ADVANCED TOPICS IN ADAPTIVE AND NONLINEAR CONTROL: FINAL PROJECTS EECS 290B

by

Professor S. S. Sastry

Memorandum No. UCB/ERL M95/8

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ELECTRONICS RESEARCH LABORATORY

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Advanced Topics in Adaptive and Nonlinear Control: Final Projects

EECS 290B*

This class was taught by Prof. S. S. Sastry in the Fall of 1994 in three parts:

1. Exterior Differential Systems, Nonholonomic Systems and Linearization by Feedback.
2. Control of Systems on Lie Groups.
3. Learning and Pattern Storage.

The students in the class wrote term projects, which were either summaries of the material taught in the class or independent research in the areas covered in the class. This collection of papers is a record of these final projects:

Exterior Differential Systems, Nonholonomy and Nonlinear Control

1. G. Pappas, C. Gerdes and D. Niemann, "Introduction to Exterior Differential Systems".
2. J. Lygeros, "Reconciliation of Different Approaches to Linearization by State Feedback".
3. P. Pagilla, "Feedback Linearization of Exterior Differential Systems: An Exterior Differential Systems Approach".

Control of Systems on Lie Groups

1. C. Tomlin, "Control of Systems on Lie Groups".
2. J. Wendlandt, "Control and Dynamics of an Ellipsoid on a Horizontal Plane".

Learning and Pattern Storage

1. L. Crawford, "Control and Steering of a Diver in the Plane".
2. A. Lindsey, "Registration of 3D shapes using least squares matching".

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Introduction to Exterior Differential Systems Notes for EECS 290B by Prof. S.S. Sastry, Fall 1994

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1 Introduction

The study of exterior differential systems may be viewed as a formulation of mechanics which unites the mathematics of geometry and algebra. While such unification is a powerful tool, the mathematical preliminaries can be rather daunting; considerable independent development of the algebraic and geometric concepts is necessary before these ideas can be linked. From a physical standpoint the mathematical development can be motivated through an example in mechanics.

Consider trajectory generation, or path planning, for purely kinematic systems with n degrees of freedom. In such a system our state space is simply the configuration space, which we can denote as $(x_1, x_2, \ldots, x_n)$. In the absence of constraints, each degree of freedom is independent, so that if we assume we can control the velocities $(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$, then trajectory generation is trivial. If, however, we impose kinematic constraints on the system's velocities or states and assume our controls influence only the remaining unconstrained directions of motion, the problem becomes significantly more interesting. Exactly how these imposed constraints influence trajectory generation depends upon the nature of the constraint.

The simplest form of constraint to impose on a system is that of the holonomic, or integrable, constraint which may be written as a function of the state variables and possibly time. For a system consisting of n states, $x_1 \ldots x_n$, such a constraint may always be put in the general form,

$$f(x_1, x_2, \ldots, x_n, t) = 0$$

Alternatively, the constraint may be expressed in differential, or Pfaffian, form,

$$a_1 dx_1 + a_2 dx_2 + \ldots + a_n dx_n + a_t dt = 0$$

where each of the a_i and a_t are functions of the states and time. The first expression is clearly the stronger of the two representations, since it implies the existence of an integrating factor to turn the differential Pfaffian form into an exact differential. More precisely, a holonomic constraint confines the motion to an integral manifold of the original system. Each holonomic constraint, therefore, reduces the dimension of the configuration space.

A simple example of a holonomic constraint is that of the spherical pendulum, a body in 3-space constrained to lie a fixed distance from the origin. The mathematical representation of such a constraint is therefore

$$x^2 + y^2 + z^2 - R^2 = 0$$

or in the Pfaffian form

$$2x dx + 2y dy + 2z dz = 0$$

Physically, the manifold on which the motion can be described is the sphere of radius R, so the configuration space has been reduced to two dimensions.

Constraints that may be written in Pfaffian form but do not possess an integrating factor, are called nonholonomic constraints and are of particular interest. Physically, these constraints represent functional dependences on the velocities of the system (as opposed to the states) and include the familiar case of a body which rolls without slipping. As an example, consider a vertical disk rolling in a horizontal plane. Assuming a unit radius, the constraint of rolling without slipping requires that the velocity of the center of the disk, v, be related in magnitude to the angular velocity,

$$v = \dot{\phi}$$

and in direction to the angle θ,

$$\begin{aligned} \dot{x} &= v \sin\theta \\ \dot{y} &= v \cos\theta \end{aligned}$$

Substituting for v, this results in two equations of constraint in terms of the state variables,

$$\begin{aligned} dx - \sin\theta d\phi &= 0 \\ dy - \cos\theta d\phi &= 0 \end{aligned}$$

From a physical standpoint, the constraint does not restrict the configuration space; hence, the entire configuration space of this system is reachable.

Returning to the issue of path planning, it should be intuitively clear that nonholonomic constraints represent special difficulties. Holonomic constraints may simply be eliminated by proper choice of coordinates and path planning performed on the resulting unconstrained, reduced-order system. General solutions for nonholonomic systems are not available; however, solutions for certain classes of these systems may be constructed. The mathematics of exterior differential systems provides a useful tool for characterizing several of these classes and developing their solutions.

Intuitively, the central concept of the exterior differential systems approach is most easily understood in relation to the more familiar approach of vector fields. Returning to the example of the rolling disc, consider the equations of motion in state-space form. Assuming that we can control the angular velocity ($u_1 = \dot{\phi} = v$) and the yaw rate ($u_2 = \dot{\theta}$), the disk becomes a drift-free two-input control system,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

Each input is associated with a vector field defining at each point the direction of the velocity or tangent vector. Pointwise, by proper choice of our control inputs, we may choose this tangent vector to lie anywhere in the two-dimensional subspace spanned by these vector fields. Taking the collection of these subspaces over all points in the configuration space, we form a distribution, fully characterizing the possible tangent vectors at all points. The path planning question can then be formulated as choosing at each point a velocity vector in the distribution such that the resultant motion takes the system from the initial to the final state.

Alternatively, we may directly consider the constraints as opposed to forming the control system dual to those constraints. In this manner, our system is described pointwise by linear, real-valued functions with tangent vectors as arguments. Taking the collection of the functions over the configuration space, we can define a co-distribution analogously to the distribution discussed above. The path planning problem then becomes finding a trajectory from the initial point to the final point such that the tangent vector at each point satisfies the constraints of the co-distribution.

The distinction between the two approaches may seem slight. After all, considering drift-free systems of this form, a dual control system may be constructed for every set of constraints in Pfaffian form by assuming that we have control over unconstrained directions. Hence, this formulation introduces no new physics. The difference, however, is that by dealing with the constraints instead of the vector fields, we may transform the problem from one of geometry to one of algebra. In so doing, we arrive at normal forms for certain classes of problems. In terms of the path planning problem, these forms provide coordinate transformations which parameterize the integral manifolds of the system and allow us to treat the nonholonomic problem in a manner akin to holonomic constraints.

These notes begin by developing the mathematical preliminaries needed in the study of exterior differential systems. Section 2 begins with a brief discussion of the mathematical notion of duality and dual spaces to vector spaces. Next, we develop the language of tensors, or multilinear real-valued functions defined on a vector space. In particular, we define the class of alternating tensors and a mathematical operation on the alternating tensors, the wedge product. This allows us to form an algebra on the space of alternating tensors for a given vector space. A discussion of algebraic structures (in particular, algebraic ideals) and their geometrical interpretations concludes the section.

Section 3 provides an introduction to the differential geometric concepts needed in the development of exterior differential systems. After a brief discussion of manifolds and tangent spaces, we present the central concept of differential forms which are simply alternating tensors defined on the tangent tangent space of a manifold. Following this, we present the exterior derivative for differential forms and discuss its properties. With the language of manifolds and forms at our disposal, we then provide a mathematical introduction to the concepts of constraints, distributions, co-distributions and integral manifolds.

Section 4 uses these preliminaries to develop exterior differential systems. We begin by discussing the problem of finding integral manifolds and motivate certain coordinate changes which are of an algebraic, rather than geometric, nature. Such transformations denote alternative representations of the system which are equivalent in the algebraic sense of ideals. From there, certain classes of systems are defined and solved in terms of normal forms. The use of these normal forms is illustrated for examples of nonholonomic path planning and feedback linearization.

2 Multilinear Algebra

2.1 The Dual Space of a Vector Space

Many of the ideas underlying the theory of multilinear algebra involve duality and the notion of the dual space to a vector space. Therefore we will begin by briefly reviewing these concepts.

Definition 1 *Let* $(V, \mathbf{R})$ *denote a vector space over* $\mathbf{R}$*. The* <u>*dual space*</u> *associated with* $(V, \mathbf{R})$ *is defined as the set of all linear mappings* $f : V \to \mathbf{R}$*. The* <u>*dual space*</u> *of* V *is denoted as* V^* *and the elements of* V^* *are called* <u>*covectors*</u>*.*

We will always deal with finite-dimensional vector spaces defined over $\mathbf{R}$. V^* can be made into a vector space in the following way:

Theorem 1 *If for all $\alpha, \beta \in V^*$ and $c \in \mathbf{R}$, we define*

$$\begin{aligned}(\alpha+\beta)(v) &= \alpha(v)+\beta(v)\\ (c\alpha)(v) &= c\cdot\alpha(v)\end{aligned}$$

then V^* is a vector space over $\mathbf{R}$ with $dim(V^*) = dim(V)$. Furthermore, if we pick a set of basis vectors $\{v_1,\ldots,v_n\}$ for V, then the set of linear functions $\phi^i : V \to \mathbf{R}$, $1 \le i \le n$, defined by:

$$\phi^i(v_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

form a basis of V^* called the <u>*dual basis.*</u>

Proof: See Munkres [1] page 220. □

Example: Let $V = \mathbf{R}^n$ with the standard basis $e_1,\ldots,e_n$ and let $\phi^1,\ldots,\phi^n$ be the dual basis. If

$$x \in \mathbf{R}^n = \sum_{j=1}^n x_j e_j$$

then evaluating each function in the dual basis at x gives

$$\phi^i(x) = \phi^i(\sum_{j=1}^n x_j e_j) = \sum_{j=1}^n x_j \phi^i(e_j) = x_i$$

Since the functions $\phi^1,\ldots,\phi^n$ form a basis for V^*, a general covector in $(\mathbf{R}^n)^*$ is of the form

$$f = \alpha_1\phi^1 + \ldots + \alpha_n\phi^n$$

Evaluating this covector at x gives

$$f(x) = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

If we think of a vector as a column matrix and a covector as a row matrix, then

$$f(x) = [\alpha_1 \ldots \alpha_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

◇

The dual space construction is useful because it allows us to define a notion of "perpendicular" subspaces.

Definition 2 *Given a subspace $W \subset V$ its annihilator is the subspace $W^\perp \subset V^*$ defined by*

$$W^\perp := \{\alpha \in V^* \mid \alpha(v) = 0 \; \forall \; v \in W\}$$

Given a subspace $X \subset V^*$, its annihilator is the subspace $X^\perp \subset V$ defined by

$$X^\perp := \{v \in V \mid \alpha(v) = 0 \; \forall \; \alpha \in X\}$$

Given a linear mapping between any two vector spaces $F : V_1 \to V_2$ we can define an induced linear mapping between their dual spaces.

Definition 3 *Given a linear mapping $F : V_1 \to V_2$, its dual map is the linear mapping $F^* : V_2^* \to V_1^*$ defined by*

$$(F^*(\alpha))(v) = \alpha(F(v)), \; \forall \; \alpha \in V_2^*, \; v \in V_1$$

Since V^* is a vector space, it also has a dual space which we denote V^{**}. There exists a "natural" identification $i : V \to V^{**}$ which is defined for all $v \in V$ and $\alpha \in V^*$ by

$$(i(v))(\alpha) = \alpha(v)$$

For all finite-dimensional vector spaces, this fact allows us to treat V and V^{**} as essentially the same object. For example, we could have defined the annihilator as

$$W^\perp := \{\alpha \in V^* \mid \alpha(v) = 0 \; \forall \; v \in W \subset V\}.$$

Using this definition, the annihilator of a subspace $W^* \subset V^*$ is defined as a subspace of V^{**}. However, we can use the natural identification to map this subspace back to V in which case we recover our original definition.

2.2 Tensors

Let $V_1, \ldots, V_k$ be a collection of real vector spaces. A function $f : V_1 \times, \ldots, \times V_k \to \mathbb{R}$ is said to be linear in the ith variable if the function $T : V_i \to \mathbb{R}$ defined with fixed $v_j \neq v_i$ as

$$T(v) = f(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k)$$

is linear. A function is called multilinear if it is linear in each variable.

A multilinear function $T : V^k \to \mathbb{R}$ is said to be a covariant tensor of order k or simply a k-tensor. The set of all k-tensors on V is denoted $\mathcal{L}^k(V)$. For $k = 1$ we have, $\mathcal{L}^k(V) = V^*$, the dual space of V. Therefore, we can think of covariant tensors as generalized covectors.

Example: A typical example of a multilinear function is the inner product of two vectors. By the definition of the inner product we have for vectors $x, y, z \in \mathbb{R}^n$

$$< a \cdot x, y >=< x, a \cdot y >= a \cdot < x, y >$$

$$< x+z, y > = < x, y > + < x, z > = < x, y+z >$$

An important example of a multilinear function is the determinant. If $x_1, x_2, \ldots, x_n$ are n column vectors in $\mathbb{R}^n$ then

$$det[x_1 \; x_2 \; \ldots \; x_n]$$

is multilinear by the properties of the determinant. ◇

As in the case of V^*, each $\mathcal{L}^k(V)$ can be made into a vector space.

Theorem 2 *If for $S, T \in \mathcal{L}^k(V)$ and $c \in \mathbb{R}$ we define*

$$\begin{aligned} (S+T)(v_1, \ldots, v_k) &= S(v_1, \ldots, v_k) + T(v_1, \ldots, v_k) \\ (cT)(v_1, \ldots, v_k) &= c \cdot T(v_1, \ldots, v_k) \end{aligned}$$

then the set of all k-tensors on V, $\mathcal{L}^k(V)$, is a real vector space.

Proof: See Munkres [1] page 220. □

Because of their multilinear structure, two tensors are equal if they agree on any set of basis elements.

Theorem 3 *Let $a_1, \ldots, a_n$ be a basis for V. Let $f, g : V^k \to \mathbb{R}$ be k-tensors on V. If $f(a_{i_1}, \ldots, a_{i_k}) = g(a_{i_1}, \ldots, a_{i_k})$ for every k-tuple (multi-index) $I = (i_1, \ldots, i_k) \in \{1, 2, \ldots, n\}^k$, then $f = g$.*

Proof: See Munkres [1] page 221. □

Theorem 3 allows us to construct a basis for the space $\mathcal{L}^k(V)$.

Theorem 4 *Let $a_1, \ldots, a_n$ be a basis for V. Let $I = (i_1, \ldots, i_k) \in \{1, 2, \ldots, n\}^k$. Then there is a unique tensor ϕ^I on V such that for every k-tuple $J = (j_1, \ldots, j_k) \in \{1, 2, \ldots, n\}^k$*

$$\phi^I(a_{j_1}, \ldots, a_{j_k}) = \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J \end{cases}$$

and the collection of all the ϕ^I forms a basis for $\mathcal{L}^k(V)$.

Proof: Uniqueness follows from Theorem 3. To construct the functions ϕ^I, we start with a basis for V^*, $\phi^i : V \to \mathbb{R}$, defined by

$$\phi^i(a_j) = \delta_{ij}.$$

We then define each ϕ^I as

$$\phi^I = \phi^{i_1}(v_1) \cdot \phi^{i_2}(v_2) \cdot \ldots \cdot \phi^{i_k}(v_k)$$

and claim that these ϕ^I form a basis for $\mathcal{L}^k(V)$. To show this, we select an arbitrary k-tensor $f \in \mathcal{L}^k(V)$, and define the scalars

$$\alpha_I := f(a_{i_1}, \ldots, a_{i_k}).$$

Next, we define a k-tensor

$$g = \sum_J \alpha_J \phi^J$$

where

$$J \in \{1, \ldots, n\}^k.$$

Then by Theorem 3, $f \equiv g$. □

Since there are n^k distinct k-tuples from the set $\{1, \ldots, n\}$ the space $\mathcal{L}^k(V)$ has dimension n^k.

Example: Let $V = \mathbf{R}^n$ with the standard basis $e_1, \ldots, e_n$,and let $\phi^1, \ldots, \phi^n$ be the dual basis. If

$$x \in \mathbf{R}^n = \sum_{j=1}^n x_j e_j,$$

then evaluating each function in the dual basis at x gives

$$\phi^i(x) = \phi^i(\sum_{j=1}^n x_j e_j) = \sum_{j=1}^n x_j \phi^i(e_j) = x_i$$

Likewise, if $I = (i_1, \ldots, i_k)$ then evaluating the basis vectors for $\mathcal{L}^k(V)$ at $(x_1, \ldots, x_k)$ gives

$$\phi^I(x_1, \ldots, x_k) = \phi^{i_1}(x_1) \cdot \phi^{i_2}(x_2) \cdot \ldots \cdot \phi^{i_k}(x_k)$$

$$= x_{i_1 1} \cdot \ldots \cdot x_{i_k k}.$$

Since the tensors $\phi^1, \ldots, \phi^n$ form a basis for V^*, evaluating a general 1-tensor $f \in (\mathbf{R}^n)^*$ at $x \in \mathbf{R}^n$ gives

$$f(x) = \alpha_1 x_1 + \ldots + \alpha_n x_n.$$

Likewise, evaluating a general 2-tensor at $(x, y) \in \mathbf{R}^2$ gives

$$g(x, y) = \sum_{i,j=1}^n \alpha_{ij} x_i y_j = x^T D y$$

and evaluating a general k-tensor at $(x_1 \ldots, x_k) \in \mathbf{R}^k$ gives

$$g(x_1, x_2, \ldots, x_k) = \sum_{i_1, \ldots, i_k = 1}^n \alpha_{i_1, \ldots, i_k} x_{i_1} \ldots x_{i_k}$$

◇

2.2.1 Tensor Products

We now introduce a product operation into the set of all tensors on V and outline its basic properties.

Definition 4 *Let $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^l(V)$. The tensor product $f \otimes g$ of f and g is a tensor in $\mathcal{L}^{k+l}(V)$ and is defined by*

$$(f \otimes g)(v_1, \ldots, v_{k+l}) := f(v_1, \ldots, v_k) \cdot g(v_{k+1}, \ldots, v_{k+l})$$

Theorem 5 *Let f, g, h be tensors on V and $c \in \mathbf{R}$. Then we have,*

1. *Associativity* $f \otimes (g \otimes h) = (f \otimes g) \otimes h$
2. *Homogeneity* $cf \otimes g = c(f \otimes g) = f \otimes cg$
3. *Distributivity* $(f + g) \otimes h = f \otimes h + g \otimes h$
4. *Given a basis $a_1, \ldots, a_n$ for V, the basis tensors satisfy* $\phi^I = \phi^{i_1} \otimes \phi^{i_2} \otimes, \ldots, \otimes \phi^{i_k}$

Proof: See Munkres [1] page 224. □

We can also define the tensor product of two subspaces $U, W \subset V^*$ as

$$U \otimes W := span\{x \in \mathcal{L}^2(V) \mid x = u \otimes w,\ u \in U,\ w \in W\}.$$

Therefore, from Theorem 4 we can conclude that

$$V^* \otimes V^* = \mathcal{L}^2(V).$$

More generally we have

$$\underbrace{V^* \otimes \ldots \otimes V^*}_{k-times} = \bigotimes^k V^* = \mathcal{L}^k(V)$$

2.3 Alternating Tensors

In this section we introduce the concept of an alternating tensor. In order to do this, we need to know some facts about permutations.

2.3.1 Permutations

Definition 5 *A permutation of the set of integers $\{1, 2, \ldots, k\}$ is a one-to-one function σ mapping this set onto itself.*

Theorem 6 *The set of all permutations σ is a group under function composition called the symmetric group on $\{1, \ldots, k\}$ and is denoted by S_k.*

Proof: The composition of permutations is also a permutation. Composition is associative. The identity function acts as the unit permutation and the inverses of permutations exist since permutations are one-to-one, onto functions. Thus all group axioms are satisfied. □

Permutations simply reshuffle the elements of a finite set. As a result, the number of permutations in S_k is $k!$.

Definition 6 *Given $1 \leq i < k$, a permutation e_i is called elementary if given some $i \in \{1, 2, \ldots, k\}$ we have*

$$\begin{aligned} e_i(j) &= j \quad for \quad j \neq i, i+1 \\ e_i(i) &= i + 1 \\ e_i(i+1) &= i \end{aligned}$$

An elementary permutation leaves the set intact except for consecutive elements i and $i+1$ which are switched. The space S_k can be constructed from the elementary permutations.

Theorem 7 *Every permutation $\sigma \in S_k$ can be written as the composition of elementary permutations.*

Proof: See Munkres [1] page 227. □

Definition 7 *Let $\sigma \in S_k$. Consider the set of all pairs of integers i, j from the set $\{1, \ldots, k\}$ for which $i < j$ and $\sigma(i) > \sigma(j)$. Each such pair is called an inversion in σ. The sign of σ is defined to be the number -1 if the number of inversions is odd and $+1$ if it is even. We call σ an odd or even permutation respectively. The sign of σ is denoted by $sgn(\sigma)$.*

The following theorem helps us calculate the sign of permutations.

Theorem 8 *Let $\sigma, \tau \in S_k$. Then*

1. *If σ equals the composite of m elementary permutations then $sgn(\sigma) = (-1)^m$*
2. $sgn(\sigma \circ \tau) = sgn(\sigma) \cdot sgn(\tau)$
3. $sgn(\sigma^{-1}) = sgn(\sigma)$
4. *If $p \neq q$, and if τ is the permutation that exchanges p and q and leaves all other integers fixed, then $sgn(\tau) = -1$*

Proof: See Munkres [1] page 228. □

2.3.2 Alternating Tensors

We are now ready to define alternating tensors.

Definition 8 *Let f be an arbitrary k-tensor on V. If σ is a permutation of $\{1, \ldots, k\}$, we define f^σ by the equation*

$$f^\sigma(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \tag{1}$$

Since f is linear in each of its variables, so is f^σ. The tensor f is said to be symmetric if $f = f^e$ for each elementary permutation e, and it is said to be alternating if $f = -f^e$ for every elementary permutation e.

In other words, f is symmetric if for all i

$$f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k) \tag{2}$$

and alternating if

$$f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k) \tag{3}$$

We will denote the set of all alternating k-tensors on V by $\Lambda^k(V^*)$. The reason for this notation will be apparent when we introduce the wedge product in the next section.

One can check that the sum of two alternating tensors as well as a scalar multiple of it is alternating. Therefore, $\Lambda^k(V^*)$ is a linear subspace of the space $\mathcal{L}^k(V)$ of all k-tensors on V.

In the special case of $\mathcal{L}^1(V)$, elementary permutations cannot be performed and therefore every 1-tensor is vacuously alternating. Therefore $\Lambda^1(V^*) = \mathcal{L}^1(V) = V^*$. Furthermore, for completeness, we define $\Lambda^0(V^*) = \mathbf{R}$.

Examples: Elementary tensors are not alternating but the following linear combination

$$f = \phi^i \otimes \phi^j - \phi^j \otimes \phi^i$$

is alternating. To see this, let $V = \mathbf{R}^n$ and let ϕ^i be the usual dual basis. Then

$$f(x,y) = x_i y_j - x_j y_i = det \begin{bmatrix} x_i & y_i \\ x_j & y_j \end{bmatrix}$$

and it is easily seen that $f(x,y) = -f(y,x)$. Similarly, the function

$$g(x,y,z) = det \begin{bmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{bmatrix}$$

is an alternating 3-tensor. $\diamond$

We are interested in obtaining a basis for the linear space $\Lambda^k(V^*)$. We start with the following lemma.

Lemma 1 *Let f be a k-tensor on V and $\sigma,\tau \in S_k$. Then*

1. *The transformation $f \longrightarrow f^\sigma$ is a linear transformation from $\mathcal{L}^k(V^*)$ to $\mathcal{L}^k(V^*)$. It has the property that for all $\sigma, \tau \in S_k$*

$$(f^\sigma)^\tau = f^{\sigma\circ\tau}$$

2. *The tensor f is alternating if and only if $f^\sigma = sgn(\sigma) \cdot f$ for all $\sigma \in S_k$.*

3. *If f is alternating and if $v_p = v_q$ with $p \neq q$ then $f(v_1, \ldots, v_k) = 0$.*

Proof: The linearity property is obvious since $(af + bg)^\sigma = af^\sigma + bg^\sigma$. Now

$$\begin{aligned} (f^\sigma)^\tau(v_1,\ldots,v_k) &= f^\sigma(v_{\tau(1)},\ldots,v_{\tau(k)}) \\ &= f^\sigma(w_1,\ldots,w_k) \qquad w_i = v_{\tau(i)} \\ &= f(w_{\sigma(1)},\ldots,w_{\sigma(k)}) \\ &= f(v_{\tau(\sigma(1))},\ldots,v_{\tau(\sigma(k))}) \\ &= f^{\sigma\circ\tau}(v_1,\ldots,v_k) \end{aligned}$$

Let σ be an arbitrary permutation. We can therefore write it as

$$\sigma = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_m$$

where each σ_i is an elementary permutation. Then

$$\begin{aligned} f^{\sigma} &= f^{\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_m} \\ &= ((\ldots(f^{\sigma_m})\ldots)^{\sigma_2})^{\sigma_1} \\ &= (-1)^m \cdot f \\ &= sgn(\sigma) \cdot f \end{aligned}$$

Now suppose $v_p = v_q$ and $p \neq q$. Let τ be a permutation that exchanges p and q. Since $v_p = v_q$,

$$f^{\tau}(v_1, \ldots, v_k) = f(v_1, \ldots, v_k)$$

but since τ is an alternating tensor

$$f^{\tau}(v_1, \ldots, v_k) = -f(v_1, \ldots, v_k)$$

since $sgn(\tau) = -1$. Therefore $f(v_1, \ldots, v_k) = 0$.□

As a result of the Lemma 1 we get that if $k > n$, the space $\Lambda^k(V^*)$ is trivial since one of the basis elements must appear in the k-tuple more than once. Hence for $k > n$, $\Lambda^k(V^*) = 0$. We have also seen that for $k = 1$ we have $\Lambda^1(V^*) = \mathcal{L}^1(V) = V^*$ and therefore one can use the dual basis as a basis for $\Lambda^1(V^*)$. We are therefore left with the case where $1 < k \leq n$. The key argument here is that in order to specify an alternating tensor we simply need to define it on an ascending k-tuple of basis elements since, by using the Lemma 1, every other combination can be obtained by permuting the k-tuple.

Theorem 9 *Let $a_1, a_2, \ldots, a_n$ be a basis for V. If f, g are alternating k-tensors on V and if*

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$$

for every <u>ascending</u> k-tuple of integers $\{1, 2, \ldots, n\}$ then $f = g$.

Proof: See Munkres [1] page 231. □

Theorem 10 *Let $a_1, \ldots, a_n$ be a basis for V. Let $I = (i_1, \ldots, i_k) \in \{1, 2, \ldots, n\}^k$ be an ascending k-tuple. There is a unique alternating k-tensor ψ^I on V such that for every ascending k-tuple $J = (j_1, \ldots, j_k) \in \{1, 2, \ldots, n\}^k$*

$$\psi^I(a_{j_1}, \ldots, a_{j_k}) = \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J \end{cases}$$

The tensors ψ^I form a basis for $\Lambda^k(V^*)$. The tensors ψ^I also satisfy the formula

$$\psi^I = \sum_{\sigma \in S_k} sgn(\sigma)(\phi^I)^{\sigma}$$

Proof: In Munkres [1] pages 232-33.

The tensors ψ^I are called elementary alternating k-tensors on V corresponding to the basis $a_1, \dots, a_n$ for V. Therefore every alternating k-tensor f may be uniquely expressed by

$$f = \sum_J d_J \psi^J$$

where J indicates that summation extends over all ascending k-tuples.

The dimension of $\Lambda^1(V^*)$ is simply n since one can use the standard basis for V^*. If $k > 1$, then we need to find the number of possible ascending k-tuples from the set $\{1, 2, \dots, n\}$. But if we choose k elements from a set of n elements, there is only one way to put them in ascending order. Therefore the number of ascending k-tuples and the dimension of $\Lambda^k(V^*)$ is

$$dim(\Lambda^k(V^*)) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

2.3.3 The Wedge Product

Just as we defined the tensor product operation in the set of all tensors on a vector space V, we can define an analogous product operation, the wedge product, in the space of alternating tensors. The tensor product alone does not suffice, since even if $f \in \Lambda^k(V^*)$ and $g \in \Lambda^l(V^*)$ are alternating, their tensor product $f \otimes g \in \mathcal{L}^{k+l}(V)$ need not be alternating. We therefore construct an alternating operator taking k-tensors to alternating k-tensors.

Theorem 11 *For any tensor $f \in \mathcal{L}^k(V)$, define $Alt : \mathcal{L}^k(V) \longrightarrow \Lambda^k(V^*)$ by:*

$$Alt(f) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) f^\sigma.$$

Then $Alt(f) \in \Lambda^k(V^*)$ and if $f \in \Lambda^k(V^*)$ then $Alt(f) = f$.

Proof: The fact that $Alt(f) \in \Lambda^k(V^*)$ is a consequence of Lemma 1, parts (1) and (2). Simply expanding the summation for $f \in \Lambda^k(V^*)$ yields that $Alt(f) = f$. □

Example: Let $f(x, y)$ be any 2-tensor. By using the alternating operator we obtain,

$$Alt(f) = \frac{1}{2}(f(x,y) - f(y,x))$$

which is clearly alternating. Similarly for any 3-tensor g(x,y,z) we have

$$Alt(g) = \frac{1}{6}(g(x,y,z) + g(y,z,x) + g(z,x,y) - g(y,x,z) - g(z,y,x) - g(x,z,y))$$

which can be easily checked to be alternating. ◇

Definition 9 *Given $f \in \Lambda^k(V^*)$ and $g \in \Lambda^l(V^*)$, we define the wedge or exterior product, $f \wedge g \in \Lambda^{k+l}(V^*)$ by*

$$f \wedge g = \frac{(k+l)!}{k!l!} Alt(f \otimes g).$$

Therefore given two tensors, the wedge product first obtains their tensor product, then uses the alternating operator in order to obtain an alternating tensor and then normalizes it. The are two reasons for the somewhat complicated normalization constant. The first reason is so that if f is alternating then $Alt(f) = f$. The second reason is that we want the wedge product to be associative. The normalizing coefficient ensures both properties. Since alternating tensors of order zero are elements of **R**, we define the wedge product of an alternating 0-tensor and any alternating k-tensor by the usual multiplication. The following theorem lists some important properties of the wedge product.

Theorem 12 *Let $f \in \Lambda^k(V^*)$,$g \in \Lambda^l(V^*)$ and $h \in \Lambda^m(V^*)$. Then*

1. *Associativity* $f \wedge (g \wedge h) = (f \wedge g) \wedge h$
2. *Homogeneity* $cf \wedge g = c(f \wedge g) = f \wedge cg$
3. *Distributivity* $(f + g) \wedge h = f \wedge h + g \wedge h$
   $h \wedge (f + g) = h \wedge f + h \wedge g$
4. *Skew-commutativity* [1], $g \wedge f = (-1)^{kl} f \wedge g$

Proof: Properties (2), (3) and (4) follow directly from the definitions of the alternating operator and the tensor product. Associativity, property (1), requires a few more manipulations (see Spivak [2] pages 80-81). □

Example: Let $f(x) \in \Lambda^1(V^*)$ and $g(y, z) \in \Lambda^2(V^*)$. Then

$$\begin{aligned} f \wedge g &= \frac{(2+1)!}{2!1!}\frac{1}{3!}(f(x) \otimes g(y,z) + f(y) \otimes g(z,x) + \\ &+ f(z) \otimes g(x,y) - f(y) \otimes g(x,z) - f(z) \otimes g(y,x) - f(x) \otimes g(z,y)) \end{aligned}$$

We can also check that

$$f \wedge f = \frac{(1+1)!}{1!1!}\frac{1}{2!}(f(x) \otimes f(x) - f(x) \otimes f(x)) = 0$$

which can also been seen from the skew commutativity of exterior multiplication. ◇

We can now formulate a basis for $\Lambda^k(V^*)$ more elegantly in terms of the dual basis for V.

