonlinearity are given by:

$$\left.\begin{array}{rcl}
\frac{dv_1}{dt} & = & \frac{1}{C_1}\left(\frac{v_2-v_1}{R_2} - g(v_1)\right) \\
\frac{dv_2}{dt} & = & \frac{1}{C_2}\left(\frac{v_1-v_2}{R_2} + i_3\right) \\
\frac{di_3}{dt} & = & \frac{1}{L_3}\left(-v_2 - \frac{R_4 i_3}{1+G_3R_4} + \frac{v_4}{1+G_3R_4}\right) \\
\frac{dv_4}{dt} & = & \frac{1}{C_4}\left(-\frac{G_3 v_4}{1+G_3R_4} - \frac{i_3}{1+G_3R_4} + i_5\right) \\
 & \vdots & \\
\frac{di_{2j-1}}{dt} & = & \frac{1}{L_{2j-1}}\left(-v_{2j-2} - \frac{R_{2j} i_{2j-1}}{1+G_{2j-1}R_{2j}} + \frac{v_{2j}}{1+G_{2j-1}R_{2j}}\right) \\
\frac{dv_{2j}}{dt} & = & \frac{1}{C_{2j}}\left(-\frac{G_{2j-1} v_{2j}}{1+G_{2j-1}R_{2j}} - \frac{i_{2j-1}}{1+G_{2j-1}R_{2j}} + i_{2j+1}\right) \\
 & \vdots & \\
\frac{dv_n}{dt} & = & \frac{1}{C_n}\left(-\frac{i_{n-1}}{1+G_{n-1}R_n} - \frac{G_{n-1} v_n}{1+G_{n-1}R_n}\right)
\end{array}\right\} \quad (23)$$

when n is even, and

$$\left.\begin{array}{rcl}
\frac{dv_1}{dt} & = & \frac{1}{C_1}\left(\frac{v_2-v_1}{R_2} - g(v_1)\right) \\
\frac{dv_2}{dt} & = & \frac{1}{C_2}\left(\frac{v_1-v_2}{R_2} + i_3\right) \\
\frac{di_3}{dt} & = & \frac{1}{L_3}\left(-v_2 - \frac{R_4 i_3}{1+G_3R_4} + \frac{v_4}{1+G_3R_4}\right) \\
\frac{dv_4}{dt} & = & \frac{1}{C_4}\left(-\frac{G_3 v_4}{1+G_3R_4} - \frac{i_3}{1+G_3R_4} + i_5\right) \\
 & \vdots & \\
\frac{di_{2j-1}}{dt} & = & \frac{1}{L_{2j-1}}\left(-v_{2j-2} - \frac{R_{2j} i_{2j-1}}{1+G_{2j-1}R_{2j}} + \frac{v_{2j}}{1+G_{2j-1}R_{2j}}\right) \\
\frac{dv_{2j}}{dt} & = & \frac{1}{C_{2j}}\left(-\frac{G_{2j-1} v_{2j}}{1+G_{2j-1}R_{2j}} - \frac{i_{2j-1}}{1+G_{2j-1}R_{2j}} + i_{2j+1}\right) \\
 & \vdots & \\
\frac{di_n}{dt} & = & \frac{1}{L_n}\left(-v_{n-1} - \frac{i_n}{G_n}\right)
\end{array}\right\} \quad (24)$$

when n is odd. We define $G_2 = 1/R_2$. For n odd, $\frac{1}{G_n} = 0$ is allowed. The state variable v_j is the voltage accross capacitor C_j and the state variable i_j is the current through inductor L_j. The function $g(\cdot)$ is a continuous real-valued function describing the v-i characteristic of the Chua's diode [R. Madan (Guest Editor), 1993] nonlinear resistor.

By using a state transformation, the following dimensionless form is obtained:

$$\left.\begin{array}{rcl}
\frac{dx_1}{dt} & = & k\alpha_1(x_2 - x_1 - h(x_1)) \\
\frac{dx_2}{dt} & = & k(x_1 - x_2 + x_3) \\
\frac{dx_3}{dt} & = & k\alpha_3(-x_2 + \beta_3(-\alpha_4 x_3 + x_4)) \\
\frac{dx_4}{dt} & = & k\beta_4(\beta_3(-x_3 - \gamma_3 x_4) + x_5) \\
 & \vdots & \\
\frac{dx_{2i-1}}{dt} & = & k\alpha_{2i-1}(-x_{2i-2} + \beta_{2i-1}(-\alpha_{2i}x_{2i-1} + x_{2i})) \\
\frac{dx_{2i}}{dt} & = & k\beta_{2i}(\beta_{2i-1}(-x_{2i-1} - \gamma_{2i-1}x_{2i}) + x_{2i+1}) \\
 & \vdots &
\end{array}\right\} \quad (25)$$

where the last equation is:

$$\frac{dx_n}{dt} = k\beta_n\beta_{n-1}(-x_{n-1} - \gamma_{n-1}x_n)$$

if n is even and

$$\frac{dx_n}{dt} = k\alpha_n\left(-x_{n-1} - \frac{x_n}{\gamma_n}\right)$$

if n is odd. The state transformation is given by:

$$E \triangleq 1V, x_1 = \frac{v_1}{E}, x_{2j} = \frac{v_{2j}}{E}, x_{2j+1} = \frac{i_{2j+1}R_2}{E}, \quad \text{for } j = 1, 2, \cdots, \lfloor \tfrac{n}{2} \rfloor$$

$$k = \frac{1}{C_2R_2}, \alpha_1 = \frac{C_2}{C_1}, h(x) = \frac{R_2 g(xE)}{E}$$

$$\alpha_{2j} = \tfrac{R_{2j}}{R_2}, \quad \beta_{2j} = \tfrac{C_2}{C_{2j}}, \quad \text{for } j = 2, 3, \cdots, \lfloor \tfrac{n}{2} \rfloor$$

$$\alpha_{2j-1} = \tfrac{C_2R_2^2}{L_{2j-1}}, \quad \beta_{2j-1} = \tfrac{1}{1+G_{2j-1}R_{2j}} = \tfrac{1}{1+\gamma_{2j-1}\alpha_{2j}}, \quad \gamma_{2j-1} = R_2G_{2j-1}, \quad \text{for } j = 2, 3, \cdots, \lceil \tfrac{n}{2} \rceil$$

where $\lfloor x \rfloor$ is the largest integer less than x and $\lceil x \rceil$ is the smallest integer larger than x. If we also rescale time, then we can assume $k = +1$ or $k = -1$ depending on whether C_2R_2 is positive or negative.

We will write (Eqs. (23-24)) in the form of Eq. (1) as:

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \frac{G_a - G_b}{C_1} f(\mathbf{e}_1^T \tilde{\mathbf{x}})\mathbf{e}_1 \tag{26}$$

where

$$\tilde{\mathbf{x}} = \begin{pmatrix} v_1 \\ v_2 \\ i_3 \\ v_4 \\ i_5 \\ \vdots \\ v_n \end{pmatrix}$$

and

$$\tilde{\mathbf{A}} = \begin{pmatrix} -\frac{G_2+G_a}{C_1} & \frac{G_2}{C_1} & & & & & \\ \frac{G_2}{C_2} & -\frac{G_2}{C_2} & \frac{1}{C_2} & & & & \\ & -\frac{1}{L_3} & -\frac{R_4}{L_3(1+G_3R_4)} & \frac{1}{L_3(1+G_3R_4)} & & & \\ & & -\frac{1}{C_4(1+G_3R_4)} & \frac{-G_3}{C_4(1+G_3R_4)} & \frac{1}{C_4} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & -\frac{1}{C_n(1+G_{n-1}R_n)} & -\frac{G_{n-1}}{C_n(1+G_{n-1}R_n)} \end{pmatrix} \tag{27}$$

when n is even, and

$$\tilde{\mathbf{x}} = \begin{pmatrix} v_1 \\ v_2 \\ i_3 \\ v_4 \\ i_5 \\ \vdots \\ i_n \end{pmatrix}$$

and

$$\tilde{\mathbf{A}} = \begin{pmatrix} -\frac{G_2+G_a}{C_1} & \frac{G_2}{C_1} & & & & \\ \frac{G_2}{C_2} & -\frac{G_2}{C_2} & \frac{1}{C_2} & & & \\ & -\frac{1}{L_3} & -\frac{R_4}{L_3(1+G_3R_4)} & \frac{1}{L_3(1+G_3R_4)} & & \\ & & -\frac{1}{C_4(1+G_3R_4)} & \frac{-G_3}{C_4(1+G_3R_4)} & \frac{1}{C_4} & \\ & & & \ddots & \ddots & \ddots \\ & & & & -\frac{1}{L_n} & -\frac{1}{L_nG_n} \end{pmatrix} \tag{28}$$

when n is odd.

The function f in Eq. (26) is related to the function g in Eqs. (23-24) as follows:

$$g(v_1) = G_a v_1 + (G_b - G_a) f(v_1) \tag{29}$$

Consider two tridiagonal matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ defined as follows:

$$\mathbf{A}_0 = \begin{pmatrix} a_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix} \tag{30}$$

$$\mathbf{A}_1 = \begin{pmatrix} \overline{a}_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix} \tag{31}$$

Thus $\mathbf{A}_0$ and $\mathbf{A}_1$ only differ in the first entry. Let the characteristic polynomials of $\mathbf{A}_0$ and $\mathbf{A}_1$ be written as:

$$\begin{aligned} \chi(\mathbf{A}_0) &= \lambda^n + r_{n-1}\lambda^{n-1} + \cdots + r_1\lambda + r_0 \\ \chi(\mathbf{A}_1) &= \lambda^n + s_{n-1}\lambda^{n-1} + \cdots + s_1\lambda + s_0 \end{aligned} \tag{32}$$

The following Lemma is proved in [Kocarev *et al.*, 1993]:

Lemma 9 *Define $\kappa_i = a_{i,i}$ for $i = 1, \ldots, n$ and $\rho_i = a_{i,i+1}a_{i+1,i}$ for $i = 1, \ldots, n-1$. Except for a set of measure zero, the values of $\overline{a}_{1,1}$, κ_i and ρ_i are uniquely determined by r_i and s_i in (32) and vice versa.*

The following $(n+1)$-step algorithm[2] is given in [Kocarev *et al.*, 1993] for computing $\overline{a}_{1,1}$, κ_i and ρ_i from r_i and s_i:

Algorithm 1

Step 0: Calculate

$$\begin{aligned} A_{i,1} &= (-1)^i r_{n-i} \quad \text{for } i = 1, \ldots, n \\ A_{i,2} &= (-1)^i \frac{r_{n-i-1} - s_{n-i-1}}{r_{n-1} - s_{n-1}} \quad \text{for } i = 1, \ldots, n-1 \end{aligned}$$

Step 1: Calculate

$$\begin{aligned} \kappa_1 &= A_{1,1} - A_{1,2} \\ \overline{a}_{1,1} &= -s_{n-1} - A_{1,2} \\ \rho_1 &= -A_{2,1} + A_{2,2} + \kappa_1 A_{1,2} \\ A_{j,3} &= \frac{-A_{j+2,1} + A_{j+2,2} + \kappa_1 A_{j+1,2}}{\rho_1} \quad \text{for } j = 1, \ldots, n-2 \\ A_{n-2,3} &= \frac{-A_{n,1} + \kappa_1 A_{n-1,2}}{\rho_1} \end{aligned}$$

Step 2: Calculate

$$\begin{aligned} \kappa_2 &= A_{1,2} - A_{1,3} \\ \rho_2 &= -A_{2,2} + A_{2,3} + \kappa_2 A_{1,3} \\ A_{j,4} &= \frac{-A_{j+2,2} + A_{j+2,3} + \kappa_2 A_{j+1,3}}{\rho_2} \quad \text{for } j = 1, \ldots, n-4 \\ A_{n-3,4} &= \frac{-A_{n-1,2} + \kappa_2 A_{n-2,3}}{\rho_2} \end{aligned}$$

Step k, for $k = 3, \ldots, n-3$: Calculate

$$\begin{aligned} \kappa_k &= A_{1,k} - A_{1,k+1} \\ \rho_k &= -A_{2,k} + A_{2,k+1} + \kappa_k A_{1,k+1} \\ A_{j,k+2} &= \frac{-A_{j+2,k} + A_{j+2,k+1} + \kappa_k A_{j+1,k+1}}{\rho_k} \quad \text{for } j = 1, \ldots, n-k-2 \\ A_{n-k-1,k+2} &= \frac{-A_{n-k+1,k} + \kappa_k A_{n-k,k+1}}{\rho_k} \end{aligned}$$

[2]The algorithm shown here corrects some typographical errors in [Kocarev *et al.*, 1993].