Theorem 13 *Given a basis $a_1, \ldots, a_n$ for vector space V, let $\phi^1, \ldots, \phi^n$ denote its dual basis and ψ^I the corresponding elementary alternating tensors. Then if $I = (i_1, \ldots, i_k)$ is any ascending k-tuple of integers,*

$$\psi^I = \phi^{i_1} \wedge \phi^{i_2} \wedge \ldots \wedge \phi^{i_k}.$$

[1] also called anti-commutativity

Proof: May be deduced from the construction of the elementary alternating tensors in Theorem 10. □

By Theorems 13, any alternating k-tensor $f \in \Lambda^k(V^*)$ may be expressed in terms of the dual basis $\phi^1, \ldots, \phi^n$ as:

$$f = \sum_J d_{j_1,\ldots,j_k} \phi^{j_1} \wedge \phi^{j_2} \wedge \ldots \wedge \phi^{j_k} \tag{4}$$

for all ascending k-tuples $J = (j_1, \ldots, j_k)$ and some scalars, $d_{j_1,\ldots,j_k}$. If we require the coefficients to be skew-symmetric,

$$d_{i_1,\ldots,i_l,i_{l+1},\ldots,i_k} = -d_{i_1,\ldots,i_{l+1},i_l,\ldots,i_k}, \ \forall\, l \in \{1, \ldots, k-1\}$$

then we can extend this summation over all k-tuples.

$$f = \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{n} d_{i_1,\ldots,i_k} \phi^{i_1} \wedge \phi^{i_2} \wedge \ldots \wedge \phi^{i_k} \tag{5}$$

The definition of the wedge product is interesting but it is the properties of the wedge product which make it a powerful tool. As a result, one does not really work with the definition of the wedge product but with its properties.

The wedge product provides a nice way to check whether a set of 1-tensors is linearly independent.

Theorem 14 *If $\omega^1, \ldots, \omega^k$ are 1-tensors over V then*

$$\omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^k = 0$$

if and only if $\omega^1, \ldots, \omega^k$ are linearly dependent.

Proof: Suppose that $\omega^1, \ldots, \omega^k$ are linearly independent, and pick $\alpha^{k+1}, \ldots, \alpha^n$ to complete a basis for V^*. From Theorem 13 we know that

$$\omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^k$$

is a basis element for $\Lambda^k(V^*)$. Therefore, it must be nonzero.

If $\omega^1, \ldots, \omega^k$ are linearly dependent, then at least one of the them can be written as a linear combination of the rest. Without loss of generality, assume that ω^k is linearly dependent. We then have

$$\omega^k = \sum_{i=1}^{k-1} c_i \omega^i$$

From this we get that

$$\omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^k = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^{k-1} \wedge \left(\sum_{i=1}^{k-1} c_i \omega^i\right) = 0$$

by the skew-commutativity of the wedge product. □

This result allows us to give a geometric interpretation to a nonzero k-form

$$\omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^k \neq 0$$

by associating it with the subspace

$$W := span\{\omega^1, \ldots, \omega^k\} \subset V^*.$$

An obvious question which arises is what happens if we select a different basis for W.

Theorem 15 *Given a subspace $W \subset V^*$ and two sets of 1-tensors which span W,*

$$W = span\{\omega^1, \ldots, \omega^k\} = span\{\alpha^1, \ldots, \alpha^k\}$$

there exists a nonzero scalar $c \in \mathbf{R}$ such that

$$c \cdot \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^k = \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k \neq 0$$

Proof: Each α^i can be written as a linear combination of the ω^i

$$\alpha^i = \sum_{j=1}^{k} a_{ij}\omega^j.$$

Therefore, the product

$$\alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k = (\sum_{j=1}^{k} a_{1j}\omega^j) \wedge \ldots \wedge (\sum_{j=1}^{k} a_{kj}\omega^j)$$

Multiplying this out gives

$$\alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k = \sum_{i_1,\ldots,i_k=1}^{n} b_{i_1,\ldots,i_k}\omega^{i_1} \wedge \omega^{i_2} \wedge \ldots \wedge \omega^{i_k}.$$

Finally, using Theorem 14, and the skew commutativity of the wedge product we get

$$c \cdot \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^k = \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k \neq 0. \ \Box$$

Therefore, the k-fold wedge product of all sets of linearly independent 1-tensors which spans a subspace of $W \subset V^*$ differ by only a scalar constant. We can therefore define an equivalence class of basis sets for W

Definition 10 *Let $\xi = x^1 \wedge \ldots \wedge x^k$. We define an equivalence class*

$$[x^1 \ldots x^k] := \{\pi \in \Lambda^k(V^*) \mid \pi = c \cdot \xi, \text{ for some nonzero } c \in \mathbf{R}\}$$

called the Grassmann coordinate of ξ.

The set of all such equivalence classes can be put in one-to-one correspondence with the set of all k-dimensional subspaces of V^*. This set of subspaces is called the Grassmann manifold of k-planes in V^* and is denoted as $G(V^*, k)$.

Definition 11 *A k-tensor $\xi \in \Lambda^k(V^*)$ is* <u>*decomposable*</u> *if there exist $x^1, x^2, \ldots, x^k \in \Lambda^1(V^*)$ such that $\xi = x^1 \wedge x^2 \wedge \ldots \wedge x^p$*

In order for an alternating k-tensor ξ to be decomposable, it must be expressible as a monomial

$$\xi = \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k$$

with respect to some set of basis vectors

$$\{\alpha^1, \alpha^2, \ldots, \alpha^n\}$$

This is not always possible. To see this consider the following example.

Example: Let $\xi = \phi^1 \wedge \phi^2 + \phi^3 \wedge \phi^4 \in \Lambda^2((\mathbf{R}^4)^*)$. If ξ is decomposable, then we must have $\xi \wedge \xi = 0$. The reason for this is that if ξ can be expressed as

$$\xi = \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k$$

then it follows that

$$\xi \wedge \xi = \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k \wedge \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^k = 0$$

In our case we have

$$\xi \wedge \xi = 2\phi^1 \wedge \phi^2 \wedge \phi^3 \wedge \phi^4 \neq 0$$

Therefore ξ is not decomposable. Notice that this is a necessary but not sufficient condition and, in particular, it does not apply to odd alternating tensors. ◇

If an alternating k-tensor ξ is not decomposable, it may still be possible to factor out a 1-tensor from every term in the summation which defines it.

Example: Let $\xi = \phi^1 \wedge \phi^2 \wedge \phi^5 + \phi^3 \wedge \phi^4 \wedge \phi^5 \in \Lambda^3((\mathbf{R}^5)^*)$. From the previous example, we know that this form is not decomposable, but the 1-tensor ϕ^5 can clearly be factored from every term

$$\xi = (\phi^1 \wedge \phi^2 + \phi^3 \wedge \phi^4) \wedge \phi^5 = \hat{\xi} \wedge \phi^5$$

◇

Definition 12 *Let $\xi \in \Lambda^k(V^*)$. We define a subspace $L_\xi \subset V^*$*

$$L_\xi := \{\omega \in V^* \mid \xi = \hat{\xi} \wedge \omega \text{ for some } \hat{\xi} \in \Lambda^{k-1}(V^*)\}$$

called the <u>*divisor space*</u> *of ξ. Any $\omega \in L_\xi$ is called a* <u>*divisor*</u> *of ξ.*

Theorem 16 *A 1-tensor $\omega \in V^*$ is a divisor of $\xi \in \Lambda^k(V^*)$ if and only if*

$$\omega \wedge \xi \equiv 0$$

Proof: Pick a basis $\phi^1, \phi^2, \ldots, \phi^n$ for V^* such that $\omega = \phi^1$. With respect to this basis, ξ can be written as

$$\xi = \sum_J^n d_{j_1,\ldots,j_k} \phi^{j_1} \wedge \phi^{j_2} \wedge \ldots \wedge \phi^{j_k} \tag{6}$$

for all ascending k-tuples $J = (j_1, \ldots, j_k)$ and some scalars, $d_{j_1,\ldots,j_k}$. If ω is a divisor of ξ, then it must be contained in each nonzero term of this summation. Therefore $\omega \wedge \xi$ must be identically 0.

If $\omega \wedge \xi \equiv 0$, then every nonzero term of ξ must contain ω. Otherwise, we would have for $j_1, \ldots, j_k \neq 1$

$$\omega \wedge \phi^{j_1} \wedge \ldots \wedge \phi^{j_k} = \phi^1 \wedge \phi^{j_1} \wedge \ldots \wedge \phi^{j_k}$$

which is a basis element of $\Lambda^{k+1}(V^*)$ and therefore nonzero. This contradicts the assumption $\omega \wedge \xi \equiv 0$. □

If we select a basis $\phi^1, \phi^2, \ldots, \phi^n$ for V^* such that

$$span\{\phi^1, \phi^2, \ldots, \phi^l\} = L_\xi$$

then, ξ can be written as

$$\xi = \hat{\xi} \wedge \phi^1 \wedge \ldots \wedge \phi^l$$

where $\hat{\xi} \in \Lambda^{k-l}(V^*)$ is not decomposable and involves only the one forms $\phi^{l+1}, \ldots, \phi^n$.

2.3.4 The Interior Product

A second useful operation on tensors in called the interior product.

Definition 13 *The* <u>*interior product*</u> *is a linear mapping* $\neg : V \times \mathcal{L}^k(V) \to \mathcal{L}^{k-1}(V)$ *which operates on a a vector* $v \in V$ *and a tensor* $T \in \mathcal{L}^k(V)$ *and produces a tensor* $(v \neg T) \in \mathcal{L}^{k-1}(V)$ *define by*

$$(v \neg T)(v_1, \ldots, v_{k-1}) := T(v, v_1, \ldots, v_{k-1})$$

The interior product has the following basic properties.

Theorem 17 *Let* a, b, c, d *be real numbers and* $v, w \in V, g, h \in \mathcal{L}^l(V)$, $r \in \Lambda^s(V^*)$, *and* $f \in \Lambda^m(V^*)$. *Then we have*

1. *Bilinearity* $$\begin{aligned} (av + bw) \neg g &= a(v \neg g) + b(w \neg g) \\ v \neg (cg + dh) &= c(v \neg g) + d(v \neg h) \end{aligned}$$

2. $v \neg (f \wedge r) = (v \neg f) + (-1)^m f \wedge (v \neg g)$

Proof: See Yang [3] page 12. □

The next result illustrates a useful property of the interior product.

Theorem 18 *Let* $a_1, \ldots, a_n$ *be a basis for* V. *Then the value of an alternating* k*-tensor* $\omega \in \Lambda^k(V^*)$ *is independent of a basis element* a_i *if and only if* $a_i \neg \omega \equiv 0$.

Proof: Let $\phi^1, \ldots, \phi^n$ be the dual basis to $a_1, \ldots, a_n$. Then ω can be written with respect to the dual basis as

$$\omega = \sum_J d_J \phi^{j_1} \wedge \phi^{j_2} \wedge \ldots \wedge \phi^{j_k} = \sum_J d_J \psi^J$$

where the sum is taken oven all ascending k-tuples J. If a basis element ψ^J does not contain ϕ_i, then clearly

$$a_i \neg \psi^J \equiv 0$$

If a basis element contains ϕ_i, then

$$a_i \neg\ \phi^{j_1} \wedge \phi^{j_2} \wedge \ldots \wedge \phi^{j_k} \not\equiv 0$$

because a_i can always be matched with ϕ_i through a permutation which only affects the sign. Consequently, $(a_I \neg \omega) \equiv 0$ if and only if the coefficients d_J of all the terms containing ϕ^i are zero. □

Definition 14 *Let $\omega \in \Lambda^k(V^*)$. The associated space of ω is defined as*

$$A_\omega := \{v \in V | v \neg \omega \equiv 0\}$$

The dual associated space of ω is defined as $A_\omega^\perp$.

Recall that the divisor space L_ω of an alternating k-tensor ω contains all the 1-tensors which can be factored from every term of ω. The dual associated space $A_\omega^\perp$ contains all the 1-tensors which are contained in at least one term of ω. Therefore, $L_\omega \subset A_\omega^\perp$. The following result ties these notions together.

Theorem 19 *The following statements are equivalent:*

1. *An alternating k-tensor $\omega \in \Lambda^k(V^*)$ is decomposable.*
2. *The divisor space L_ω has dimension k.*
3. *The dual associated space $A_\omega^\perp$ has dimension k.*
4. $L_\omega = A_\omega^\perp$.

Proof:

(1) $\Leftrightarrow$ (2). If ω is decomposable, then there exist a set of basis vectors $\phi^1, \phi^2, \ldots, \phi^n$ for V^* such that

$$\omega = \phi^1 \wedge \ldots \wedge \phi^k$$

Therefore $L_\omega = span\{\phi^1, \phi^2, \ldots, \phi^k\}$ which has dimension k. Conversely, if L_ω has dimension k, then k terms can be factored from ω. Since ω is k-tensor, it must be decomposable.

(1) $\Leftrightarrow$ (3). Let $a_1, \ldots, a_n$ be the basis of V which is dual to $\phi^1, \phi^2, \ldots, \phi^n$. Since

$$\omega = \phi^1 \wedge \ldots \wedge \phi^k,$$

ω is not a function of $a_{k+1}, \ldots, a_n$. Therefore,

$$A_\omega = span\{a_{k+1}, \ldots, a_n\}.$$

This implies that $A_\omega^\perp$ has dimension k. Conversely, if $A_\omega^\perp$ has dimension k, then A_ω has dimension $n-k$ which means that ω is an alternating k-tensor which is a function of k variables. Therefore, it must have the form

$$\omega = \phi^1 \wedge \ldots \wedge \phi^k,$$

for some linearly independent $\phi^1, \phi^2, \ldots, \phi^k$ in V^*.

(2)&(3) $\Leftrightarrow$ (4). It is always true that $L_\omega \subset A_\omega^\perp$. Therefore if $dim(L_\omega) = dim(A_\omega^\perp)$ then $L_\omega = A_\omega^\perp$. It is also always true that $0 \leq dim(L_\omega) \leq k$ and $k \leq dim(A_\omega^\perp) \leq n$. Therefore, $L_\omega = A_\omega^\perp$ implies that $dim(L_\omega) = dim(A_\omega^\perp) = k$. □

2.4 The Pull Back of a Linear Transformation

The pull back of a linear transformation is a generalization of the dual map which we introduced in the section on dual spaces.

Let T be a linear map from a vector space V to a vector space W. Furthermore, let's assume that there exists a multilinear function f on W. Using the above, on can define a multilinear function on V as follows. First, given any vector v on V, use the map T to get to W and then apply the multilinear function f to $T(v)$. More formally,

Definition 15 *Let $T : V \to W$ be a linear transformation. The dual or pull back transformation*

$$T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$$

is defined for all $f \in \mathcal{L}^k(W)$ by

$$(T^*f)(v_1, \ldots, v_k) := f(T(v_1), \ldots, T(v_k))$$

Note that T^*f is multilinear since T is a linear transformation. The pull back map T^* also has the following properties

Theorem 20 *Let $T : V \to W$ be a linear transformation, and let*

$$T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$$

be the dual transformation. Then

1. *T^* is linear.*

2. *$T^*(f \otimes g) = T^*f \otimes T^*g$.*

3. *If $S : W \to X$ is linear, then $(S \circ T)^*f = T^*(S^*f)$.*

Proof: See Munkres [1] page 225. □

The following theorem ensures us that the pull back of an alternating tensor is alternating.

Theorem 21 *Let $T : V \longrightarrow W$ be a linear transformation. If f is an alternating tensor on W then T^*f is an alternating tensor on V, and*

$$T^*(f \wedge g) = T^*f \wedge T^*g$$

Proof: See Munkres [1] page 234. □

2.5 Contravariant Tensors

Up to this point, all the tensors which we have worked with have been defined as multilinear functions over the vector space V. If we simply replace V with V^* in all our definitions, then nothing is changed, and we can define an identical set of tensors over V^*.

A multilinear function $T : (V^*)^k \rightarrow \mathbb{R}$ is said to be a contravariant tensor of order k. The set of all k-tensors on V^* is denoted by $\mathcal{L}^k(V^*)$ or $\underbrace{V \otimes \ldots \otimes V}_{k-times}$. Note that in this notation we are implicitly using the natural identification between V^{**} and V. For $k = 1$ we have $\mathcal{L}^k(V^*) = V$, the vector space itself. For this reason, contravariant tensors are sometimes called multivectors.

2.6 Exterior Algebra

In section 2.4, we introduced the wedge product and interior product and demonstrated some of their properties. We now want to look more closely at the algebraic properties these operations give to the space of all alternating tensors.

2.6.1 Algebras and Ideals

We begin by introducing some algebraic structures which will be used in the development of the exterior algebra.

Definition 16 *An algebra is a vector space V together with a multiplication operation $\odot$: $V \times V \rightarrow V$ which for every scalar $\alpha \in \mathbb{R}$ and $a, b \in V$ satisfies $\alpha(a \odot b) = (\alpha a) \odot b = a \odot (\alpha b)$.*

Definition 17 *Given an algebra $(V, \odot)$, a subspace $W \subset V$ is called an algebraic ideal*[2] *if $x \in W, y \in V$ implies that $x \odot y, y \odot x \in W$*

Note that if W is an ideal and $x, y \in W$ then $x + y \in W$ since W is a subspace.

[2]For readers with knowledge of algebra, the algebraic ideal is the ideal of the algebra thought of as a ring. Furthermore since our ring has an identity we can show that any ideal must be a subspace of the algebra thought of as a vector space.

Example: The set of all polynomials with real-valued coefficients, $\mathbf{R}[s]$, is a vector space over $\mathbf{R}$ with vector addition and scalar multiplication defined by

$$(P_1 + P_2)(s) = P_1(s) + P_2(s)$$

$$(\alpha \cdot P)(s) = \alpha \cdot P(s)$$

If we define multiplication by

$$(P_1 \cdot P_2)(s) = P_1(s) \cdot P_2(s)$$

then $\mathbf{R}[s]$ is also an algebra.

In $\mathbf{R}[s]$, the set of all polynomials with a zero at $s = -2$ is an algebraic ideal. This is true because for all $P_1(s), P_2(s) \in \mathbf{R}[s]$ which satisfy

$$P_1(-2) = P_2(-2) = 0$$

we have that

$$P_1(-2) + P_2(-2) = 0, \quad \alpha \cdot P_1 = 0, \quad P_1(-2) \cdot P_2(-2) = 0$$

so this set is a subspace of $\mathbf{R}[s]$ which is closed under multiplication. Furthermore for all $P(s), R(s) \in \mathbf{R}[s]$ with $R(-2) = 0$ we have that

$$P(-2) \cdot R(-2) = 0$$

which verifies that the set of all polynomials with a root at -2 is an ideal of $\mathbf{R}[s]$. $\diamond$

Definition 18 *Let $(V, \odot)$ be an algebra. Let the set $A := \{a_i \in V, 1 \leq i \leq K\}$ be any finite collection of linearly independent elements in V. Let S be the set of all ideals containing A*

$$S := \{I \subset V | I \text{ is an ideal and } A \subset I\}.$$

The ideal I_A generated by A is defined as

$$I_A = \bigcap_{I \in S} I$$

and is the minimal ideal in S containing A.

If $(V, \odot)$ is an algebra, and there exists an element $e \in V$ such that for all $x \in V, x \odot e = e \odot x = x$ then e is called a unity element and is unique. If $(V, \odot)$ is an algebra with a unity element, then the ideal generated by a finite set of elements can be represented in a simple form.

Theorem 22 *Let $(V, \odot)$ be an algebra, $A := \{a_i \in V, 1 \leq i \leq K\}$ a finite collection of elements in V, and I_A the ideal generated by A. Then for each $x \in I_A$, there exist vectors $v_1, \ldots, v_K$ such that*

$$x = v_1 \odot a_1 + v_2 \odot a_2 + \ldots + v_K \odot a_K$$

Proof: See Hungerford [4] pages 123-124. $\square$

Example: The polynomial $(s+2)$ generates an ideal in $\mathbf{R}[s]$ which is equal to the set of all polynomials with a zero at $s = -2$. We will denote this set as I_{-2}. In the previous example we verified that this set is an ideal, and the polynomial $(s+2)$ is clearly contained in I_{-2}. Therefore, in order to verify that I_{-2} is the ideal generated by $(s+2)$ we only need to show that any other ideal which contains $(s+2)$ also contains I_{-2}. Because a real root can always be factored, I_{-2} can be written as

$$I_{-2} := \{P(s) \in \mathbf{R}[s] \mid P(s) = R(s)(s+2) \; \forall \, R(s) \in \mathbf{R}[s]\} \tag{7}$$

If I is any other ideal containing $(s+2)$, then

$$(\forall \, R(s) \in \mathbf{R}[s]) \; R(s)(s+2) \in I$$

because of the definition of an ideal. Therefore, $I_{-2} \subset I$. Consequently, I_{-2} must be the ideal generated by $(s+2)$.

This result also follows directly from Theorem 22, since $\mathbf{R}[s]$ has the constant polynomial **1** as a unity element. In order to see the importance of the unity element, suppose that we had taken as our algebra the set I_{-3} of all polynomials in $\mathbf{R}[s]$ with a root at -3. It is easy to verify that this is an algebra, and that the set $I_{-3,-2}$ of all polynomials with roots at -2 and -3 is an ideal. However,

$$I_{-3,-2} \neq \{P(s) \in \mathbf{R}[s] \mid P(s) = R(s)(s+2)(s+3) \; \forall \, R(s) \in I_{-3}\}$$

because the set on the right contains only polynomials with roots of order 2 and higher at -3. $\diamond$

Example: The two polynomials $P_1(s) = (s+2)(s+4)$ and $P_2(s) = (s+2)(s+3)$ generate an ideal in $\mathbf{R}[s]$

$$I_{P_1,P_2} := \{P(s) \in \mathbf{R}[s] \mid P(s) = Q(s)P_1(s) + R(s)P_2(s), \;\; R(s), Q(s) \in \mathbf{R}[s]\}.$$

Although this ideal is generated by two linearly independent vectors, it is equivalent to the ideal generated by the single vector $P_3(s) = (s+2)$. To demonstrate this fact, let $P(s) \in I_{P_1,P_2}$. Then

$$P(s) = (Q(s)(s+3) + R(s)(s+4))(s+2).$$

which implies that $P(s) \in I_{-2}$.

Now suppose $P(s) \in I_{-2}$ Then

$$P(s) = Q(s)(s+2)$$

for some $Q(s) \in \mathbf{R}[s]$. From the coprime factorization property of polynomials it can be shown that there exists polynomials $N(s), M(s) \in \mathbf{R}[s]$ such that

$$1 = N(s)(s+3) + M(s)(s+4).$$

Using this identity we get that

$$P(s) = Q(s) \cdot 1 \cdot (s+2) = Q(s)N(s)(s+3)(s+2) + Q(s)M(s)(s+4)(s+2).$$

which implies that $P(s) \in I_{P_1,P_2}$. This example shows that if we are given an arbitrary set of generators, it may be possible to find a smaller set of generators which will generate the same ideal. $\diamond$

Definition 19 *Let $(V, \odot)$ be an algebra, and $I \subset V$ an ideal. Two vectors $x, y \in V$ are said to be equivalent mod I if and only if $x - y \in I$. This equivalence is denoted*

$$x \equiv y \; mod \; I$$

From the definition above we can see that

$$x \equiv y \; mod \; I$$

if and only if

$$x - y \in I$$

which simply means that

$$x - y = \sum_{i=1}^{K} \theta_K \odot \alpha_K$$

for some $\theta_K \in V$. It is customary to abuse notation and denote this as

$$x \equiv y \; mod \; \alpha_1, \ldots, \alpha_K$$

where the mod operation is implicitly performed over the ideal generated by $\alpha_1, \ldots, \alpha_K$.

2.6.2 The Exterior Algebra of a Vector Space

Although the space $\Lambda^k(V^*)$ is a vector space with a multiplication operation, namely the wedge product, the wedge product of two alternating k-tensors is, almost always, not a k-tensor. Therefore $\Lambda^k(V^*)$ can not be made into an algebra under the wedge product. If we consider, however, the direct sum of the space of all alternating 0-tensors, 1-tensors, 2-tensors etc. we obtain,

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \Lambda^2(V^*) \oplus \ldots \oplus \Lambda^n(V^*)$$

where $\Lambda^0(V^*) = \mathbb{R}$, $\Lambda^1(V^*) = V^*$. Again the wedge product acts as the multiplication operator. This is clearly a vector space, and is also closed under exterior multiplication. It is therefore an algebra, and is called the exterior algebra over V^*. In this notation, any $\xi \in \Lambda(V^*)$ may be written as

$$\xi = \xi_0 + \xi_1 + \ldots + \xi_n$$

where each $\xi_p \in \Lambda^p(V^*)$.

Since $(\Lambda(V^*), \wedge)$ has the unity element $1 \in \Lambda^0(V^*)$, Theorem 22 implies that the ideal generated by a finite set of elements $\Sigma := \{\alpha^i \in \Lambda(V^*), 1 \leq i \leq K\}$ can be written as

$$I_S = \{\pi \in \Lambda(V^*) | \; \pi = \sum_{i=1}^{K} \theta^i \wedge \alpha^i, \theta^i \in \Lambda(V^*)\}$$

Given an arbitrary set of linearly independent generators Σ, it may also be possible to generate I_Σ with a smaller set of generators Σ'. In the next section we will use these ideas to study systems of exterior equations.

2.6.3 Systems of Exterior Equations

In the preceding sections we have developed an algebra of alternating multilinear functions over a vector space. We are now going to apply these ideas to solving a system of equations in the form

$$\alpha^1 = 0, \ldots, \alpha^K = 0$$

where each $\alpha^i \in \Lambda(V^*)$ The first thing we need to clarify is exactly what a "solution" to these equations means.

Definition 20 *A system of exterior equations on V is a finite set of linearly independent equations*

$$\alpha^1 = 0, \ldots, \alpha^K = 0$$

where each $\alpha^i \in \Lambda^k(V^*)$ for some $1 \le k \le n$. A solution to a system of exterior equations is any subspace $W \subset V$ such that

$$\alpha^1|_W \equiv 0, \ldots, \alpha^K|_W \equiv 0$$

where $\alpha|_W$ means that the arguments of $\alpha(v_1, \ldots, v_k)$ satisfy $v_1, \ldots, v_k \in W$

A system of exterior equations generally does not have a unique solution since any subspace $W_1 \subset W$ will satisfy $\alpha|_{W_1} \equiv 0$ if $\alpha|_W \equiv 0$.

A central fact concerning systems of exterior equations is given by the following theorem:

Theorem 23 *Given a system of exterior equations*

$$\alpha^1 = 0, \ldots, \alpha^K = 0 \tag{8}$$

and the corresponding ideal I_Σ generated by the collection of alternating tensors

$$\Sigma := \{\alpha^1, \ldots, \alpha^K\} \tag{9}$$

a subspace W solves the system of exterior equations if and only if it also satisfies $\pi|_W \equiv 0$ for every $\pi \in I_A$.

Proof: If $\pi \in I_A$, then $\pi = \sum_{i=1}^{K} \theta^i \wedge \alpha^i$. Furthermore, if W satisfies $\pi|_W = 0$ then

$$\pi = \sum_{i=1}^{K} \theta^i \wedge \alpha^i \equiv 0$$

for some set of $\theta^i \in \Lambda(V^*)$. This equation must hold for every $\pi \in I_A$. Since the α^i are assumed to be linearly independent, they must all satisfy,

$$\alpha^1|_W \equiv 0, \ldots, \alpha^K|_W \equiv 0. \tag{10}$$

Similarly, if equations 10 hold, then $\pi|_W \equiv 0$ for all $\pi \in I_A$. □

This result lets us treat the system of exterior equations, the set of generators, and the algebraic ideal as essentially equivalent objects. We may sometimes abuse notation and confuse a system of equations with its corresponding generator set. We may also confuse a generator set with its corresponding ideal. When it is important to distinguish, we will write out the system of exterior equations, and will denote the set of generators as Σ and the ideal which they generate as I_Σ.

Recall that an algebraic ideal was defined in a coordinate free way as a subspace of the algebra satisfying certain closure properties. Thus the ideal has an intrinsic geometric meaning, and we can think of two sets of generators as representing the same system of exterior equations if they generate the same algebraic ideal.

Definition 21 *Two sets of generators, Σ_1 and Σ_2, are said to be algebraically equivalent if and only if they generate the same ideal. i.e. $I_{\Sigma_1} = I_{\Sigma_2}$.*

We will exploit this notion of equivalence to represent a system of exterior equations in a simplified form. In order to do this, we need a few preliminary definitions.

Definition 22 *Let Σ be a system of exterior equations and I_Σ the ideal which it generates. The associated space of the ideal I_Σ is defined as*

$$A(I_\Sigma) := \{v \in V | v \neg \alpha \in I_\Sigma \; \forall \; \alpha \in I_\Sigma\}$$

The dual associated space or retracting space of the ideal is defined as $A(I_\Sigma)^\perp$ and denoted by $C(I_\Sigma)$.

Once we have determined $C(I_\Sigma)$, we can find an algebraically equivalent system Σ' which is a subset of $\Lambda(C(I_\Sigma))$.

Theorem 24 *Let Σ be a system of exterior equations and I_Σ its corresponding algebraic ideal. Then there exists an algebraically equivalent system Σ' such that $\Sigma' \subset \Lambda(C(I_\Sigma))$.*

Proof: Let $v_1, \ldots, v_n$ be a basis for V, and $\phi^1, \ldots, \phi^n$ be the dual basis, selected such that $v_{r+1}, \ldots, v_n$ span $A(I_\Sigma)$. Consequently $\phi^1, \ldots, \phi^r$ must span $C(I_\Sigma)$. Let α be any generator in $\Lambda^1(V^*)$. With respect to this basis, α can be written as

$$\alpha = \sum_{i=1}^{n} a_i \phi^i$$

Since $v \neg \alpha \equiv 0$ for all $v \in A(I_\Sigma)$, we must have $a_i = 0$ for $i = r+1, \ldots, n$. Therefore,

$$\alpha = \sum_{i=1}^{r} a_i \phi^i.$$

So all the 1-forms in Σ are contained in $\Lambda(C(I_\Sigma))$.

Now suppose that the result is true for all forms of degree $\leq k$. Let α be a $k+1$ form. Consider the form

$$\alpha' = \alpha - \phi^{r+1} \wedge (v_{r+1} \neg \alpha)$$

The second term on the right must be in $\Lambda(C(I_\Sigma))$ because of the induction hypothesis. Furthermore,

$$v_{r+1}\neg\alpha' = v_{r+1}\neg\alpha - v_{r+1}\neg\alpha - \phi^{r+1}\wedge(v_{r+1}\neg(v_{r+1}\neg\alpha)) \equiv 0$$

Therefore, α' has no terms involving ϕ^{r+1}.

If we now replace α with α' the ideal generated will be unchanged since

$$\theta\wedge\alpha = \theta\wedge\alpha' + \theta\wedge\phi^{r+1}\wedge(v_{r+1}\neg\alpha)$$

and $v_{r+1}\neg\alpha \in I$.