Step $n-2$**:** Calculate

$$\begin{aligned}
\kappa_{n-2} &= A_{1,n-2} - A_{1,n-1} \\
\rho_{n-2} &= -A_{2,n-2} + A_{2,n-1} + \kappa_{n-2}A_{1,n-1} \\
A_{1,n} &= \frac{-A_{3,n-2} + \kappa_{n-2}A_{2,n-1}}{\rho_{n-2}}
\end{aligned}$$

Step $n-1$**:** Calculate

$$\begin{aligned}
\kappa_{n-1} &= A_{1,n-1} - A_{1,n} \\
\rho_{n-1} &= -A_{2,n-1} + \kappa_{n-1}A_{1,n}
\end{aligned}$$

Step n**:** Calculate

$$\kappa_n = A_{1,n}$$

End of Algorithm 1

The matrix $\tilde{\mathbf{A}}$ defined in (27) or (28) is tridiagonal and differs from $\tilde{\mathbf{A}} + \frac{G_a - G_b}{C_1}\mathbf{e}_1\mathbf{e}_1^T$ in only the first entry. Therefore we can apply Lemma 9 to the matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}} + \frac{G_a - G_b}{C_1}\mathbf{e}_1\mathbf{e}_1^T$. Applying Algorithm 1 to find $\overline{a}_{1,1}$, κ_i and ρ_i and then solving for the circuit parameters in $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}} + \frac{G_a - G_b}{C_1}\mathbf{e}_1\mathbf{e}_1^T$, we obtain the following theorem:

Theorem 5 *Assume that the parameters $\overline{a}_{1,1}$, κ_i and ρ_i are calculated as in Algorithm 1. Suppose that the following inequalities are satisfied:*

$$\begin{aligned}
\overline{a}_{1,1} &\neq \kappa_1 && (33)\\
\kappa_2 &\neq 0 && (34)\\
\rho_i &\neq 0 \quad \textit{for } i = 1, \ldots, n-1 && (35)\\
\rho_{2i-1} &\neq \kappa_{2i-1}\kappa_{2i} \quad \textit{for } i = 1, \ldots, \lfloor \tfrac{n}{2} \rfloor && (36)
\end{aligned}$$

Then the following values for the circuit parameters will give a matrix $\tilde{\mathbf{A}}$ (Eq. (27) or (28)) such that

$$\begin{aligned}
\chi\left(\tilde{\mathbf{A}}\right) &= \lambda^n + r_{n-1}\lambda^{n-1} + \cdots + r_1\lambda + r_0 \\
\chi\left(\tilde{\mathbf{A}} + \frac{G_a - G_b}{C_1}\mathbf{e}_1\mathbf{e}_1^T\right) &= \lambda^n + s_{n-1}\lambda^{n-1} + \cdots + s_1\lambda + s_0
\end{aligned}$$

$$
\left.\begin{array}{lcl}
C_1 & = & 1 \\
G_2 & = & -\frac{C_1 \rho_1}{\kappa_2} \\
G_a & = & -\kappa_1 C_1 - G_2 \\
G_b & = & -\overline{a}_{1,1} C_1 - G_2 \\
C_2 & = & -\frac{G_2}{\kappa_2} \\
L_3 & = & -\frac{1}{\rho_2 C_2} \\
G_3 & = & \frac{\kappa_4}{L_3(\rho_3 - \kappa_3 \kappa_4)} \\
R_4 & = & -\frac{\kappa_3 L_3(\rho_3 - \kappa_3 \kappa_4)}{\rho_3} \\
C_4 & = & -\frac{\rho_3}{L_3(\rho_3 - \kappa_3 \kappa_4)^2} \\
L_{2i-1} & = & -\frac{1}{\rho_{2i-2} C_{2i-2}} \quad \textit{for } i = 2, \ldots, \lceil \frac{n}{2} \rceil \\
G_{2i-1} & = & \frac{\kappa_{2i}}{L_{2i-1}(\rho_{2i-1} - \kappa_{2i-1}\kappa_{2i})} \quad \textit{for } i = 2, \ldots, \lfloor \frac{n}{2} \rfloor \\
R_{2i} & = & -\frac{\kappa_{2i-1} L_{2i-1}(\rho_{2i-1} - \kappa_{2i-1}\kappa_{2i})}{\rho_{2i-1}} \quad \textit{for } i = 2, \ldots, \lfloor \frac{n}{2} \rfloor \\
C_{2i} & = & -\frac{\rho_{2i-1}}{L_{2i-1}(\rho_{2i-1} - \kappa_{2i-1}\kappa_{2i})^2} \quad \textit{for } i = 2, \ldots, \lfloor \frac{n}{2} \rfloor \\
G_n & = & -\frac{1}{L_n \kappa_n} \quad \textit{if } n \textit{ is odd}
\end{array}\right\} \tag{37}
$$

4 Topological Conjugacy Between Chua's Oscillator and Vector Fields in C

We are now in a position to combine the results in the previous sections into the main result in this paper which shows that by choosing appropriate parameters the n-dimensional Chua's oscillator is topologically conjugate to almost every n-dimensional vector field in C. The statement of the next theorem also gives an algorithm for choosing the parameters of Chua's oscillator.

Theorem 6 *Consider a* proper *vector field of C written in the form (1). Let the characteristic polynomials of* $\mathbf{A}$ *and* $\mathbf{A} + \mathbf{b}\mathbf{w}^T$ *be written as Eq. (7). Suppose the inequalities (33)-(36) are satisfied, then the Chua's oscillator defined in (23-24) with the parameters specified by (37) and (29) is topologically conjugate to system (1).*

Proof: It can easily be shown that given the conditions in the theorem, the matrix $\tilde{\mathbf{A}}$ defined in (27-28) together with $\mathbf{e}_1$ satisfies condition K. The theorem then follows from Theorem 2 and Theorem 5. ■

Remark 6 *Lemma 1 implies that almost all vector fields in C' are topologically conjugate to Chua's oscillator.*

Remark 7 *Lemma 5 implies that the eigenvalues of the Jacobian matrix in Chua's oscillator (Eqs. (23-24)) and the vector field in C (when written as Eq. (2)) will be the same at corresponding points where the Jacobian matrix is defined.*

Remark 8 *The set of vector fields in C where inequalities (33)-(36) are not satisfied or is not proper is of measure zero. For these vector fields, in general it is possible to perturb the system slightly to obtain a system in C outside this set which generates similar behavior.*

Remark 9 *In dimensionless form (25), the parameters for Chua's oscillator are given by:*

$$\left.\begin{array}{lcl} k & = & -\kappa_2 \\ \alpha_1 & = & \frac{\rho_1}{\kappa_2^2} \\ \alpha_3 & = & -\frac{\rho_2}{\kappa_2^2} \\ \gamma_{2i-1} & = & -\frac{\kappa_{2i-1}\kappa_{2i}}{\rho_{2i-1}} \quad \textit{for } i = 2, \ldots, \lceil \frac{n}{2} \rceil \\ \beta_{2i-1} & = & \frac{1}{1+\gamma_{2i-1}} \quad \textit{for } i = 2, \ldots, \lceil \frac{n}{2} \rceil \\ \alpha_{2i+1} & = & \frac{\rho_{2i}\rho_{2i-1}\rho_1\alpha_{2i-1}}{(\rho_{2i-1}-\kappa_{2i-1}\kappa_{2i})^2} \quad \textit{for } i = 2, \ldots, \lfloor \frac{n}{2} \rfloor \\ \beta_{2i} & = & -\frac{(\rho_{2i-1}-\kappa_{2i-1}\kappa_{2i})^2}{\kappa_2^2\rho_{2i-1}\alpha_{2i-1}} \quad \textit{for } i = 2, \ldots, \lfloor \frac{n}{2} \rfloor \\ \alpha_{2i} & = & \frac{\kappa_{2i-1}(\rho_{2i-1}-\kappa_{2i-1}\kappa_{2i})}{\kappa_2\rho_{2i-1}\alpha_{2i-1}} \quad \textit{for } i = 2, \ldots, \lfloor \frac{n}{2} \rfloor \\ h(x) & = & \left(\frac{\kappa_1\kappa_2}{\rho_1} - 1\right) x + \frac{\kappa_2}{\rho_1}(\overline{a}_{1,1} - \kappa_1) f(x) \end{array}\right\} \tag{38}$$

5 Continuous Piecewise-Linear Vector Fields

For piecewise-linear vector fields, the Jacobian matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ in neighboring regions must satisfy a consistent variation property $\mathbf{J}_1 - \mathbf{J}_2 = \delta \mathbf{c}\mathbf{a}^T$. Piecewise-linear vector fields in C are such that the vectors $\mathbf{c}$ and $\mathbf{a}$ are the same at all boundaries. Thus we consider the subclass of piecewise-linear vector fields, where the Jacobian matrix in the different regions differ by a scalar multiple of a fixed rank 1 matrix:

Definition 7 *The class $\mathcal{P}$ is the class of vector fields which are:*

- *continuous,*
- *piecewise-linear with a countable number of regions,*
- *the boundary planes are parallel planes of the form $\mathbf{w}^T\mathbf{x} = d_i$,*
- *there is at least one equilibrium point (which can be real or virtual),*
- *there exists a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ such that the Jacobian matrix in each region is of the form $\mathbf{A} + \mu_i \mathbf{b}\mathbf{w}^T$.*

Lemma 10 *By a change of coordinates, the vector fields in class $\mathcal{P}$ form a subclass of C and can be written in the form*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \left(f + e\mathbf{e}_1^T\mathbf{x} + \sum_{i=1}^{\infty} c_i \left|\mathbf{e}_1^T\mathbf{x} - d_i\right|\right)\mathbf{b} \tag{39}$$

Proof: The proof is similar to Lemma 1, where an equilibrium point is translated to the origin. Without loss of generality we can assume that $\mathbf{w} = \mathbf{e}_1$. It is clear that vector fields in class $\mathcal{P}$ can be written in the form

$$\dot{\mathbf{x}} = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \qquad \tilde{d}_{i-1} \leq \mathbf{e}_1^T\mathbf{x} \leq \tilde{d}_i \tag{40}$$

Suppose that there exists an equilibrium point due to the linearization in region k, i.e. there exists $\mathbf{f}_k$ such that $\mathbf{A}_k\mathbf{f}_k + \mathbf{b}_k = \mathbf{0}$. The difference between the Jacobian matrices in two different regions are of the form $\delta\mathbf{b}\mathbf{e}_1^T$. If $\tilde{d}_i \neq 0$, by continuity, we must have

$$\mathbf{A}_{i+1} - \mathbf{A}_i = \frac{1}{\tilde{d}_i}(\mathbf{b}_i - \mathbf{b}_{i+1})\mathbf{e}_1^T$$

If $\tilde{d}_i = 0$, by continuity we have $\mathbf{b}_i - \mathbf{b}_{i+1} = \mathbf{0}$. Thus in either case, we have $(\mathbf{b}_i - \mathbf{b}_{i+1}) = \delta\mathbf{b}$ for some constant δ. So we can write Eq. (40) as

$$\dot{\mathbf{x}} = \mathbf{A}_i\mathbf{x} + \mathbf{b}_k + \delta_i\mathbf{b}, \qquad \tilde{d}_{i-1} \leq \mathbf{e}_1^T\mathbf{x} \leq \tilde{d}_i \tag{41}$$

where δ_i are real numbers and $\delta_k = 0$. Using the transformation $\mathbf{y} = \mathbf{x} - \mathbf{f}_k$, we obtain

$$\dot{\mathbf{y}} = \dot{\mathbf{x}} = \mathbf{A}_i\mathbf{y} + \mathbf{A}_i\mathbf{f}_k + \mathbf{b}_k + \delta_i\mathbf{b}, \qquad \tilde{d}_{i-1} - \mathbf{e}_1^T\mathbf{f}_k \leq \mathbf{e}_1^T\mathbf{y} \leq \tilde{d}_i - \mathbf{e}_1^T\mathbf{f}_k \tag{42}$$

Observing that $\mathbf{A}_i\mathbf{f}_k + \mathbf{b}_k = (\mathbf{A}_i - \mathbf{A}_k)\mathbf{f}_k + \mathbf{A}_k\mathbf{f}_k + \mathbf{b}_k = (\mathbf{A}_i - \mathbf{A}_k)\mathbf{f}_k = \kappa_i\mathbf{b}\mathbf{e}_1^T\mathbf{f}_k = \kappa_i\mathbf{e}_1^T\mathbf{f}_k\mathbf{b}$ for some κ_i, we see that system (42) is in the form

$$\dot{\mathbf{y}} = \mathbf{A}_i\mathbf{y} + \hat{\delta}_i\mathbf{b}, \qquad d_{i-1} \leq \mathbf{e}_1^T\mathbf{y} \leq d_i \tag{43}$$

where $i \in \{0, 1, \ldots, n\}$, $\hat{\delta}_i \in \mathbb{R}$ and $\hat{\delta}_k = 0$ for some k. Noting that $\mathbf{A}_i = \mathbf{A} + \mu_i\mathbf{b}\mathbf{e}_1^T$ and by using the canonical piecewise-linear equation [Chua and Kang, 1977], this can be written as Eq. (39). ■

Thus the class $\mathcal{P}$ can be reduced to a subclass of $\mathcal{C}$ and we can apply Theorem 6 to these vector fields. For vector fields in $\mathcal{P}$, Theorem 3 has the following interpretation. If two vector fields in $\mathcal{P}$ have the same boundary planes, and the eigenvalues in corresponding regions are identical, and the equilibrium points are matched, then the two systems with these two vector fields are topologically conjugate (except for a measure zero set in $\mathcal{P}$).

Consider the 4-dimensional 3-region piecewise-linear system considered in [Matsumoto *et al.*, 1986] which exhibits hyperchaos:

$$\begin{aligned} \frac{dv_1}{dt} &= \frac{f(v_2 - v_1) - i_3}{C_1} \\ \frac{dv_2}{dt} &= \frac{-f(v_2 - v_1) - i_4}{C_2} \\ \frac{di_3}{dt} &= \frac{v_1 + Ri_3}{L_3} \\ \frac{di_4}{dt} &= \frac{v_2}{L_4} \end{aligned} \tag{44}$$

where

$$f(x) = m_1x + \frac{1}{2}(m_0 - m_1)\left(|x+1| - |x-1|\right)$$

For the parameters $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{20}$, $L_3 = 1$, $L_4 = \frac{2}{3}$, $R = 1$, $m_0 = -0.2$, and $m_1 = 3$, there exists a real equilibrium point in each of the three regions.

Applying Theorem 5 we obtain the following values for the 4-dimensional Chua's oscillator:

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= -6.2599109091 \times 10^5 \\
C_4 &= -7.3076305375 \times 10^5 \\
L_3 &= -6.7877793494 \times 10^{-7} \\
R_2 &= 2.4121749046 \times 10^{-4} \\
G_3 &= 7.7704985725 \times 10^5 \\
R_4 &= -9.9757154563 \times 10^{-8} \\
G_a &= -4.1500363636 \times 10^3 \\
G_b &= -4.0796363636 \times 10^3
\end{aligned}$$

with $g(\cdot)$ defined as

$$g(x) = G_b x + \frac{1}{2}(G_a - G_b)\left(|x+1| - |x-1|\right)$$

For these parameters, both system (23) and system (44) have an equilibrium point at the origin, both systems are odd-symmetric 3 segment piecewise-linear, and the eigenvalues in corresponding regions are matched. Figure 4 shows a projection of the resulting attractor from Chua's oscillator.