We can continue this process for $v_{r+2},\ldots,v_n$ to produce an $\hat{\alpha}$ which is generated by elements of $\Lambda(C(I_\Sigma))$. □

Example : Let $v_1,\ldots,v_6$ be a basis for $\mathbb{R}^6$, and let $\theta^1,\ldots,\theta^6$ be the dual basis. Consider the system of exterior equations

$$\alpha^1 = \theta^1\wedge\theta^3 = 0,$$
$$\alpha^2 = \theta^1\wedge\theta^4 = 0,$$
$$\alpha^3 = \theta^1\wedge\theta^2 - \theta^3\wedge\theta^4 = 0,$$
$$\alpha^4 = \theta^1\wedge\theta^2\wedge\theta^5 - \theta^3\wedge\theta^4\wedge\theta^6 = 0$$

The set of generators Σ is given by

$$\Sigma = \{\alpha^1,\alpha^2,\alpha^3,\alpha^4\},$$

and the ideal I_Σ is given by

$$I_\Sigma := \{\xi\in\Lambda(V^*)\mid \xi = \sum_{i=1}^{4}\pi^i\wedge\alpha^i,\ \pi^i\in\Lambda(V^*)\}.$$

The associated space of I_Σ is defined by

$$A(I_\Sigma) := \{v\in\mathbb{R}^6 \mid v\neg\pi\in I_\Sigma\ \forall\,\pi\in I_\Sigma\}.$$

Because I_Σ contains no 1-forms, we can infer that

$$v\neg\alpha^1 = 0,\ v\neg\alpha^2 = 0,\ \text{and}\ v\neg\alpha^3 = 0,\ \forall\, v\in A(I_\Sigma)$$

Expanding the first equation, we get

$$v\neg\alpha^1 = v\neg(\theta^1\wedge\theta^3) = (v\neg\theta^1)\wedge\theta^3 + (-1)^1\theta^1\wedge(v\neg\theta^3) = \theta^1(v)\theta^3 - \theta^3(v)\theta^1 = 0$$

which implies that $\theta^1(v) = 0$ and $\theta^3(v) = 0$. Similarly,

$$v\neg\alpha^2 = \theta^1(v)\theta^4 - \theta^4(v)\theta^1 = 0$$

$$v\neg\alpha^3 = \theta^1(v)\theta^2 - \theta^2(v)\theta^1 - \theta^3(v)\theta^4 + \theta^4(v)\theta^3 = 0$$

implying that $\theta^2(v) = 0$ and $\theta^4(v) = 0$. Therefore, we can conclude that

$$A(I_\Sigma) \subset span\{v_5, v_6\}.$$

Evaluating the equation $v \neg \alpha^4$ gives

$$\begin{aligned} v\neg\alpha^4 &= (v\neg(\theta^1 \wedge \theta^2)) \wedge \theta^5 + (-1)^2(\theta^1 \wedge \theta^2) \wedge (v\neg\theta^5) \\ &\quad -(v\neg(\theta^3 \wedge \theta^4)) \wedge \theta^6 - (-1)^2(\theta^3 \wedge \theta^4) \wedge (v\neg\theta^6) \\ &= \theta^5(v)\theta^1 \wedge \theta^2 - \theta^6(v)\theta^3 \wedge \theta^4 \\ &= a(\theta^1 \wedge \theta^3) + b(\theta^1 \wedge \theta^4) + c(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) \end{aligned}$$

Equating coefficients, we find that

$$\theta^5(v) = \theta^6(v) = c, \ \forall\, v \in A(I_\Sigma).$$

Now v must be of the form $v = xv_5 + yv_6$, so we get

$$\theta^5(xv_5 + yv_6) = x = c$$

$$\theta^6(xv_5 + yv_6) = y = c.$$

Therefore, $A(I_\Sigma) = span\{(v_5 + v_6)\}$. If we select as a new basis for $\mathbf{R}^6$ the vectors

$$w_i = v_i, \ i = 1, \ldots, 4, \ w_5 = v_5 - v_6, \ w_6 = v_5 + v_6$$

then the new dual basis becomes

$$\gamma^i = \theta^i, \ i = 1, \ldots, 4, \ \gamma^5 = \frac{\theta^5 - \theta^6}{2}, \ \gamma^6 = \frac{\theta^5 + \theta^6}{2}.$$

With respect to this new basis, the retracting space $C(I_\Sigma)$ is given by

$$C(I_\Sigma) = span\{\gamma^1, \ldots, \gamma^5\}$$

In these coordinates, the generator set becomes

$$\Sigma' = \{\gamma^1 \wedge \gamma^3, \ \gamma^1 \wedge \gamma^4, \ \gamma^1 \wedge \gamma^2 - \gamma^3 \wedge \gamma^4, \ \gamma^1 \wedge \gamma^2 \wedge \gamma^5\} \subset \Lambda(C(I_\Sigma))$$

◇

We conclude this section on exterior algebra with a theorem which will allows us to find the dimension of the retracting space in the special case where the generators of the ideal are a collection of 1-tensors together a single alternating 2-tensor.

Theorem 25 *Let I_Σ be an ideal generated by the set*

$$\Sigma = \{\omega^1, \ldots, \omega^s, \Omega\}$$

where $\omega^i \in V^*$ and $\Omega \in \Lambda^2(V^*)$. Let r be the smallest integer such that

$$(\Omega)^{r+1} \wedge \omega^1 \wedge \ldots \wedge \omega^s = 0$$

Then the retracting space $C(I_\Sigma)$ is of dimension $2r + s$.

Proof: See [5] pages 11-12. □

3 Differential Geometry and Forms

The multilinear algebra presented in the previous section can be applied to the tangent space of a differentiable manifold. We first review some basic facts from differential geometry. The reader may wish to consult numerous books on the subject such as [6, 7, 1].

3.1 Differentiable Manifolds

A manifold is simply a space which locally looks like $\mathbf{R}^n$.

Definition 23 *A manifold M of dimension n is a metric space* [3] *which is locally homeomorphic to* $\mathbf{R}^n$.

A simple example of a manifold, which is of great interest to us, is $\mathbf{R}^n$ itself. Other examples are the circle S^1 and the sphere S^2. The circle S^1 is locally homeomorphic to $\mathbf{R}$ while the sphere is locally homeomorphic to $\mathbf{R}^2$. Therefore the circle is a one dimensional manifold while the sphere is a two dimensional manifold.

A subset N of manifold M which is itself a manifold is called a submanifold of M. Any open subset N of a manifold M is clearly a submanifold since if M is locally homeomorphic to $\mathbf{R}^n$ then so is N.

We now wish to perform calculus on manifolds. Since we know how to differentiate and integrate on $\mathbf{R}^n$, and since manifolds look locally like $\mathbf{R}^n$, the way to differentiate and integrate on manifolds is to first locally flatten the manifold and then perform the desired operation on the flattened space which looks like $\mathbf{R}^n$. In order to do the above procedure our manifold must have, what is called, a differentiable structure.

A coordinate chart on a manifold M is a pair (U, x) where U is an open set of M and x is a homeomorphism of U on an open set of $\mathbf{R}^n$. The function x is also called a coordinate function and can also writen as $(x^1, \ldots, x^n)$ where $x^i : M \longrightarrow \mathbf{R}$. If $p \in U$ then $x(p) = (x^1(p), \ldots, x^n(p))$ is called the set of local coordinates in the chart (U, x).

When doing operations on a manifold, we must ensure that our results are consistent regardless of the particular chart we use. We must therefore impose some conditions. Two charts (U, x) and (V, y) with $U \cap V \neq \emptyset$, are called C^∞ compatible if the map

$$y \circ x^{-1} : x(U \cap V) \subset \mathbf{R}^n \longrightarrow y(U \cap V) \subset \mathbf{R}^n$$

is a C^∞ function. A C^∞ atlas on a manifold M is a collection of charts (U_α, x_α) with $\alpha \in A$ which are C^∞ compatible and such that the open sets U_α cover the manifold M, so $M = \bigcup_{\alpha \in A} U_\alpha$. An atlas is called maximal if it is not contained in any other atlas.

Definition 24 *A differentiable or smooth manifold is a manifold with a maximal, C^∞ atlas.*

---

[3] for readers with knowledge of topology replace metric space with Hausdorff, second countable topological space

Now that we have imposed this differential structure on our manifold M we can perform calculus on M. In particular let $f : M \longrightarrow \mathbf{R}$ be a map. If (U,x) is a chart on M then the function

$$\hat{f} = f \circ x^{-1} : x(U) \subset \mathbf{R}^n \longrightarrow \mathbf{R}$$

is called the local representative of f in the chart (U,x). We therefore define the map f to be C^∞ or smooth if its local representative $\hat{f}$ is C^∞ Notice if f is C^∞ in one chart, then it must be C^∞ in every chart since we required our charts to be C^∞ compatible and our atlas to be maximal. Therefore our results intrinsic to the manifold and do not depend on the particular homeomorphism we use. Similarly, if we have a map $f : M \longrightarrow N$, where M,N are differentiable manifolds, the local representation of f given a chart (U,x) of M and (V,y) of N is

$$\hat{f} = y \circ f \circ x^{-1}$$

which makes sense only if $f(U) \cap V \neq \emptyset$. Again f is a C^∞ map if $\hat{f}$ is a C^∞ map.

Let $f : M \longrightarrow N$ be a map between two manifolds. The map f is called a diffeomorphism if both f and f^{-1} are smooth. In this case, manifolds M and N are called diffeomorphic.

Example: We have seen that $\mathbf{R}^n$ is an example of a trivial but important manifold. The differentiable structure on $\mathbf{R}^n$ consists of the chart $(\mathbf{R}^n, i)$ where i is the identity function on $\mathbf{R}^n$ as well as all other charts that are C^∞ compatible with it.

The sphere, S^2 can be given a differentiable structure as follows. Consider the charts (U_N, p_N) and (U_S, p_S) where U_N is the sphere minus the North pole, U_S is the sphere minus the South pole and p_N, p_S are the stereographic projections of the sphere to the plane from the North and South poles respectively. One can show that these charts are compatible. We can then extend our atlas to a maximal one by consider all other charts that are compatible with $(U_N, p_N), (U_S, p_S)$. $\diamond$

3.2 Tangent Spaces

Let p be a point on a manifold M. Let $C^\infty(p)$ denote the set of all smooth functions in a neighborhood of p. The set $C^\infty(p)$ is a vector space over $\mathbf{R}$ since the sum of two smooth functions and the scalar multiple of a smooth function are smooth function themselves.

Definition 25 *A tangent vector X_p at $p \in M$ is an operator from $C^\infty(p)$ to $\mathbf{R}$ which satisfies for $f, g \in C^\infty(p)$ and $a, b \in \mathbf{R}$, the following properties,*

1. *Linearity* $X_p(a \cdot f + b \cdot g) = a \cdot X_p(f) + b \cdot X_p(g)$
2. *Derivation* $X_p(f \cdot g) = f(p) \cdot X_p(g) + X_p(f) \cdot g(p)$

The set of all tangent vectors at $p \in M$ is called the tangent space of M at p and is denoted by T_pM.

The tangent space T_pM becomes a vector space over $\mathbf{R}$ if for tangent vectors X_p, Y_p and real numbers c_1, c_2 we define

$$(c_1 \cdot X_p + c_2 \cdot Y_p)(f) = c_1 \cdot X_p(f) + c_2 \cdot Y_p(f)$$

for any smooth function f in the neighborhood of p. The collection of all tangent spaces of the manifold,

$$TM = \bigcup_{p \in M} T_pM$$

is called the tangent bundle.

Example: Given the standard differentiable structure on $\mathbf{R}^n$, the standard tangent vectors of $\mathbf{R}^n$ at any point p are

$$\frac{\partial}{\partial r^1} \cdots \frac{\partial}{\partial r^n}$$

Thus given any smooth function $f(r^1, \ldots, r^n) : U \longrightarrow \mathbf{R}$ where U is a neighborhood of p, we have

$$\frac{\partial}{\partial r^i}(f) = \frac{\partial f}{\partial r^i}$$

for $i = 1, \ldots, n$. $\diamond$

Now let M be a manifold and let (U, x) be a chart containing the point p. In this chart we can associate the following tangent vectors

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$$

defined by

$$\frac{\partial}{\partial x^i}(f) = \frac{\partial (f \circ x^{-1})}{\partial r^i}$$

for any smooth function $f \in C^\infty(p)$.

Theorem 26 *Let M be an n dimensional manifold and let T_pM be the tangent space at $p \in M$. Then T_pM is an n-dimensional vector space and if (U, x) is a local chart around p then the tangent vectors*

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$$

form a basis for T_pM.

Proof: See Spivak [6] pages. $\square$

From the above theorem we can see that if X_p is a tangent vector at p then

$$X_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}$$

where $a_1, \ldots, a_n$ are real numbers. From the above formula we can see that a tangent vector is an operator which simply takes the directional derivative of function in the direction of $[a_1, \ldots, a_n]$.

Now let M and N be smooth manifolds and $f : M \longrightarrow N$ be a smooth map. Let $p \in M$ and let $q = f(p) \in N$. We wish to transport tangent vectors from T_pM to T_qN using the map f. The natural way to do this is by defining a map $f_* : T_pM \longrightarrow T_qN$ by

$$(f_*(X_p))(g) = X_p(g \circ f)$$

for smooth functions g in the neighborhood of q. One can easily check that $f_*(X_p)$ is a linear operator and a derivation and thus a tangent vector. The map $f_* : T_pM \longrightarrow T_{f(p)}N$ is called the push forward map of f.

Proposition 1 *The push forward map $f_* : T_pM \longrightarrow T_{f(p)}N$ is a linear map.*

Proof: Let X_p and Y_p be two tangent vectors in T_pM. Then

$$\begin{aligned}(f_*(X_p + Y_p))(g) &= (X_p + Y_p)(g \circ f) \\ &= X_p(g \circ f) + Y_p(g \circ f) \\ &= (f_*(X_p))(g) + (f_*(Y_p))(g)\end{aligned}$$

and also for real number c,

$$\begin{aligned}(f_*(c \cdot X_p))(g) &= (c \cdot X_p)(g \circ f) \\ &= c \cdot X_p(g \circ f) \\ &= c \cdot (f_*(X_p))(g)\end{aligned}$$

which completes the proof. □

Proposition 2 *Let $f : M \longrightarrow N$ and $g : N \longrightarrow K$. Then*

$$(g \circ f)_* = g_* \circ f_*$$

Proof: See Spivak [6] page 101. □

We now arrive at the important concept of a vector field on a manifold.

Definition 26 *Let M be a manifold. A vector field on M is a continuous function F which places at each point p of M a tangent vector from T_pM. Such functions are called sections of the tangent bundle TM. If F is of class C^∞ it is called a smooth section of TM.*

Recall that in a coordinate chart (U, x) of a point $p \in M$ a tangent vector is expressed as

$$X_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}$$

Therefore since a vector field, F, places at each point p a tangent vector $F(p)$ we have that in the chart (U, x) the local expression for the vector field F is

$$F(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x^i}$$

We can easily see from the above that the vector field is C^∞ if and only if the scalar functions $a_i : M \longrightarrow \mathbf{R}$ are C^∞.

An integral curve of a vector field is a mapping $c : (-\varepsilon, \varepsilon) \longrightarrow M$ whose tangent at each point is identically equal to the vector field at that point.

3.3 Tensor Fields

Since the tangent space of manifold at a point is a vector space we can apply all the multilinear algebra that we presented in the previous section. The dual space of T_pM at each $p \in M$ is called the cotangent space of the manifold M at p and is denoted by T_p^*M. The collection of all cotangent spaces,

$$T^*M := \bigcup_{p \in M} T_p^*M$$

is called the cotangent bundle. Similarly, we can form the spaces $\mathcal{L}^k(T_pM)$ and $\Lambda^k(T_p^*M)$ over each $p \in M$ as well as the bundles

$$\mathcal{L}^k(M) := \bigcup_{p \in M} \mathcal{L}^k(T_pM)$$

$$\Lambda^k(M) := \bigcup_{p \in M} \Lambda^k(T_p^*M)$$

One can construct tensor fields on a manifold M by assigning to each point p of the manifold a tensor. A k-tensor field on M is a function ω assigning to every $p \in M$ a k-tensor,

$$\omega(p) \in \mathcal{L}^k(T_pM)$$

Therefore a k-tensor field is a section of $\mathcal{L}^k(M)$. At each point $p \in M$, $\omega(p)$ is a function mapping k-tuples of tangent vectors of T_pM to $\mathbf{R}$. That is

$$\omega(p)(X_1, X_2, \ldots, X_k) \in \mathbf{R}$$

is a multi-linear function of tangent vectors $X_1, \ldots, X_k \in T_pM$. In particular, if ω is a section of $\Lambda^k(M)$ then ω is called a differential form of order k or k-form on M. In this case, $\omega(p)$ is an alternating k-tensor at each point $p \in M$. The space of all k-forms on a manifold M will be denoted by $\Omega^k(M)$ and the space of all forms on M is simply

$$\Omega(M) := \Omega^0(M) \oplus \ldots \oplus \Omega^n(M)$$

At each point $p \in M$, let

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$$

be the basis for T_pM. Let the 1-forms ϕ^i be the dual basis of the basis tangent vectors of T_pM. Therefore,

$$\phi^i(p)(\frac{\partial}{\partial x^j}) = \delta_{ij}$$

Recall that the forms

$$\phi^I = \phi^{i_1} \otimes \phi^{i_2} \otimes \ldots \otimes \phi^{i_k}$$

for multi-index $I = (i_1, \ldots, i_k)$ form a basis for $\mathcal{L}^k(T_pM)$. Similarly, given an ascending multi-index $I = (i_1, \ldots, i_k)$, the k-forms

$$\psi^I = \phi^{i_1} \wedge \phi^{i_2} \wedge \ldots \wedge \phi^{i_k}$$

form a basis for $\Lambda^k(T_pM)$.

If ω is a k-tensor on M, it can be uniquely written as

$$\omega(p) = \sum_I b_I(p)\phi^I(p)$$

for multi-index I and scalar functions $b_I(p)$ whereas the k-form α can be written uniquely as

$$\alpha(p) = \sum_I c_I(p)\psi^I(p)$$

for ascending multi-index I and scalar functions c_I. The k-tensor ω and k-form α are of class C^∞ iff the functions b_I and c_I are of class C^∞ respectively. Given two forms $\omega \in \Omega^k(M), \theta \in \Omega^l(M)$, we have,

$$\omega = \sum_I b_I\psi^I$$

$$\theta = \sum_I c_I\psi^I$$

$$\omega \wedge \theta = \sum_I \sum_J b_I c_I \psi^I \wedge \psi^J$$

Recall that we have defined $\Lambda^0(T_pM) = \mathbb{R}$. As a result, the space of differential forms of order 0 on M is simply the space of all functions $f : M \longrightarrow \mathbb{R}$ and the wedge product of $f \in \Omega^0(M)$ and $\omega \in \Omega^k(M)$, is defined as

$$(w \wedge f)(p) = (f \wedge w)(p) = f(p) \cdot w(p)$$

3.4 The Exterior Derivative

Recall that a 0-form on a manifold M is a function $f : M \rightarrow \mathbb{R}$. The differential df of a 0-form f is defined pointwise as the 1-form,

$$df(p)(X_p) = X_p(f)$$

and is therefore the directional derivative of f in the direction of X_p at p. The operator d is linear on 0-forms, that is

$$d(af + bg) = a \cdot df + b \cdot dg$$

and this follows from the fact that X_p is a linear operator.

Using this operator d, we obtain a new way of expressing the elementary 1-forms $\phi^i(p)$ on T_pM. Let $x : M \longrightarrow \mathbb{R}^n$ be the coordinate function in a neighborhood of p. Then consider the differentials of the coordinate functions

$$dx^i(p)(X_p) = X_p(x^i)$$

If we evaluate the differentials dx^i at the basis tangent vectors of T_pM we obtain,

$$dx^i(p)(\frac{\partial}{\partial x^j}) = \delta_{ij}$$

and therefore the $dx^i(p)$ are the dual basis of T_pM and therefore $dx^i(p) = \phi^i(p)$ since the $\phi^i(p)$ are also the dual basis. Thus the differentials $dx^i(p)$ span $\mathcal{L}^1(T_pM)$ and therefore using our previous results any k-tensor ω can be uniquely writen as

$$\omega(p) = \sum_I b_I(p) dx^I(p)$$

for multi-index I while any k-form can be uniquely writen as

$$\omega(p) = \sum_I b_I(p) dx^I(p)$$

for ascending multi-index I. Therefore any k-tensor ω can be expressed in the chart (U, x) containing p as

$$\omega(p) = \sum_{i=1}^{n} b_I(p) dx^1 \otimes \ldots \otimes dx^n$$

while the k-form α is expressed as

$$\alpha(p) = \sum_{i=1}^{n} c_I(p) dx^1 \wedge \ldots \wedge dx^n$$

Using this basis now we have that for a 0-form,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

We have therefore defined an operator d which takes 0-forms to 1-forms. We now proceed to inductively define an operator $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. We will do so by proving that there exists a unique operator d which is compatible with the operator which takes 0-forms to 1-forms while also satisfying some additional properties.

Definition 27 *Let ω be a k-form on a manifold M. Its representation in a chart (U, x) is*

$$\omega = \sum_I \omega_I dx^I$$

for ascending multi-index I. The exterior derivative or differential operator, d, is a linear map taking the k-form ω to the (k+1)-form $d\omega$ by

$$d\omega = \sum_I d\omega_I \wedge dx^I$$

Notice that the ω_I are smooth functions, and thus 0-forms whose differential $d\omega_I$ has already been defined as

$$d\omega_I = \sum_{j=1}^{n} \frac{\partial \omega_I}{\partial x^j} dx^j$$

and therefore we get that for any k-form

$$d\omega = \sum_I \sum_{j=1}^{n} \frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I$$

One can see from the definition that this operator is certainly linear. We now show that this differential operator is a true generalization of the operator taking 0-forms to 1-forms, satisfies some important properties and is the unique operator with those properties.

Theorem 27 *Let M be a manifold and let $p \in M$. Then the exterior derivative is the unique linear operator*

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

for $k \geq 0$, that satisfies,

1. *If f is a 0-form, then df is the 1-form*

$$df(p)(X_p) = X_p(f)$$

2. *If $\omega^1 \in \Omega^k(M), \omega^2 \in \Omega^l(M)$ then*

$$d(\omega^1 \wedge \omega^2) = d\omega^1 \wedge \omega^2 + (-1)^k \omega^1 \wedge d\omega^2$$

3. *For every form ω, $d(d\omega) = 0$*

Proof: Property (1) can be easily checked from the definition of the exterior derivative. We now prove property (2). Because of linearity of the exterior derivative it suffices to consider the case $\omega^1 = f dx^I$ and $\omega^2 = g dx^J$ in some chart (U, x). We have

$$\begin{aligned}
d(\omega^1 \wedge \omega^2) &= d(fg) \wedge dx^I \wedge dx^J \\
&= g df \wedge dx^I \wedge dx^J + f dg \wedge dx^I \wedge dx^J \\
&= d\omega^1 \wedge \omega^2 + (-1)^k f dx^I \wedge dg \wedge dx^J \\
&= d\omega^1 \wedge \omega^2 + (-1)^k \omega^1 \wedge d\omega^2
\end{aligned}$$

We now prove property (3). Again it suffices to consider the case $\omega = f dx^I$ because of linearity. Since f is a 0-form,

$$d(df) = d(\sum_{j=1}^{n} (D_j f) dx^j) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_i(D_j f) dx^i \wedge dx^j$$

$$= (D_i(D_j f) - D_j(D_i f)) dx^i \wedge dx^j = 0$$

where $D_i f$ is the standard derivative $\frac{\partial f}{\partial x^i}$. If $\omega = f dx^I$ is a k-form, then $d\omega = df \wedge dx^I$, and since

$$d(dx^I) = d(1 \wedge dx^I) = d(1) \wedge dx^I = 0$$

we get

$$d(d\omega) = d(df) \wedge dx^2 - df \wedge d(dx^I) = 0$$

We now show that d is the unique linear operator with the above properties. To show this let's assume that d' is another linear operator with the same properties. We will show that $d = d'$.

Consider again a k-form $\omega = f dx^I$. Since d' satisfies property (2) we have

$$d'(f dx^I) = d' f \wedge dx^I + f \wedge d'(dx^I)$$

From the above formula we see that if we can show that $d'(dx^I) = 0$ then we will get

$$d'(f dx^I) = d'f \wedge dx^I = d(f dx^I)$$

which will complete the proof. We therefore want to show that

$$d'(dx^1 \wedge \ldots \wedge dx^k) = 0 \tag{11}$$

But since both d and d' satisfy property (1) we have

$$dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k} = d'x^{i_1} \wedge \ldots \wedge d'x^{i_k} = d'x^I$$

since the coordinate functions x^i are 0-forms.

We prove equation (11) by induction. It can be easily checked to hold for $k = 0$. Assume that equation (11) holds for $k-1$. Then define

$$\eta = dx^2 \wedge \ldots \wedge dx^k$$

Then

$$d'(dx^I) = d'(d'x^{i_1} \wedge d'x^{i_k}) = d'(d'x^{i_1}) \wedge \eta - d'\omega_{i_1} \wedge d'\eta = 0$$

since d' also satisfies property (3) and by the induction hypothesis. □

Now let $f : M \longrightarrow N$ be a smooth map between two manifolds. We have seen that the push forward map, f_*, is a linear transformation from T_pM to $T_{f(p)}N$. Therefore given tensors or forms on $T_{f(p)}N$ we can use the pull back transformation, f^*, in order to define tensors or forms on T_pM. [4] The next theorem shows that the exterior derivative and the pull back transformation commute.

Theorem 28 *Let $f : M \longrightarrow N$ be a smooth map between manifolds. If ω is a k-form on N then*

$$f^*(d\omega) = d(f^*\omega)$$

Proof: See Spivak [6] pages 295-6. □

3.5 The Exterior Derivative and the Grad, Div, Curl Operators

From a historical standpoint, the language of forms arose from the study of integration on manifolds. In addition to generalizing certain notions of vector calculus in $\mathbb{R}^3$ to higher dimensions and arbitrary manifolds, forms provided an elegant reformulation of many of the original theorems regarding vector and scalar fields in $\mathbb{R}^3$. As a result, considerable physical insight can be gained in this context by studying the relationships between vector fields and scalar fields in $\mathbb{R}^3$ and differential forms. In the context of exterior differential systems, however, forms have an inherent physical interpretation as constraints and not as the analog of some vector or scalar field. The parallels drawn in this section should therefore be viewed as an alternate application of the mathematics of differential forms.

[4] to be consistent with our previous notation one must write $(f_*)^*$ to denote the pull back of f_*. However notation is abused and we simply denote it by f^*.

Given a scalar field, f on $\mathbf{R}^3$, there is an obvious trivial transformation to a 0-form $f \in \Omega^0(\mathbf{R}^3)$. Similarly, any vector field in $\mathbf{R}^3$ can be expressed as

$$G(x) = (x; g_1(x)e_1 + g_2(x)e_2 + g_3(x)e_3)$$

and a corresponding 1-form

$$\omega = g_1(x)dx^1 + g_2(x)dx^2 + g_3(x)dx^3$$

may be constructed in a very straightforward manner. Perhaps less obvious is that scalar fields may also be identified with 3-forms and vector fields with 2-forms.

Theorem 29 *The following transformations from vector and scalar fields to forms*

$$\begin{aligned} T_0 &: \textit{Scalar fields} \longrightarrow \Omega^0(\mathbf{R}^3) \\ T_1 &: \textit{Vector fields} \longrightarrow \Omega^1(\mathbf{R}^3) \\ T_2 &: \textit{Vector fields} \longrightarrow \Omega^2(\mathbf{R}^3) \\ T_3 &: \textit{Scalar fields} \longrightarrow \Omega^3(\mathbf{R}^3) \end{aligned}$$

given by:

$$\begin{aligned} T_0 &: f \longrightarrow f \\ T_1 &: (x; \sum g_i e_i) \longrightarrow \sum g_i dx^i \\ T_2 &: (x; \sum g_i e_i) \longrightarrow g_1 dx^2 \wedge dx^3 + g_2 dx^3 \wedge dx^1 + g_3 dx^1 \wedge dx^2 \\ T_3 &: f \longrightarrow f dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

are vector space isomorphisms.

Proof: See Munkres [1]. □

This may be generalized to $\mathbf{R}^n$, with vector fields isomorphic to 1-forms and (n-1) forms and scalar fields isomorphic to 0-forms and n-forms.

Drawing from the familiar results of vector calculus, the three primary operations on vector fields and scalar fields in $\mathbf{R}^3$, the gradient, divergence and the curl, may be defined.

Definition 28 *Let $f(x)$ be a scalar field on $\mathbf{R}^3$ and $G(x) = (x; \sum g_i(x)e_i)$ a vector field. The gradient of f, ∇f or $gradf$, is the vector field given by:*

$$\nabla f(x) = (x; \sum D_i f(x) e_i)$$

The divergence of G, $\nabla \cdot G$ or $divG$ is the scalar field given by:

$$\nabla \cdot G(x) = \sum D_i g_i(x)$$

The curl of G, $\nabla \times G$ or $curlG$, is the vector field given by:

$$\nabla \times G(x) = (x; (D_2 g_3 - D_3 g_2)e_1 + (D_3 g_1 - D_1 g_3)e_2 + (D_1 g_2 - D_2 g_1)e_3)$$

The gradient and divergence may be formed analogously in $\mathbf{R}^n$ by extending the summation over all n indices; the curl has no real equivalent outside of $\mathbf{R}^3$.

With the above transformations, these three operations be translated into the language of forms.

Theorem 30 *The transformations form the following diagram:*

$$\begin{array}{rcl} ScalarFields & \xrightarrow{T_0} & \Omega^0(\mathbf{R}^n) \\ \nabla \downarrow & & \downarrow d \\ VectorFields & \xrightarrow{T_1} & \Omega^1(\mathbf{R}^n) \\ (\nabla \times) \downarrow & & \downarrow d \\ VectorFields & \xrightarrow{T_2} & \Omega^2(\mathbf{R}^n) \\ (\nabla \cdot) \downarrow & & \downarrow d \\ ScalarFields & \xrightarrow{T_3} & \Omega^3(\mathbf{R}^n) \end{array}$$

Equivalently,

$$\begin{aligned} df &= T_1(\nabla f) \\ d(T_1 G) &= T_2(\nabla \times G) \\ d(T_2 G) &= T_3(\nabla \cdot G) \end{aligned}$$

Proof: Follows immediately from the previous theorem and definitions. □

This is a rather remarkable result. The d-operator, combined with the appropriate transformation to differential forms, supplants all three operators of vector calculus. Arguably, such an approach is far more elegant both notationally and conceptually. Nowhere is this more evident then in the derivation of the generalized Stokes' theorem, which, combined with the discussion above, may be viewed as a single compact representation of Green's theorem, Stokes' theorem and the divergence theorem. This theorem, however, is a substantial topic in its own right and the reader is referred to treatments by Munkres [1] or Spivak [2].

3.6 Closed and Exact Forms

The d-operator may be used to define two classes of forms of particular interest.

Definition 29 *A k-form $\omega \in \Omega^k(M)$ is said to be closed if $d\omega \equiv 0$.*

Definition 30 *A k-form $\omega \in \Omega^k(M)$ with $k > 0$ is exact if there exists a (k-1)-form θ such that $\omega = d\theta$. A 0-form is exact on any open set if it is constant on that set.*

Clearly, exactness is the stronger condition, since for any form θ, $d(d\theta) = 0$. Any exact form must therefore be closed. The converse does not necessarily hold, though further conditions may be found to ensure this.

The concept of exactness for forms is equivalent to the more familiar concept of exactness in differential equations. From a physical standpoint, exactness may be used to translate certain statements of mechanics into the language of forms. Given a constraint in Pfaffian form,

$$a_1 dx^1 + a_2 dx^2 + \ldots + a_n dx^n = 0$$

the question of whether or not this constraint is holonomic may be equivalently posed as whether or not the 1-form

$$a_1 dx^1 + a_2 dx^2 + \ldots + a_n dx^n$$

is exact. Similarly, properties of vector fields may be expressed in this notation under the vector space isomorphisms of Section 3.5. For instance, determining if the vector field

$$F(x) = (x; f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3)$$

is derivable from a scalar potential (i.e. $F = \nabla f$) is equivalent to determining if the 1-form

$$\omega = f_1(x)dx_1 + f_2(x)dx_2 + f_3(x)dx_3$$

is exact.

3.7 The Interior Product

We can define the interior product of a tensor field and a vector field pointwise as the interior product of a tensor and a tangent vector.

Definition 31 *Given a k-form $\omega \in \Omega^k(M)$ and a vector field X the interior product or anti-derivation of ω with X is a $(k-1)$ form defined pointwise by,*

$$(X(p) \neg \omega(p))(v_1, \ldots, v_{k-1}) = \omega(p)(X(p), v_1, \ldots, v_{k-1})$$

Therefore an antiderivation of a k-form ω simply substitutes the first argument with the given vector and thus results in a (k-1)-form.

Definition 32 *Given a function $h : M \longrightarrow \mathbf{R}$, the Lie derivative of h along the vector field X is denoted as $L_X h$ and is defined by*

$$L_X h = X(h) = X \neg dh$$

The Lie derivative is simply the directional derivative of h along X.

Definition 33 *Given two vector fields X and Y, their Lie bracket is defined to be the vector field such that for each $h \in C^\infty(p)$ we have*

$$[X, Y](h) = X(Y(h)) - Y(X(h)) = X \neg d(Y \neg dh) - Y \neg d(X \neg dh)$$

In particular, if we choose the coordinate functions x^i, we get

$$[X, Y](x^i) = [X, Y]_i = \sum_j \frac{\partial Y_i}{\partial x^j} X_j - \sum_j \frac{\partial X_i}{\partial x^j} Y_j$$

and we therefore obtain

$$[X,Y](x) = \frac{\partial Y}{\partial x}X(x) - \frac{\partial X}{\partial x}Y(x)$$

The Lie bracket is skew symmetric

$$[X,Y] = -[Y,X]$$

and also satisfies the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

The following lemma establishes a relation between the exterior derivative and Lie brackets.