6 2-Segment Continuous Piecewise-Linear Vector Fields

In this section we study the subclass of vector fields in $\mathcal{P}$ which is piecewise-linear with 2 segments. Thus the system has the form:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}_0\mathbf{x} + \mathbf{b}_0, & \mathbf{e}_1^T\mathbf{x} \leq d \\ \mathbf{A}_1\mathbf{x} + \mathbf{b}_1, & \mathbf{e}_1^T\mathbf{x} \geq d \end{cases} \tag{45}$$

Without loss of generality we can assume that there exists a (real or virtual) equilibrium point by linearizing in the region $\mathbf{e}_1^T\mathbf{x} \leq d$.[3]

The following corollary to Lemma 10 transforms system (45) into a more simplified form:

Corollary 2 *Assume system (45) is in $\mathcal{P}$. System (45) is then topologically conjugate to one of the following systems:*

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}_0\mathbf{x}, & \mathbf{e}_1^T\mathbf{x} \leq 1 \\ \mathbf{A}_1\mathbf{x} + \mathbf{b}_2, & \mathbf{e}_1^T\mathbf{x} \geq 1 \end{cases} \tag{46}$$

[3]Thus we assume that there is either a real equilibrium point in the region $\mathbf{e}_1^T\mathbf{x} \leq d$ or a virtual equilibrium point in the region $\mathbf{e}_1^T\mathbf{x} > d$. If this assumption is not satisfied, it will be satisfied after the application of the transformation $\mathbf{y} = -\mathbf{x}$ since we assume the existence of at least one equilibrium point for a vector field in P.

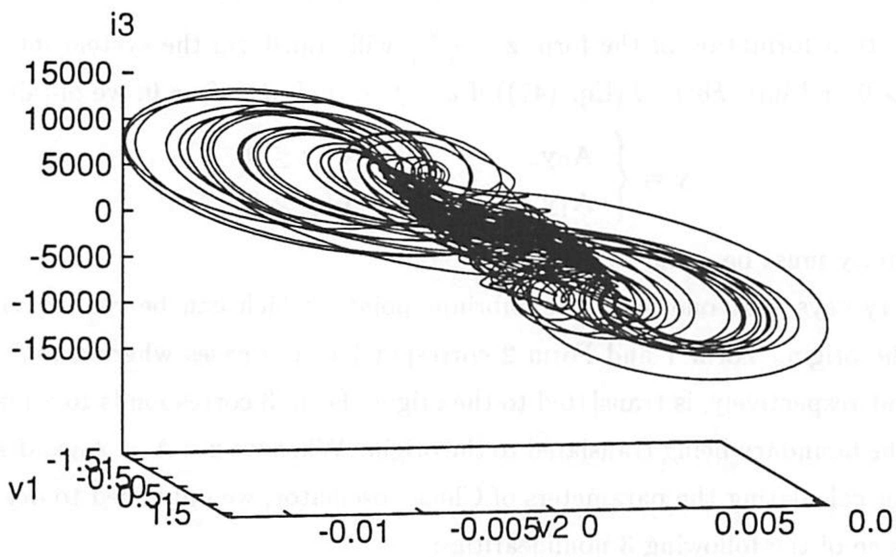


Figure 4: Attractor from the 4-dimensional Chua's oscillator which is topologically conjugate to the system in Eq. (44) with parameters $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{20}$, $L_3 = 1$, $L_4 = \frac{2}{3}$, $R = 1$, $m_0 = -0.2$ and $m_1 = 3$. The attractor is projected onto the v_1-v_2-i_3 plane.

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}_1\mathbf{x} + \mathbf{b}_2, & \mathbf{e}_1^T\mathbf{x} \leq 1 \\ \mathbf{A}_0\mathbf{x}, & \mathbf{e}_1^T\mathbf{x} \geq 1 \end{cases} \tag{47}$$

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}_0\mathbf{x}, & \mathbf{e}_1^T\mathbf{x} \leq 0 \\ \mathbf{A}_1\mathbf{x}, & \mathbf{e}_1^T\mathbf{x} \geq 0 \end{cases} \tag{48}$$

for some vector $\mathbf{b}_2$.

Proof: We will refer to Eqns. (46)-(48) as *Form 1* to *Form 3* respectively. Consider system (45). Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$ be such that $\mathbf{A}_0\mathbf{f} + \mathbf{b}_0 = \mathbf{0}$. Using the transformation $\mathbf{y} = \mathbf{x} - \mathbf{f}$, we get

$$\dot{\mathbf{y}} = \begin{cases} \mathbf{A}_0\mathbf{y}, & \mathbf{e}_1^T\mathbf{y} \leq d - f_1 \\ \mathbf{A}_1\mathbf{y} + \mathbf{b}_1 + \mathbf{A}_1\mathbf{f}, & \mathbf{e}_1^T\mathbf{y} \geq d - f_1 \end{cases}$$

If $d - f_1 \neq 0$, a transformation of the form $\mathbf{z} = \frac{\mathbf{y}}{d - f_1}$ will transform the system into *Form 1* (Eq. (46)) if $d - f_1 > 0$ and into *Form 2* (Eq. (47)) if $d - f_1 < 0$. If $d - f_1 = 0$, we obtain the system

$$\dot{\mathbf{y}} = \begin{cases} \mathbf{A}_0\mathbf{y}, & \mathbf{e}_1^T\mathbf{y} \leq 0 \\ \mathbf{A}_1\mathbf{y} + \mathbf{b}_1 + \mathbf{A}_1\mathbf{f}, & \mathbf{e}_1^T\mathbf{y} \geq 0 \end{cases}$$

which by continuity must be equal to *Form 3*. ■

This corollary says that one of the equilibrium points, which can be real or virtual, can be translated to the origin. Form 1 and Form 2 correspond to the cases where a real and a virtual equilibrium point respectively, is translated to the origin. Form 3 corresponds to a real equilibrium point lying on the boundary being translated to the origin. When we use $\mathbf{A} = \mathbf{A}_0$ and $\mathbf{A} + \mathbf{b}\mathbf{e}_1^T = \mathbf{A}_1$ in Theorem 5 for calculating the parameters of Chua's oscillator, we only need to deal with Chua's oscillator with one of the following 3 nonlinearities:

$$g(x) = \frac{1}{2}(G_a - G_b) + \frac{1}{2}(G_a + G_b)x + \frac{1}{2}(G_b - G_a)\,|x - 1| \qquad \text{for Form 1} \tag{49}$$

$$g(x) = \frac{1}{2}(G_a - G_b) + \frac{1}{2}(G_a + G_b)x + \frac{1}{2}(G_a - G_b)\,|x - 1| \qquad \text{for Form 2} \tag{50}$$

$$g(x) = \frac{1}{2}(G_a + G_b)x + \frac{1}{2}(G_b - G_a)\,|x| \qquad \text{for Form 3} \tag{51}$$

Note that using these three forms for $g(\cdot)$ results in Chua's oscillator having the same number of parameters as the 3-segment odd-symmetric Chua's oscillator.

Let us illustrate the above by transforming two chaotic systems, which have 2-segment piecewise-linear vector fields, into equivalent Chua's oscillators.

[Nishio *et al.*, 1992] introduced two simple circuits which exhibit hyperchaos. The state equations are given by:

$$\begin{aligned} \dot{x}_1 &= x_3 - f(x_1) \\ \dot{x}_2 &= \gamma_C(x_4 - x_3) \\ \dot{x}_3 &= x_2 - x_1 \\ \dot{x}_4 &= -\gamma_L x_2 + \alpha x_4 \end{aligned} \tag{52}$$

and

$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= \gamma_C(x_4 - x_3 - f(x_2)) \\
\dot{x}_3 &= x_2 - x_1 \\
\dot{x}_4 &= -\gamma_L x_2 + \alpha x_4
\end{aligned}
\tag{53}
$$

where $f(x) = \frac{1}{2\epsilon}(|x-1| + x - 1)$.

Consider Eq. (52) with parameters $\gamma_L = 0.5$, $\gamma_C = 0.5$, $\alpha = 0.18$, $\epsilon = 0.005$. There exists a real equilibrium point in each of the two regions, so we obtain Form 1. Using Theorem 5, we obtain the following parameters for the 4-dimensional Chua's oscillator:

$$
\begin{aligned}
C_1 &= 1 \\
C_2 &= -1.2980742146337069 \times 10^{33} \\
C_4 &= -2.5961484292674138 \times 10^{33} \\
L_3 &= -1.5407439555097887 \times 10^{-33} \\
R_2 &= -2.7755575615628914 \times 10^{-17} \\
G_3 &= 4.6730671726813447 \times 10^{32} \\
R_4 &= 4.2764235361475130 \times 10^{-50} \\
G_a &= 3.6028797018963968 \times 10^{16} \\
G_b &= 3.6028797018964168 \times 10^{16}
\end{aligned}
$$

where $g(\cdot)$ is defined by Eq. (49). Note that some ill-conditioning causes G_a to be very close to G_b. Figure 5 shows a projection of the resulting attractor from Chua's oscillator.

Consider Eq. (53) with parameters $\gamma_L = 1$, $\gamma_C = 1.545$, $\alpha = 0.26$, $\epsilon = 0.005$. Again there exists a real equilibrium point in each of the two regions. Using Theorem 5, we obtain the following parameters for the 4-dimensional Chua's oscillator:

$$
\begin{aligned}
C_1 &= 1 \\
C_2 &= -1.8284023669 \times 10^{2} \\
C_4 &= -2.1568595807 \times 10^{2} \\
L_3 &= -1.1321166763 \times 10^{-2} \\
R_2 &= 4.2071197411 \times 10^{-2} \\
G_3 &= 7.1307692308 \times 10^{1} \\
R_4 &= -2.5374219121 \times 10^{-3} \\
G_a &= -2.3769230769 \times 10^{1} \\
G_b &= 2.8523076923 \times 10^{2}
\end{aligned}
$$

where $g(\cdot)$ is defined by Eq. (49). Figure 6 shows a projection of the resulting attractor from Chua's oscillator.

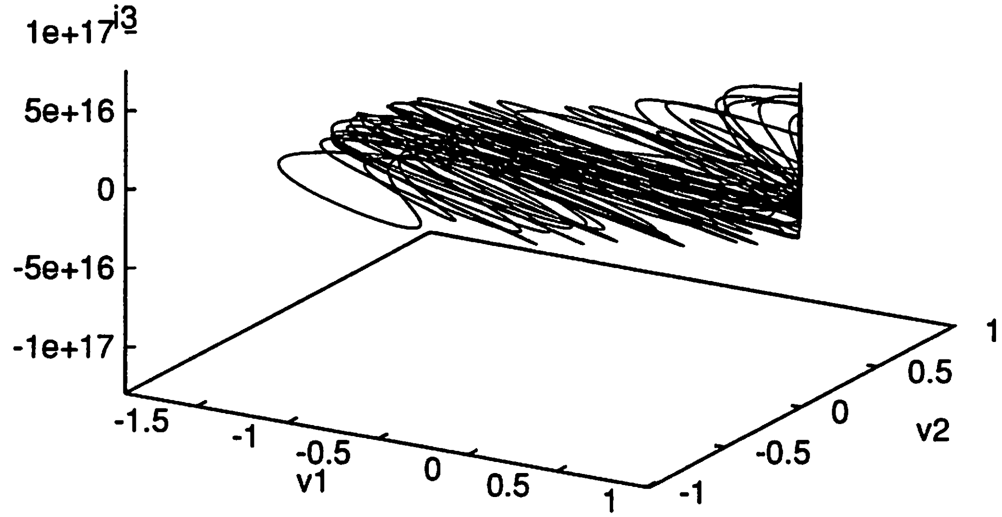


Figure 5: Attractor from the 4-dimensional Chua's oscillator which is topologically conjugate to the system in Eq. (52) with parameters $\gamma_L = 0.5$, $\gamma_C = 0.5$, $\alpha = 0.18$, $\epsilon = 0.005$. The attractor is projected onto the v_1-v_2-i_3 plane.

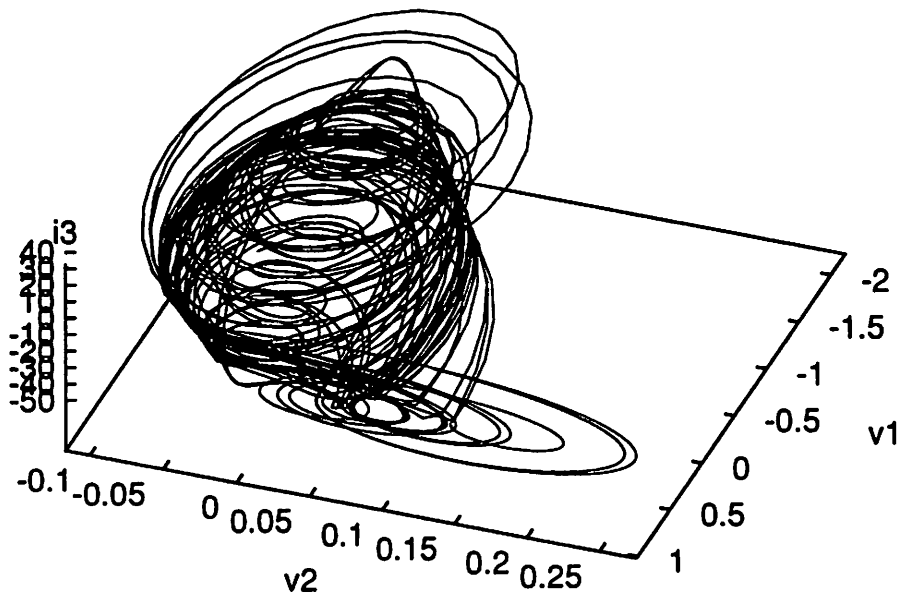


Figure 6: Attractor from the 4-dimensional Chua's oscillator which is topologically conjugate to the system in Eq. (53) with parameters $\gamma_L = 1$, $\gamma_C = 1.545$, $\alpha = 0.26$, $\epsilon = 0.005$. The attractor is projected onto the v_1-v_2-i_3 plane.

7 Linear Conjugacy of Vector Fields with Multiple Scalar Nonlinearities

We now extend the results in Section 2 to vector fields with several scalar nonlinearities. We consider the following class of vector fields:

Definition 8 *The class $C_n(\mathbf{w})$ consists of vector fields of the form:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{i=1}^{\infty} f_i\left(\mathbf{w}^T\mathbf{x}\right)\mathbf{b}_i \tag{54}$$

where $\mathbf{A}$ is an $n \times n$ matrix and $f_i(\cdot) : \mathbb{R} \to \mathbb{R}$ are real-valued continuous functions. We define $C_n = \bigcup_{\mathbf{w}} C_n(\mathbf{w})$.

If $\mathbf{w} = \mathbf{0}$, this reduces to the class of affine systems. Vector fields in class $C_n(\mathbf{w})$ can be considered to be n-dimensional nonlinear vector fields where the nonlinearity occurs only in the direction $\mathbf{w}$. This is stated more precisely in the following lemma.

Lemma 11 *For $\mathbf{w} \neq \mathbf{0}$, the class $C_n(\mathbf{w})$ is equal to the class of continuous vector fields $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{u} + \mathbf{g}(\mathbf{v})$ for all $\mathbf{x}$ where $\mathbf{x}$ is decomposed as $\mathbf{x} = \mathbf{u} + \mathbf{v}$ and $\mathbf{v} = \frac{\mathbf{w}^T\mathbf{x}}{\mathbf{w}^T\mathbf{w}}\mathbf{w}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{w}$.*

Proof: Since $\mathbf{w}^T\mathbf{x} = \mathbf{w}^T\mathbf{v}$, a system in the form (54) can be written as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} + \sum_{i=1}^{\infty} f_i\left(\mathbf{w}^T\mathbf{v}\right)\mathbf{b}_i$$

So one direction is clear. Consider a system of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{u} + \mathbf{g}(\mathbf{v})$. Since $\mathbf{v} = \left(\frac{\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\right)\mathbf{w}^T\mathbf{x}$, $\mathbf{g}(\mathbf{v})$ can be written as $\tilde{\mathbf{g}}(\mathbf{w}^T\mathbf{x})$. Thus we can write the system as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{v} + \tilde{\mathbf{g}}(\mathbf{w}^T\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{A}\frac{\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w}^T\mathbf{x} + \tilde{\mathbf{g}}(\mathbf{w}^T\mathbf{x})$$

This can be written as (54) if we define $\mathbf{b}_i = \mathbf{e}_i$ and

$$f_i(\mathbf{w}^T\mathbf{x}) = \mathbf{e}_i^T\left(\tilde{\mathbf{g}}(\mathbf{w}^T\mathbf{x}) - \mathbf{A}\frac{\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w}^T\mathbf{x}\right)$$

for $i = 1, \cdots, n$. ∎

Definition 9 *A vector field in C_n in the form (54) is said to be* proper *if $(\mathbf{A}, \mathbf{w})$ satisfy condition K.*

The following theorem is the analog of Theorem 2 for the class $C_n(\mathbf{w})$.

Theorem 7 *Consider the systems*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{i=1}^{\infty} f_i\left(\mathbf{w}^T\mathbf{x}\right)\mathbf{b}_i \tag{55}$$

and

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \sum_{i=1}^{\infty} f_i\left(\tilde{\mathbf{w}}^T\tilde{\mathbf{x}}\right)\tilde{\mathbf{b}}_i \tag{56}$$

Assume that both systems are proper. Then system (55) and system (56) are linearly conjugate if the eigenvalues of $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *are the same and the eigenvalues of* $\mathbf{A} + \mathbf{b}_i\mathbf{w}^T$ *and* $\tilde{\mathbf{A}} + \tilde{\mathbf{b}}_i\tilde{\mathbf{w}}^T$ *are the same for each* i*.*

Proof: The proof follows the same steps as in Theorem 2. ■

The next theorem is the analog of Theorem 3 for the class C_n.

Theorem 8 *Consider two systems in* C_n

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{i=1}^{\infty} f_i(\mathbf{w}^T\mathbf{x})\mathbf{b}_i \tag{57}$$

and

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \sum_{i=1}^{\infty} g_i(\tilde{\mathbf{w}}^T\mathbf{x})\tilde{\mathbf{b}}_i \tag{58}$$

where f_i *and* g_i *are differentiable functions for all* i*. Suppose that these two systems are proper and have at least one matched (real or virtual) equilibrium point and their Jacobian matrices at each point have the same eigenvalues, then these systems are linearly conjugate.*

Proof: The proof is similar to the proof of Theorem 3.

As in Theorem 3 we can assume that without loss of generality $\mathbf{A}$ and $\tilde{\mathbf{A}}$ have the same eigenvalues. Consider $\mathbf{K} = \mathbf{K}(\mathbf{A}, \mathbf{w})$ and $\tilde{\mathbf{K}} = \mathbf{K}(\tilde{\mathbf{A}}, \tilde{\mathbf{w}})$ which are nonsingular by hypothesis. Since $\mathbf{K}^{-1}\mathbf{e}_i$ and $\tilde{\mathbf{K}}^{-1}\mathbf{e}_i$ forms two bases of $\mathbb{R}^n$, it is easy to see that without loss of generality we can assume that the two systems can be rewritten in the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \sum_{i=1}^{n} f_i(\mathbf{w}^T\mathbf{x})\mathbf{K}^{-1}\mathbf{e}_i && (59) \\ \dot{\mathbf{x}} &= \tilde{\mathbf{A}}\mathbf{x} + \sum_{i=1}^{n} g_i(\tilde{\mathbf{w}}^T\mathbf{x})\tilde{\mathbf{K}}^{-1}\mathbf{e}_i && (60) \end{aligned}$$

Similar to the proof of Theorem 2, by using the transformations $\mathbf{y} = \mathbf{K}\mathbf{x}$ and $\mathbf{y} = \tilde{\mathbf{K}}\mathbf{x}$ for the two systems respectively, we get

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + \sum_{i=1}^{n} f_i(\mathbf{e}_1^T\mathbf{y})\mathbf{e}_i$$

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + \sum_{i=1}^{n} g_i(\mathbf{e}_1^T\mathbf{y})\mathbf{e}_i$$

The Jacobian matrices of these systems at $\mathbf{y}$ are respectively $\hat{\mathbf{A}} + \sum_{i=1}^{n} f_i'(\mathbf{e}_1^T\mathbf{y})\mathbf{e}_i\mathbf{e}_1^T$ and $\hat{\mathbf{A}} + \sum_{i=1}^{n} g_i'(\mathbf{e}_1^T\mathbf{y})\mathbf{e}_i\mathbf{e}_1^T$. By Lemma 6, $f_i'(\mathbf{e}_1^T\mathbf{y})\mathbf{e}_i = g_i'(\mathbf{e}_1^T\mathbf{y})\mathbf{e}_i$ for all $\mathbf{y}$. This implies that $f_i'(\mathbf{e}_1^T\mathbf{y}) = g_i'(\mathbf{e}_1^T\mathbf{y})$ for all $\mathbf{y}$ and thus there exists a constant c_i such that $f_i(\cdot) = g_i(\cdot) + c_i$. Thus the two systems simplify to

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + \sum_{i=1}^{n} \left(g_i(\mathbf{e}_1^T\mathbf{y}) + c_i\right)\mathbf{e}_i$$

and

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + \sum_{i=1}^{n} g_i(\mathbf{e}_1^T\mathbf{y})\mathbf{e}_i$$

Now if they share an equilibrium point $\mathbf{y}^*$, then $\hat{\mathbf{A}}\mathbf{y}^* + \sum_{i=1}^{n}(d_i + c_i)\mathbf{e}_i = \mathbf{0}$ for the first system, and $\hat{\mathbf{A}}\mathbf{y}^* + \sum_{i=1}^{n} d_i\mathbf{e}_i = \mathbf{0}$ for the second system, where $d_i = g_i(\mathbf{e}_1^T\mathbf{y}^*)$ if the equilibrium point is real, and $d_i = g_i'(\mathbf{e}_1^T\hat{\mathbf{y}})\mathbf{e}_1^T(\mathbf{y}^* - \hat{\mathbf{y}}) + g_i(\mathbf{e}_1^T\hat{\mathbf{y}})$ if the equilibrium point is virtual due to linearization around $\hat{\mathbf{y}}$. In either case, we see that $c_i = 0$ for all i, thus the two systems are linearly conjugate. ■

For completeness, we will state the following analog of Theorem 4 for class C_n.

Theorem 9 *Consider two systems in C_n*

$$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \sum_{i=1}^{\infty} f_i(\mathbf{w}_1^T\mathbf{x})\mathbf{b}_{i,1} = \mathbf{h}_1(\mathbf{x}) \tag{61}$$

and

$$\dot{\mathbf{y}} = \mathbf{A}_2\mathbf{y} + \sum_{i=1}^{\infty} g_i(\mathbf{w}_2^T\mathbf{y})\mathbf{b}_{i,2} = \mathbf{h}_2(\mathbf{y}) \tag{62}$$

where f_i and g_i are differentiable functions and $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times n$ matrices. Suppose that both systems are proper. Suppose further that $\mathbf{T}$ is an nonsingular matrix and $\mathbf{t}$ is a vector such that $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}} = \mathbf{T}\tilde{\mathbf{x}} + \mathbf{t}$ are real equilibrium points of (61) and (62) respectively and $D\mathbf{h}_1(\mathbf{x})$ share the same eigenvalues as $D\mathbf{h}_2(\mathbf{T}\mathbf{x} + \mathbf{t})$ for all $\mathbf{x}$, then the two systems (61) and (62) are topologically conjugate.

The simplest case of nonlinear vector fields in C_n are the continuous piecewise-linear vector fields where the boundary planes are parallel. Parallel boundary planes means that in the consistent variation property $\mathbf{J}_1 - \mathbf{J}_2 = \mathbf{c}\mathbf{a}^T$ the vector $\mathbf{a}$ is the same at all boundaries. It is easy to show that all piecewise-linear vector fields with parallel boundary planes belong to class C_n. Thus we get the following corollary to Theorem 8:

Corollary 3 *Two proper continuous piecewise-linear vector fields with parallel boundary planes are topological conjugate if the boundary planes and the equilibrium points are matched up and the eigenvalues are the same in corresponding regions.*

In [Komuro, 1988] it was shown that n-dimensional 2-region proper piecewise-linear vector fields are topologically conjugate if the eigenvalues in corresponding regions are the same. In [Feldmann and Schwarz, 1994] this is shown for all n-dimensional 3-region odd-symmetric piecewise-linear vector fields. Corollary 3 extends these results to the class of proper continuous piecewise-linear vector fields with parallel boundary planes.

8 Conclusions

We have identified a class of vector fields where the members are topologically conjugate whenever the Jacobian matrices have the same eigenvalues at each point and the equilibrium points are matched up. Since Chua's oscillator belongs to this class and can synthesize a large set of eigenvalue patterns, this implies that Chua's oscillator is topologically conjugate to a large class of vector fields. This extends previous results which deal only with 3-dimensional odd-symmetric 3-segment piecewise-linear vector fields. The topological conjugacy part of the main result can be extended to systems with multiple scalar nonlinearities and we use that to prove that almost all continuous piecewise-linear vector fields with parallel boundary planes are topologically conjugate if the boundaries and equilibrium points are the same and the eigenvalues in corresponding regions are the same. This result extends previously known results which are limited to 2-segment and 3-segment odd-symmetric piecewise-linear vector fields.

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Appendix 1

In this appendix we present 3-D phase projections of attractors from several well-known 3-dimensional chaotic systems and their counterparts from the 3-dimensional Chua's oscillator by applying Theorem 6. In cases where Eq. (33) is not satisfied, s_{n-1} is decreased by a small amount when applying Algorithm 1.

Chaotic Colpitts Oscillator

The state equations of the chaotic Colpitts oscillator [Kennedy, 1994] are given by:

$$\begin{aligned} C_1\tfrac{dV_{CE}}{dt} &= I_L - I_C \\ C_2\tfrac{dV_{BE}}{dt} &= -\tfrac{V_{EE}+V_{BE}}{R_{EE}} - I_L - I_B \\ L\tfrac{dI_L}{dt} &= V_{CC} - V_{CE} + V_{BE} - I_L R_L \end{aligned}$$

where

$$\begin{aligned} I_B &= \begin{cases} 0 & V_{BE} \le V_{TH} \\ \frac{V_{BE}-V_{TH}}{R_{ON}} & V_{BE} > V_{TH} \end{cases} \\ I_C &= \beta_F I_B \end{aligned}$$

For the parameters $V_{CC} = 5V$, $R_L = 35\Omega$, $L = 98.5\mu H$, $C_1 = C_2 = 54nF$, $R_{EE} = 400\Omega$, $V_{EE} = -5V$, $V_{TH} = 0.75V$, $R_{ON} = 100\Omega$, $\beta_F = 200$ of this system, the corresponding parameters for Chua's oscillator are given by

$$\begin{aligned} C_1 &= 1 \\ C_2 &= 3.9234278162 \times 10^2 \\ L_3 &= 6.7473529008 \times 10^{-17} \\ R_2 &= 8.2538319739 \times 10^{-9} \\ G_3 &= 3.1852491364 \times 10^{11} \\ G_a &= -1.2092436506 \times 10^8 \\ G_b &= -1.2110955024 \times 10^8 \\ g(x) &= \tfrac{1}{2}(G_a - G_b) + \tfrac{1}{2}(G_a + G_b)x + \tfrac{1}{2}(G_b - G_a)\,|x-1| \end{aligned}$$

The corresponding attractors from these two systems are shown in Fig. 7a and Fig. 7b respectively.