Lemma 2 *Cartan's Magic Formula). Let $\omega \in \Omega(M)$ and X, Y smooth vector fields. Then*

$$\begin{aligned} d\omega(X,Y) &= X(\omega(Y) - Y(\omega(x)) - \omega([X,Y]) \\ &= X \neg d(Y \neg \omega) - Y \neg d(X \neg \omega) - [X,Y] \neg \omega \end{aligned}$$

Proof: Because of linearity, it is adequate to consider $\omega = fdg$. The left hand side of the above formula is

$$\begin{aligned} d\omega(X,Y) &= df \wedge dg(X,Y) \\ &= df(X) \cdot dg(Y) - df(Y) \cdot dg(X) \\ &= X(f) \cdot Y(g) - Y(f) \cdot X(g) \end{aligned}$$

while the right hand side is

$$\begin{aligned} X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) &= X(fY(g)) - Y(fX(g)) - f(XY(g) - YX(g)) \\ &= X(f) \cdot Y(g) - Y(f) \cdot X(g) \end{aligned}$$

which completes the proof. □

3.8 Distributions and Codistributions

Recall that a vector field is a map that assigns a tangent vector to each point on the manifold. In the case of multiple vector fields, one may assign a number of tangent vectors at a point and look at the subspace of the tangent space spanned by these vectors. This assignment, which places at each point of the manifold a subspace of the tangent space at that point, is called a distribution and is denoted by

$$\Delta(p) = span\{f_1(p), \ldots, f_d(p)\}$$

or if we drop the dependence on the point p,

$$\Delta = span\{f_1, \ldots, f_d\}$$

Therefore, at a point p of manifold M, $\Delta(p)$ is the subspace of T_pM which is spanned by the tangent vectors assigned at p by the vector fields $f_1, \ldots, f_d$. If the vector fields are smooth, we call $\Delta(p)$ a smooth distribution. The dimension of the distribution at a point is defined to be the dimension of the subspace $\Delta(p)$. A vector field f belongs to a distribution Δ if $f(p) \in \Delta(p)$ for all p.

Since distributions are subspaces one can easily define the sum or intersection of two distributions as the sum or intersection of the respective subspaces.

A distibution is called involutive if given any two vector fields f_1 and f_2 belonging to the distribution, their Lie bracket also belongs to the distribution, i.e.

$$f_1, f_2 \in \Delta \Longrightarrow [f_1, f_2] \in \Delta$$

A distribution Δ is called integrable if there exists a submanifold N of M such that the tangent space of N at x equals $\Delta(x)$. The submanifold N is called the integral manifold of the distribution Δ. The following theorem, provides us with a condition under which a distribution is integrable.

Theorem 31 *(Frobenius Theorem for distributions) A distribution $\Delta(x)$ is integrable if and only if it is involutive.*

Proof: See Spivak [2] pages □.

In a construction similar to the one described above, one may assign to each point of the manifold a number of 1-forms. The span of these 1-forms will be, at each point, a subspace of the cotangent space T_p^*M. This assignment is called a codistribution and is denoted by

$$\Theta(p) = span\{\omega_1(p), \ldots, \omega_d(p)\}$$

or if we drop the dependence on the point p,

$$\Theta = span\{\omega_1, \ldots, \omega_d\}$$

where $\omega_1, \ldots, \omega_d$ are the 1-forms which generate this codistribution.

There is a notion of duality between distributions and codistributions which allows us to construct codistributions from distributions and vice verca.

Given a distribution Δ, for each p in a neighborhood U, consider all the 1-forms which pointwise annihilate all vectors in $\Delta(p)$,

$$\Delta^\perp(p) = span\{\omega(p) \in T_p^*M : \omega(p)(f) = 0 \; \forall f \in \Delta(p)\}$$

Clearly, $\Delta^\perp(p)$ is a subspace of T_p^*M and is therefore a codistribution. We call $\Delta^\perp$ the annihilator or dual of Δ. Conversely, given a codistribution Θ, we construct the annihilating or dual distribution pointwise as

$$\Theta^\perp(p) = span\{v \in T_pM : \omega(p)(v) = 0 \; \forall \omega(p) \in \Omega(p)\}$$

If N is an integral manifold of a distribution Δ and v is a tangent vector in the distribution Δ at a point p (and consequently in T_pN) then

$$\alpha(p)(v) = 0$$

for any $\alpha \in \Delta^{\perp}$. Notice that this must be true for integral curves as well. Therefore given a codistribution

$$\Theta = span\{\omega_1, \ldots, \omega_s\}$$

an integral curve is a curve $c(t)$ whose tangent $c'(t)$ at each point satisfies,

$$\omega_1(c(t))(c'(t)) = 0, \ldots, \omega_s(c(t))(c'(t)) = 0$$

Example Consider the following model of a unicycle

$$\begin{aligned} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \end{aligned}$$

which can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 = f_1 u_1 + f_2 u_2$$

The corresponding distribution is

$$\Delta(x) = span\{ \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$$

while the dual codistribution is

$$\Delta^{\perp} = span\{\omega\}$$

where $\omega = \sin\theta dx - \cos\theta dy + 0 d\theta$, the nonholonomic constraint of rolling without slipping. It can be easily seen that the 1-form ω annihilates both vector fields f_1 and f_2. The above example shows that vector fields and distributions describe the allowable motions of the system, forms and codistributions describe the constraints of the system. ◇

4 Exterior Differential Systems

4.1 The Exterior Algebra On a Manifold

The space of all forms on a manifold M,

$$\Omega(M) = \Omega^0(M) \oplus \ldots \oplus \Omega^n(M)$$

together with the wedge product is called the exterior algebra on M. An ideal of this algebra is defined as in Section 2.6 as a subspace I such that if $\alpha \in I$ then $\alpha \wedge \beta \in I$ for any $\beta \in \Omega(M)$.

Since we are dealing with $\Omega(M)$ we are also interested in what happens when we perform exterior differentiation on elements of the ideal.

Definition 34 *An ideal $I \subset \Omega(M)$ is said to be closed with respect to exterior differentiation if and only if*

$$\alpha \in I \Longrightarrow d\alpha \in I$$

or more compactly $dI \subset I$. A algebraic ideal which is closed with respect to exterior differentiation is called a differential ideal.

A finite collection of forms, $\Sigma := \{\alpha^1, \ldots, \alpha^K\}$ generates an algebraic ideal

$$I_\Sigma := \{\omega \in \Omega(M) \mid \omega = \sum_{i=1}^{K} \theta^i \wedge \alpha^i \text{ for some } \theta^i \in \Omega(M)\}.$$

We can also talk about the differential ideal generated by Σ.

Definition 35 *Let S_d denote the collection of all differential ideals containing Σ. The differential ideal generated by Σ is defined as*

$$\mathcal{I}_\Sigma := \bigcap_{I \in S_d} I$$

Theorem 32 *Let Σ be a finite collection of forms, and let $\mathcal{I}_\Sigma$ denote the differential ideal generated by Σ. Define the collection*

$$\Sigma' = \Sigma \cup d\Sigma$$

and denote the algebraic ideal which it generates by $I_{\Sigma'}$. Then

$$\mathcal{I}_\Sigma = I_{\Sigma'}$$

Proof: By definition, $\mathcal{I}_\Sigma$ is closed with respect to exterior differentiation, so $\Sigma' \subset \mathcal{I}_\Sigma$. Consequently, $I_{\Sigma'} \subset \mathcal{I}_\Sigma$.

The ideal $I_{\Sigma'}$ is a closed with respect to exterior differentiation and contains Σ by construction. Therefore, from the definition of $\mathcal{I}_\Sigma$ we have that $\mathcal{I}_\Sigma \subset I_{\Sigma'}$. □

The associated space and retracting space of an ideal in $\Omega(M)$ are defined pointwise as in section 2.6.3. The associated space of $\mathcal{I}_\Sigma$ is called the Cauchy characteristic distribution and is denoted $A(\mathcal{I}_\Sigma)$.

4.2 Exterior Differential Systems

In section 2.6.3 we introduced systems of exterior equations on a vector space V and characterized their solutions as subspaces of V. We are now ready to define a similar notion for a collection of differential forms defined on a manifold M. The basic problem will be to study the integral submanifolds of M which satisfy the constraints represented by the exterior differential system.

Definition 36 *An exterior differential system is a finite collection of equations*

$$\alpha^1 = 0, \ldots, \alpha^r = 0$$

where each $\alpha^i \in \Omega^k(M)$ is a smooth k-form. A solution to an exterior differential system is any submanifold N of M which satisfies $\alpha^i(x)|_{T_xN} \equiv 0$. for all $x \in N$ and all $i \in 1, \ldots, r$.

An exterior differential system can be viewed pointwise as a system of exterior equations on T_pM. In view of this, one might expect that a solution would be defined as a distribution on the manifold. The trouble with this approach is that most distributions are not integrable, and we want our solution set to be a collection of integral submanifolds. Therefore, we will restrict our solution set to integrable distributions.

Theorem 33 *Given an exterior differential system*

$$\alpha^1 = 0, \ldots, \alpha^K = 0 \tag{12}$$

and the corresponding differential ideal $\mathcal{I}_\Sigma$ generated by the collection of forms

$$\Sigma := \{\alpha^1, \ldots, \alpha^K\} \tag{13}$$

an integral submanifold N of M solves the system of exterior equations if and only if it also solves the equation $\pi = 0$ for every $\pi \in \mathcal{I}_A$.

Proof: If an integral submanifold N of M is a solution to Σ, then for all $x \in N$ and all $i \in 1, \ldots, K$

$$\alpha^i(x)|_{T_xN} \equiv 0.$$

Taking the exterior derivative gives

$$d\alpha^i(x)|_{T_xN} \equiv 0.$$

Therefore, the submanifold also satisfies the exterior differential system

$$\alpha^1 = 0, \ldots, \alpha^K = 0, d\alpha^1 = 0, \ldots, d\alpha^K = 0$$

From theorem 32 we know that the differential ideal generated by Σ is equal to the algebraic ideal generated by the above system. Therefore, from theorem 23 we know that N will also be a solution for every element of $\mathcal{I}_\Sigma$.

Conversely, if N solves the equation $\pi = 0$ for every $\pi \in \mathcal{I}_\Sigma$ then in particular it must solve Σ. □

The above theorem allows us to either work with the generators of an ideal or with the ideal itself. In fact some authors define exterior differential systems as differential ideals of $\Omega(M)$.

Because a set of generators Σ generates both a differential ideal $\mathcal{I}_\Sigma$ and a algebraic ideal I_Σ, we can define two different notions of equivalence for exterior differential systems.

Definition 37 *Two exterior differential systems, Σ_1 and Σ_2, are said to be algebraically equivalent if and only if they generate the same algebraic ideal. i.e. $I_{\Sigma_1} = I_{\Sigma_2}$.*

Definition 38 *Two exterior differential systems, Σ_1 and Σ_2, are said to be equivalent if and only if they generate the same differential ideal. i.e. $\mathcal{I}_{\Sigma_1} = \mathcal{I}_{\Sigma_2}$.*

Intuitively, we want to think of two exterior differential systems as equivalent if they have the same solution set. Therefore, we will usually discuss equivalence in terms of this second definition.

4.3 Pfaffian Systems

An exterior differential system of the form

$$\alpha^1 = \alpha^2 = \ldots = \alpha^s = 0$$

where the α^i are independent 1-forms on an n-dimensional manifold is called a Pfaffian system of codimension $n-s$. The 1-forms $\alpha^1, \ldots, \alpha^s$, generate the algebraic ideal

$$I = \{\alpha^1, \ldots, \alpha^s\}$$

which means that

$$I = \{\sigma \in \Omega(M) : \sigma = \sum_{j=1}^{s} \theta^j \wedge \alpha^j\}$$

for some $\theta^j \in \Omega(M)$ or equivalently

$$I = \{\sigma \in \Omega(M) : \sigma \wedge \alpha^1 \wedge \ldots \wedge \alpha^s = 0\}$$

The following conditions ensure that the algebraic ideal generated by the 1-forms α^i is also a differential ideal.

Definition 39 *A set of linearly independent 1-forms $\alpha^1, \ldots, \alpha^s$ in the neighborhood of a point is said to satisfy the Frobenius condition if one of the following equivalent conditions hold:*

1. $d\alpha^i$ *is a linear combination of* $\alpha^1, \ldots, \alpha^s$.
2. $d\alpha^i \wedge \alpha^1 \wedge \ldots \wedge \alpha^s = 0$ *for* $1 \leq i \leq s$.
3. $d\alpha^i = \sum_{j=1}^{s} \theta^j \wedge \alpha^j$

When $d\alpha^i$ is a linear combination of $\alpha^1, \ldots, \alpha^s$ the following expression if frequently used

$$d\alpha^i \equiv 0 \; mod \; \alpha^1, \ldots, \alpha^s \; 1 \leq i \leq s$$

where the mod operation is implicitly performed over the algebraic ideal generated by the α^i.

Example: We will illustrate the above concepts for the unicycle. Recall that the unciycle can be described by the following codistribution

$$I = \{\omega\}$$

where

$$\omega = \sin\theta dx - \cos\theta dy + 0 d\theta$$

The exterior derivative of ω is

$$d\omega = \cos\theta \; d\theta \wedge dx + \sin\theta \; d\theta \wedge dy$$

and therefore

$$d\omega \wedge \omega = -\cos^2 d\theta \wedge dx \wedge dy + \sin^2 d\theta \wedge dy \wedge dx = dx \wedge dy \wedge d\theta \neq 0$$

and therefore I is not a differential ideal since the Frobenius condition does not hold.◇

We are now ready for Frobenius Theorem for codistributions.

Theorem 34 *(Frobenius Theorem for codistributions) Let I be an algebraic ideal generated by the independent 1-forms $\alpha^1, \ldots, \alpha^s$ such that the Frobenius condition is satisfied. Then in a neighborhood of x there exist functions h^i with $1 \leq i \leq s$ such that*

$$I = \{\alpha^1, \ldots, \alpha^s\} = \{dh^1, \ldots, dh^s\}$$

Proof: See [5] pages 27-29. □

For more general exterior differential systems we have the following integrability results.

Theorem 35 *If the Cauchy characteristic distribution of I has constant dimension r in a neighborhood, then it is integrable.*

Proof: See [5] page 31. □

Theorem 36 *Let $\mathcal{I}$ be a differential ideal whose retracting space has a constant dimension $n-r$. There is a neighborhood in which there are coordinates $y^1, \ldots, y^n$ such that $\mathcal{I}$ has a set of generators which are forms in $y^1, \ldots, y^{n-r}$.*

Proof: See [5] page 31-33. □

4.4 Derived flags

If the algebraic ideal generated by a Pfaffian system does not satisfy the Frobenius conditions, then it is not a differential ideal. However, there may be a piece of the algebraic ideal which is also a differential ideal.

Let $I^{(0)} = \{\omega^1, \ldots, \omega^s\}$ be the algebraic ideal generated by independent 1-forms $\omega^1, \ldots, \omega^s$. We define $I^{(1)}$ as

$$I^{(1)} \quad = \quad \{\lambda \in I^{(0)} : d\lambda \equiv 0 \; mod \; I^{(0)}\} \subset I^{(0)}$$

The ideal $I^{(1)}$ is called the first derived system. A natural question is to ask what is the analog of the first derived system from the distribution point of view. The next theorem answers this question.

Theorem 37 *If $I^{(0)} = \Delta^{\perp}$ then $I^{(1)} = (\Delta + [\Delta, \Delta])^{\perp}$.*

Proof: Let $I^{(0)}$ be spanned by 1-forms $\omega^1, \ldots, \omega^s$ and let Δ be the annihilating distribution. By definition we have that

$$I^{(1)} = \{\omega \in I^{(0)} : d\omega \equiv 0 \; mod \; I^{(0)}\}$$

Now let $\eta \in I^{(1)}$. Therefore $d\eta \equiv 0 \; mod \; I^{(0)}$ which means that

$$d\eta = \sum_{i=1}^{s} \theta^j \wedge \omega^j$$

for some forms θ^j. Now let X be a vector field in Δ. Since Δ is the annihilating distribution of $I^{(0)}$ then it follows that

$$\eta(X) = 0$$

Likewise for another vector field $Y \in \Delta$ we have $\eta(Y) = 0$. Now since

$$d\eta(X,Y) = \sum_{i=1}^{s} \theta^j \wedge \omega^j$$

we get that

$$d\eta(X,Y) = 0$$

since $\omega^j(X) = \omega^j(Y) = 0$. By using Cartan's magic formula we obtain

$$d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]) = 0$$

and therefore

$$\eta([X,Y]) = 0$$

which means that η annihilates any vector fields belonging in $[\Delta, \Delta]$ in addition to any vector fields in Δ. Therefore $\eta \in (\Delta + [\Delta, \Delta])^{\perp}$ and thus

$$I^{(1)} \subset (\Delta + [\Delta, \Delta])^{\perp}$$

Conversely, let $\eta \in (\Delta + [\Delta, \Delta])^{\perp}$ and let X,Y be vector fields in $(\Delta + [\Delta, \Delta])$. Using Cartan's magic formula gives

$$d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]) = 0$$

and therefore $d\eta = 0 \; mod \; I^{(0)}$ which means that $\eta \in I^{(1)}$. Thus $(\Delta + [\Delta, \Delta])^{\perp} \subset I^{(1)}$ and therefore $(\Delta + [\Delta, \Delta])^{\perp} = I^{(1)}$.□

One may inductively continue this procedure of obtaining derived systems and define

$$I^{(2)} = \{\lambda \in I^{(1)} : d\lambda \equiv 0 \; mod \; I^{(1)}\} \subset I^{(1)}$$

or in general

$$I^{(k+1)} = \{\lambda \in I^{(k)} : d\lambda \equiv 0 \; mod \; I^{(k)}\} \subset I^{(k)}$$

The above procedure results in a nested sequence of codistibutions

$$I^{(k)} \subset I^{(k-1)} \subset \ldots \subset I^{(1)} \subset I^{(0)}$$

We can also generalize Theorem 37. If we define $\Delta_0 = (I^{(0)})^{\perp}$, $\Delta_1 = (I^{(1)})^{\perp}$, and in general $\Delta_k = (I^{(k)})^{\perp}$, then it is not hard to show that if $I^{(k)} = \Delta_k^{\perp}$ then $I^{(k+1)} = (\Delta_k + [\Delta_k, \Delta_k])^{\perp}$. The proof of this fact is similar to the proof of Theorem 37 but uses a more general form of Cartan's magic formula. Therefore one can show that

$$\Delta_{k+1} = \Delta_k + [\Delta_k, \Delta_k]$$

and we can therefore see that higher order derived systems of codistributions are associated with higher order Lie brackets of distributions.

Therefore to the sequence of decreasing codistributions

$$I^{(k)} \subset I^{(k-1)} \subset \ldots \subset I^{(1)} \subset I^{(0)}$$

we can associate a sequence of increasing distributions,

$$\Delta_k \supset \Delta_{k-1} \supset \ldots \supset \Delta_1 \supset \Delta_0$$

The sequence of decreasing codistributions is called the derived flag of $I^{(0)}$ while the increasing sequence of distributions is called a filtration of Δ_0. If the dimension of each codistribution is constant then the above construction will terminate in finitely many steps. There will therefore be a codistribution $I^{(N)}$ for some integer N such that $I^{(N)} = I^{(N+1)}$. This integer N is called the derived length of I. In fact, $I^{(N)}$ is always integrable by definition since

$$dI^{(N)} \equiv 0 \; mod \; I^{(N)}$$

Codistribution $I^{(N)}$ is the largest integrable subsystem in I. Therefore if $I^{(N)} \neq \{0\}$ then there exist functions $h^1, \ldots, h^r$ such that $\{dh^1, \ldots, dh^r\} \subset I$.

As a result, if a Pfaffian systems contains an integrable subsystem $I^{(N)} \neq 0$ which is spanned by the 1-forms $dh^1, \ldots, dh^r$ then the integral curves are constrained to satisfy the following equations,

$$dh^i = 0 \Longrightarrow h^i = k_i \; for \; 1 \leq i \leq r$$

and therefore trajectories of the system must lie on the manifold,

$$M = \{x : h^i(x) = k_i \; for \; 1 \leq i \leq r\}$$

Therefore if $I^{(N)} \neq 0$ it is not possible to find an integral curve from a configuration $x(0) = x_0$ to another configuration $x(1) = x_1$ unless both configurations satisfy

$$h^i(x_0) = h^i(x_1) \; for \; 1 \leq i \leq r$$

Example: Consider the following system which is slightly more sophisticated than the unicycle. Consider a rolling penny on a plane, with unit radius, which is similar to the unicycle with the additional requirement that we can specify a desired configuration of Lincoln's head. Thus in addition to the three configuration variables of the unicycle we also have an angle ϕ describing the orientation of Lincoln's head. The model in this case is,

$$\begin{aligned} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \\ \dot{\phi} &= -u_1 \end{aligned}$$

which can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 = f_1 u_1 + f_2 u_2$$

The annihilating codistribution can be easily checked to be

$$I = \Delta^{\perp} = \{\alpha^1, \alpha^2\}$$

where

$$\alpha^1 = \sin\theta dx - \cos\theta dy + 0d\theta + 0d\phi$$
$$\alpha^2 = \cos\theta dx + \sin\theta dy + 0d\theta + 1d\phi$$

Now

$$\begin{aligned}
d\alpha^1 &= \cos\theta d\theta \wedge dx + \sin\theta d\theta \wedge dy \\
d\alpha^2 &= -\sin\theta d\theta \wedge dx + \cos\theta d\theta \wedge dy \\
d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 &= d\theta \wedge dx \wedge dy \wedge d\phi \\
d\alpha^2 \wedge \alpha^1 &= \sin\theta\cos\theta(d\theta \wedge dx \wedge dy + d\theta \wedge dy \wedge dx) = 0 \\
d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 &= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
d\alpha^1 &\neq 0 \; mod \; \alpha^1, \alpha^2 \\
d\alpha^2 &= 0 \; mod \; \alpha^1, \alpha^2
\end{aligned}$$

and thus the first derived system is,

$$I^{(1)} = \{\alpha^2\}$$

It can be easily checked that

$$d\alpha^2 \wedge \alpha^2 \neq 0$$

and thus

$$I^{(2)} = \{0\}$$

The derived flag of the system is

$$\begin{aligned}
I^{(0)} &= \{\alpha^1, \alpha^2\} \\
I^{(1)} &= \{\alpha^1\} \\
I^{(2)} &= \{0\}
\end{aligned}$$

and therefore an integrable subsystem does not exist. As a result the system is not constrained to move on some hypersurface, as expected. ◇

4.5 Pfaffian Systems of Codimension $n-1$

We will now restrict to the study of Pfaffian systems of codimension $n-1$. Pfaff originally investigated systems consisting of a single equation

$$\alpha = 0$$

where α is a 1-form on a manifold M. In some chart (U, x) of a point $p \in M$ the equation can be expressed as

$$a_1(x)dx^1 + a_2(x)dx^2 + \ldots + a_n(x)dx^n = 0$$

In order to understand the integral manifolds of this equation we will first try to express α in a normal form by performing a coordinate transformation.

Definition 40 *Let $\alpha \in \Omega^1(M)$. The integer r defined by*

$$\begin{aligned}(d\alpha)^r \wedge a &\neq 0 \\ (d\alpha)^{r+1} \wedge a &= 0\end{aligned}$$

is called the *rank* *of α.*

The following theorem allows us, under a rank condition, to write α in a normal form.

Theorem 38 *(Pfaff) Given any $\alpha \in \Omega^1(M)$ with constant rank r in a neighborhood of p, there exists a coordinate chart (U, z) such that $\alpha = dz^1 + z^2dz^3 + \ldots + z^{2r}dz^{2r+1}$.*

Proof: Let $\mathcal{I}$ be the differential ideal generated by α. From theorem 25 the retraction space of $\mathcal{I}$ is of dimension $2r+1$. By theorem 36 there is a function f_1 such that

$$(d\alpha)^r \wedge \alpha \wedge df_1 = 0$$

Now let $\mathcal{I}_1$ be the differential ideal generated by $\{df_1, \alpha, d\alpha\}$. If $r = 0$ then the result follows from the Frobenius theorem. If $r > 0$, the forms df_1 and α must be linearly independent. Applying theorem 25 to $\mathcal{I}_1$, let r_1 be the smallest integer such that

$$(d\alpha)^{r_1+1} \wedge \alpha \wedge df_1 = 0$$

Clearly, $r_1 + 1 \leq r$. Furthermore, the equality sign must hold because $(d\alpha)^r \wedge \alpha \neq 0$. Applying theorem 36 to $\mathcal{I}_1$ there exists a function f_2 such that

$$(d\alpha)^{r-1} \wedge \alpha \wedge df_1 \wedge df_2 = 0$$

Repeating this process, we find r functions $f_1, f_2, \ldots, f_r$ satisfying

$$d\alpha \wedge \alpha \wedge df_1 \wedge df_2 \wedge \ldots \wedge df_r = 0$$

$$\alpha \wedge df_1 \wedge df_2 \wedge \ldots \wedge df_r \neq 0$$

Finally, let $\mathcal{I}_r$ be the ideal $\{df_1, \ldots, df_r, \alpha, d\alpha\}$. Its retraction space $C(\mathcal{I}_r)$ is of dimension $r+1$. There is a function f_{r+1} such that

$$\alpha \wedge df_1 \wedge df_2 \wedge \ldots \wedge df_{r+1} = 0$$

$$df_1 \wedge df_2 \wedge \ldots \wedge df_{r+1} \neq 0$$

By modifying α by a factor, we can write

$$\alpha = df_{r+1} + g_1 df_1 + \ldots + g_r df_r.$$

Because $(d\alpha)^r \wedge \alpha \neq 0$, the functions $f_1, \ldots, f_{r+1}, g_1, \ldots, g_r$ are independent. The result then follows by setting

$$w^1 = f_{r+1} \qquad w^{2i} = g_i \qquad w^{2i+1} = f_i$$

for $1 \leq i \leq r$. □

Example: Consider the unicycle example described by the codistribution

$$I = \{\alpha\}$$

where

$$\alpha = \sin\theta dx - \cos\theta dy$$

We can immediately see that

$$d\alpha = \cos\theta d\theta \wedge dx + \sin\theta d\theta \wedge dy$$

and that

$$\begin{aligned} d\alpha \wedge \alpha &= d\theta \wedge dy \wedge dx \neq 0 \\ (d\alpha)^2 \wedge \alpha &= 0 \end{aligned}$$

Therefore the rank of α is 1 and by Pfaff's Theorem there exist coordinates z^1, z^2, z^3 such that

$$\alpha = dz^1 + z^2 dz^3$$

In this example we trivially obtain,

$$\alpha = dy + (-\tan\theta)dx$$

◇

The following theorem is similar to Pfaff's theorem and it simply expresses the result in a more symmetric form.

Theorem 39 *Given any $\alpha \in \Omega^1(M)$ with constant rank r in a neighborhood U of p, there exist coordinates $z, y^1, \ldots, y^r, x^1, \ldots, x^r$ such that*

$$\alpha = dz + \frac{1}{2}\sum_{i=1}^{r}(y^i dx^i - x^i dy^i)$$

Proof: The following coordinate transformation

$$\begin{aligned} z^1 &= z - \frac{1}{2}\sum_{i=1}^{r} x^i y^i \\ z^{2i} &= y^i \quad 1 \leq i \leq r \\ z^{2i+1} &= x^i \quad 1 \leq i \leq r \end{aligned}$$

reduces the above theorem to Pfaff's Theorem. □

The Pfaffian system $\alpha = 0$ on a manifold M is said to have the local accessibility property if every point $x \in M$ has a neighborhood U such that every point in U can be joined to x by an integral curve. The following theorem answers the question of when does this Pfaffian system have the local accessibility property.

Theorem 40 *(Caratheodory) The Pfaffian system,*

$$\alpha = 0$$

where α has constant rank, has the local accessibility property if and only if

$$\alpha \wedge d\alpha \neq 0$$

Proof: The above condition simply says that the rank of α must be greater than or equal to 1. If α has zero rank then $d\alpha \wedge \alpha = 0$ and therefore by Frobenius Theorem we can write

$$\alpha = dh = 0$$

for some zero form h. Since $dh = 0$ the integral curves are of the form $h = c$ for arbitrary constant c. Therefore we do not have the local accessibility property since we can only join points $p, q \in M$ for which $h(p) = h(q)$.

Conversely, let α have rank $r \geq 1$. Then from Theorem 39 we can find coordinates $z, x^1, \ldots, x^r, y^1, \ldots, y^r, u^1, \ldots, u^s$ in some neighborhood U with $2r + s + 1 = dimM$ such that

$$\alpha = dz + \frac{1}{2}\sum_{i=1}^{r}(y^i dx^i - x^i dy^i) = 0$$

and therefore

$$dz = \frac{1}{2}\sum_{i=1}^{r}(x^i dy^i - y^i dx^i)$$

Given any two points $p, q \in U$ we must find integral curves $c : [0,1] \longrightarrow U$ with $c(0) = p$ and $c(1) = q$. Without loss of generality we can assume $z(p) = x^i(p) = y^i(p) = u^i(p) = 0$. Let $z(q) = z_1, x^i(p) = x^i_1, y^i(p) = y^i_1, u^i(p) = u^i_1$. The form α is independent of the u^i coordinates and therefore one can choose the curve tu^i_1 to steer the u^i coordinates.

In the (x^i, y^i) plane there are many curves $(x^i(t), y^i(t))$ which can join the origin with the desired point (x^i_1, y^i_1). We are therefore left with steering the z coordinate to z_1. However we have that

$$dz = \frac{1}{2}\sum_{i=1}^{r}(x^i dy^i - y^i dx^i)$$

and therefore

$$z(t) = \frac{1}{2}\int_0^t \sum_{i=1}^{r}(x^i \frac{dy^i}{dt} - y^i \frac{dx^i}{dt})dt$$

We can therefore impose the restriction that the curve $(x^i(t), y^i(t))$ be such that $z(1) = z_1$. The reason why such a curve must exist is that the

$$z(t) = \frac{1}{2}\int_0^t \sum_{i=1}^{r}(x^i \frac{dy^i}{dt} - y^i \frac{dx^i}{dt})dt = \frac{1}{2}\sum_{i=1}^{r} A_i$$

where A_i is the area enclosed by the curve $(x^i(t), y^i(t))$ and the cord joining the origin with the desired endpoint. Therefore we can geometrically see that we can always generate a curve $(x^i(t), y^i(t))$ linking the desired points while enclosing the desired area prescribed by z_1.

Therefore the integral curve $c(t)$ given by

$$(z(t), x^1(t), \ldots, x^r(t), y^1(t), \ldots, y^r(t), tu^1(t), \ldots, tu^s(t))$$

has $c(0) = p$ and $c(1) = q$. Therefore the systems has the local accessibility property. □

4.6 Pfaffian Systems of Codimension 2

In the previous section we considered Pfaffian systems consisting of a single equation. We now consider systems of codimension two. We are again interested in performing coordinate changes so that the generators of these Pfaffian systems are in some normal form.