Brockett's System

The state equation for Brockett's system [Brockett, 1982] is given by:

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 1.25\frac{dy}{dt} + f(y) = 0$$

where f is a 3-segment piecewise-linear function given by:

$$f(y) = \begin{cases} -ky & |y| \leq 1 \\ 2ky - 3k\mathrm{sgn}(y) & |y| \geq 1 \end{cases}$$

For $k = 1.8$, the corresponding parameters for Chua's oscillator are given by:

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= 5.3270141220 \times 10^{2} \\
L_3 &= 3.4698425778 \times 10^{-6} \\
R_2 &= -1.8666566179 \times 10^{-3} \\
G_3 &= 2.8657513826 \times 10^{5} \\
G_a &= 5.3671716962 \times 10^{2} \\
G_b &= 5.3672716962 \times 10^{2} \\
g(x) &= G_b x + \tfrac{1}{2}(G_a - G_b)\left(|x+1| - |x-1|\right)
\end{aligned}$$

The corresponding attractors from these two systems are shown in Fig. 8a and Fig. 8b respectively.

Sparrow's System

The state equations for Sparrow's system [Sparrow, 1981] are given by:

$$\begin{aligned}
\dot{x}_1 &= f(x_3) - x_1 \\
\dot{x}_2 &= x_1 - x_2 \\
\dot{x}_3 &= x_2 - x_3
\end{aligned}$$

where f is a 3-segment piecewise-linear function given by:

$$f(x_3) = \begin{cases} 8.4rx_3 + 0.96276 + 2.3872r & x_3 \leq 0.28419 \\ -8.4x_3 + 3.35 & 0.28419 \leq x_3 \leq \frac{3}{7} \\ 8.4rx_3 - 0.25 - 3.6r & x_3 \geq \frac{3}{7} \end{cases}$$

For $r = 19.0$, the corresponding parameters for Chua's oscillator are given by:

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= -6.2252592698 \times 10^{2} \\
L_3 &= 2.8731155811 \times 10^{-7} \\
R_2 &= 5.3544019103 \times 10^{-4} \\
G_3 &= 1.1601531225 \times 10^{6} \\
G_a &= -1.8646222233 \times 10^{3} \\
G_b &= -1.8645922233 \times 10^{3} \\
g(x) &= G_b x + \tfrac{1}{2}(G_a - G_b)\left(|x+1| - |x-1|\right)
\end{aligned}$$

The corresponding attractors from these two systems are shown in Fig. 9a and Fig. 9b respectively.

Ogorzałek's System

The state equations for Ogorzałek's system [Ogorzalek, 1989] are given by:

$$\begin{aligned}
\dot{x}_1 &= f(x_3) - 2x_1 + x_2 \\
\dot{x}_2 &= x_1 - 2x_2 + x_3 \\
\dot{x}_3 &= x_2 - x_3
\end{aligned}$$

where f is a 3-segment piecewise-linear function given by:

$$f(x_3) = \begin{cases} m_1 x_3 + m_1 - m_0 & x_3 \leq -1 \\ m_0 x_3 & |x_3| \leq 1 \\ m_1 x_3 + m_0 - m_1 & x_3 \geq 1 \end{cases}$$

For $m_0 = -33.03$ and $m_1 = 400$, the corresponding parameters for Chua's oscillator are given by:

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= -3.4660736904 \times 10^2 \\
L_3 &= 3.3408722323 \times 10^{-7} \\
R_2 &= 5.7696798665 \times 10^{-4} \\
G_3 &= 5.9859025418 \times 10^5 \\
G_a &= -1.7281984151 \times 10^3 \\
G_b &= -1.7281484151 \times 10^3 \\
g(x) &= G_b x + \tfrac{1}{2}(G_a - G_b)\left(|x+1| - |x-1|\right)
\end{aligned}$$

The corresponding attractors from these two systems are shown in Fig. 10a and Fig. 10b respectively.

Arnéodo's System

The state equation for Arnéodo's system [Arnéodo *et al.*, 1982] are given by:

$$\frac{d^3y}{dt^3} + \mu_2 \frac{d^2y}{dt^2} + \mu_1 \frac{dy}{dt} + \mu_0 y = \mu y^3$$

For $\mu = -1$, $\mu_0 = -5.5$, $\mu_1 = 3.5$, $\mu_2 = 1$, the corresponding parameters for Chua's oscillator are given by:

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= 8.0737820354 \times 10^1 \\
L_3 &= 1.2239479563 \times 10^{-4} \\
R_2 &= -1.1329163199 \times 10^{-2} \\
G_3 &= 7.4732911509 \times 10^3 \\
G_a &= 8.9267772512 \times 10^1 \\
G_b &= 8.9277772512 \times 10^1 \\
g(x) &= G_a x + (G_b - G_a)x^3
\end{aligned}$$

The corresponding attractors from these two systems are shown in Fig. 11a and Fig. 11b respectively.

Nishio's System

The state equations for Nishio's system [Nishio *et al.*, 1990] are given by:

$$\begin{aligned}\dot{x}_1 &= -b(f(x_1)+x_3)\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= (a-b)x_3 - x_2 - bf(x_1)\end{aligned}$$

where f is a 3-segment piecewise-linear function given by:

$$f(x_1) = \begin{cases} m_1x_1 + m_1 - m_0 & x_1 \leq -1 \\ m_0x_1 & |x_1| \leq 1 \\ m_1x_1 + m_0 - m_1 & x_1 \geq 1 \end{cases}$$

For $m_0 = -0.5$ and $m_1 = 10$, $a = 0.3$, $b = 1$, the corresponding parameters for Chua's oscillator are given by:

$$\begin{aligned}C_1 &= 1\\ C_2 &= -3.2604757879 \times 10^{-2}\\ L_3 &= -2.7603333333\\ R_2 &= -1.0111111111 \times 10^{1}\\ G_3 &= 1.0868252626 \times 10^{-1}\\ G_a &= 5.9890109890 \times 10^{-1}\\ G_b &= 1.1098901099 \times 10^{1}\\ g(x) &= G_bx + \tfrac{1}{2}(G_a - G_b)\left(|x+1| - |x-1|\right)\end{aligned}$$

The corresponding attractors from these two systems are shown in Fig. 12a and Fig. 12b respectively.

Dmitriev's System

The state equations for Dmitriev's system [Rul'kov *et al.*, 1992] are given by:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - \delta x_2 + x_3\\ \dot{x}_3 &= \gamma(F(x_1) - x_3) - \sigma x_2\end{aligned}$$

where F is given by:

$$F(x_1) = \begin{cases} 0.528\alpha & x_1 < -1.2 \\ \alpha x_1(1-x_1^2) & |x_1| \leq 1.2 \\ -0.528\alpha & x_1 \geq 1.2 \end{cases} \tag{63}$$

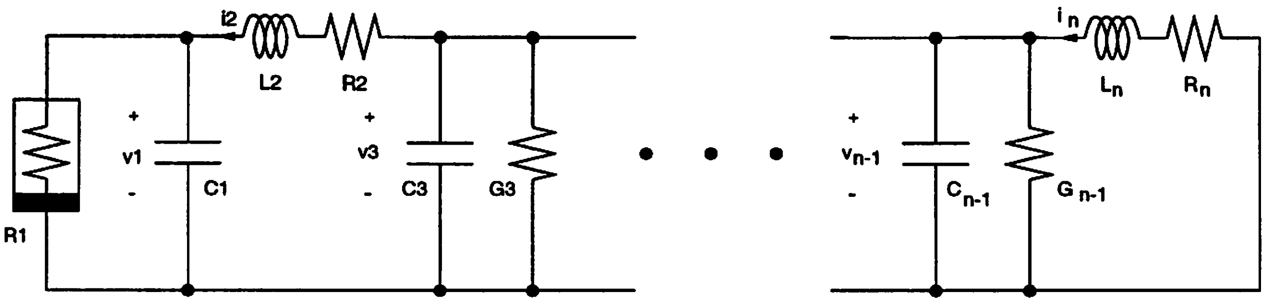
For $\alpha = 20$, $\delta = 0.43$, $\sigma = 0.71$, $\gamma = 0.1$, the corresponding parameters for Chua's oscillator are given by:

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= -8.9906258405 \times 10^1 \\
L_3 &= 5.6294509561 \times 10^{-4} \\
R_2 &= 2.2612337453 \times 10^{-2} \\
G_3 &= 3.6113480161 \times 10^3 \\
G_a &= -4.3693645701 \times 10^1 \\
G_b &= -4.3688645701 \times 10^1 \\
g(x) &= G_a x + (G_b - G_a)F(x)
\end{aligned}$$

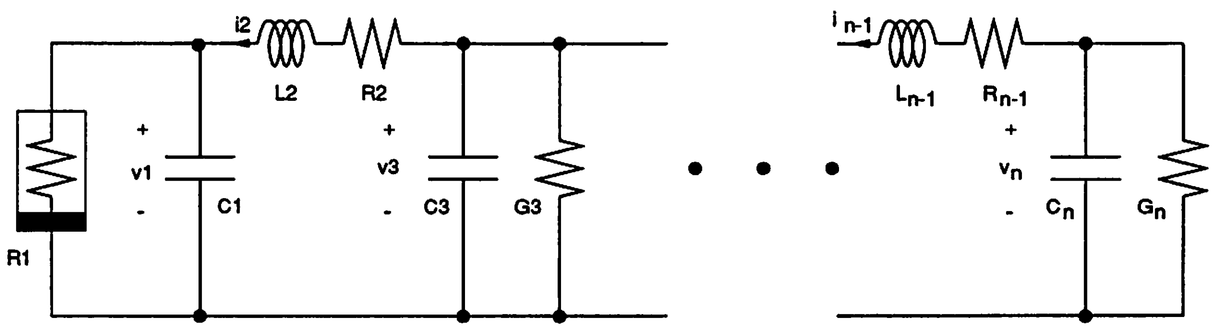
where $F(\cdot)$ is given by (63). The corresponding attractors from these two systems are shown in Fig. 13a and Fig. 13b respectively.

Appendix 2

We give in this Appendix a different n-dimensional circuit which is presented in [Kocarev *et al.*, 1993]. This circuit can also synthesize almost all eigenvalue patterns in $\mathcal{C}$. The circuit diagram for n is even and n is odd is shown in Fig. 14a and Fig. 14b respectively.



(a)



(b)

Figure 14: n-dimensional circuit given in [Kocarev *et al.*, 1993]. (a) n is even. (b) n is odd.

The state equations are given by:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} - \frac{g(\mathbf{e}_1^T\mathbf{x})}{C_1}\mathbf{e}_1$$

where $g(\cdot)$ is the v-i characteristic of the nonlinear resistor and $\mathbf{A}$ is given by:

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{C_1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{L_2} & -\frac{R_2}{L_2} & \frac{1}{L_2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{C_3} & -\frac{G_3}{C_3} & \frac{1}{C_3} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{C_{n-1}} & -\frac{G_{n-1}}{C_{n-1}} & \frac{1}{C_{n-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{L_n} & \frac{R_n}{L_n} \end{bmatrix}$$

if n is even and

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{C_1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{L_2} & -\frac{R_2}{L_2} & \frac{1}{L_2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{C_3} & -\frac{G_3}{C_3} & \frac{1}{C_3} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{L_{n-1}} & -\frac{R_{n-1}}{L_{n-1}} & \frac{1}{L_{n-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{C_n} & \frac{G_n}{C_n} \end{bmatrix}$$

if n is odd. Applying Algorithm 1, the following assignment of circuit parameters will give the desired topologically conjugate system when $g(\cdot)$ is given by (29):

$$\begin{aligned} C_1 &= 1 \\ G_a &= -\kappa_1 C_1 \\ G_b &= -\bar{a}_{1,1} C_1 \\ L_2 &= -\frac{1}{\rho_1 C_1} \\ R_2 &= -\kappa_2 L_2 \\ C_3 &= -\frac{1}{\rho_2 L_2} \\ &\vdots \\ G_{2l-1} &= -\kappa_{2l-1} C_{2l-1} \\ L_{2l} &= -\frac{1}{\rho_{2l-1} C_{2l-1}} \\ R_{2l} &= -\kappa_{2l} L_{2l} \\ C_{2l+1} &= -\frac{1}{\rho_{2l} L_{2l}} \\ &\vdots \\ R_n &= -\kappa_n L_n, \quad \text{if } n \text{ is even} \\ G_n &= -\kappa_n C_n, \quad \text{if } n \text{ is odd} \end{aligned}$$

Other circuit topologies for synthesizing eigenvalue patterns in $\mathbb{C}$ are given in [Götz *et al.*, 1993].

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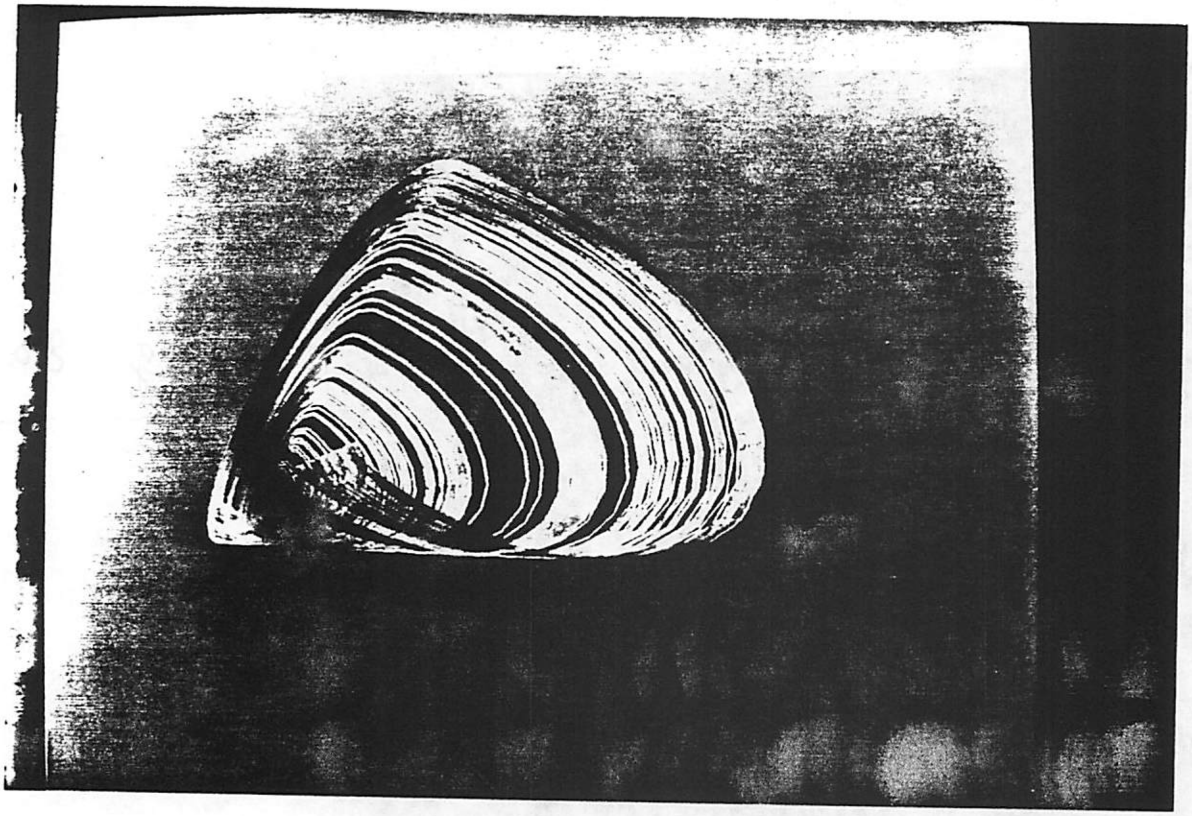


Fig. 7a

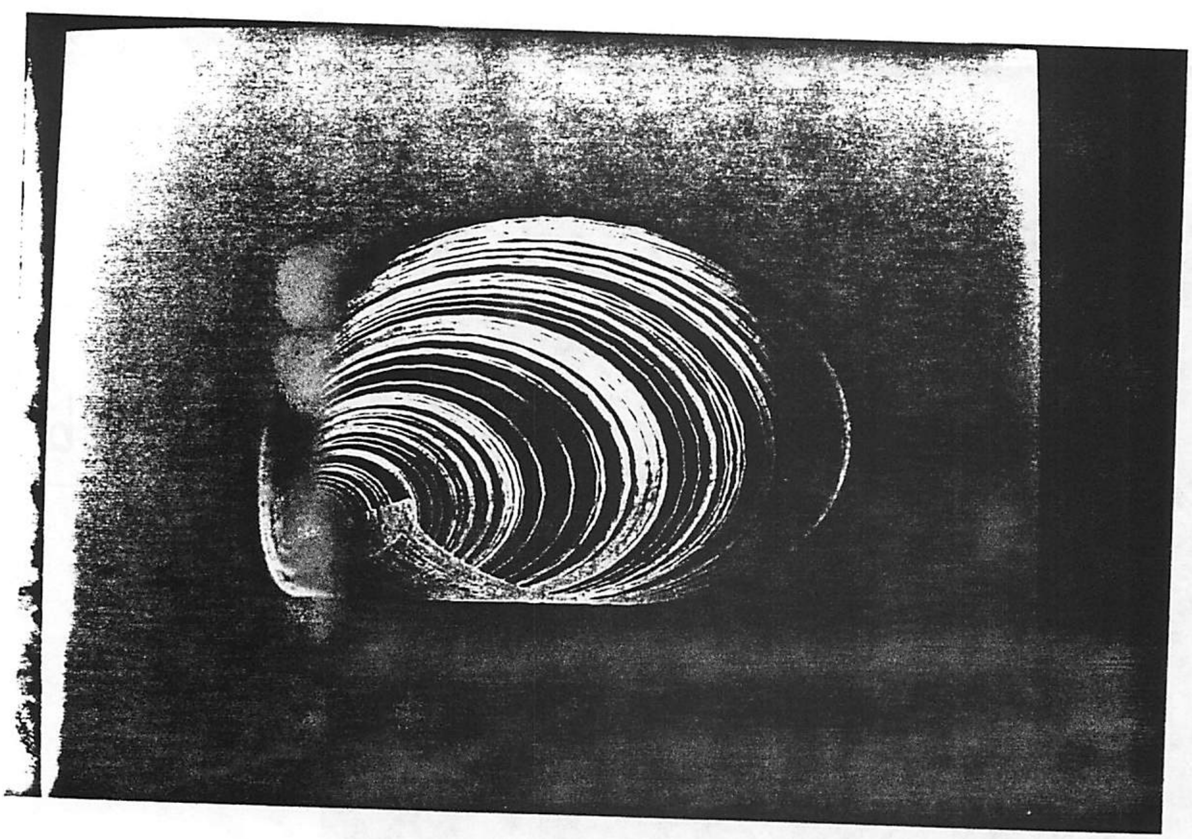


Fig. 7b

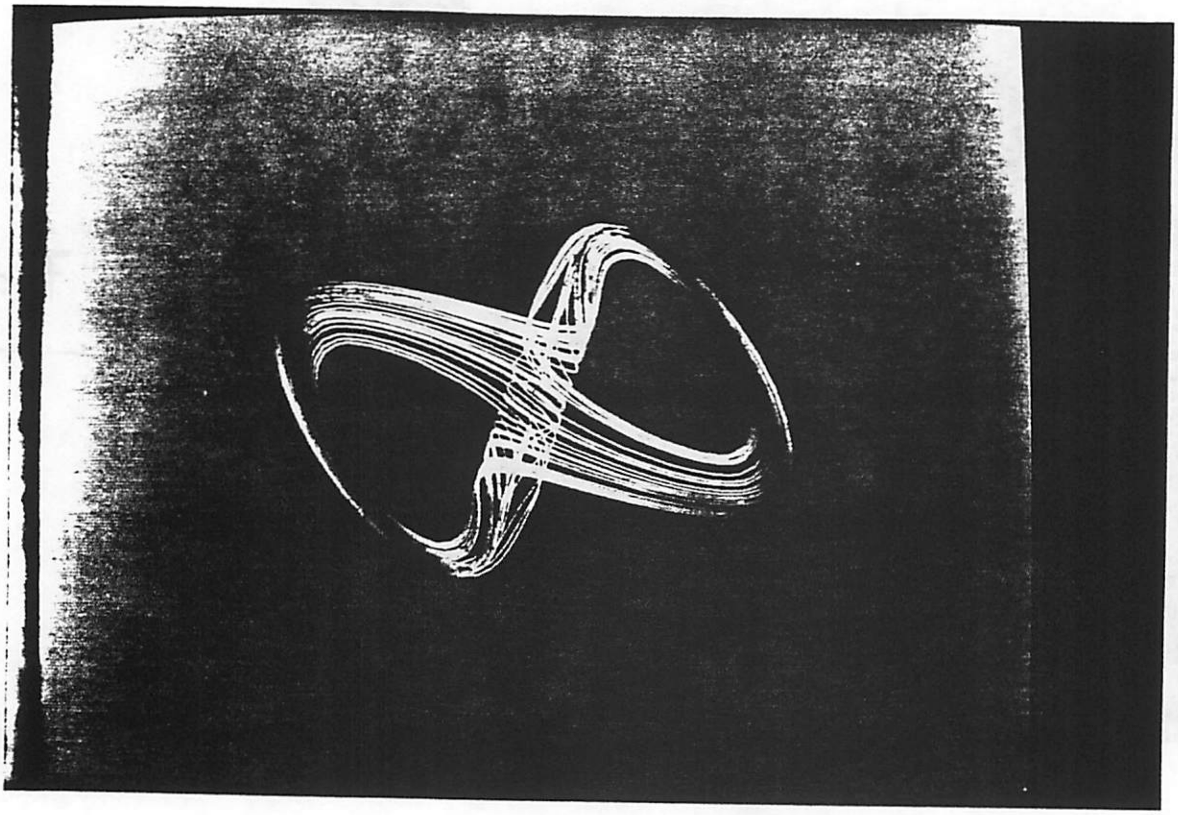
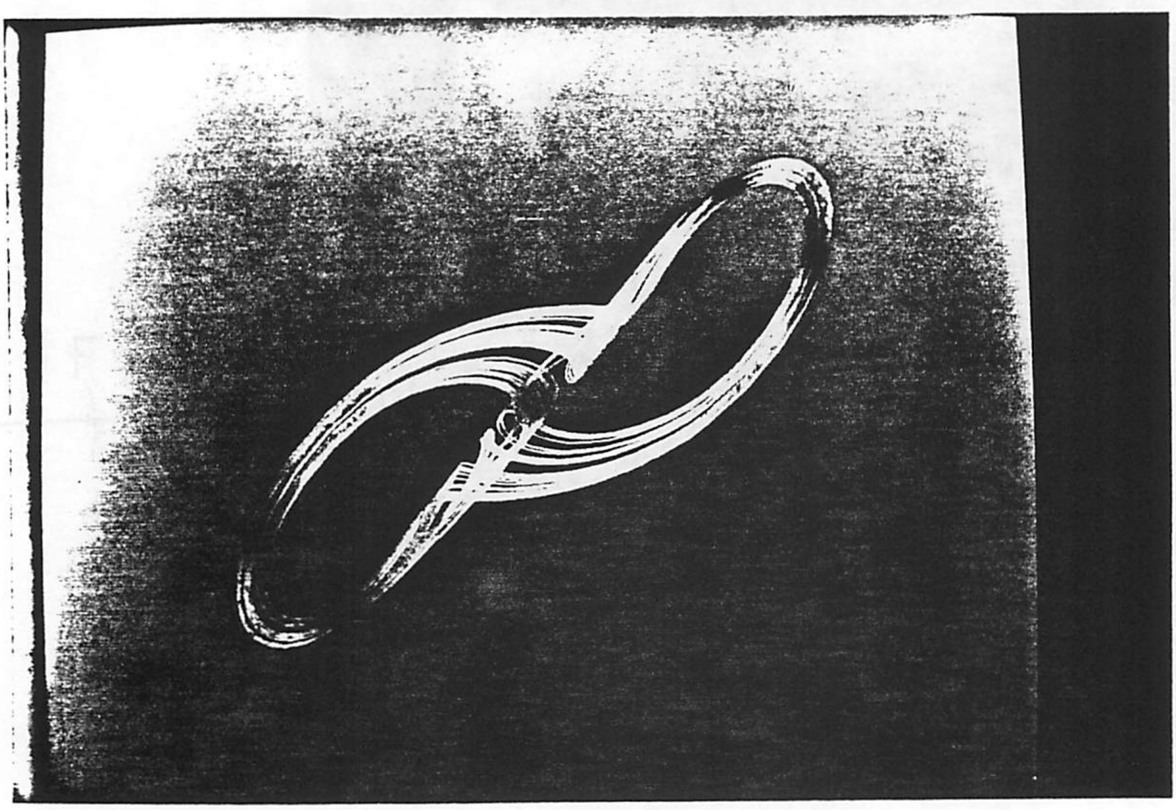