Theorem 41 *(Engel) Let I be a two dimensional codistribution*

$$I = \{\alpha^1, \alpha^2\}$$

of four variables. If the derived flag satisfies

$$dim I^{(1)} = 1$$

$$dim I^{(2)} = 0$$

then there exist coordinates z^1, z^2, z^3, z^4 such that

$$I = \{dz^4 - z^3 dz^1, dz^3 - z^2 dz^1\}$$

Proof: Choose a basis such that $I^{(0)} = I = \{\alpha^1, \alpha^2\}$ $I^{(1)} = \{\alpha^1\}$ and $I^{(2)} = \{0\}$. Choose α^3 and α^4 to complete the basis. Since $I^{(2)} = \{0\}$ we have

$$d\alpha^1 \wedge \alpha^1 \neq 0$$

while

$$(d\alpha^1)^2 \wedge \alpha^1 = 0$$

since it is a 5-form on a 4-dimensional space. Therefore the rank of α^1 is 1. By Pfaff's Theorem, we know that there exists a coordinate change so that

$$\alpha^1 = dz^4 - z^3 dz^1$$

and thus

$$d\alpha^1 = -dz^3 \wedge dz^1 = dz^1 \wedge dz^3$$

Now since $\alpha^1 \in I^{(1)}$ we have

$$d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 = 0$$

and thus

$$dz^1 \wedge dz^3 \wedge \alpha^1 \wedge \alpha^2 = 0$$

Therefore α^2 is a linear combination of dz^1, dz^3 and α^1 or likewise

$$\alpha^2 = a(x)dz^3 + b(x)dz^1 \ mod \ \alpha^1$$

By definition this means that

$$\alpha^2 + \lambda(x)\alpha^1 = a(x)dz^3 + b(x)dz^1$$

Now $a(x)$ and $b(x)$ cannot be both zero since $\alpha^2 \neq 0$. If $a(x) \neq 0$ then we have

$$\frac{1}{a(x)}\alpha^2 + \frac{\lambda(x)}{a(x)}\alpha^1 = dz^3 + \frac{b(x)}{a(x)}dz^1$$

and if we set $z^2 = -\frac{b(x)}{a(x)}$ then

$$\frac{1}{a(x)}\alpha^2 + \frac{\lambda(x)}{a(x)}\alpha^1 = dz^3 - z^2dz^1$$

and thus

$$I = \{\alpha^1, \alpha^2\} = \{\alpha^1, \frac{1}{a(x)}\alpha^2 + \frac{\lambda(x)}{a(x)}\alpha^1\} = \{dz^4 - z^3dz^1, dz^3 - z^2dz^1\}$$

The case $b(x) \neq 0$ is similar. □

Example: Consider again the penny rolling on a plane which is described by the codistribution,

$$I = \{\alpha^1, \alpha^2\}$$

with

$$\begin{aligned} \alpha^1 &= \cos\theta dx + \sin\theta dy - d\phi \\ \alpha^2 &= \sin\theta dx - \cos\theta dy \end{aligned}$$

In a previous example we have already seen that the derived flag is

$$\begin{aligned} I^{(0)} &= \{\alpha^1, \alpha^2\} \\ I^{(1)} &= \{\alpha^1\} \\ I^{(2)} &= \{0\} \end{aligned}$$

and thus satisfies the conditions of Engel's Theorem. After some calculations we obtain

$$\begin{aligned} d\alpha^1 &= -\sin\theta d\theta \wedge dx + \cos\theta d\theta \wedge dy \\ d\alpha^1 \wedge \alpha^1 &= -dx \wedge dy \wedge d\theta + \sin\theta d\theta \wedge dx \wedge dy - \cos\theta d\theta \wedge dy \wedge dy \end{aligned}$$

and thus the rank of α^1 is 1. Following the proof of Pfaff's Theorem we know that there exists a function f_1 such that

$$d\alpha^1 \wedge \alpha^1 \wedge df_1 = 0$$

We can easily see that the function $f_1 = \theta$ will satisfy the above equation. Since the rank of α is 1, we must now search for a function f_2 such that

$$\alpha^1 \wedge df_1 \wedge df_2 = 0$$

Let $f_2 = f_2(x, y, \theta, \phi)$. Then

$$df_2 = \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial \theta}d\theta + \frac{\partial f_2}{\partial \phi}d\phi$$

Combining the last two equations and algebraic manipulations result in the following partial differential equations

$$\begin{aligned} -\cos\theta\frac{\partial f_2}{\partial y} + \sin\theta\frac{\partial f_2}{\partial x} &= 0 \\ \frac{\partial f_2}{\partial x} + \cos\theta\frac{\partial f_2}{\partial \phi} &= 0 \\ \frac{\partial f_2}{\partial y} + \sin\theta\frac{\partial f_2}{\partial \phi} &= 0 \end{aligned}$$

A solution to the above system of equations is

$$f_2(x, y, \theta, \phi) = x\cos\theta + y\sin\theta - \phi$$

which can checked easily. Therefore, following the proof of Pfaff's Theorem we may now choose $z^1 = f_1$ and $z^4 = f_2$ so that

$$\alpha^1 = dz^1 - z^3 dz^4$$

where z^3 can be solved from the above equation to be

$$z^3 = -x\sin\theta + y\cos\theta$$

We will now try to transform α^2 in this normal form. Following the proof of Engel's Theorem we have that

$$\alpha^2 = [a(x)dz^3 + b(x)dz^1] \; mod \; \alpha^1$$

where we must now determine the functions $a(x), b(x)$. Expanding the above equation gives,

$$\sin\theta dx - \cos\theta dy = [a(x)(-dx\sin\theta + x\cos\theta d\theta + dy\cos\theta - y\sin\theta d\theta) + b(x)d\theta] \; mod \; \alpha^1$$

Simple calculations show that the following choices

$$\begin{aligned} a(x) &= -1 \\ b(x) &= x\cos\theta - y\sin\theta \end{aligned}$$

will satisfy the equation. Therefore by Engel's Theorem, if we set

$$z^2 = -x\cos\theta + y\sin\theta$$

we may express α^1,α^2 in the following normal form

$$\begin{aligned} \alpha^1 &= dz^4 - z^3 dz^1 \\ \alpha^2 &= dz^3 - z^2 dz^1 \end{aligned}$$

If we look at the annihilating distribution of our codistribution expressed in these new coordinates we obtain

$$\dot{z}^4 = z^3\dot{z}^1$$
$$\dot{z}^3 = z^3\dot{z}^1$$

and z^1, z^2 are free. If we set $\dot{z}^1 = u_1, \dot{z}^2 = u_2$ the distribution has the form

$$\begin{aligned}
\dot{z}^4 &= z^3 u_1 \\
\dot{z}^3 &= z^3 u_1 \\
\dot{z}^2 &= u_2 \\
\dot{z}^1 &= u_1
\end{aligned}$$

We can now see that the advantage of performing this coordinate transformation is that our system can be expressed in this simple form. In particular if we set $u_1 = 1$ then the system has been transformed to a linear system. This allows us to utilize the powerful analysis tools that exist for linear systems. This method of transforming a system to a linear one through a coordinate transformation is called feedback linearization. Engel's Theorem therefore states the conditions under which we can linearize a system of four configuration variables with two constraints. ◇

Engel's theorem can be generalized to system with n configuration variables and $n-2$ constraints. This powerful theorem was proved by Goursat.

Theorem 42 *(Goursat Normal Form) Let I be a Pfaffian system spanned by s 1-forms,*

$$I = \{\alpha^1, \ldots, \alpha^s\}$$

on a space of dimension $n = s + 2$. Suppose that there exists an integrable form π with $\pi \neq 0$ mod I satisfying the Goursat congruence,

$$\begin{aligned}
d\alpha^i &\equiv -\alpha^{i+1} \wedge \pi \ mod\ \alpha^1, \ldots, \alpha^i \quad 1 \leq i \leq s-1 \\
d\alpha^s &\neq 0 \ mod\ I
\end{aligned}$$

then there exists a coordinate system $z^1, z^2, \ldots, z^n$ such that

$$I = \{dz^3 - z^2 dz^1, dz^4 - z^3 dz^1, \ldots, dz^n - z^{n-1} dz^1\}$$

Proof: The Goursat congruences can be reformulated using the derived flag of I. Expanding the Goursat congruences gives

$$\begin{aligned}
d\alpha^1 &\equiv -\alpha^2 \wedge \pi \ mod\ \alpha^1 \\
d\alpha^2 &\equiv -\alpha^3 \wedge \pi \ mod\ \alpha^1, \alpha^2 \\
&\vdots \\
d\alpha^{s-1} &\equiv -\alpha^s \wedge \pi \ mod\ \alpha^1, \alpha^2, \ldots, \alpha^s \\
d\alpha^s &\equiv -\alpha^s \wedge \pi \ mod\ \alpha^1, \alpha^2, \ldots, \alpha^s
\end{aligned}$$

and therefore the derived flag of I must be

$$\begin{aligned}
I^{(0)} &= \{\alpha^1, \alpha^2, \ldots, \alpha^s\} \\
I^{(1)} &= \{\alpha^1, \ldots, \alpha^{s-1}\} \\
&\vdots \\
I^{(s-1)} &= \{\alpha^1\} \\
I^{(s)} &= \{0\}
\end{aligned}$$

From the Goursat congruences we have that

$$d\alpha^1 = -\alpha^2 \wedge \pi \; mod \; \alpha^1$$

which means that

$$d\alpha^1 = -\alpha^2 \wedge \pi + \alpha^1 \wedge \eta$$

for some form η. But then we have that

$$\begin{aligned} d\alpha^1 \wedge \alpha^1 &= -\alpha^2 \wedge \pi \wedge \alpha^1 \neq 0 \\ (d\alpha^1)^2 \wedge \alpha^1 &= -\alpha^2 \wedge \pi \wedge \alpha^1 = 0 \end{aligned}$$

which means that α^1 has rank 1. We can therefore apply Pfaff's Theorem and express α^1 as

$$\alpha^1 = dz^n - z^{n-1} dz^1$$

Furthermore by Engel's Theorem we can express α^2 as

$$\alpha^2 = dz^{n-1} - z^{n-2} dz^1 \tag{14}$$

In these new coordinates we have

$$d\alpha^1 \wedge \alpha^1 = -dz^{n-1} \wedge dz^1 \wedge dz^n$$

Now we have that

$$\pi \wedge (-\alpha^2 \wedge \pi \wedge \alpha^1) = \pi \wedge (-dz^{n-1} \wedge dz^1 \wedge dz^n) = 0$$

and therefore π is a linear combination of dz^1, dz^{n-1}, dz^n and thus

$$\pi = adz^1 + bdz^{n-1} + cdz^n = adz^1 + bz^{n-2} dz^1 + cz^{n-1} dz^1$$

or equivalently

$$\pi = (a + bz^{n-2} + cz^{n-1}) dz^1 \; mod \; \alpha^1, \alpha^2 = \psi \; mod \; \alpha^1, \alpha^2$$

where $\psi = a + bz^{n-2} + cz^{n-1}$ is nonzero since we have assumed that $\pi \neq 0 \; mod \; I$. From the Goursat congruences we have that

$$d\alpha^2 = -\alpha^3 \wedge \pi \; mod \; \alpha^1, \alpha^2$$

while from equation (14) we have

$$d\alpha^2 = -dz^{n-2} \wedge dz^1$$

and thus

$$-dz^{n-2} \wedge dz^1 = -\alpha^3 \wedge \pi \; mod \; \alpha^1, \alpha^2$$

which means that

$$\alpha^3 = \lambda(x) dz^{n-2} \; mod \; dz^1, \alpha^1, \alpha^2$$

for nonzero function $\lambda(x)$. Therefore we can rewrite this as

$$\alpha^3 = dz^{n-2} - \frac{1}{\lambda(x)} dz^1 \; mod \; \alpha^1, \alpha^2$$

and if we set $z^{n-3} = \frac{1}{\lambda(x)}$ we have

$$\alpha^3 = dz^{n-2} - z^{n-3} dz^1 \; mod \; \alpha^1, \alpha^2$$

and we can therefore let

$$\alpha^3 = dz^{n-2} - z^{n-3} dz^1$$

If we inductively continue this procedure using the Goursat congruences we obtain

$$\begin{aligned} \alpha^4 &= dz^{n-3} - z^{n-3} dz^1 \\ &\vdots \\ \alpha^s &= dz^3 - z^2 dz^1 \end{aligned}$$

Now from the Goursat congruences we have that

$$d\alpha^s \neq 0 \; mod \; I$$

and therefore

$$\alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^s \wedge \; d\alpha^s \neq 0$$

If we substitute the α^i in the above expression we obtain

$$dz^1 \wedge dz^2 \wedge \ldots \wedge dz^n \neq 0$$

and therefore the functions $z^1, \ldots, z^n$ can serve as a local coordinate system. □

The following example illustrates the power of the Goursat's Theorem by applying it in order to feedback linearize a nonlinear system. A more systematic approach to the this problem can be found in [8].

Example: Consider the following nonlinear system with s configuration variables and a single input,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_s, u) \\ \dot{x}_2 &= f_2(x_1, \ldots, x_s, u) \\ &\vdots \\ \dot{x}_s &= f_s(x_1, \ldots, x_s, u) \end{aligned}$$

Equivalently we can look at following Pfaffian system,

$$I = \{dx^i - f_i(x^1, \ldots, x^s, u)dt\} \;\; 1 \leq i \leq s$$

The system is of codimension 2 since we have s constraints and $s + 2$ variables, namely $x^1, \ldots, x^s, u, t$. Assume that the form $\pi = dt$ satisfies the Goursat congruences. Then by Goursat's Theorem there exists a coordinate transformation such that I is generated by

$$I = \{dz^3 - z^2 dz^1, dz^4 - z^3 dz^1, \ldots, dz^s - z^{s-1} dz^1\}$$

The annihilating distribution of the above codistribution is

$$\begin{aligned} \dot{z}^1 &= u_1 \\ \dot{z}^2 &= u_2 \\ \dot{z}^3 &= z^2 u_1 \\ &\vdots \\ \dot{z}^s &= z^{s-1} u_1 \end{aligned}$$

which, if we set $u_1 = 1$, is clearly a linear system. We can therefore use Goursat's Theorem in order to linearize nonlinear systems satisfying the Goursat congruences. Although this approach holds for single input systems, that is systems of codimension 2, more general versions of the above theorem can be found in [9, 10]. This approach has had success in the case of steering cars with n trailers. In [10], this system was successfully converted to Goursat normal form and then standard techniques such as those described in [11] were utilized in order to steer the system to a desired configuration. ◇

5 Conclusion

Exterior differential systems offer a new way of looking at systems of differential equations. This approach is more algebraic compared to the standard vector field approach which is more geometric. The main advantage of looking at systems using differential forms instead of tangent vectors is exactly the algebraic power of exterior systems. As a result, although certain facts may be less intuitive due to their non-geometric nature they can be easily proved because of the strong structure of the underlying algebra.

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Reconciliation of Different Approaches to Linearization by State Feedback

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Abstract

Several techniques that can be used for linearization by state feedback and coordinate transformation are presented. Both static and dynamic state feedback are considered and a number of different approaches for each case are outlined. The similarities and differences between them are highlighted. A list of the problems not addressed by any of the techniques and may be of interest or form the topic of further research are also provided.

1 Introduction

Linear systems have so far been the most popular choice among control engineers for system modeling. One of the main reasons for this is the fact that linear dynamics are very predictable and therefore linear systems are very easy to analyze and control. This is not the case for nonlinear systems however which can, in general, display very rich and complicated behavior. This fact led considerable research effort in an attempt to determine conditions under which the behavior of a nonlinear system can be linked to that of a linear system. This report presents a brief overview of a number of techniques for establishing such a link. An attempt was made to indicate the points that these techniques have in common and to contrast the points on which they differ. A list of unresolved issues related to either the connection between the various techniques or questions that remain unanswered by them is also given.

Consider the nonlinear dynamical system:

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and f a smooth vector field[1]:

$$\begin{aligned} f : \mathbf{R}^n &\longrightarrow T\mathbf{R}^n \\ x &\longrightarrow f(x) \in T_x\mathbf{R}^n \end{aligned}$$

A very important special case of system 1 is the one where the input enters affinely in the dynamics:

$$\dot{x} = f(x) + g(x)u \tag{2}$$

where $g(x) = [g_1(x) \dots g_m(x)]$ and $g_i(x)$ are smooth vector fields. Most of the results presented here will be concerned with systems belonging to this class, even though some can be extended to the the more general case 1.

We would like to establish conditions under which the dynamics of 1 and 2 are adequately described by those of a linear system:

$$\dot{\hat{x}} = A\hat{x} + B\hat{u} \tag{3}$$

where $\hat{x} \in \mathbf{R}^{\hat{n}}$, $\hat{u} \in \mathbf{R}^{\hat{m}}$, $A \in \mathbf{R}^{\hat{n}\times\hat{n}}$ and $B \in \mathbf{R}^{\hat{n}\times\hat{m}}$ with $\hat{n} \geq n, \hat{m} \geq m$.

The literature on this problem can roughly be divided into three distinct classes:

1. Jacobian linearization and extensions
2. Linearization by static state feedback and coordinate transformation
3. Linearization by dynamic state feedback and coordinate transformation

It should be noted here that the results in the first class (polynomial approximations, etc.) are primarily concerned with approximate linearization while those in the other two classes deal with exact linearization. This report will be restricted to the exact linearization results. A brief discussion of the approximate techniques, as well as some attempts of extending the exact results to systems that are "almost" linearizable will be given in the last section. Despite considerable progress in this area there are still quite a few issues that remain unresolved. Some of the most prominent ones will also be presented in the last section.

Due to restrictions of space and time this report will be rather terse and the pace will be quite intense. In particular, I will assume that the reader is familiar with the basic definitions and concepts of differential topology, nonlinear dynamical systems and exterior differential systems. For background definitions and results the reader can refer to [1, 2, 3, 4].

2 Linearization by Static State Feedback

2.1 Problem Statement

Following the notation of [1] the problem of exact linearization by static state feedback and coordinate transformation can be stated as follows:

[1]Most of the techniques presented here can be generalized to the case where the state evolves on a manifold. $\mathbf{R}^n$ will be used however to simplify the calculations

Problem 1 (State Space Exact Linearization Problem)
Given a control system of the form 1 and an initial state x°, find, if possible, a neighborhood U of x°, a pair of feedback functions $a(x)$ and $b(x)$, a coordinate transformation $z = \Phi(x)$ all defined on U and matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, such that:

$$\left[\frac{\partial \Phi}{\partial x}(f(x, a(x)))\right]_{x=\Phi^{-1}(z)} = Az \tag{4}$$

$$\left[\frac{\partial \Phi}{\partial x}\frac{\partial(f(x, a(x) + b(x)v)}{\partial v}\right]_{x=\Phi^{-1}(z)} = B \tag{5}$$

$$rank(B \;\; AB \;\; \ldots \;\; A^{n-1}B) = n \tag{6}$$

In the special case of systems affine in the inputs, the problem simplifies to:

Problem 2 *Given a control system of the form 2 and an initial state x°, find, if possible, a neighborhood U of x°, a pair of feedback functions $a(x)$ and $b(x)$, a coordinate transformation $z = \Phi(x)$ all defined on U and matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, such that:*

$$\left[\frac{\partial \Phi}{\partial x}(f(x) + g(x)a(x))\right]_{x=\Phi^{-1}(z)} = Az \tag{7}$$

$$\left[\frac{\partial \Phi}{\partial x}(g(x)b(x))\right]_{x=\Phi^{-1}(z)} = B \tag{8}$$

$$rank(B \;\; AB \;\; \ldots \;\; A^{n-1}B) = n \tag{9}$$

Note that the last condition of both problem statements implies that, if the Exact Linearization Problem is solvable, we can assume, without loss of generality, that the resulting linear system will be in Brunovsky canonical form.

2.2 The Vector Field Approach

The standard results on linearization by static state feedback and coordinate transformation concern systems of the form 2. The relevant theorems can be found in [1, 2]. I will be using the notation and definitions of [1].

Theorem 1 *For the control system 2 define the filtration:*

$$\begin{aligned} G_0 &= span\{g_1, \ldots, g_m\} \\ G_{i+1} &= G_i + span\{[f, G_i]\} = span\{ad_f^k g_j : 0 \leq k \leq i+1, 1 \leq j \leq m\} \end{aligned}$$

Suppose the matrix $g(x^\circ)$ has rank m. Then, the State Space Exact Linearization Problem is solvable if and only if:

1. *for each $0 \leq i \leq n-1$ the distribution G_i has constant dimension near x°*
2. *The distribution G_{n-1} has dimension n*
3. *for each $0 \leq i \leq n-2$ the distribution G_i is involutive*

If the system has only one input ($m = 1$) Theorem 1 simplifies to:

Theorem 2 *The single input State Space Exact Linearization Problem is solvable if and only if:*

1. *the matrix* $[g(x^\circ)\ ad_f g(x^\circ)\ \ldots\ ad_f^{n-1} g(x^\circ)]$ *has rank* n
2. *the distribution* $G_{n-2} = span\{g,\ ad_f g,\ \ldots,\ ad_f^{n-2} g\}$ *is involutive near* x°.

The conditions for the single input case may seem slightly less restrictive, but in fact it turns out that the involutivity of G_{n-2} implies the involutivity of all G_i for $0 \leq i \leq n-2$ in this case. Even for the multi input case the involutivity of certain distributions (namely those corresponding to the Kronecker indices of the resulting linear system) implies the involutivity of others. However, an equivalent statement of Theorem 1 that takes this fact into account would be unnecessarily complicated.

2.3 The Pfaffian System Approach

The problem of linearization can also be approached from the point of view of exterior differential systems. Let $\Omega^p(\mathbb{R}^n)$ denote the set of smooth exterior p-forms on $\mathbb{R}^n$ and define $\Omega^*(\mathbb{R}^n) = \bigoplus \Omega^p(\mathbb{R}^n)$, the set of smooth exterior differential forms of all orders.

Definition 1 *An* **exterior differential system** *is an ideal* $\mathcal{I} \subset \Omega^*(\mathbb{R}^n)$ *that is closed under exterior differentiation. In particular a* **Pfaffian system of co-dimension** $n - s$ *is an exterior differential system on* $\mathbb{R}^n$ *generated by a set of linearly independent one-forms* $I = \{\alpha^1, \ldots, \alpha^s\}$, *hence* $\mathcal{I} = \{\omega \wedge \theta : \omega \in I, \theta \in \Omega^*(\mathbb{R}^n)\}$.

A class of Pfaffian systems is of particular interest here:

Definition 2 *A Pfaffian system* I *on* $\mathbb{R}^{n+m+1}$ *of co-dimension* $m+1$ *is in* **Extended Goursat Normal Form** *if it is generated by* n *one-forms of the form:*

$$I = \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \ldots, s_j; j = 1, \ldots, m\}$$

where $s_1 + s_2 + \ldots + s_m = n$.

The term **towers** is used to describe the m chains of one forms $dz_i^j - z_{i+1}^j dz^0$, each s_j "deep". The class of Pfaffian systems that can be brought to Extended Goursat Normal Form via a coordinate transformation is characterized by two equivalent theorems:

Theorem 3 *The co-dimension* $m+1$ *Pfaffian system* I *can be brought into Extended Goursat Normal Form if and only if there exists a set of generators* $\{\alpha_j^i : i = 1, \ldots s_j; j = 1, \ldots, m\}$ *for* I *and an integrable one-form* π *such that the system satisfies the* **Goursat congruences**, *i.e. for all* j:

$$\begin{aligned} d\alpha_i^j &\equiv -\alpha_{i+1}^j \wedge \pi \ mod\ I^{(s_j - i)} \quad i = 1, \ldots s_j - 1 \\ d\alpha_{s_j}^j &\not\equiv 0\ mod\ I \end{aligned}$$

Theorem 4 *For the co-dimension $m+1$ Pfaffian system I define the* **derived flag***:*

$$\begin{aligned} I^{(0)} &= \{\alpha_j^i : i = 1, \ldots s_j; j = 1, \ldots, m\} \\ I^{(i+1)} &= \{\alpha \in I^{(i)} : d\alpha \equiv 0 \ mod \ I^{(i)}\} \end{aligned}$$

The system can be converted to Extended Goursat Normal Form if and only if:

1. $I^{(N)} = \{0\}$ *for some* N
2. *There exists a one-form* π *such that* $\{I^{(k)}, \pi\}$ *is integrable for* $k = 0, \ldots, N-1$.

An implicit assumption of the above Theorem is that the dimension of $I^{(i)}$ is constant for all i. The proofs of both these theorems can be found in [5].

An equivalent formulation of the conditions of Theorem 3 involving the annihilating distributions is given in [6]. The result is restricted to Pfaffian systems of co-dimension two.

Theorem 5 *Given a 2-dimensional distribution* Δ *construct two filtrations:*

$$\begin{array}{ll} E_0 = \Delta & F_0 = \Delta \\ E_{i+1} = E_i + [E_i, E_i] & F_{i+1} = F_i + [F_i, F_0] \end{array}$$

If all the distributions are of constant rank and:

$$dim \ E_i = dim \ F_i = i + 2 \quad i = 0, \ldots, n-2$$

there exists a local basis $\{\alpha^1, \ldots, \alpha^s\}$ *and a one-form* π *such that the Goursat congruences are satisfied for the differential system* $I = \Delta^\perp$.

In [6] this Theorem is shown to be equivalent to Theorem 3 (and consequently Theorem 4). It should be noted that the conditions involving the distributions are probably easier to check than the Goursat congruences.

2.4 The connection between the two approaches

Note that any control system in of the form 1 can also be thought of as a Pfaffian system of co-dimension $m+1$ in $\mathbb{R}^{n+m+1}$. The corresponding ideal is generated by the co-distribution:

$$I = \{dx_i - f_i(x,u)dt : i = 1, \ldots, n\} \tag{10}$$

Note that the $n+m+1$ variables for the Pfaffian system are the n states, m inputs and the time t. For the special case of the affine system 2 the co-distribution becomes:

$$I = \{dx_i - (f_i(x) + \sum_{j=1}^{m} g_{ij}(x)u_j)dt : i = 1, \ldots, n\} \tag{11}$$

In this light the Extended Goursat Normal Form looks remarkably similar to the Brunovsky Normal Form with Kronecker indices $s_j, j = 1, \ldots, m$. Indeed if we identify coordinates $z^0, z^j_{s_j+1}, j = 1, \ldots, m$ in the Goursat Normal Form with $t, u_j, j = 1, \ldots, m$, the Pfaffian system becomes equivalent (in vector field notation) to a collection of m chains of integrators, each one $s_j, j = 1, \ldots, m$ "deep" and terminating with an input in the right hand side. With this in mind, Theorems 3 and 4, that provide conditions under which a Pfaffian system can be transformed to Extended Goursat Normal Form, can be viewed as linearization theorems. Indeed:

Proposition 1 *The control system 2 satisfies the conditions of Theorem 1 if and only if the corresponding Pfaffian system 11 satisfies the conditions of Theorems 3 or 4 for $\pi = dt$.*

In order to prove this we need the following fact:

Lemma 1 *Consider a co-distribution $I^{(0)}$ and the corresponding annihilating distribution $(I^{(0)})^\perp = \Delta$. Define $I^{(1)} = \{\alpha \in I^{(0)} : d\alpha \equiv 0 \bmod I^{(0)}\}$. Then $(I^{(1)})^\perp = \{\Delta + [\Delta, \Delta]\}$.*

In other words the construction of the derived flag induces a filtration of the annihilating distributions where the next step is spanned by the vector fields of the current step and their Lie brackets.

Proof (of Proposition): Consider control system 2 and the equivalent Pfaffian system 11 and, for simplicity, assume $m = 2$. Let:

$$\begin{aligned} I^{(0)} &= \{dx_i - (f_i(x) + g_{i1}(x)u_1 + g_{i2}(x)u_2)dt : i = 1, \dots, n\} \\ \Delta_0 &= (I^{(0)})^\perp = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ f + g_1u_1 + g_2u_2 \end{bmatrix} \right\} \end{aligned}$$

As the notation suggests the first three entries in the distribution are scalars (corresponding to t, u_1 and u_2) while the last entry is a column vector of dimension n. We will construct the derived flag $I^{(0)} \supset I^{(1)} \supset \dots \supset I^{(N)}$ and the corresponding orthogonal filtration $\Delta_0 \subset \Delta_1 \subset \dots \subset \Delta_N$. We will denote by $\hat{I}^{(i)} = \{I^{(i)}, dt\}$ and $\hat{\Delta}_i = (\hat{I}^{(i)})^\perp$. We will go through the conditions of Theorem 4 step by step, assuming $\pi = dt$:

Step 0: As above:

$$\begin{aligned} I^{(0)} &= \{dx_i - (f_i(x) + g_{i1}(x)u_1 + g_{i2}(x)u_2)dt : i = 1, \dots, n\} \\ \Delta_0 &= (I^{(0)})^\perp = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ f + g_1u_1 + g_2u_2 \end{bmatrix} \right\} \\ &= \{v_1, v_2, v_3\} \end{aligned}$$

The condition of Theorem 4 requires that $\hat{I}^{(0)}$ be integrable. This is equivalent to:

$$\hat{\Delta}_0 = \{v_1, v_2\}$$

being involutive, which is true ($[v_1, v_2] = 0$ since both vector fields are constant).

Step 1: It is easy to show that:

$$[v_1, v_2] = 0 \quad [v_1, v_3] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_1 \end{bmatrix} \quad [v_2, v_3] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_2 \end{bmatrix}$$

Therefore:

$$\begin{aligned} I^{(1)} &= \{\alpha \in I^{(0)} : d\alpha \equiv 0 \bmod I^{(0)}\} \\ \Delta_1 &= (I^{(1)})^{\perp} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ f + g_1 u_1 + g_2 u_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_2 \end{bmatrix} \right\} \\ &= \{v_1, v_2, v_3, v_4, v_5\} \end{aligned}$$

The condition of Theorem 4 requires that $\hat{I}^{(1)}$ be integrable. This is equivalent to:

$$\hat{\Delta}_1 = \{v_1, v_2, v_4, v_5\}$$

being integrable. Now:

$$[v_1, v_2] = [v_1, v_4] = [v_1, v_5] = [v_2, v_4] = [v_2, v_5] = 0$$

and:

$$[v_4, v_5] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ [g_1, g_2] \end{bmatrix}$$

Therefore $\hat{\Delta}_1$ is involutive if and only if $[g_1, g_2]$ is in the span of $\{g_1, g_2\}$. Overall the condition of Theorem 4 for the the first iteration of the derived flag holds if and only if distribution G_0 of Theorem 1 is involutive.

<u>Step 2:</u>

$$\begin{aligned} [v_3, v_4] &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial f}{\partial x} g_1 + \frac{\partial g_1}{\partial x} g_1 u_1 + \frac{\partial g_2}{\partial x} g_1 u_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial g_1}{\partial x} f + \frac{\partial g_1}{\partial x} g_1 u_1 + \frac{\partial g_1}{\partial x} g_2 u_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{\partial f}{\partial x} g_1 - \frac{\partial g_1}{\partial x} f\right) + \left(\frac{\partial g_2}{\partial x} g_1 u_2 - \frac{\partial g_1}{\partial x} g_2 u_2\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ ad_f g_1 - [g_1, g_2] u_2 \end{bmatrix} \end{aligned}$$

Similarly for $[v_3, v_5]$. Therefore, assuming that the conditions of Step 1 hold and in particular that $[g_1, g_2] \in \text{span}\{g_1, g_2\}$:

$$I^{(2)} = \{\alpha \in I^{(1)} : d\alpha \equiv 0 \bmod I^{(1)}\}$$

$$
\begin{aligned}
\Delta_2 &= (I^{(2)})^{\perp} = \Delta_1 + \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ ad_f g_1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ ad_f g_2 \end{bmatrix} \right\} \\
&= \{v_i : i = 1, \ldots 7\}
\end{aligned}
$$

The condition of Theorem 4 requires that $\hat{I}^{(2)}$ be integrable. This is equivalent to:

$$\hat{\Delta}_2 = \{v_1, v_2, v_4, v_5, v_6, v_7\}$$

being involutive. As before the only pairs that can cause trouble are the ones not involving v_1 and v_2, i.e. the condition is equivalent to:

$$\{g_1, g_2, ad_f g_1, ad_f g_2\}$$

being involutive. Overall the condition of Theorem 4 for the the second iteration of the derived flag holds if and only if distribution G_1 of Theorem 1 is involutive.

Step i: Assume that:

$$\Delta_{i-1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ f + g_1 u_1 + g_2 u_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ ad_f^k g_1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ ad_f^k g_2 \end{bmatrix} \right\}$$

for $0 \leq k \leq i-2$. Also assume that $\hat{I}^{(k)}$, $0 \leq k \leq i-1$ are integrable, or, equivalently, that $\hat{\Delta}_k$ for $0 \leq k \leq i-1$ (which is the same as Δ_k without the third vector field) are involutive, or, equivalently, that $G_k = \{ad_f^l g_j : 0 \leq l \leq k, j = 1, 2\}$ for $0 \leq k \leq i-2$ are involutive. Construct $\Delta^i = \Delta^{i-1} + [\Delta^{i-1}, \Delta^{i-1}]$. By involutivity of $\hat{\Delta}_{i-1}$ and the construction of the filtration the only terms not already in Δ^{i-1} are ones of the form:

$$
\begin{aligned}
&\left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ f + g_1 u_1 + g_2 u_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ ad_f^{i-2} g_1 \end{bmatrix} \right] = \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial f}{\partial x} ad_f^{i-2} g_1 + \frac{\partial g_1}{\partial x} ad_f^{i-2} g_1 u_1 + \frac{\partial g_2}{\partial x} ad_f^{i-2} g_1 u_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial ad_f^{i-2} g_1}{\partial x} f + \frac{\partial ad_f^{i-2} g_1}{\partial x} g_1 u_1 + \frac{\partial ad_f^{i-2} g_1}{\partial x} g_2 u_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{\partial f}{\partial x} ad_f^{i-2} g_1 - \frac{\partial ad_f^{i-2} g_1}{\partial x} f \right) + \left(\frac{\partial g_1}{\partial x} ad_f^{i-2} g_1 - \frac{\partial ad_f^{i-2} g_1}{\partial x} g_1 \right) u_1 + \left(\frac{\partial g_2}{\partial x} ad_f^{i-2} g_1 - \frac{\partial ad_f^{i-2} g_1}{\partial x} g_2 \right) u_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ ad_f^{i-1} g_1 + [g_1, ad_f^{i-2} g_1] u_1 + [g_2, ad_f^{i-2} g_1] u_2 \end{bmatrix}
\end{aligned}
$$

$$(\vec{q}, d_{ms}) = \Phi(P, X)$$

where d_{ms} is the mean square point matching error. The notation $\vec{q}(P)$ is used to denote the point set P after transformation by the registration vector $\vec{q}$. Thus we have found an estimation or "initial point" of our desired transformation and this will be used to guide a search process for the local minimum of the function $f(\Theta) = tr(M\Theta^T)$.