Fig. 8a



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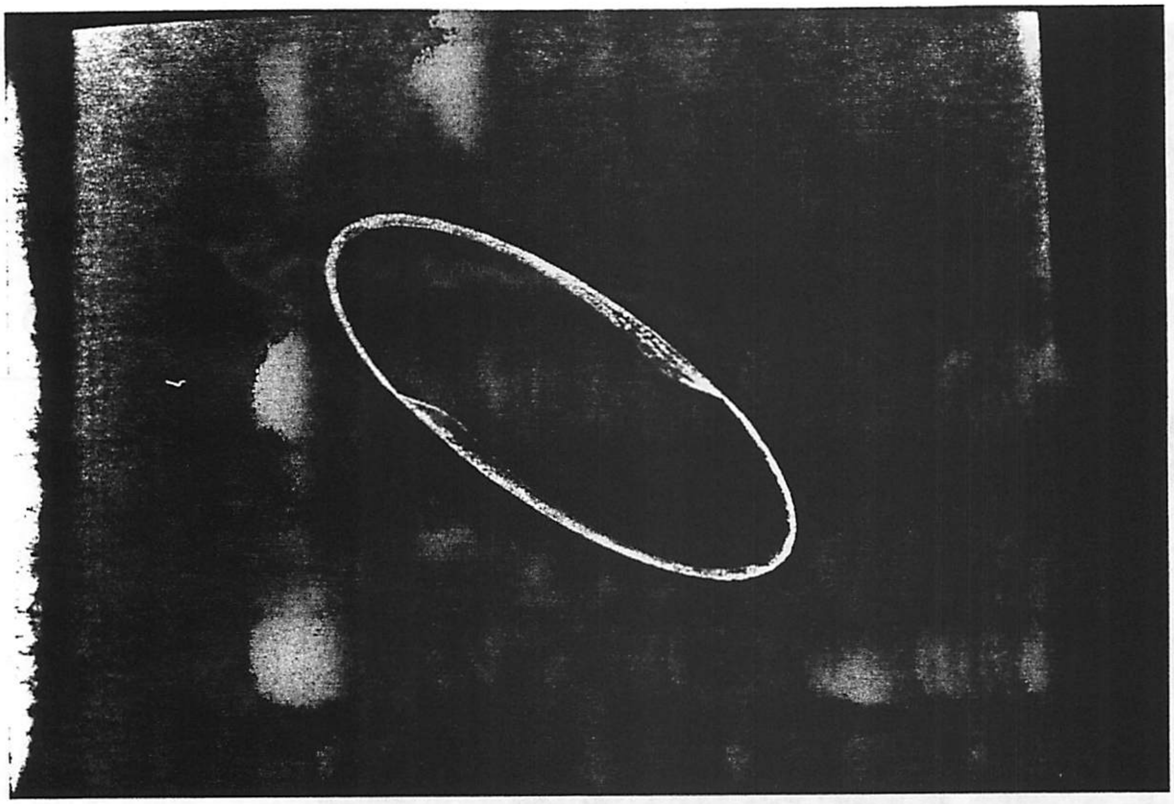
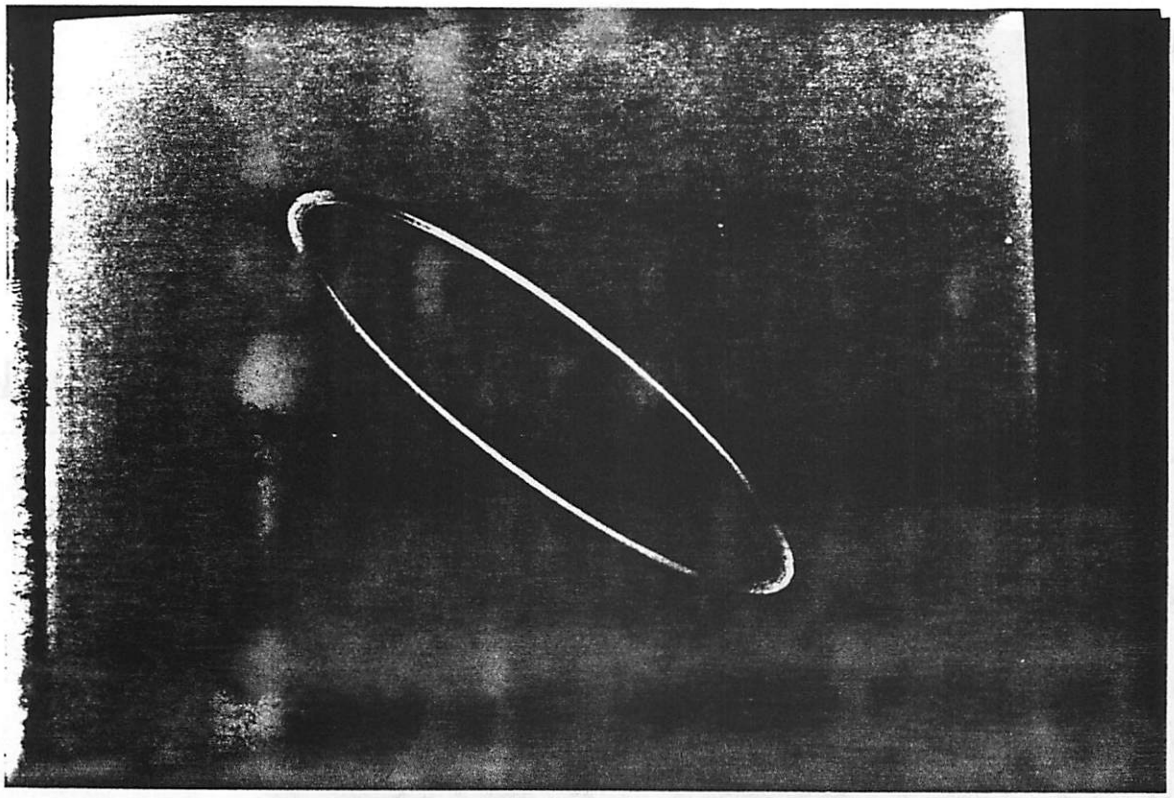


Fig 9a



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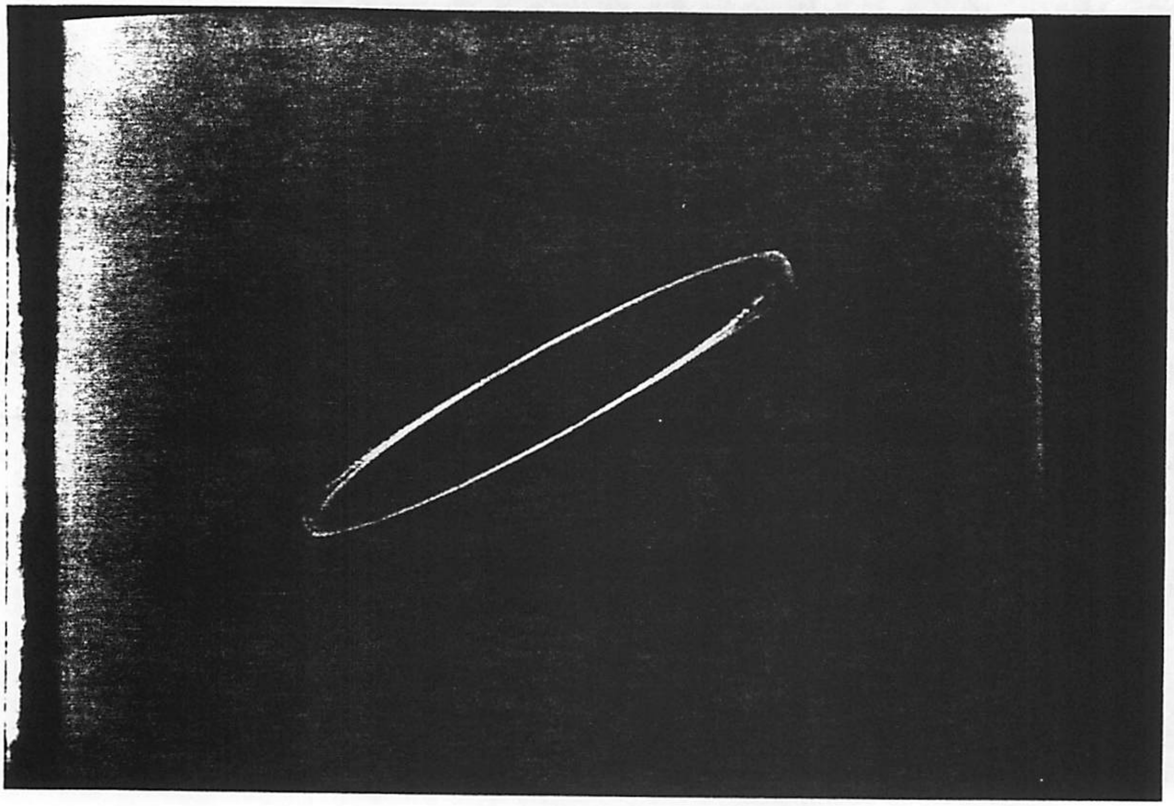
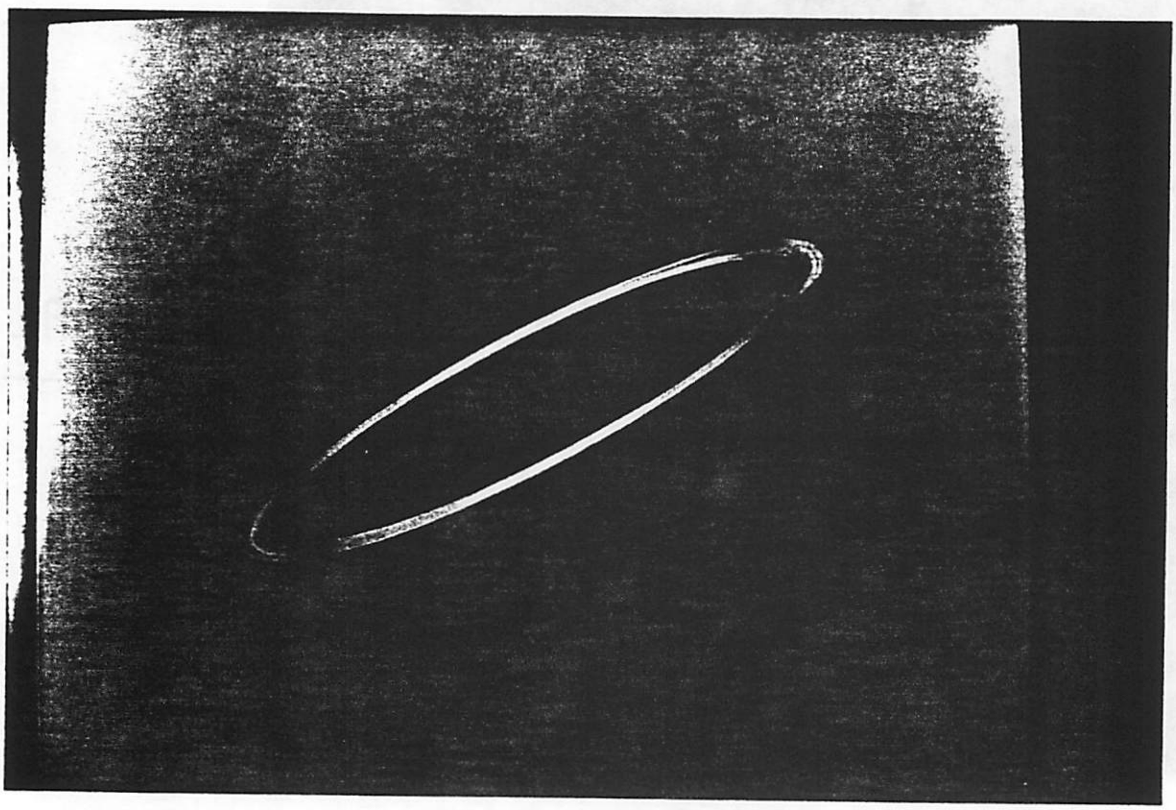


FiG. 10a



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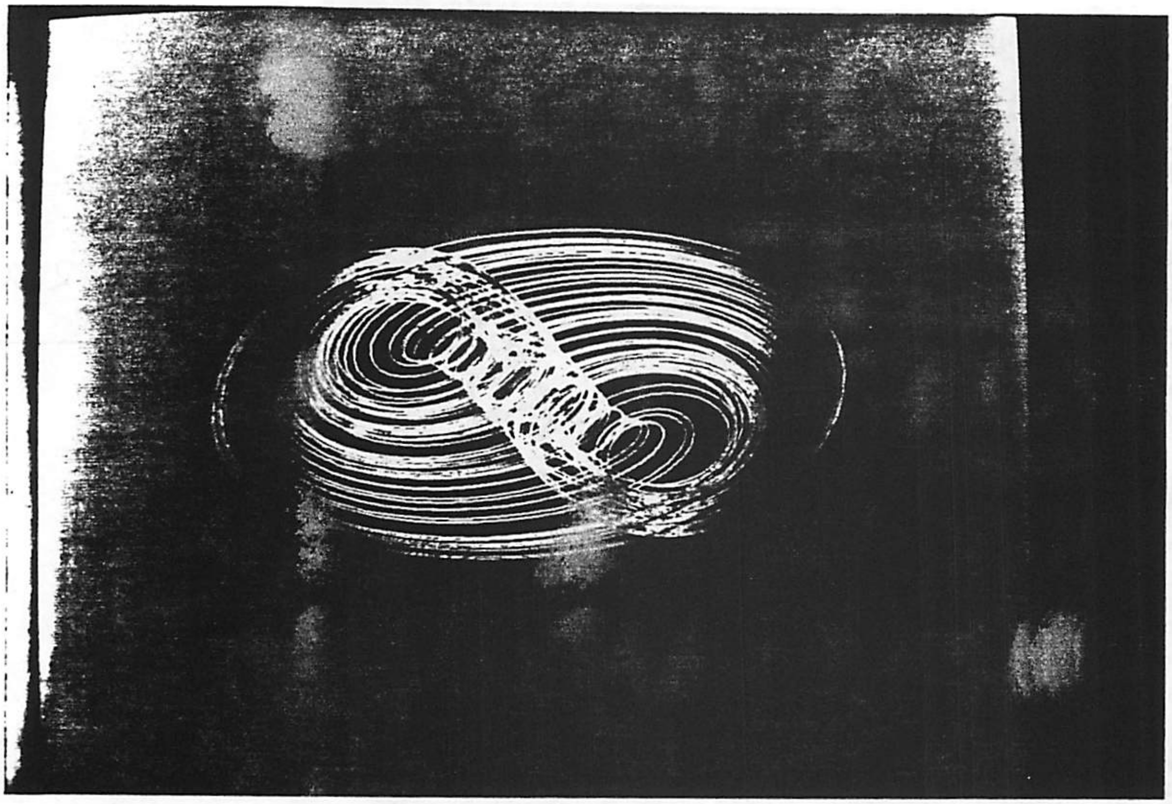