5 Steepest Descent

We will use steepest descent to find a local minimum of the function, $f(\Theta)$. The basic idea behind descent algorithm is that it begins at an initial point in the domain of the function and it searches a neighbourhood of the initial point for a new point at which the value of the function is less. The algorithm replaces the initial point by the new point and then repeats the cycle. The algorithm terminates when no significant reduction in the value of the error function can be achieved. One disadvantage of descent is that its performance is difficult to predict. The descent is not guaranteed to terminate after a fixed prespecified number of iterations. Worst still, the algorithm may become snagged in a local minimum of the error function or it may spend a long time in regions where the gradient of the error function is low.

One of the advantages of representing rotations by unit quaternions is that it is easier to preserve the normalization during the descent. The descent algorithm will be employed to find the minimum over the five-dimensional space formed by the product of the three-dimensional space of unit quaternions **q** and the two-dimensional space of unit translation vectors, **t**. The global minimum is not unique this is because $V(q,t) = V(-q,-t)$. The four choices of sign yield two rigid displacements {R,t}, {R,-t}.

The strategy in the descent algorithms is to begin with an estimate (q,t) of the rigid displacement and to calculate the effects of small perturbations Δq, Δt on the value of V. The perturbation producing the greatest decrease in the value of V(q,t) is used to update (q,t) according to the scheme [1]

$$(q, t) \mapsto (q + \Delta q, t + \Delta t)$$

In addition to reducing the value of V the update is chosen so that

$$||q + \Delta q|| = ||q|| = 1$$

$$||t + \Delta t|| = ||t|| = 1$$

The Taylor series expansion of the error function V about the point (q,t) to the second order is

$$V(q+\delta q, t+\delta t) = V(q,t) + l\delta q + m\delta t + \delta q^T L \delta q + 2\delta q^T M \delta t + \delta t^T N \delta t + R_3 \quad (5)$$

where l, m are vectors and L, M, N are matrices. The term R_3 in (5) is third order in δt, δq.

Their are two closely related forms of descent. Both forms are based on (5). First order descent uses these terms on the right-hand side of (5) up to first order in δq or δt and second order descent uses the terms up to second order in δq or δt.

In order to find the descent equation for our registration problem it is necessary to find the gradient vector field on SO(n). Thus we define the Riemannian metric on SO(n) by restricting the Frobenious norm on $R^{n \times n}$ to the tangent space, $T_x SO(n)$, $\forall x \in SO(n)$. Since each tangent vector in $T_x SO(n)$ can be written as $x\Omega$ where $\Omega \in so(n)$ is skew symmetric the norm becomes

$$< \Omega_1, \Omega_2 > = tr(\Omega_1^T \Omega_2)$$

This inner product defines a Riemannian metric on the manifold GL(n) and by using this, it becomes possible to pass from the linear term of (4) to a vector field on GL(n). By restricting Θ to the orthogonal group, O(n), we find that the linear function $< \Theta^T M, \bullet >$ represents the gradient of $f(\Theta)$. Projecting this linear functional onto the tangent space at Θ and setting $\Theta^T \dot{\Theta}$ equal to this results in the descent equation for $f(\Theta)$:

$$\dot{\Theta} = (\Theta M^T \Theta - M) \qquad (6)$$

Since O(n) is a compact group, the solution of (6) will approach a stationary point. In fact, because of stability considerations the equilibrium point will be a stationary point as well as a local minimum of $f(\Theta)$.

Let $M = Q\Theta^T N$ where Q and N are fixed $n \times n$ symmetric matrices then (6) becomes

$$\dot{\Theta} = -(-\Theta N \Theta^T Q \Theta + Q \Theta N) \qquad (7)$$

By using the following change of variables we can rewrite equation (7) as one which evolves in the space of symmetric matrices [5]. Let $H = \Theta^T Q \Theta$. Then

$$\begin{aligned} \dot{H} &= -(-\Theta^T Q(\Theta N \Theta^T Q \Theta - Q \Theta N) - (\Theta^T Q \Theta N \Theta^T - N \Theta^T Q) Q \Theta) \\ &= -(-HNH + H^2 N - HNH + NH^2) \\ &= -[H, [H, N]] \end{aligned}$$

where [A, B] = AB - BA denotes the matrix Lie bracket.

6 Simulation and Results

In order to implement matching problem A we used the data below. The model point set, $\{x_i\}$, and data point set, $\{p_i\}$, used consisted of vertices of a rectangle in Euclidean 3-space. The correspondence between the two point sets is known and the data point set was aligned with the model point set via the techniques described in this paper.

$\{x_i\}$	$\{p_i\}$
(0,0,0)	(3,1,1)
(0,2,0)	(1,1,1)
(2,2,0)	(1,3,1)
(2,0,0)	(3,3,1)
(0,0,2)	(3,1,3)
(0,2,2)	(1,1,3)
(2,2,2)	(1,3,3)
(2,0,2)	(3,3,3)

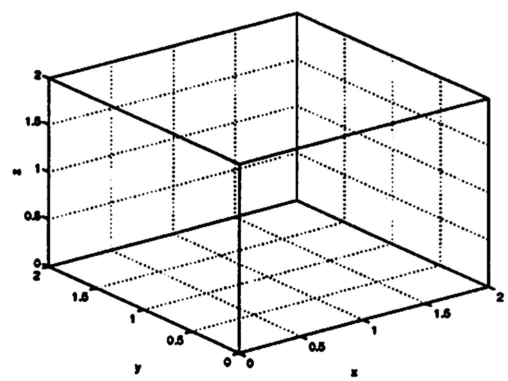

Figure 3: Rectangle reconstructed from point sets

In order to test the effectiveness of our methods the points of the data and model sets were chosen to represent a rotation of 90 degrees and a translation of 1 units along the x,y and z-axes. See figure 3. The rotation matrix and translation vector corresponding to the data point set was estimated using unit quaternions and found to be the following:

$$R(q_R) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q_T = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

The descent equation (6) for $f(\Theta) = tr(M\Theta^T)$ was solved using 2nd and 3rd order Runge-Kutta formulas provided by Matlab. The elements of $R(q_R)$ were used to form an initial value column-vector needed to solve the matrix equation. The resulting transformation was found to be the following:

$$R = \begin{pmatrix} -.5771 & -.2116 & -.7884 \\ .5775 & -.5771 & -.5771 \\ -.5771 & -.7884 & .2116 \end{pmatrix}, \quad t = \begin{pmatrix} 3.1539 \\ 2.5767 \\ 3.1539 \end{pmatrix}$$

where R was constructed from the resulting column-vector and t computed using equation (4). Finally, these values of R and t were used in equation (3) and minimized the function $V(R,t)$ to the value of 38.7471. Next we wished to solve the descent equation (6) analytically for a equilibrium point by setting the right-hand side equal to zero. Noting that $f(\Theta) = tr(M\Theta^T) = tr(M^T\Theta)$ descent equation (6) was rewritten as

$$\dot{\Theta} = (\Theta M \Theta - M^T). \tag{8}$$

Consider the following orthogonal matrix

$$\Theta = (M^T M)^{1/2} M^{-1}, \quad M = \sum p_i x_i^T \tag{9}$$

where the square root is the symmetric, positive definite square root. Substituting Θ into equation (8) we obtain the following

$$\begin{aligned} \dot{\Theta} &= (M^T M)^{1/2} M^{-1} M (M^T M)^{1/2} M^{-1} - M^T \\ &= (M^T M)^{1/2} I_3 (M^T M)^{1/2} M^{-1} - M^T \end{aligned}$$

$$= (M^T M)M^{-1} - M^T$$
$$= M^T - M^T$$
$$= 0$$

Hence the Θ of equation (9) is an equilibrium point of equation (8) and thus a local minimum of $f(\Theta)$. Using the points from our data and model sets we found that

$$M = \begin{pmatrix} 16 & 24 & 16 \\ 8 & 16 & 16 \\ 16 & 16 & 24 \end{pmatrix},$$

Finally computing Θ via equation (9) we found that

$$\Theta = \begin{pmatrix} .5774 & -.5774 & .5774 \\ .7887 & .5774 & -.2113 \\ -.2113 & .5774 & .7887 \end{pmatrix}, \; t = \begin{pmatrix} 1.4226 \\ .8453 \\ .8453 \end{pmatrix}$$

where the translation vector t was computed using equation (4). These values of Θ and t were used in equation (3) and minimized the function $V(\Theta, t)$ to the value of 13.5248.

In order to test matching problem B the below data was used. The model and data sets are defined as before. However, the correspondence between the two point sets is unknown. In order to find the permutation π which satisfies Theorem 1 we first simplified the problem by partitioning both the model and data sets into two disjoint subsets. The first subset consisted of vertices on the top face of the rectangle and the second subset consisted of those on the bottom face. Since the axis of rotation is the z-axis the points on these two faces are invariant. We then matched the corresponding subsets by finding two permutations π_t and π_b.

$\{x_t\}$	$\{x_b\}$	$\{p_t\}$	$\{p_b\}$
(0,2,0)	(2,0,2)	(3,1,1)	(3,1,3)
(2,2,0)	(2,2,2)	(1,1,1)	(1,1,3)
(2,0,0)	(0,2,2)	(1,3,1)	(1,3,3)
(0,0,0)	(0,0,2)	(3,3,1)	(3,3,3)

We found that

$$\pi_t(1) = 3, \quad \pi_b(1) = 1$$
$$\pi_t(2) = 4, \quad \pi_b(2) = 4$$
$$\pi_t(3) = 1, \quad \pi_b(3) = 3$$
$$\pi_t(4) = 2, \quad \pi_b(4) = 2$$

Thus the permutation matrix is

$$\Pi = (e_3 \; e_4 \; e_1 \; e_2 \; e_5 \; e_8 \; e_7 \; e_6)$$

where e_i denotes the i-th unit column vector in E^8. After computing Q_i and N_i as described in theorem 1 and equating Θ to Π we minimized the function $n(\Theta)$ to the value of 72.

7 Conclusion

We have shown how to apply the matrix differential equation $\dot{H} = [H,[H,N]]$ to solve a matching problem in the following computer vision scenario: Given 3-D data in a sensor coordinate system, which describes a data shape that may correspond to a model shape, and given a model shape in a model coordinate system in a different geometric shape representation, estimate the optimal rotation and translation that aligns, or registers, the model shape and the data shape minimizing the distance between the shapes and thereby allowing determination of the equivalence of the shapes via a **least squares matching** criterion. We used a rigidity constraint to simplify the registration and **unit quaternions** to represent the rotation matrix associated with a matching transformation which resulted in a good initial point of an error function. A **descent equation** was used to find a local minimum for this function and thus give rise to the geometric matching transformation. Our simulation shows that using the stationary point $\Theta = (M^T M)^{1/2} M^{-1}$ in matching problem A gives a better alignment than iteratively solving the descent equation. In addition, matching problem B was simplified by partitioning the model and data sets and finding permutations for corresponding subsets. The partition was based on information about the geometry of the point sets.

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FEEDBACK LINEARIZATION OF NONLINEAR CONTROL SYSTEMS: AN EXTERIOR DIFFERENTIAL SYSTEMS APPROACH

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1. Introduction

In this report we review the existing results in the feedback linearization of nonlinear systems. In particular, we look at the body of papers which use tools from theory of exterior differential systems to study the feedback linearization problem. The use of thoery of exterior differential systems as compared to the traditional differential geometric thoery to study the dynamic feedback linearization problem gives a different angle to the problem.

Consider a nonlinear control system with n states and p controls

$$\frac{dx}{dt} = f(x,u,t) \quad x \in \Re^n, \quad u \in \Re^p. \tag{1.1}$$

where the state $x \in \Re^n$, the controls $u \in \Re^p$ and the derivative of the state is taken with respect to time $t \in \Re$. The static state feedback linearization ($\mathcal{SFL}$) and the dyanamic state feedback linearization ($\mathcal{DFL}$) problem may be stated as follows

$\mathcal{SFL}$: *For the control system (*1.1*), find static state feedback controls $u = \alpha(x) + \beta(x)v$ and a coordinate transformation $z = \phi(x)$, such that in the new coordinates z, the system can be expressed in Brunovsky normal form with v as the new set of inputs.*

$\mathcal{DFL}$: *For the control system (*1.1*), find a general dynamic compensator of the form*

$$\dot{w} = a(x,w) + B(x,w)v$$
$$u = \alpha(x,w) + \beta(x,w)v$$

and an extended state space diffeomorphism $z = \phi(x,w)$, such that in these extended coordinates the control system can be expressed in Brunovsky normal form with v as the new set of inputs.

The $\mathcal{SFL}$ problem was first solved in [9]. For the systems which are not static feedback linearizable, the problem of partial feedback linearization has been solved in [6], where the largest static feedback linearizable subsystem contained in 1.1 is obtained. Necessary and sufficient conditions for $\mathcal{SFL}$ problem are given in [10]. In the exterior differential systems formulation, a simple algorithm has been proposed to solve the $\mathcal{SFL}$ problem by [1]. This algorithm can find the necessary feedback transformations to solve the $\mathcal{SFL}$ problem and is algebraically and computationally much simpler. In this work, the control system is considered to be autonomous and affine in inputs, and it is shown that the resulting linearizing controls are also autonomous and affine in the new inputs. We give an overview of the procedure followed in [1] and explain the algorithm with an example.

When the linearization problem cannot be solved using static feedback, the use of dynamic feedback provides an appealing alternative. The $\mathcal{DFL}$ problem was first stated in full generality in [11]. [12] establishes that the $\mathcal{DFL}$ problem is equivalent to solving the $\mathcal{SFL}$ problem when the number of inputs in the control systems is one i.e., $p = 1$. Necessary and sufficient conditions when $p = n - 1$ are also obtained. Although necessary and sufficient conditions for the $\mathcal{DFL}$ problem are not available in general, several variations of this problem are analysed and a significant amount of literature is available. One important variation that has been studied extensively is the dynamic extension. A dynamic extension of a nonlinear control system is an augmented systems with integrators added to the inputs. A differential algebraic theory is used to study the dynamic extension problem in [2]. The dynamic extension problem is also stated in [12]. In [3], [8] notions such as prolongations and absolute equivalence from the exterior differential systems theory are used to study the feedback linearization problem. This provides a different approach to looking at the dynamic feedback linearization problem as compared to the traditional differential geometry approach. The dynamic extension problem is equivalent to prolongation by differentiation in exterior differential systems. We review the existing literature for solving the $\mathcal{DFL}$ problem in a exterior differential systems setting.

The remainder of the report is organized as follows. In section 2 we give an overview of the GS algorithm and show an example to explicitly compute the input transformations for solving the $\mathcal{SFL}$ problem. In section 3 we define $\mathcal{DFL}$ problem in an exterior differential equations setting and give the tools which would be used to analyze this problem.

2. The GS Algorithm

The control system (1.1) generates a Pfaffian system I on $\Re^{n+p+1}$

$$I = \{dx^1 - f^1(x,u)dt, dx^2 - f^2(x,u)dt, \cdots, dx^n - f^n(x,u)dt\} \tag{2.1}$$

Definition 2.1. *A control system is a Brunovsky normal if there are integers* $k_1 \geq \cdots \geq k_p > 0$ *(called the Kronecker indexes) and independent functions*

$$t, x_1^1, \cdots, x_{k_1}^1, x_1^2, \cdots, x_{k_2}^2, \cdots, x_1^p, \cdots, x_{k_p}^p, u^1, \cdots, u^p$$

such that the associated Pfaffian system I has generators of form

$$\omega_1^1 = dx_1^1 - x_2^1 dt, \omega_2^1 = dx_2^1 - x_3^1 dt, \cdots, \omega_{k_1}^1 = dx_{k_1}^1 - u^1 dt$$
$$\omega_1^2 = dx_1^2 - x_2^2 dt, \omega_2^2 = dx_2^2 - x_3^2 dt, \cdots, \omega_{k_2}^2 = dx_{k_2}^2 - u^2 dt$$

$$\cdots$$

$$\omega_1^p = dx_1^p - x_2^p dt, \omega_2^p = dx_2^p - x_3^p dt, \cdots, \omega_{k_p}^p = dx_{k_p}^p - u^p dt$$

The goal is to find a transformation that puts the original system into the Brunovsky normal form. The Brunovsky normal form is a special case of extended Goursat normal form [5] with independence condition dt.

Definition 2.2. *Consider a Pfaffian system $I = \{\omega^1, \omega^2, \cdots, \omega^s\}$. The first derived system is the collection of 1-forms $\gamma \in I$ which satisfy the Frobenius conditions $d\gamma \equiv 0 \ mod \ I$*

$$I^{(1)} = \{\gamma \in I ; d\gamma \equiv 0 \ mod \ I\}$$

The process can be continued until at some finite integer N for which $I^{(N+1)} = I^{(N)}$. $I^{(N)}$ is called the bottom derived system and N is called the derived length.

The derived flag of (2.1) can be analyzed by calculating the exterior derivatives of the generators. For $1 \leq i \leq p$ and $1 \leq j \leq k_i - 1$

$$\begin{aligned} d\omega_j^i &= dt \wedge dx_{j+1}^i \\ &= dt \wedge (dx_{j+1}^i - x_{j+2}^i dt) \\ &= dt \wedge \omega_{j+1}^i \end{aligned}$$

and for $j = k_i$

$$d\omega_{k_i}^i = dt \wedge du^i$$

and the derived flag can be expressed as a collection of p towers, and the jth tower is defined to be

$$\begin{array}{llll} \omega_1^j & \omega_2^j & \cdots & \omega_{k_j}^j \\ & \vdots & & \\ \omega_1^j & \omega_2^j & & \\ \omega_1^j. & & & \end{array}$$

From the p towers, the top row of each tower taken together generate I, and the succeeding rows generate the derived flag of I. Notice that the derived length of I is k_1 by construction. Now define an integer m_j as

$$\begin{aligned} m_j &= \dim I^{(j)}/I^{(j+1)} \\ &= \text{number of towers with atleast } j+1 \text{ rows.} \end{aligned}$$

Therefore m_j is the largest integer such that $k_{m_j} \geq j + 1$. Notice that each k_j is determined by the integers m_j.

Theorem 2.1. *A control system can be put into Brunovsky normal form by a nonlinear feedback transformation if and only if there is a set of generators for its associated Pfaffian system satisfying the following congruences modulo $I^{(j)}$ for $1 \leq j \leq k_1$.*

$$
\begin{aligned}
d\omega^1_{k_1-j} &\equiv dt \wedge \omega^1_{k_1-j+1} \\
\cdots \quad & \quad \cdots \\
d\omega^p_{k_{m_j}-j} &\equiv dt \wedge \omega^p_{k_{m_j}-j+1}.
\end{aligned}
$$

The proof of this theorem is given in detail in [1]. Notice that the congruences in the theorem (2.1) are exactly the extended Goursat congruences with an independence condition dt. Now consider the control system

$$
\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \\ \dot{x}^4 \\ \dot{x}^5 \end{bmatrix} = \begin{bmatrix} sinx^2 \\ sinx^3 \\ (x^4)^3 \\ x^5 + (x^4)^3 - (x^1)^{10} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \tag{2.2}
$$

The associated Pfaffian system has generators

$$I = \{\omega^1, \omega^2, \omega^3, \omega^4, \omega^5\}$$

where

$$
\begin{aligned}
\omega^1 &= dx^1 - sinx^2 dt \\
\omega^2 &= dx^2 - sinx^3 dt \\
\omega^3 &= dx^3 - ((x^4)^3 + u^1)dt \\
\omega^4 &= dx^4 - (x^5 + (x^4)^3 - (x^1)^{10})dt \\
\omega^5 &= dx^5 - u^2 dt.
\end{aligned}
$$

Step 1: The derived flag of I is given by

$$I^{(1)} = \{\omega^1, \omega^2, \omega^4\}, \quad I^{(1)} = \{\omega^1\}, \quad I^{(3)} = \{0\}$$

The Kronecker indices can be obtained by looking at the derived flag of I. Since there are two control inputs, there are two towers i.e., two Kronecker indices k_1 and k_2. Since there are 5 states $k_1 + k_2 = 5$. The first tower has two generators in its second row, but dim $I^{(1)} = 3$ and dim $I^{(2)} = 1$, yielding $k_2 = 2$ and hence $k_3 = 3$.

Step 2 : We compute the two towered structure determining the congruences. First we start with $I^{(2)}$, since $\omega^1 \in I^{(2)}$ set $\omega^1_1 = \omega^1$, so

$$d\omega^1_1 = dt \wedge cosx^2 dx^2 = dt \wedge cosx^2 \omega^2$$

Now set second term in the above as $\omega^1_2 = cosx^2 \omega^2$. We started with $I^{(2)}$ and since the dimension of $I^{(1)} = 3$, we have to start a new tower with a generator which is a complement of $\{\omega^1, \omega^2\} \in I^{(1)}$ namely $\omega_1^2 = \omega^4$. To obtain ω_3^1 and ω_2^2, we calculate the exterior derivatives, $d\omega_1^2$ and $d\omega_2^1$ modulo $I^{(1)}$. Thus,

$$
\begin{aligned}
d\omega_2^1 &= dt \wedge (-sinx^2 sinx^3 dx^2 + cosx^2 cosx^3 dx^3) \\
&\equiv dt \wedge (cosx^2 cosx^3)\omega^3 \mod I^{(1)} \\
d\omega_1^2 &= dt \wedge (dx^5 + 3(x^4)^2 dx^4 - 10(x^1)^9 dx^1) \\
&\equiv dt \wedge \omega^5 \mod \mod I^{(1)}.
\end{aligned}
$$

Set $\omega_3^1 = (cosx^2 cosx^3)\omega^3$ and $\omega_2^2 = \omega^5$.

Step 3: Modify generators ω_j^i to obtain new generators for which the congruences are replaced by equalities, so that the two towers are decoupled. The first congruence for $d\omega_1^1$ is already an equality so ω_2^1 need not be changed. Now to make $d\omega_2^1$ and $d\omega_1^2$ into equalities we have

$$
\begin{aligned}
d\omega_2^1 &= dt \wedge (-sinx^2 sinx^3 \omega_2^1 + cosx^2 cosx^3)\omega_3^1) \\
d\omega_1^2 &= dt \wedge (\omega_2^2 + 3(x^4)^2 \omega_1^2 - 10(x^1)^9 \omega_1^1).
\end{aligned}
$$

Now ω_3^1 is modified to

$$\bar{\omega}_3^1 = (-sinx^2 sinx^3 \omega_2^1 + cosx^2 cosx^3)\omega_3^1)$$

and ω_2^2 is modified to

$$\bar{\omega}_2^2 = (\omega_2^2 + 3(x^4)^2 \omega_1^2 - 10(x^1)^9 \omega_1^1).$$

Step 4: Obtain the linearizing coordinates and linearizing controls. Since $\omega_1^1 = dx^1 - sinx^2 dt$, we set the first two linearizing coordinates as $y_1^1 = x_1^1$ and $y_2^2 = sinx^2$. Now write $\omega_2^1 = dy^1)_2 - y_3^1 dt$, so $y_3^1 = cosx^2 sinx^3$. To find the first linearizing control, write $\bar{\omega}_3^1 = dy_3^1 - v^1 dt$. Therefore, the coefficient of dt in $\bar{\omega}_3^1$ is the first linearizing control, which is

$$v^1 = sinx^2 sin^2 x^3 - cosx^2 cosx^3((x^4)^3 + u^1).$$

. In a similar fashion, the linearizing coordinates can be found to be $y_1^2 = x^4$ and $y_2^2 = x^5 + (x^4)^3 - (x^1)^{10}$, and the linearizing control is

$$v^2 = 3(x^4)^2(x^5 + (x^4)^3 - (x^1)^{10}) - 10(x^1)^9 sinx^2 + u^2.$$

3. Dynamic Feedback Linearization

There are several tools from exterior differential systems that are used to study the $(\mathcal{DFL})$ problem. The notions such as prolongations and absolute equivalence are essential for studying this problem.

Definition 3.1. *Let I be a Pfaffian system on a manifold M. Let π be a canonical projection such that $\pi : M \times \Re^p \to M$. A Pfaffian system J on $M \times \Re^p$ is a Cartan prolongation of the system if the following hold.*

(1) $\pi^*(I) \subset J$

(2) *For every solution curve* $c : (-\epsilon, \epsilon) \to M$ *of* I *there exists a unique solution curve* $\tilde{c} : (-\epsilon, \epsilon) \to M \times \Re^p \to M$ *of* J *with* $\pi \circ \tilde{c} = c$.

Notice that Cartan prolongations can be used to study the equivalence between systems of differential equations on manifolds of different dimensions. Also there will be a one-to-one correspondence between solution trajectories of the original system and the prolonged system as long as the solutions are smooth. In exterior differential systems the independent variable time in the control systems becomes another coordinate on the base manifold along with the dependent variable. So dt is introduced as an independence condition. An independence condition is a one-form that is not allowed to vanish along the solution trajectories of the system. Notice that if the original system I has an independence condition τ, then $\pi^*\tau$ is the independence condition for the prolonged system. The extended coordinates, $y \in \Re^p$, are called the fiber coordinates.

If I is a Pfaffian system with an independence condition dt on a manifold M and du an integrable one-form in the complement of I, then $J = \{I, du - ydt\}$ is called a first-order prolongation by differentiation. If I is a Pfaffian system of codimension $p+1$ then the p-th order prolongation by differentiation is called the total prolongation. Recall that dynamic extension of a control system is an augmented system with integrators added to the inputs. So prolongation by differentiation is exactly dynamic extension in the exterior differential systems perspective.

Definition 3.2. *Two Pfaffian systems* I_1, I_2 *are called absolutely equivalent if they have Cartan prolongations* J_1, J_2 *that are equivalent i.e., there exists a diffeomorphism* ϕ *such that* $\phi^*(J_2) = J_1$.

Another variation of the $(\mathcal{DFL})$ problem is the endogenous dynamic dynamic feedback problem. The dynamic feedback given in $(\mathcal{DFL})$ is said to be endogenous if w and v can be expressed as functions of x, u, t and a finite number of derivatives of u.

We state two results for endogenous feedback from [3].

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Control of Systems on Lie Groups

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1 Introduction

These notes were developed from the second part of an Advanced Topics in Control Theory course taught at U.C. Berkeley in the fall of 1994.

The first chapter describes some of the mathematics of matrix Lie groups in a self-contained manner. The second chapter introduces control systems with left-invariant vector fields on matrix Lie groups. The examples are restricted to $SO(3)$ and $SE(3)$, although a section about the Wei-Norman formula discusses how one may deal with the higher dimension case. Some recent work by Walsh, Sarti, and Sastry [8] about steering algorithms on SO(3) is also described.

2 Mathematical Preliminaries

This chapter describes some of the main topics in the mathematics of matrix Lie groups. The coverage is by no means exhaustive; its purpose is to provide a good base for the applications in the next chapter.

2.1 Groups, Fields, and Algebras

We begin with a set of definitions.

Definition 1 (Group) *A group G is a set with a binary operation $(\cdot)$: $G \times G \rightarrow G$, such that, $\forall a, b, c$ in G, the following properties are satisfied:*

1. *associativity:* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. $\exists$ *an identity* $e \ni a \cdot e = e \cdot a = a$
3. $\exists$ *an inverse* $a^{-1} \ni a \cdot a^{-1} = a^{-1} \cdot a = e$.

A group G is called *abelian* if $a \cdot b = b \cdot a$, $\forall a, b$ in G.

Definition 2 (Homomorphism) *A homomorphism between groups, ϕ : $G \rightarrow H$, is a map which preserves the group operation:*

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b).$$

Definition 3 (Isomorphism) *An isomorphism is a homomorphism which is bijective.*

Definition 4 (Field) *A field K is a set with two binary operations: addition $(+)$, and multiplication $(\cdot)$, such that:*

1. *K is an abelian group under $(+)$, with identity* 0
2. *$K - \{0\}$ is an (abelian) group under $(\cdot)$, with identity* 1
3. *$(\cdot)$ distributes over $(+)$* $\ni a \cdot (b + c) = a \cdot b + a \cdot c$.

Some examples of fields are presented below.

$\mathbf{R}$ is a field with addition and multiplication defined in the usual way.

$\mathbf{R}^2$, with addition defined in the usual way and with multiplication defined as:

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_2)$$

for $\{x_1, x_2, y_1, y_2\}$ in $\mathbf{R}$, is not a field. Why not? If it were, we would have:

$$\begin{aligned}
(1,0)\cdot(0,1) &= (0,0)\\
(1,0)^{-1}\cdot(1,0)\cdot(0,1) &= (1,0)^{-1}\cdot(0,0)\\
(0,1) &= (0,0).
\end{aligned}$$

This is clearly a contradiction. $\mathbf{R}^2$ can be made into a field if we define $(\cdot)$ as $(x_1,x_2)\cdot(y_1,y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)$. We denote this field as $\mathbf{C}$, the set of complex numbers, where $(x_1,x_2) = x_1 + \imath x_2$.

If we relax the requirement that $K - \{0\}$ be an abelian group under multiplication, we may define the *quaternions* as a field. This field is denoted $\mathbf{H}$, for *Hamiltonian field.*

The quaternions $\mathbf{H}$ are the set of 4-tuples $(x_1,x_2,x_3,x_4) = (x_1 + \imath x_2 + \jmath x_3 + k x_4)$ with addition defined in the usual way, and multiplication defined according to the following table:

$(\cdot)$	1	$\imath$	$\jmath$	k
1	1	$\imath$	$\jmath$	k
$\imath$	$\imath$	-1	k	$-\jmath$
$\jmath$	$\jmath$	$-k$	-1	$\imath$
k	k	$\jmath$	$-\imath$	-1

In the following we will be defining similar constructions for each of the fields $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$. For ease of notation, we denote the set as $\mathbf{K} \in \{\mathbf{R},\mathbf{C},\mathbf{H}\}$.

We write $\mathbf{K}^n$ as the set of all n-tuples whose elements are in $\mathbf{K}$. If we denote $\psi : \mathbf{K}^n \to \mathbf{K}^n$ as a linear map, then ψ has matrix representation $M_n(\mathbf{K}) \in \mathbf{K}^{n\times n}$. $\mathbf{K}^n$ and $M_n(\mathbf{K})$ are both vector spaces over $\mathbf{K}$.

Definition 5 (Algebra) *An algebra is a vector space with a multiplication operation which distributes over addition.*

$M_n(\mathbf{K})$ is an algebra with multiplication defined as the usual multiplication of matrices: for $A, B, C \in M_n(\mathbf{K})$,

$$\begin{aligned}
A(B+C) &= AB + AC\\
(B+C)A &= BA + CA.
\end{aligned}$$

Definition 6 (Unit) *If $\mathcal{A}$ is an algebra, $x \in \mathcal{A}$ is a unit if there exists $y \in \mathcal{A}$ such that $xy = yx = 1$.*

If $\mathcal{A}$ is an algebra with an associative multiplication operation, and $U \in \mathcal{A}$ is the set of units in $\mathcal{A}$, then U is a group with respect to this multiplication operation.

2.2 Matrix Groups

The class of groups whose elements are $n \times n$ matrices is introduced in this section.

General and Special Linear Groups

- The group of units of $M_n(\mathbf{K})$ is the set of matrices M for which $\det(M) \neq 0$, where 0 is the additive identity of $\mathbf{K}$. This group is called the *general linear group* and denoted by $GL(n, \mathbf{K})$.
- $SL(n, \mathbf{K}) \subset GL(n, \mathbf{K})$ is the subgroup of $GL(n, \mathbf{K})$ whose elements have determinant 1. $SL(n, \mathbf{K})$ is called the *special linear group.*

Orthogonal Matrix Groups

- $O(n, \mathbf{K}) \subset GL(n, \mathbf{K})$ is the subgroup of $GL(n, \mathbf{K})$ whose elements matrices A satisfy the orthogonality condition: $\bar{A}^T = A^{-1}$, where $\bar{A}^T$ is the complex conjugate transpose of A.