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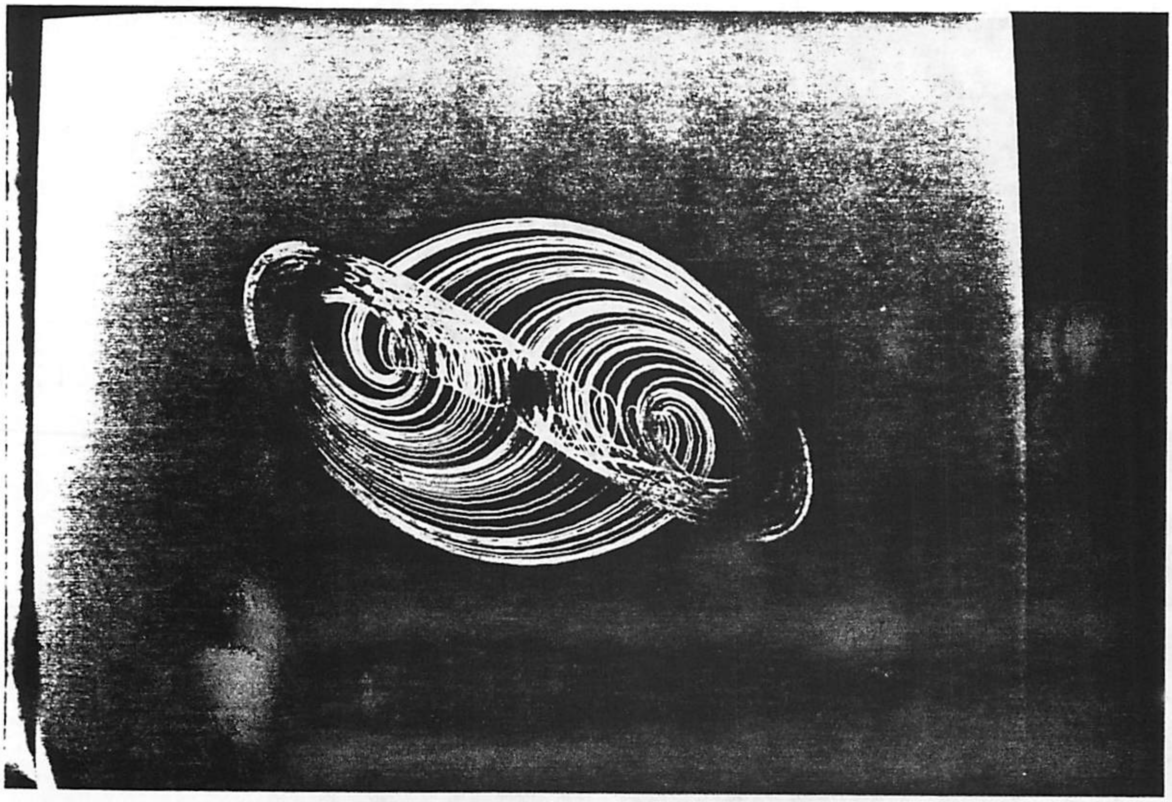
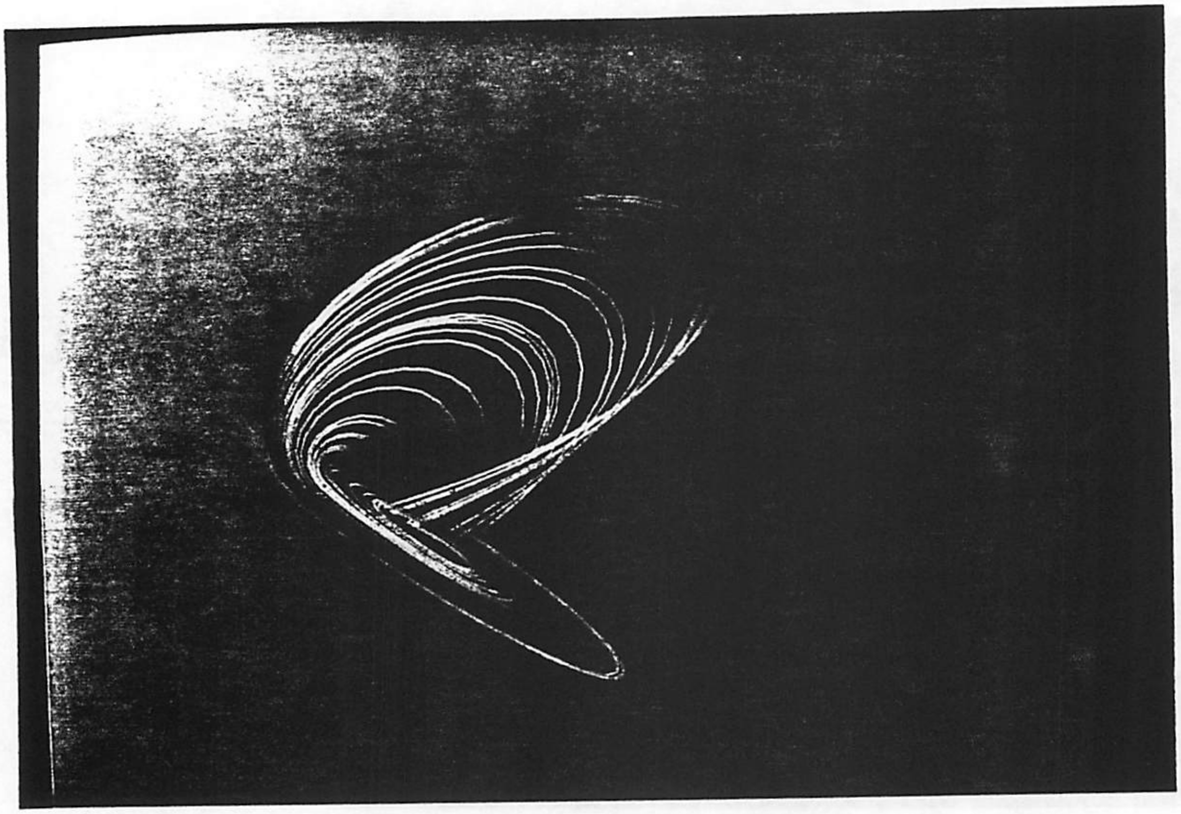


Fig 11b



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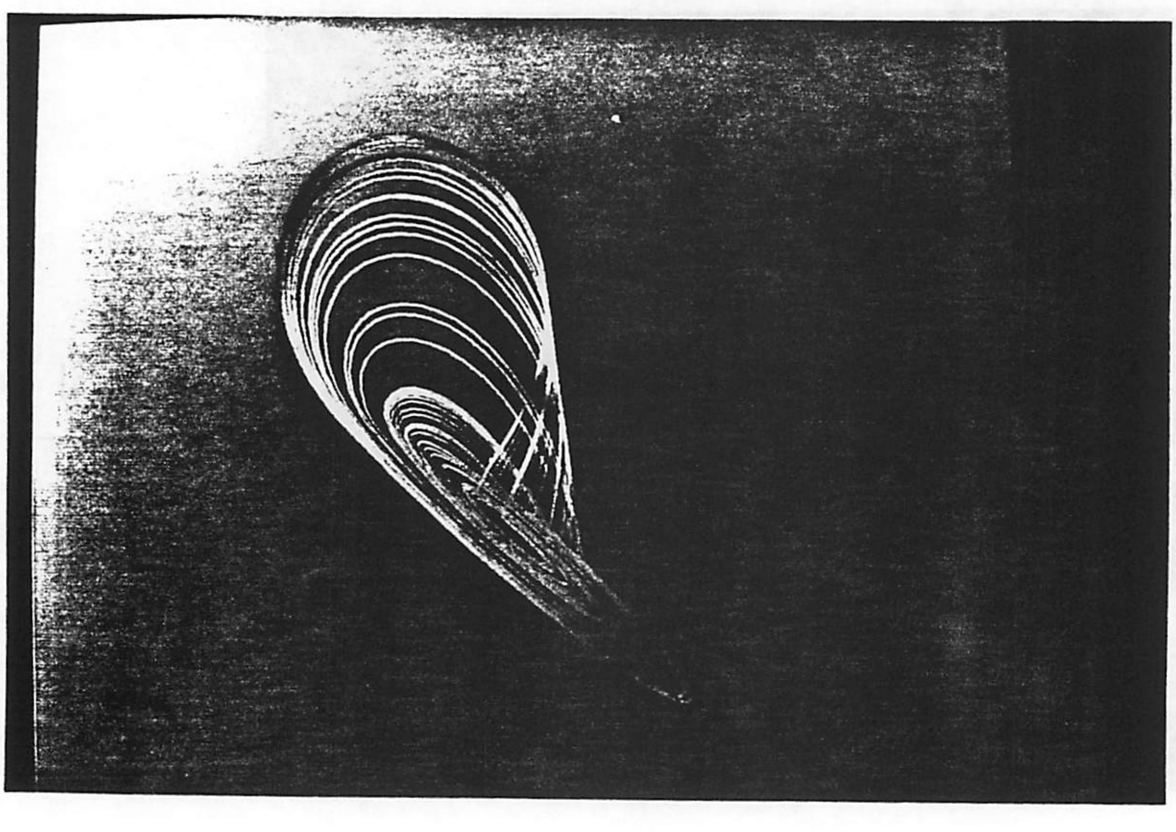

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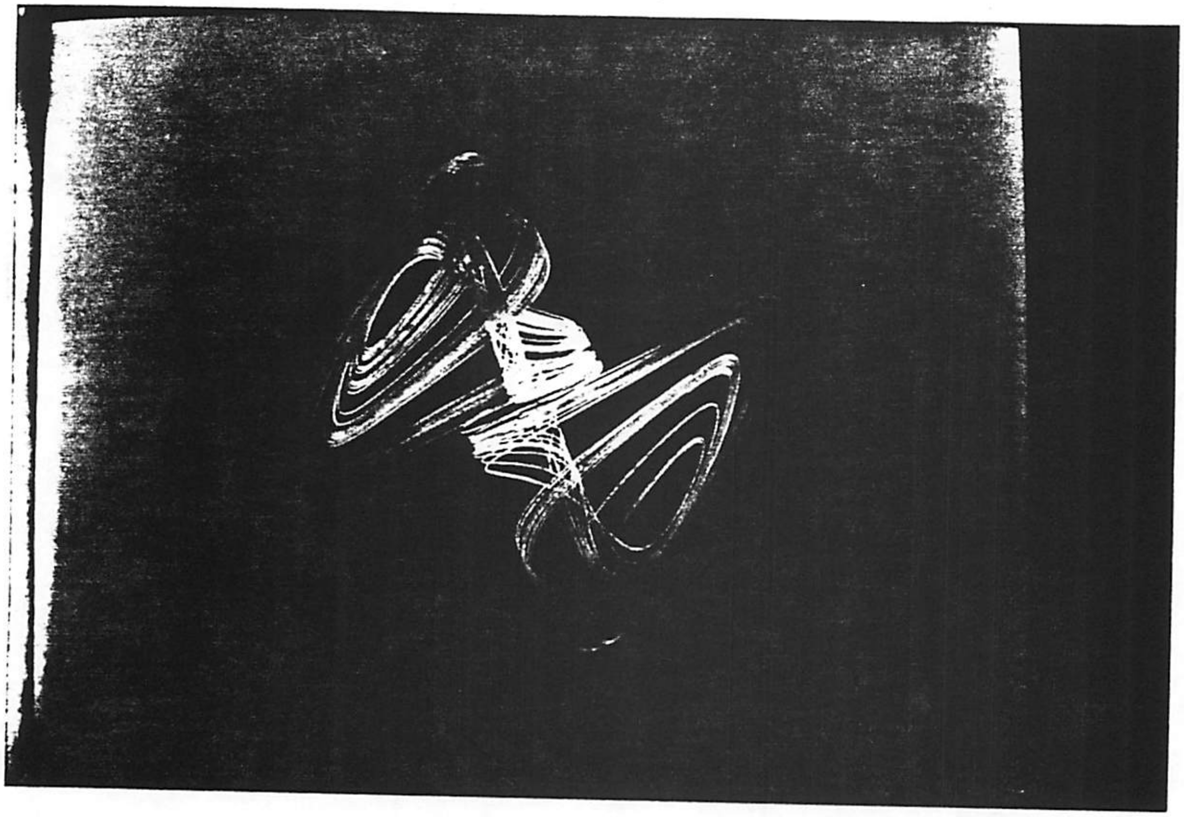

Fig. 13 a

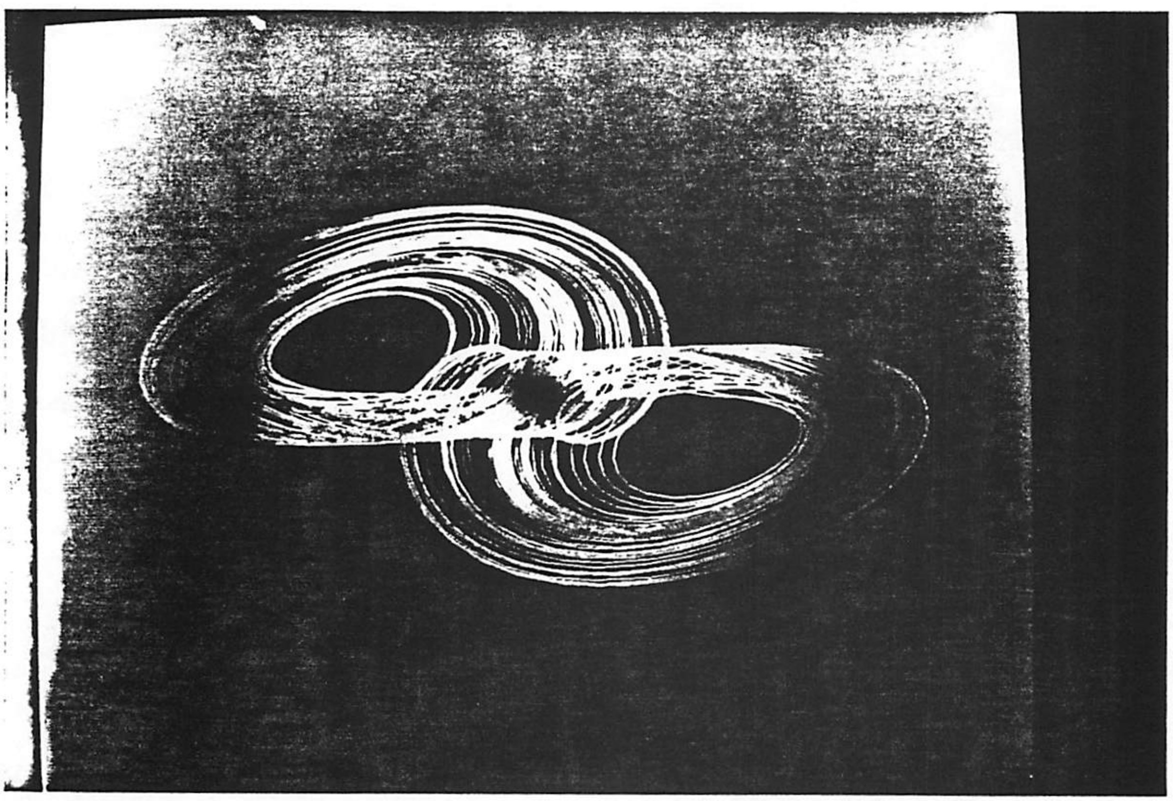

Fig. 13 b