Examples of orthogonal matrix groups are:

$O(n) \equiv O(n, \mathbf{R})$ is called the *orthogonal group.*

$U(n) \equiv U(n, \mathbf{C})$ is called the *unitary group.*

$Sp(n) \equiv Sp(n, \mathbf{H})$ is called the *symplectic group.*

Note that for $A \in GL(n, \mathbf{H})$, $\bar{A}$ denotes the complex conjugate of the quaternion, defined by conjugating each element, using

$$\overline{x + \imath y + \jmath z + kw} = x - \imath y - \jmath z - kw.$$

An equivalent way of defining the symplectic group is as a subset of $GL(2n, \mathbf{C})$, such that

$$Sp(n) = \{B \in GL(2n, \mathbf{C}) : B^T J B = J; \bar{B}^T = B^{-1}\},$$

where the matrix J is called the *infinitesimal symplectic matrix*, and is written as:

$$J = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix}.$$

Special Orthogonal Matrix Groups

- $SO(n) = O(n) \cap SL(n, \mathbf{R})$ is the set of all orthogonal matrices of determinant 1. It is called the *special orthogonal group.*
- $SU(n) = U(n) \cap SL(n, \mathbf{C})$, the set of all unitary matrices of determinant 1, is called the *special unitary group.*

Euclidean Matrix Groups

- The *Euclidean group* is the set of matrices $E(n)$ such that

$$E(n) = \{A \in \mathbf{R}^{(n+1)\times(n+1)} : A = \begin{bmatrix} R & p \\ 0^{1\times n} & 1 \end{bmatrix}, R \in GL(n), p \in \mathbf{R}^n\}.$$

- The *special Euclidean group* is the set of matrices $SE(n)$ such that

$$SE(n) = \{A \in \mathbf{R}^{(n+1)\times(n+1)} : A = \begin{bmatrix} R & p \\ 0^{1\times n} & 1 \end{bmatrix}, R \in SO(n), p \in \mathbf{R}^n\}.$$

Proposition 1 *Let $G \subset M_n(\mathbf{K})$ be a matrix group. Let $\gamma : [a, b] \rightarrow G$ be a curve with $0 \in (a, b)$ and $\gamma(0) = I$. Let T be the set of all tangent vectors $\gamma'(0)$ to curves γ. Then T is a real subspace of $M_n(\mathbf{K})$.*

Proof: If γ and σ are two curves in G, then $\gamma'(0)$ and $\sigma'(0)$ are in T. Also, $\gamma\sigma$ is a curve in G with $(\gamma\sigma)(0) = \gamma(0)\sigma(0) = I$.

$$\begin{aligned} \frac{d}{du}(\gamma(u)\sigma(u)) &= \gamma'(u)\sigma(u) + \gamma(u)\sigma'(u) \\ (\gamma\sigma)'(0) &= \gamma'(0)\sigma(0) + \gamma(0)\sigma'(0) \\ &= \gamma'(0) + \sigma'(0). \end{aligned}$$

Since $\gamma\sigma$ is in G, $(\gamma\sigma)'(0)$ is in T. Therefore, $\gamma'(0)\sigma(0) + \gamma(0)\sigma'(0)$ is in T, and T is closed under vector addition.

Also, if $\gamma'(0) \in T$ and $r \in \mathbf{R}$, if we let $\sigma(u) = \gamma(ru)$, then $\sigma(0) = \gamma(0) = I$ and $\sigma'(0) = r\sigma'(0)$. Therefore, $r\sigma'(0) \in T$, and T is closed under scalar multiplication. □

Definition 7 (Dimension of a Matrix Group) *The dimension of the matrix group G is the dimension of the vector space T of tangent vectors to G at I.*

We now introduce a family of matrices which we will use to determine the dimensions of our matrix groups. Let $so(n)$ denote the set of all skew-symmetric matrices in $M_n(\mathbf{R})$,

$$so(n) = \{A \in M_n(\mathbf{R}) : A^T + A = 0\}.$$

Similarly, the set

$$su(n) = \{A \in M_n(\mathbf{C}) : \bar{A}^T + A = 0\}$$

denotes the skew-hermitian matrices, and the set

$$sp(n) = \{A \in M_n(\mathbf{H}) : \bar{A}^T + A = 0\}$$

denotes the skew-symplectic matrices. We also define

$$sl(n) = \{A \in M_n(\mathbf{R}) : \text{trace}(A) = 0\},$$

and

$$se(n) = \{A \in \mathbf{R}^{(n+1)\times(n+1)} : A = \begin{bmatrix} \hat{w} & p \\ 0 & 0 \end{bmatrix}, \hat{w} \in SO(n), p \in \mathbf{R}^n\}.$$

Now consider the orthogonal matrix group. Let $\gamma : [a, b] \rightarrow O(n)$, such that $\gamma(u) = A(u)$, where $A(u) \in O(n)$. Therefore, $A^T(u)A(u) = I$. Taking the derivative of this identity with respect to u, we have:

$$A'^T(u)A(u) + A^T(u)A'(u) = 0.$$

Since $A(0) = I$,

$$A'^T(0) + A'(0) = 0.$$

Thus, the vector space $T_I O(n)$ of tangent vectors to $O(n)$ at I is a subset of the set of skew-symmetric matrices, $so(n)$:

$$T_I O(n) \subset so(n).$$

Similarly, we can derive

$$\begin{aligned} T_I U(n) &\subset su(n) \\ T_I Sp(n) &\subset sp(n) \end{aligned}$$

Armed with our definition of the dimension of a matrix group, we conclude that:

$$\begin{aligned} \dim O(n) &\leq \dim so(n) \\ \dim U(n) &\leq \dim su(n) \\ \dim Sp(n) &\leq \dim sp(n) \end{aligned}$$

We will now show that these inequalities are actually equalities.

Definition 8 (Exponential and Logarithm) *The matrix exponential function, $exp : M_n(\mathbf{K}) \rightarrow M_n(\mathbf{K})$, is defined in terms of the Taylor series expansion of the exponential:*

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

The matrix logarithm $log : M_n(\mathbf{K}) \rightarrow M_n(\mathbf{K})$ is defined only for matrices near the identity matrix I:

$$\log X = (X - I) - \frac{(X - I)^2}{2} + \frac{(X - I)^3}{3} - \ldots$$

Proposition 2 $A \in so(n) \Longrightarrow e^A \in SO(n)$.

Proof: $(e^A)^T = e^{A^T} = e^{A^{-1}} = (e^A)^{-1}$, therefore, $e^A \in O(n)$. Using $\det(e^A) = e^{\text{trace}(A)}$, we have $\det(e^A) = e^0 = 1$. □

Similarly,

$$\begin{aligned} A \in su(n) &\Longrightarrow e^A \in U(n); \\ A \in sp(n) &\Longrightarrow e^A \in Sp(n); \\ A \in sl(n) &\Longrightarrow e^A \in SL(n); \\ A \in se(n) &\Longrightarrow e^A \in SE(n). \end{aligned}$$

Proposition 3 $X \in SO(n) \Longrightarrow \log(X) \in so(n)$.

Proof: Noting that $\log(XY) = \log(X) + \log(Y)$ iff $XY = YX$, we take the logarithm on both sides of the equation: $XX^T = X^TX = I$. Thus, $\log(X) + \log(X^T) = 0$, so $\log(X) \in so(n)$. □

Similarly,

$$\begin{aligned} X \in U(n) &\Longrightarrow \log(X) \in su(n); \\ X \in Sp(n) &\Longrightarrow \log(X) \in sp(n); \\ X \in SL(n) &\Longrightarrow \log(X) \in sl(n); \\ X \in SE(n) &\Longrightarrow \log(X) \in se(n). \end{aligned}$$

The logarithm and exponential thus define maps which send a matrix group G to its tangent space T, and vice versa.

Definition 9 (One Parameter Subgroup) *A* <u>*one parameter subgroup*</u> γ *of a matrix group* G *is a smooth homomorphism* $\gamma : \mathbf{R} \to G$.

The group operation in $\mathbf{R}$ is addition, thus, $\gamma(u+v) = \gamma(u) \cdot \gamma(v)$. Since $\mathbf{R}$ is an abelian group under addition, we have that

$\gamma(u+v) = \gamma(v+u) = \gamma(u) \cdot \gamma(v) = \gamma(v) \cdot \gamma(u)$.

Note that by defining γ on some small neighbourhood U of $0 \in \mathbf{R}$, γ is defined over all $\mathbf{R}$, since for any $x \in \mathbf{R}$, some $\frac{1}{n}x \in U$ and $\gamma(x) = (\gamma(\frac{1}{n}x))^n$.

Proposition 4 *If $A \in M_n(\mathbf{K})$, then e^{Au} is a one parameter subgroup.*

Proof: Noting that $e^{X+Y} = e^X e^Y$ iff $XY = YX$, we have

$$e^{A(u+v)} = e^{Au+Av} = e^{Au}e^{Av},$$

since A commutes with itself. □

Proposition 5 *Let γ be a one parameter subgroup of $M_n(\mathbf{K})$. Then there exists $A \in M_n(\mathbf{K})$ such that $\gamma(u) = e^{Au}$.*

Proof: Define $A = \sigma'(0)$, where $\sigma(u) = \log \gamma(u)$, (ie. $\gamma(u) = e^{\sigma(u)}$). We need to show that $\sigma(u) = Au$, a line through 0 in $M_n(\mathbf{K})$.

$$\begin{aligned}
\sigma'(u) &= \lim_{v \to 0} \frac{\sigma(u+v) - \sigma(u)}{v} \\
&= \lim_{v \to 0} \frac{\log \gamma(u+v) - \log \gamma(u)}{v} \\
&= \lim_{v \to 0} \frac{\log \gamma(u)\gamma(v) - \log \gamma(u)}{v} \\
&= \lim_{v \to 0} \frac{\log \gamma(v)}{v} \\
&= \sigma'(0) \\
&= A.
\end{aligned}$$

Therefore, $\sigma(u) = Au$. □

So, given any element in the tangent space of G at I, its exponential belongs to G.

Proposition 6 *Let $A \in T_I O(n, \mathbf{K})$, the tangent space at I to $O(n, \mathbf{K})$. Then there exists a unique one parameter subgroup γ in $O(n, \mathbf{K})$ with $\gamma'(0) = A$.*

Proof: $\gamma(u) = e^{Au}$ is a one parameter subgroup of $GL(n, \mathbf{K})$, and γ lies in $O(n, \mathbf{K})$ since $\gamma(u)^T\gamma(u) = (e^{Au})^T e^{Au} = I$. □

Thus,

$$\dim O(n, \mathbf{K}) \geq \dim so(n, \mathbf{K}).$$

But we have shown using our definition of the dimension of a matrix group that

$$\dim O(n, \mathbf{K}) \leq \dim so(n, \mathbf{K}).$$

Therefore,

$$\dim O(n, \mathbf{K}) = \dim so(n, \mathbf{K}),$$

and the tangent space, at I, to $O(n, \mathbf{K})$ is exactly the set of skew-symmetric matrices.

Now the dimension of $so(n, \mathbf{R})$ is easily computable: we simply find a basis. Let E_{ij} be the matrix whose entries are all zero except the ij^{th} entry, which is 1, and the ji^{th} entry, which is -1. Then E_{ij}, for $i < j$, form a basis for $so(n)$. There are $\frac{n(n-1)}{2}$ of these basis elements. Therefore, $\dim O(n) = \frac{n(n-1)}{2}$.

Similarly,

$$\begin{aligned} \dim SO(n) &= \frac{n(n-1)}{2} \\ \dim U(n) &= n^2 \\ \dim SU(n) &= n^2 - 1 \\ \dim Sp(n) &= n(2n+1). \end{aligned}$$

2.3 Matrix Lie Groups and their Lie Algebras

We start our discussion of matrix Lie groups with some definitions from differential geometry.

Definition 10 (Topological Space) *A* <u>*topological space*</u> *is a set M with a collection of subsets $\mathcal{T}$ of M having the properties:*

1. *$\emptyset$ and M are in $\mathcal{T}$;*
2. *the union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$;*
3. *the intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*

$\mathcal{T}$ *is called a* topology *on* M.

Definition 11 (Homeomorphism) *A homeomorphism f between two topological spaces M and N is a bijective, continuous map $f : M \rightarrow N$ with a continuous inverse $f^{-1} : N \rightarrow M$.*

Definition 12 (Manifold) *An n-manifold is a topological space M with the property that, if $x \in M$, then there is some neighbourhood U of x such that U is homeomorphic to $\mathbf{R}^n$.*

Definition 13 (Chart, Atlas, Maximal Atlas) *A chart (φ, U) on an n-manifold M is an open set U of M and a homeomorphism $\varphi : U \rightarrow \mathbf{R}^n$. Two charts, (φ, U) and (ϕ, V), are said to have smooth overlap if the maps $\phi^{-1} \circ \varphi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ and $\varphi^{-1} \circ \phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ are smooth. A family of charts which covers M and whose members have smooth overlap is called an atlas. A maximal atlas for M is an atlas which contains the maximum number of charts.*

Definition 14 (Differentiable Manifold) *A differentiable manifold is a manifold with an associated maximal atlas.*

The atlas allows us to perform calculus on the manifold: the charts in the atlas provide explicit homeomorphisms which refer the manifold to $\mathbf{R}^n$, a space in which we know how to integrate and differentiate. The requirement that the charts have smooth overlap guarantees that these operations are well-defined over the whole manifold.

We may characterize a smooth function $f : M \rightarrow N$, where M is an m-manifold and N is an n-manifold, according to the corresponding atlases on M and N. Let (φ, U) be a chart on M and (ϕ, V) be a chart on N. Let $p \in M$ such that U is an open neighbourhood of p and V is an open neighbourhood of $f(p)$. Then f is said to be smooth at p if

$$\phi \circ f \circ \varphi^{-1} : \mathbf{R}^m \rightarrow \mathbf{R}^n$$

is smooth at $\varphi(p)$.

We now introduce the concept of tangent vectors to manifolds. In $\mathbf{R}^n$, tangent vectors to smooth surfaces are easy to picture, however smooth surfaces and tangent vectors in an arbitrary manifold are not as intuitive to visualize.

Let M be a differentiable m-manifold with $p \in M$. Let

$$A(p) = \{(W, f) : p \in W, W \text{ open in } M, f : W \to \mathbf{R} \text{ is smooth}\}.$$

Define vector addition and scalar multiplication on $A(p)$ as, for $(W_1, f_1), (W_2, f_2) \in A(p)$ and $r \in \mathbf{R}$,

$$\begin{aligned} (W_1, f_1) + (W_2, f_2) &= (W_1 \cap W_2, f_1 + f_2) \\ r(W_1, f_1) &= (W_1, r f_1). \end{aligned}$$

Under these operations, $A(p)$ is a real vector space. We make $A(p)$ into an algebra by defining vector multiplication as

$$(W_1, f_1)(W_2, f_2) = (W_1 \cap W_2, f_1 f_2).$$

Definition 15 (Tangent Vectors) *A tangent vector ξ to M at p is a linear map $\xi : A(p) \to \mathbf{R}$ satisfying, for $f, g \in A(p)$:*

1. *if $f = g$ in a neighbourhood of p, then $\xi(f) = \xi(g)$;*
2. *$\xi(fg) = f(p)\xi(g) + \xi(f)g(p)$.*

The second condition above is called the *derivation law.*

Consequences of this definition:

- If f is a constant function ($f(q) \equiv r, \forall q \in U$), then $\forall$ tangent vectors ξ, $\xi(f) = 0$.

  Proof:

$$\begin{aligned} \xi(f \cdot g) &= f(p)\xi(g) + \xi(f)g(p) \\ \xi(r \cdot g) &= r\xi(g) + \xi(f)g(p) \end{aligned}$$

  Therefore, $\xi(f)g(p) = 0$ since $\xi(r \cdot g) = r\xi(g)$. Since this holds $\forall g$, we have $\xi(f) = 0$.

- If $f(p) = g(p) = 0$ then $\xi(fg) = 0$.

Tangent vectors are operators which act on functions: if γ is a smooth curve in a manifold M, then γ gives rise to a linear function

$$\xi \equiv \gamma_*(t) : A(p) \to \mathbf{R}$$

defined by

$$\xi(t)(f) = \gamma_*(t)(f) = (f \circ \gamma)'(t),$$

which may be described as the directional derivative of f at p in the direction of γ. The *tangent space* of M at a point p, denoted T_pM, is the set of all tangent vectors to M at p.

Proposition 7 *T_pM is a real vector space of dimension m, the dimension of M.*

Proof: With the definition of vector addition and scalar multiplication on tangent vectors as follows, for $\xi, \eta \in T_p(M), r \in \mathbf{R}$,

$$\begin{aligned} (\xi + \eta)(f) &= \xi(f) + \eta(f) \\ (r\xi)(f) &= r\xi(f), \end{aligned}$$

it is easy to verify that T_pM is a real vector space.

We now prove that the dimension of T_pM is m.

Let (φ, U) be a chart on M, where $p \in \varphi(U) \subset M$ and $U \in \mathbf{R}^m$. Suppose that M sits in the ambient space $\mathbf{R}^N$, and assume that $\varphi(0) = p$. The best approximation to $\varphi : U \to M$ at 0 is the map:

$$\varphi(u) = \varphi(0) + d\varphi_0(u) = x + d\varphi_0(u).$$

Recall that φ^{-1} is a smooth map from M to $\mathbf{R}^m$ Choose an open set W in $\mathbf{R}^N$ and a smooth map $\Phi' : \mathbf{R}^N \to \mathbf{R}^m$ that extends φ^{-1}. Thus $\Phi' \circ \varphi$ is the identity map of U, so, by the chain rule,

$$\mathbf{R}^m \xrightarrow{d\varphi_0} T_p(M) \xrightarrow{d\Phi'_p} \mathbf{R}^m$$

is the identity map of $\mathbf{R}^m$. Therefore, $d\varphi_0 : \mathbf{R}^m \to T_p(M)$ is an isomorphism, and the dimension of $T_p(M)$ is m. □

Definition 16 (Smooth Vector Fields) *A smooth vector field X on a manifold M is an assignment of $X_p \in T_pM$ for each $p \in M$, such that, if $f : M \to \mathbf{R}$ is a smooth function, then*

$$(Xf)_p \equiv X_p(f) : M \to \mathbf{R}$$

is smooth over p.

The smooth vector fields on M form a real vector space. Indeed, it is easy to check that if X and Y are smooth vector fields, then $X + Y$ is a smooth vector field, since $(X + Y)(f) = X(f) + Y(f)$. If X is a smooth vector field and $r \in \mathbf{R}$ then rX is smooth, where $rX(f) = r(X(f))$.

Definition 17 (Integral Curve) *Let $c : [0,1] \to M$ be a curve on the differential manifold M, and let X be a smooth vector field on M. The curve c is said to be an integral curve of the vector field X if*

$$\dot{c} = X(c(t)).$$

Vector fields thus represent differential equations on manifolds.

The space of smooth vector fields becomes an algebra under the appropriate multiplication operation. If we have two smooth vector fields X and Y, let us define

$$(X \circ Y)(p)(f) \equiv X_p((Yf)_p).$$

Now $(Yf)_p$ is a smooth function from M to $\mathbf{R}$, thus

$$X_p(Yf)_p : A(p) \to \mathbf{R}.$$

However, $X_p(Yf)_p$ may not necessarily be a tangent vector. Consider an example in which

$$\begin{aligned} M &= \mathbf{R}^n, \\ X &= \frac{\partial}{\partial x_1}, \\ Y &= \frac{\partial}{\partial x_2}. \end{aligned}$$

Thus,

$$X_p(Yf)_p = \frac{\partial^2 f}{\partial x_1 \partial x_2}|_p;$$

and, for this example, the derivation law is not satisfied. Therefore, $(X \circ Y)$ is not a tangent vector, so the vector space of smooth vector fields is not closed under this operation.

The candidate multiplication operation under which the vector space of smooth vector fields becomes an algebra must therefore somehow cancel these mixed partial derivatives.

Proposition 8 *For smooth vector fields X, Y, the operator*

$$f \longmapsto X_p(Yf) - Y_p(Xf)$$

is a tangent vector.

Proof: We prove only that the operator defined above satisfies the derivation law:

$$\begin{aligned} X_p(Y(fg)) &= X_p(f(Y(g)) + Y(f)g) \\ &= X_p(f)Y_p(g) + f(p)X_p(Y(g)) + X_p(Y(f))g(p) + Y_p(f)X_p(g). \end{aligned}$$

There is a symmetric formula for $Y_p(X(fg))$. Thus

$$X_p(Y(fg)) - Y_p(X(fg)) = (X_pY - Y_pX)(f)g(p) + f(p)(X_pY - Y_pX)(g). \quad \square$$

We now have a multiplication operation which makes smooth vector fields on M into algebras. Let $[X, Y] = XY - YX$ denote the vector field defined by $[X, Y]_p = X_pY - Y_pX$.

Definition 18 (Lie Algebra) *A Lie algebra is a real vector space, V, with a multiplication operation $[\ ,\ ]$ which satisfies, for $A, B \in V$,*

1. $[A, B] = -[B, A]$;
2. $[A, B + C] = [A, B] + [A, C]$,
   $[A + B, C] = [A, C] + [B, C]$;

3. *for* $r \in \mathbf{R}$, $r[A,B] = [rA,B] = [A,rB]$;

4. $[A,[B,C]] + [B,[A,C]] + [C,[A,B]] = 0$.

The fourth condition is called the *Jacobi Identity.*

Proposition 9 *The set* $\mathcal{L}(M)$ *of smooth vector fields on a differentiable manifold* M *forms a Lie Algebra under* $[\ ,\ ]$.

Proof: We have shown that $\mathcal{L}(\mathcal{M})$ is a vector space, and it is a matter of substitution to show that $[X,Y]_p = X_pY - Y_pX$ satisfies the four properties listed above. □

The multiplication operation $[X,Y]$ is called the *Lie bracket* of X and Y.

Definition 19 (Lie Group) *A* *Lie group* *is a group* G *which is also a differentiable manifold such that, for* $a,b \in G$,

1. $(a,b) \longmapsto ab$

2. $a \longmapsto a^{-1}$

are smooth functions.

All finite dimensional Lie groups may be represented as matrix groups. For example, since the function $det : \mathbf{R}^{n^2} \to \mathbf{R}$ is continuous, the matrix group $GL(n,\mathbf{R}) = \det^{-1}(\mathbf{R} - \{0\})$ is open. It can be given a differentiable structure which makes it an open submanifold of $\mathbf{R}^{n^2}$. Multiplication of matrices in $GL(n,\mathbf{R})$ is continuous, and smoothness of the inverse map follows from Cramer's Rule. Thus, $GL(n,\mathbf{R})$ is a Lie group. Similarly, $O(n), SO(n), E(n)$, and $SE(n)$ are Lie groups.

In order to study the algebras associated with matrix Lie groups, the concepts of differential maps and left translations are first introduced.

Let M,N be differentiable manifolds and let $M \xrightarrow{\psi} N$ be a smooth map. Then ψ induces a linear map $T_p(M) \xrightarrow{d\psi} T_{\psi(p)}N$:

$$(d\psi \circ \xi)(f) = \xi(f \circ \psi),$$

for $\xi \in T_pM, f \in A(\psi(p))$. The map $d\psi$ is called the *differential* of ψ.

Let G be a Lie group with identity I, and let X_I be a tangent vector to G at I. We may construct a vector field defined on all of G in the following way. For any $g \in G$, define the *left translation* by g to be a map $L_g : G \to G$ such that $L_g(x) = gx$, where $x \in G$. Since G is a Lie group, L_g is a diffeomorphism of G for each g. Taking the differential of L_g at e results in a map from the tangent space of G at e to the tangent space of G at g:

$$dL_g : T_eG \to T_gG$$

such that

$$X_g = dL_g(X_e).$$

The vector field formed by assigning $X_g \in T_gG$ for each $g \in G$ is called a *left invariant* vector field.

Proposition 10 *If X and Y are left invariant vector fields on G, then so is $[X, Y]$.*

Proof: Let $g \in G$ and $f \in A(g)$.

$$\begin{aligned} dL_g[X,Y]_e(f) &= [X,Y]_e(f \circ L_g) \\ &= X_e(Y(f \circ L_g)) - Y_e(X(f \circ L_g)) \\ &= dL_gX_e(Yf) - dL_gY_e(Xf)) \\ &= X_g(Yf) - Y_g(Xf) \\ &= [X,Y]_g(f). \quad \square \end{aligned}$$

Also, if X and Y are left invariant vector fields, then $X + Y$ and $rX, r \in \mathbf{R}$ are also left invariant vector fields on G. Thus, the left invariant vector fields of G form an algebra under $[\ ,\ ]$, which is called the *Lie algebra* of G and denoted $\mathcal{L}(G)$. $\mathcal{L}(G)$ is actually a subalgebra of the Lie algebra of all smooth vector fields on G.

With this notion of a Lie group's associated Lie algebra, we can now look at the Lie algebras associated with some of our matrix Lie groups. We first look at three examples, and then, in the next section, study the general map from a Lie algebra to its associated Lie group.

Examples:

- The Lie algebra of $GL(n, \mathbf{R})$ is denoted $gl(n, \mathbf{R})$, the set of all $n \times n$ real matrices. The tangent space of $GL(n, \mathbf{R})$ at the identity can be identified with $\mathbf{R}^{n^2}$ since $GL(n, \mathbf{R})$ is an open submanifold of $\mathbf{R}^{n^2}$. The Lie bracket operation is simply $[A, B] = AB - BA$, matrix multiplication.

- The special orthogonal group $SO(n)$ is a submanifold of $GL(n, \mathbf{R})$, so $SO(n)_I$ is a subspace of $GL(n, \mathbf{R})_I$. The Lie algebra of $SO(n)$, denoted $so(n)$, may thus be identified with a certain subspace of $\mathbf{R}^{n^2}$. We have shown in the previous section that the tangent space at I to $SO(n)$ is the set of skew- symmetric matrices; it turns out that we may identify $so(n)$ with this set. For example, for $SO(3)$, the Lie algebra is:

$$so(3) = \left\{ \hat{w} \equiv \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}, w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\}.$$

  The Lie bracket on $so(n)$ is defined as $[\hat{w}_a, \hat{w}_b] = (\widehat{w_a \times w_b})$, the skew-symmetric matrix form of the vector cross product.

- The Lie algebra of $SE(3)$, called $se(3)$, is defined as follows:

$$se(3) = \left\{ \hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \quad w, v \in \mathbf{R}^3 \right\}.$$

  The Lie bracket on $se(3)$ is defined as

$$[\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_1\hat{\xi}_2 - \hat{\xi}_2\hat{\xi}_1 = \begin{bmatrix} (\widehat{w_1 \times w_2}) & w_1 \times v_2 - w_2 \times v_1 \\ 0 & 0 \end{bmatrix}.$$

2.4 The Exponential Map

In computing the dimension of $O(n, \mathbf{K})$ in Section 2.2, we showed that for each matrix A in $O(n, \mathbf{K})_I$, there is a unique one parameter subgroup γ in $O(n, \mathbf{K})$, with $\gamma(u) = e^{Au}$, such that $\gamma'(0) = A$. In this section we introduce a function

$$\exp : T_eG \rightarrow G,$$

for a general Lie group G. This map is called the *exponential map* of the Lie algebra $\mathcal{L}(G)$ into G. We then apply this exponential map to the Lie algebras of the matrix Lie groups discussed in the previous section.

Consider a general Lie group G with identity e. For every $\xi \in T_eG$, let $\phi_\xi : \mathbf{R} \to G$ denote the integral curve of the left invariant vector field X_ξ passing through e at $t = 0$. Thus,

$$\phi_\xi(0) = e$$

and

$$\frac{d}{dt}\phi_\xi(t) = X_\xi(\phi_\xi(t)).$$

One can show that $\phi_\xi(t)$ is a one parameter subgroup of G. Now the *exponential map* of the Lie algebra $\mathcal{L}(G)$ into G is defined as $\exp : T_eG \to G$ such that for $s \in \mathbf{R}$,

$$\begin{aligned} \exp(\xi s) &= \phi_\xi(s) \\ \exp(\xi) &= \phi_\xi(1). \end{aligned}$$

Thus, a line ξs in $\mathcal{L}(G)$ is mapped to a one parameter subgroup $\phi_\xi(s)$ of G.

We differentiate the map $\exp(\xi s) = \phi_\xi(s)$ with respect to s at $s = 0$ to obtain $d(\exp) : T_eG \to T_eG$ such that:

$$d(\exp)(\xi) = \phi'_\xi(0) = \xi,$$

thus, $d(\exp)$ is the identity map on T_eG.

By the inverse function theorem,

$$\exp : \mathcal{L}(G) \to G$$

is a local diffeomorphism from a neighbourhood of zero in $\mathcal{L}(G)$ onto a neighbourhood of e in G, which is denoted as G_0, the *identity component* of G.

We now discuss the conditions under which the exponential map is surjective onto the Lie group.

Definition 20 (Path Connected) *For any two points x and y of a topological space X, a path in X from x to y is a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. X is said to be path connected if every pair of points of X can be joined by a path in X.*

G_0 is path connected by construction: the one parameter subgroup $\exp(\xi s) = \phi_\xi(s)$ defines a path between any two elements in G_0.

Proposition 11 *If G is a path connected Lie group and H is a subgroup which contains an open neighbourhood U of e in G, then $H = G$.*

Proof: See Curtis [1].

We may thus conclude that if G is a path connected Lie group, then $\exp : \mathcal{L}(G) \to G$ is surjective. If G is not path connected, $\exp(\mathcal{L}(G))$ is the identity component G_0 of G.

For matrix Lie groups, the exponential map is just the matrix exponential function, e^A, where A is a matrix in the associated Lie algebra.

- For $G = SO(3)$, the exponential map $\exp \hat{w}$, $\hat{w} \in so(3)$, is given by

$$e^{\hat{w}} = I + \hat{w} + \frac{\hat{w}^2}{2!} + \frac{\hat{w}^3}{3!} + \dots,$$

  which can be written in closed form solution as:

$$e^{\hat{w}} = I + \frac{\hat{w}}{\| w \|} \sin \| w \| + \frac{\hat{w}^2}{\| w \|^2}(1 - \cos \| w \|).$$

  This is known as *Rodrigues' formula.*

- For $G = SE(3)$, the exponential map $\exp \hat{\xi}$, $\hat{\xi} \in se(3)$ is given by

$$e^{\hat{\xi}} = \begin{bmatrix} I & v \\ 0 & 1 \end{bmatrix},$$

  for $w = 0$, and

$$e^{\hat{\xi}} = \begin{bmatrix} e^{\hat{w}} & Av \\ 0 & 1 \end{bmatrix},$$

  for $w \neq 0$, where

$$A = I + \frac{\hat{w}}{\| w \|^2}(1 - \cos \| w \|) + \frac{\hat{w}^2}{\| w \|^3}(\| w \| - \sin \| w \|).$$

2.5 Canonical Coordinates on Matrix Lie Groups

Let $\{X_1, X_2, \ldots, X_n\}$ be a basis for the Lie algebra $\mathcal{L}(G)$. Since

$$\exp : \mathcal{L}(G) \to G$$

is a local diffeomorphism, the mapping $\sigma : \mathbf{R}^n \to G$ defined by

$$g = \exp\{\sigma_1 X_1 + \ldots + \sigma_n X_n\}$$

is a local diffeomorphism between $\sigma \in \mathbf{R}^n$ and $g \in G$ for g in a neighbourhood of the identity e of G. Therefore, $\sigma : U \to \mathbf{R}^n$, where $U \subset G$ is a neighbourhood of e, may be considered a coordinate mapping with coordinate chart (σ, U). Using the left translation L_g, we can construct an atlas for the Lie group G from this single coordinate chart. The functions σ_i are called the *Lie-Cartan coordinates of the first kind* relative to the basis $\{X_1, X_2, \ldots, X_n\}$.

A different way of writing coordinates on a Lie group using the same basis is to define $\theta : \mathbf{R}^n \to G$ by:

$$g = \exp X_1\theta_1 \exp X_1\theta_2 \ldots \exp X_n\theta_n$$

for g in a neighbourhood of e. The functions $(\theta_1, \theta_2, \ldots \theta_n)$ are called the *Lie-Cartan coordinates of the second kind.*

An example of a parameterization of $SO(3)$ using the Lie-Cartan coordinates of the second kind is just the product of exponentials formula:

$$\begin{aligned} R &= e^{\hat{z}\theta_1} e^{\hat{y}\theta_2} e^{\hat{x}\theta_3} \\ &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) \end{bmatrix}, \end{aligned}$$

where $R \in SO(3)$ and

$$\hat{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \hat{y} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \hat{z} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is known as the ZYX Euler angle parameterization. Similar parameterizations are the YZX Euler angles, and the ZYZ Euler angles.

A *singular configuration* of a parameterization is one in which there does not exist a solution to the problem of calculating the Lie-Cartan coordinates from the matrix element of the Lie group. For example, the ZYX Euler angle parameterization for $SO(3)$ is singular when $\theta_2 = -\pi/2$. The ZYZ Euler angle parameterization is singular when $\theta_1 = -\theta_3$ and $\theta_2 = 0$, in which case $R = I$, illustrating that there are infinitely many representations of the identity rotation in this parameterization.

2.6 The Campbell-Baker-Hausdorff Formula

The exponential map may be used to relate the algebraic structure of the Lie algebra $\mathcal{L}(G)$ of a Lie group G with the group structure of G. The relationship is described through the Campbell-Baker-Hausdorff (CBH) formula which is introduced in this section. The notions of conjugation and adjoint maps are first described. The structure of a Lie algebra in terms of its structure constants is also presented.

If M is a differentiable manifold and G is a Lie group, we define a *left action* of G on M as a smooth map $\Phi : G \times M \to M$ such that

1. $\Phi(e, x) = x$ for all $x \in M$
2. for every $g, h \in G$ and $x \in M$, $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$.

The left action of G on itself defined by $C_g : G \to G$:

$$C_g(h) = ghg^{-1} = R_{g^{-1}} L_g h$$

is called the *conjugation map* associated with g.

The derivative of the conjugation map at e is called the *adjoint map*, defined as $Ad_g : \mathcal{L}(G) \to \mathcal{L}(G)$ such that, for $\xi \in \mathcal{L}(G)$, $g \in G$,

$$Ad_g(\xi) = (T_e(C_g))(\xi) = T_e(R_{g^{-1}} L_g)(\xi).$$

If $G \subset GL(n, \mathbb{C})$, then $Ad_g(\xi) = g\xi g^{-1}$.

The lower-case adjoint map $ad_\xi : \mathcal{L}(G) \to \mathcal{L}(G)$ is defined as

$$ad_\xi(\eta) = [\xi, \eta].$$

Lemma 1 (Campbell-Baker-Hausdorff Formula) *If $x, y \in \mathcal{L}(G)$, then*

$$\begin{aligned} Ad_{e^x}y &= e^x y e^{-x} = y + [x,y] + \frac{1}{2!}[x,[x,y]] + \frac{1}{3!}[x[x,[x,y]]] + \dots \\ &= y + ad_x y + \frac{1}{2!}ad_x^2 y + \frac{1}{3!}ad_x^3 y + \dots \end{aligned}$$

The CBH formula is a measure of how much x and y fail to commute over the exponential: if $[x,y] = 0$, then $Ad_{e^x}y = y$.

If $\{X_1, X_2, \dots, X_n\}$ is a basis for the Lie algebra $\mathcal{L}(G)$, the *structure constants* of $\mathcal{L}(G)$ with respect to $\{X_1, X_2, \dots, X_n\}$ are the values $c_{ij}^k \in \mathbf{R}$ defined by:

$$[X_i, X_j] = \sum_k c_{ij}^k X_k.$$

Lemma 2 *Consider $\mathcal{L}(G)$ with basis $\{X_1, X_2, \dots, X_n\}$ and structure constants c_{ij}^k with respect to this basis. Then*

$$\prod_{j=1}^{r} \exp(p_j X_j) X_i \prod_{j=r}^{1} \exp(-p_j X_j) = \sum_{k=1}^{n} \xi_{ki} X_k,$$

where $p_j \in \mathbf{R}$ and $\xi_{ki} \in \mathbf{R}$.

Proof of 2: We first prove the lemma for $r = 1$. Using the CBH formula, write:

$$\exp(p_1 X_1) X_i \exp(-p_1 X_1) = X_i + \sum_{k=1}^{\infty} \frac{ad_{X_1}^k X_i}{k!} p_1^k.$$

The terms $ad_{X_1}^k X_i$ are calculated using the structure constants:

$$\begin{aligned} ad_{X_1} X_i &= \sum_{n_1=1}^{n} c_{1i}^{n_1} X_{n_1} \\ ad_{X_1}^2 X_i &= \sum_{n_1=1}^{n} \sum_{n_2=1}^{n} c_{1i}^{n_1} c_{1n_1}^{n_2} X_{n_2} \\ &\ \ \vdots \\ ad_{X_1}^k X_i &= \sum_{n_1=1}^{n} \sum_{n_2=1}^{n} \cdots \sum_{n_k=1}^{n} c_{1i}^{n_1} c_{1n_1}^{n_2} c_{1n_2}^{n_3} \cdots c_{1n_{k-1}}^{n_k} X_{n_k}. \end{aligned}$$

Substitute the above formula for $ad^k_{X_1}X_i$ into $X_i + \sum_{k=1}^{\infty} \frac{ad^k_{X_1}X_i}{k!} p_1^k$, and note that since each of the c^k_{ij} is finite, the infinite sum is bounded. The ξ_{ki} are consequently bounded and are functions of c^k_{ij}, $k!$, and p_1^k. The proof is similar for $r > 1$. □

3 Left Invariant Control Systems on Matrix Lie Groups

This chapter uses the mathematics developed in the previous chapter to describe control systems with left-invariant vector fields on matrix Lie groups. For an n-dimensional Lie group G, the type of system described in this section has state which can be represented as an element $g \in G$. The time differential equation which describes the evolution of g can be written as:

$$\dot{g} = g(\sum_{i=1}^{n} X_i u_i),$$

where the u_i are the inputs, and the X_i are a basis for the Lie algebra $\mathcal{L}(g)$. In the above equation, gX_i is the notation for the left invariant vector field associated with X_i. The equation represents a *driftless* system, since if $u_i = 0$ for all i, $\dot{g} = 0$.

In the first section, the state equation describing the motion of a rigid body on $SE(3)$ is developed. The second section develops a relationship, called the Wei-Norman formula, between the inputs u_i and the Lie-Cartan coordinates of the group. In the third section, the problem of steering a control system on $SO(3)$ is studied through a specific example.

3.1 Frenet-Serret Equations: A Control System on $SE(3)$

In this section, arc-length parameterization of a curve describing the path of a rigid body in $\mathbf{R}^3$ is used to derive the state equation of the motion of this left invariant system.

Consider a curve

$$\alpha(s) : [0,1] \rightarrow \mathbf{R}^3,$$

representing the motion of a rigid body in 3-space. Represent the tangent to the curve as

$$t(s) = \alpha'(s).$$

Constrain the tangent to have unity norm, $\| t(s) \| = 1$, so that

$$< t(s), t(s) >= 1.$$

Now taking the derivative of the above with respect to s, we have

$$< t'(s), t(s) > + < t(s), t'(s) >= 0,$$

so that $t'(s) \perp t(s)$. Denote the norm of $t'(s)$ as

$$\| t'(s) \| = \kappa(s),$$

where $\kappa(s)$ is called the *curvature* of the motion: it measures how quickly the curve is pulling away from the tangent. Let us assume $\kappa > 0$. Denoting the unit normal vector to the curve $\alpha(s)$ as $n(s)$, we have that

$$t'(s) = \kappa(s)n(s),$$

and also

$$< n(s), n(s) >= 1$$

so that $n'(s) \perp n(s)$.

The *binormal* to the curve at s is denoted as $b(s)$, where

$$b(s) = t(s) \times n(s),$$

or equivalently,

$$n(s) = b(s) \times t(s).$$

Let

$$n'(s) = \tau(s)b(s),$$

where $\tau(s)$, called the *torsion* of the motion, measures how quickly the curve is pulling out of the plane defined by $n(s)$ and $b(s)$. Thus

$$
\begin{aligned}
b'(s) &= t'(s) \times n(s) + t(s) \times n'(s) \\
&= \kappa(s)n(s) \times n(s) + t(s)\tau(s)b(s) \\
&= \tau(s)t(s) \times b(s) \\
&= \tau(s)n(s),
\end{aligned}
$$

where $n(s) \times n(s) = 0$.

Similarly,

$$
\begin{aligned}
n'(s) &= b'(s) \times t(s) + b(s) \times t'(s) \\
&= \tau(s)n(s) \times t(s) + \kappa(s)b(s) \times n(s) \\
&= -\tau(s)b(s) - \kappa(s)t(s).
\end{aligned}
$$

We thus have:

$$
\begin{aligned}
\alpha'(s) &= t(s) \\
t'(s) &= \kappa(s)n(s) \\
n'(s) &= -\tau(s)b(s) - \kappa(s)t(s) \\
b'(s) &= \tau(s)n(s).
\end{aligned}
$$

Since $t(s), n(s)$, and $b(s)$ are all orthogonal to each other, the matrix with these vectors as its columns is an element of $SO(3)$:

$$
[t(s),\ n(s),\ b(s)] \in SO(3).
$$

Thus,

$$
\left[\begin{array}{ccc|c} t(s) & n(s) & b(s) & \alpha(s) \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \in SE(3),
$$

and

$$
\frac{d}{ds}\left[\begin{array}{ccc|c} t(s) & n(s) & b(s) & \alpha(s) \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} t(s) & n(s) & b(s) & \alpha(s) \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 0 & -\kappa(s) & 0 & 1 \\ \kappa(s) & 0 & \tau(s) & 0 \\ 0 & -\tau(s) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].
$$

These are known at the *Frenet-Serret* equations of a curve. The evolution of the Frenet-Serret frame in $\mathbf{R}^3$ is given by

$$\dot{g} = gX,$$

where $g \in SE(3)$ and X is an element of the Lie algebra $se(3)$. We may regard the curvature $\kappa(s)$ and the torsion $\tau(s)$ as inputs to the system, so that if

$$\begin{aligned} u_1 &= \kappa(s) \\ u_2 &= -\tau(s), \end{aligned}$$

then

$$\dot{g} = g \left[\begin{array}{ccc|c} 0 & -u_1 & 0 & 1 \\ u_1 & 0 & -u_2 & 0 \\ 0 & u_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right],$$

which is a special case of the general form describing the state evolution of a left invariant control system in $SE(3)$.

An example of the general form of a left invariant control system in $SE(3)$ is given by an aircraft flying in $\mathbf{R}^3$:

$$\dot{g} = g \left[\begin{array}{ccc|c} 0 & -u_3 & u_2 & u_4 \\ u_3 & 0 & -u_1 & 0 \\ -u_2 & u_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right].$$

The inputs u_1, u_2, and u_3 control the *roll*, *pitch*, and *yaw* of the aircraft, and the input u_4 controls the velocity in the forward direction.

Specializing the above to $SE(2)$, we have the example of the unicycle rolling on the plane:

$$\dot{g} = g \left[\begin{array}{cc|c} 0 & -u_2 & 1 \\ u_2 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right].$$

In this case, the input u_2 controls the angle of the wheel.

The previous formulation describes *kinematic* steering problems since it is assumed that we have direct control of the velocities of the rigid bodies. In the control of physical systems, though, we generally only have access to the forces and torques which drive the motion. A more realistic approach would therefore be to formulate the steering problem with a *dynamic* model of the rigid body, which uses these forces and torques as inputs. Dynamic models are more complex than their kinematic counterparts, and the control problem is harder to solve.

3.2 The Wei-Norman Formula

In this section we derive the Wei-Norman formula, which describes a relationship between the open loop inputs to a system and the Lie-Cartan coordinates used to parameterize the system.

Consider the state equation of a left-invariant control system on a Lie group G with state $g \in G$:

$$\dot{g} = g(\sum_{i=1}^{n} X_i u_i),$$

where the u_i are inputs and the X_i are a basis of the Lie algebra $\mathcal{L}(g)$.

We may express g in terms of its Lie-Cartan coordinates of the 2^{nd} kind:

$$g(t) = exp(\gamma_1(t)X_1)exp(\gamma_2(t)X_2)\dots exp(\gamma_n(t)X_n).$$

Thus,

$$\begin{aligned}
\dot{g} &= \sum_{i=1}^{n} \gamma_i'(t) \prod_{j=1}^{i-1} \exp(\gamma_j X_j) X_i \prod_{j=i}^{n} \exp(\gamma_j X_j) \\
&= g \sum_{i=1}^{n} \gamma_i'(t) (\prod_{j=1}^{n} \exp(\gamma_j X_j))^{-1} X_i (\prod_{j=1}^{n} \exp(\gamma_j X_j)) \\
&= g \sum_{i=1}^{n} \gamma_i'(t) \sum_{k=1}^{n} \xi_{ki}(\gamma) X_k,
\end{aligned}$$

Where the last equation results from Lemma 2. If we compare this equation with the state equation, we may generate a formula for the inputs to the system in terms of the Lie-Cartan coordinates:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ u_n(t) \end{bmatrix} = \begin{bmatrix} & & \\ & \xi_{ij}(\gamma) & \\ & & \end{bmatrix} \begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \cdot \\ \cdot \\ \gamma_n'(t) \end{bmatrix}$$

so that

$$\begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \cdot \\ \cdot \\ \gamma_n'(t) \end{bmatrix} = \begin{bmatrix} & & \\ & \xi_{ij}(\gamma) & \\ & & \end{bmatrix}^{-1} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ u_n(t) \end{bmatrix}.$$

The above is known as the *Wei-Norman formula*. It transforms the problem from a differential equation on Lie groups to one in $\mathbf{R}^n$: steering from an initial configuration g_i to a final configuration g_f is converted into steering from $\gamma(0)$ to $\gamma(1)$, both vectors in $\mathbf{R}^n$.

3.3 Steering a Satellite on $SO(3)$

This section introduces a set of algorithms for steering on $SO(3)$. The simple structure of the group is exploited to solve for the control inputs directly, instead of using the Wei-Norman formula of the previous section to transform the problem. The algorithms and results are described in detail in the paper by Walsh, Sarti, and Sastry [8].

A satellite is a rigid body floating in space. There are *rotors* or *momentum wheels* attached to its body which create linearly independent momentum fields which rotate the satellite to any configuration in $SO(3)$. The satellite may be modelled as a drift free system on $SO(3)$:

$$\dot{g} = g\hat{b}_1 u_1 + g\hat{b}_2 u_2 + g\hat{b}_3 u_3,$$

where $g \in SO(3)$, $\hat{b}_i \in so(3)$, and the $u_i \in \mathbf{R}$ are scalars. The vector $b_i \in \mathbf{R}^3$ describes the direction and magnitude of the momentum field created by the i^{th} momentum wheel on the satellite. Given an initial state g_i and a desired final state g_f, we wish to find control inputs $u_1(t), u_2(t), u_3(t)$ which will steer the system from g_i to g_f in finite time T.

First, consider the case in which $u_i \neq 0$ for $i \in \{1,2,3\}$. Since we have assumed that the momentum fields are linearly independent, the input vector fields $\hat{b}_1 u_1$, $\hat{b}_2 u_2$, and $\hat{b}_3 u_3$ span the tangent space $so(3)$ at every point of $SO(3)$. If we assume that the inputs are constant (u_1, u_2, u_3) and applied over one second, the solution to the state equation is

$$g_f = g_i \exp(\hat{b}_1 u_1 + \hat{b}_2 u_2 + \hat{b}_3 u_3).$$

Since the exponential map is surjective onto $SO(3)$, for any initial configuration g_i and final configuration g_f, inputs (u_1, u_2, u_3) may be calculated which steer the system from g_i to g_f, hence the system is completely controllable. This is the easy case.

The second case that we consider is the two input system in which $u_1 = 0$. We claim that even without the first vector field, the system is completely controllable. An intuitive way to see why this claim is valid is to consider the ZYZ Euler angle parameterization of $SO(3)$. If rotation about the y axis corresponds to *pitch*, and rotation about the z axis corresponds to *roll*, then rotation about the x axis (*yaw*) may be generated from the two vector fields corresponding to pitch and roll. To see this, let $R \in SO(3)$ be defined as $R = g_i^{-1} g_f$. We may parameterize R as

$$R = e^{\hat{z}\theta_1} e^{\hat{y}\theta_2} e^{\hat{z}\theta_3}.$$

Recalling that the ZYZ Euler angle parameterization is singular at $R = I$, we perturb the representation about $R = I$ with respect to the angles θ_2 and θ_3 to obtain:

$$\begin{aligned}
\left.\frac{dR}{d\theta_3}\right|_{R=I} &= \hat{z} \\
\left.\frac{dR}{d\theta_2}\right|_{R=I} &= (e^{-\hat{z}\theta_3}\hat{y}e^{\hat{z}\theta_3}) \\
&= (Ad_{e^{-\hat{z}\theta_3}}\hat{y}) \\
&= \hat{y} + ad_{-\hat{z}\theta_3}\hat{y} + \frac{1}{2!}ad^2_{-\hat{z}\theta_3}\hat{y} + \dots \\
&= \hat{y} + -\theta_3[\hat{z}, \hat{y}] + \dots
\end{aligned}$$

Where the last equation results from the CBH formula. Since

$$[\hat{z}, \hat{y}] = (\widehat{z \times y}) = -\hat{x},$$

perturbations of R with respect to θ_2 produce motion in the direction of x.

Considering the same control problem as in the three input case, we proceed to calculate the required inputs $(u_1(t), u_2(t))$ through a series of steps.

If the momentum vectors b_2 and b_3 are not orthogonal, the first step is to orthogonalize their actions using the Gram-Schmidt algorithm. If v_2 corresponds to pitch and v_3 corresponds to roll, then Gram-Schmidt produces:

$$\begin{bmatrix} u_3 \\ u_2 \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{22} \end{bmatrix} \begin{bmatrix} v_3 \\ v_2 \end{bmatrix},$$

where

$$\begin{aligned} \beta_{11} &= (\| b_3 \|)^{-1}, \\ \beta_{22} &= (\| b_2 - b_2^T b_3 \beta_{11}^2 b_3 \|)^{-1}, \\ \beta_{12} &= -b_2^T b_3 \beta_{11}^2 \beta_{22}. \end{aligned}$$

Thus, defining a_1, a_2, and a_3 as the lengths of time that v_3 (roll) is applied, v_2 (pitch) is applied, and then v_3 applied again, the equation to solve is:

$$g_i^{-1} g_f = \exp(\beta_{11} \hat{b}_3 a_1) \exp(\beta_{12} \hat{b}_3 a_2 + \beta_{22} \hat{b}_2 a_2) \exp(\beta_{11} \hat{b}_3 a_3).$$

The second step is to transform the system so that the input vector fields are the canonical ones for $\mathbf{R}^3$. We construct the matrix $K \in SO(3)$:

$$K = [\beta_{11} b_3 \quad (\beta_{12} b_3 + \beta_{22} b_2) \quad (\beta_{11} b_3 \times (\beta_{12} b_3 + \beta_{22} b_2))].$$

Now $K^{-1} \beta_{11} b_3 = e_1$ and $K^{-1}(\beta_{12} b_3 + \beta_{22} b_2) = e_2$, where e_1 and e_2 are the standard first two basis vectors for $\mathbf{R}^3$. Defining the similarity transform as

$$\tilde{g}(t) = K^{-1} g_i^{-1} g(t) K,$$

and taking the derivative of the above, results in

$$\dot{\tilde{g}}(t) = \tilde{g}(t)(\hat{e}_1 v_1 + \hat{e}_2 v_2),$$

which is in the desired canonical representation.

The third step is to solve the general form of the roll-pitch-roll equation for the coordinates (a_1, a_2, a_3). Denoting

$$\sin a_1 \equiv s_{a_1}$$
$$\cos a_2 \equiv c_{a_2}$$

etc., we obtain

$$\begin{aligned}
\tilde{g}_f \equiv (g_i K)^{-1} g_f K &= \exp(\hat{e}_1 a_1)\exp(\hat{e}_2 a_2)\exp(\hat{e}_1 a_3) \\
&= \begin{bmatrix} c_{a_2} & s_{a_2} s_{a_3} & s_{a_2} c_{a_3} \\ s_{a_1} s_{a_2} & c_{a_1} c_{a_3} - s_{a_1} c_{a_2} s_{a_3} & -c_{a_1} s_{a_3} - s_{a_1} c_{a_2} c_{a_3} \\ -c_{a_1} s_{a_2} & s_{a_1} c_{a_3} + c_{a_1} c_{a_2} s_{a_3} & -s_{a_1} s_{a_3} + c_{a_1} c_{a_2} c_{a_3} \end{bmatrix}.
\end{aligned}$$

Denoting the elements of $\tilde{g}_f$ as $\tilde{g}_{ij}$, we may solve for (a_1, a_2, a_3):

$$\begin{aligned}
a_1 &= \text{atan2}(\tilde{g}_{21}, -\tilde{g}_{31}) \text{ if } \tilde{g}_{31} \neq 0 \\
&= \text{acot2}(-\tilde{g}_{31}, \tilde{g}_{21}) \text{ else} \\
a_2 &= \text{atan2}(\tilde{g}_{11} \sin(a_1), \tilde{g}_{21}) \text{ if } \tilde{g}_{21} \neq 0 \\
&= \text{atan2}(\tilde{g}_{11} \cos(a_1), -\tilde{g}_{31}) \text{ else} \\
a_3 &= \text{atan2}(\tilde{g}_{12}, \tilde{g}_{13}) \text{ if } \tilde{g}_{13} \neq 0 \\
&= \text{acot2}(\tilde{g}_{13}, \tilde{g}_{12}) \text{ else}
\end{aligned}$$

Finally, in the fourth step, the a_i calculated in the previous step are used to compute the actual controls. Assuming the system is steered from g_i to g_f in the time duration T, the controls below are applied each for a duration of $\frac{T}{3}$:

$$\begin{aligned}
(u_1, u_2)_1 &= (3\beta_{11}\frac{a_1}{T}, 0) \\
(u_1, u_2)_2 &= (3\beta_{12}\frac{a_2}{T}, 3\beta_{22}\frac{a_2}{T}) \\
(u_1, u_2)_3 &= (3\beta_{11}\frac{a_3}{T}, 0).
\end{aligned}$$

3.4 Concluding Remarks

The representation of a system as one with left-invariant vector fields on matrix Lie groups:

$$\dot{g} = g(\sum_{i=1}^{n} X_i u_i),$$

is a natural one for systems describing the motion of a rigid body with respect to a coordinate frame attached to the body. Aircraft, underwater vehicles, and satellites such as the one modelled in the previous section are all important examples of systems which may be modelled and controlled using matrix Lie groups.

The appeal of this theory is that it is mathematically simple: once the system has been modelled using matrix Lie groups, the computation of the controls required to place the system in a desired final configuration is no harder than the corresponding problem in linear system theory.

The theory of Lie groups and Lie algebras is of current interest in optimal control theory, in which a control solution is sought to minimize a prespecified cost function. The optimal control problem reduces to solving two differential equations - a problem which is theoretically simple on $\mathbf{R}^n$ but becomes very complicated on a differentiable manifold, since the differential equations are only defined locally on each coordinate chart. If the manifold is a Lie group, however, the optimal control problem may be simplified. Using the exponential map from $\mathcal{L}(G)$ to G, the problem may be formulated on the Lie algebra $\mathcal{L}(G)$ of the group. For groups such as $SO(3)$ and $SE(3)$, the Lie algebras are isomorphic to R^n, allowing for a global definition of the differential equations. Optimal control on Lie groups therefore not very much more complicated than optimal control on R^n.

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EECS290B Report:
Control and Dynamics of an Ellipsoid on a Horizontal Plane
Fall 1994

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1 Introduction

The goal of this research is to control the rolling ellipsoid on the plane between an initial point and final point in configuration space. This example is chosen since it is a dynamic system that evolves on a Lie Group and involves nonholonomic constraints. The hope of this report is to learn more about controlling nonholonomic mechanical systems and controlling systems on Lie Groups. A dynamic model of the nonholonomic system is presented and a controller is developed to control the ellipsoid to points in plane. This controller controls to points in the plane and does not control the orientation of the ellipsoid. One can take a fibre bundle point of view with the orientation being the base space and the position being the fiber direction. This controller then generates locomotion in the fiber direction by motions in the base space. A controller is then presented which controls the orientation of the ellipsoid relative to a fixed spatial frame. These two controllers are then combined in an attempt to simultaneously control both position and orientation. The controlled system is simulated using a simulation program created by Brian Mirtich and John Canny called *Impulse-Based Dynamic Simulation* [1]. The simulation results for the three control methods are presented. The report concludes with a discussion of future research on the controlled ellipsoid.

2 Dynamic Equations

The configuration space of the free rigid ellipsoid is $SE(3)$, the Euclidean Group. The system is shown in Figure 1. The spatial reference frame is fixed to the horizontal plane. For the controller design, it is assumed that the ellipsoid is in contact with the planar surface. This constraint is holonomic and reduces the dimension of the configuration space by one. The controller design also assumes that the ellipsoid is rolling without slip. The rolling constraints are two nonholonomic constraints. The configuration of the ellipsoid in contact with the plane is given by the (x, y) position of the center of mass and the orientation of a body frame fixed to the ellipsoid. The body frame is chosen to be a reference frame fixed to the ellipsoid and centered at the center of mass. The orientation of the body frame relative to the spatial frame is given by $R \in SO(3)$. Given a point in the body frame, p_b, the coordinates of the point in the spatial frame, p_s, is given by $p_s = R\, p_b + p_{cm}$, where p_{cm} is the location of the center of mass in the spatial frame.

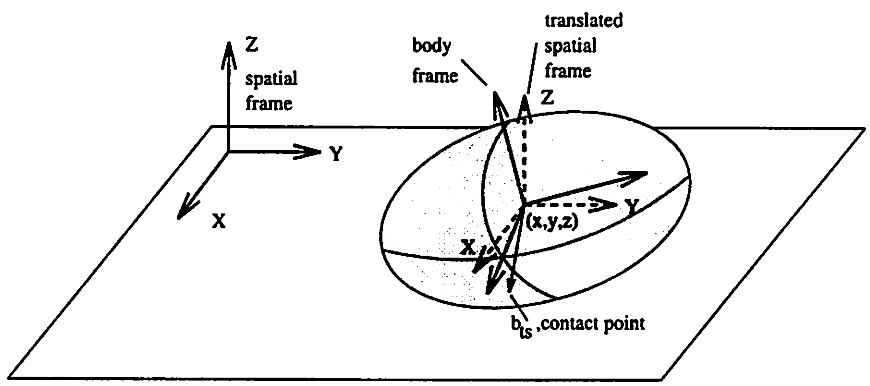

Figure 1: Rolling Ellipsoid

The control inputs are the torques about the three axes of the body frame. The method of generating these torques is not given. The dynamic equations are shown in equations (1)-(3). In the following equations, $\hat{}: \Re^3 \to so(3)$ and $\check{}: so(3) \to \Re^3$.

$$\dot{R} = R\hat{\omega}_b \tag{1}$$

$$\mathcal{I}\dot{\omega}_b = \omega_b \times \mathcal{I}\omega_b + \tau_c + \tau_f + \tau_g \tag{2}$$

$$\begin{pmatrix} \dot{x}_{cm} \\ \dot{y}_{cm} \end{pmatrix} = A(R)\,\omega_s, \tag{3}$$

where

$$A(R) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{b}_s(R),$$

$\omega_s = (\dot{R}R^T)\check{}$ is the spatial angular velocity, $\omega_b = (R^T\dot{R})\check{}$ is the body angular velocity, $\mathcal{I}$ is the diagonal inertia matrix, and b_{ts} is the contact point written in terms of the translated spatial frame, a frame with the same orientation as the spatial frame but centered at the origin of the body frame. Since the contact point is written in terms of the translated spatial frame, the contact point is only a function of the orientation. One can find the contact point in the translated frame by minimizing the Z coordinate subject to the equations of the ellipsoid surface written in the translated spatial frame. If the equation for the ellipsoid surface written in the body frame coordinates is $x_b^T E x_b = 1$, then the equation in the translated spatial frame coordinates is $x_{ts}^T R E R^T x_{ts} = 1$. Minimizing the Z coordinate subject to the equation for the ellipse gives the following equation involving b_{ts}:

$$RE^{-1}R^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b_{ts} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} b_{ts}. \tag{4}$$

Solving equation (4) for b_{ts} gives the contact point in terms of the translated spatial frame coordinates. The torque on the body is given by the controller torque, τ_c, the torque from friction at the contact point, τ_f, and the torque from gravity, τ_g. The simulation program provides the contact point location, b_{ts}, as well as τ_f and τ_g. One can calculate the messy expressions for τ_f and τ_g but this is not done in this report.

3 Controller

3.1 Position Control

The position controller attempts to drive the ellipsoid from an initial center of mass position to a final center of mass position. The orientation at the initial and final position is not controlled. Based on the current position, the controller determines a desired center of mass velocity. The controller then uses a pseudo inverse on equation (3) to determine a desired spatial angular velocity. The controller is designed to bring the spatial angular velocity to the desired angular velocity through body torques. The body torque is determined through equations (5) - (8).

$$\begin{pmatrix} \dot{x}_{cm}^{des} \\ \dot{y}_{cm}^{des} \end{pmatrix} = k_{cm}^{p} \begin{pmatrix} x_{cm}^{des} - x_{cm} \\ y_{cm}^{des} - y_{cm} \end{pmatrix} \tag{5}$$

$$\omega_s^{des} = A^{\dagger}(R) \begin{pmatrix} \dot{x}_{cm}^{des} \\ \dot{y}_{cm}^{des} \end{pmatrix} \tag{6}$$

$$\omega_b^{des} = R^T \omega_s^{des} \tag{7}$$

$$\tau_c = \mathcal{I} K_\omega^p \left(\omega_b^{des} - \omega_b \right) \tag{8}$$

Equation (5) defines the desired positional velocity in the spatial coordinates and uses the controller gain, k_{cm}^p. Equation (6) uses the pseudo inverse of $A(R)$, denoted $A^{\dagger}(R)$, and the desired center of mass velocity to calculate the desired spatial angular velocity. Equation (7) calculates the corresponding desired body angular velocity. The last equation calculates the torque based on an error between the desired and actual body angular velocities, the inertia matrix, and a controller gain matrix, K_ω^p.

The conjecture for this controller is that it drives the ellipsoid to a position "close" to the desired position. The torques from gravity and friction often prevent the controller from reaching the desired position. The desired position may be reached by "cancelling" the contribution of the torque from friction and gravity in equation (2) with the controller torque.

3.2 Orientation Control

The orientation controller drives the ellipsoid through body torques to a desired orientation from an initial orientation. The orientation of the ellipsoid is represented by quaternions, and the controller attempts to drive the error in quaternions to zero. Quaternion representation is a four parameter, two to one covering of $SO(3)$ that is free from singularities. This representation is also convenient for numerical simulation.

The desired orientation is represented by a desired quaternion value. The controller calculates an error between the desired orientation, q_{des}, and the current orientation, q. Since q_{des} represents the same orientation as $-q_{des}$, two errors are calculated. One error is based on $-q_{des}$, and the other is based on q_{des}. The error with the smaller norm is used to calculate a desired velocity in quaternion space, $\dot{q}_{des}$.

The relationship between the body angular velocity and the velocity in quaternion space is shown in equation (9).

$$\omega_b = 2 \begin{bmatrix} -q_x & q_s & q_z & -q_y \\ -q_y & -q_z & q_s & q_x \\ -q_z & q_y & -q_x & q_s \end{bmatrix} \begin{bmatrix} \dot{q}_s \\ \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix} := W(q)\,\dot{q}, \tag{9}$$

where $q = (q_s, q_x, q_y, q_z)$. This expression is used to calculate the desired body angular velocity, w_b^{des}, from a desired quaternion velocity, $\dot{q}_{des}$.

The same expression for τ_c in equation (8) is used to drive ω_b to the desired value, w_b^{des}. The controller is outlined as follows:

1. $e_1 = q_{des} - q$ and $e_2 = -q_{des} - q$.
2. $e = e_1$ if $||e_1|| \leq ||e_2||$ else $e = e_2$.
3. $\dot{q}_{des} = k_q^p e$.
4. $w_b^{des} = W(q)\,\dot{q}_{des}$.
5. $\tau_c = \mathcal{I} K_\omega^p \left(\omega_b^{des} - \omega_b\right)$

The controller gains in the orientation controller are k_q^p and K_ω^p, the same gain used in the position controller.

The claim for this controller is that it brings the orientation "close" to the desired orientation. The torque from gravity and friction prevents the controller from reaching the desired orientation. The desired orientation may be reached by "cancelling" the contribution of the torque from friction and gravity in equation (2) with the controller torque. The orientation controller is conjectured to stabilize orientation in the presence of no external torques, such as those due to friction and gravity.

3.3 Position and Orientation Control

The goal of this report is to control to a point in configuration space using the three body torques. The controller must control to a desired orientation of the ellipsoid at a desired position in the plane. The desired configuration is five dimensional and there are only three inputs.

A first attempt at controlling both position and orientation is to manually switch between the two control modes. The simulation results of one switching strategy is shown in the next section.

4 Simulations

The simulation results are presented in this section. The ellipsoid has the long axis along the body x-axis. The longest distance between two points along the x-axis is 20cm. The length of

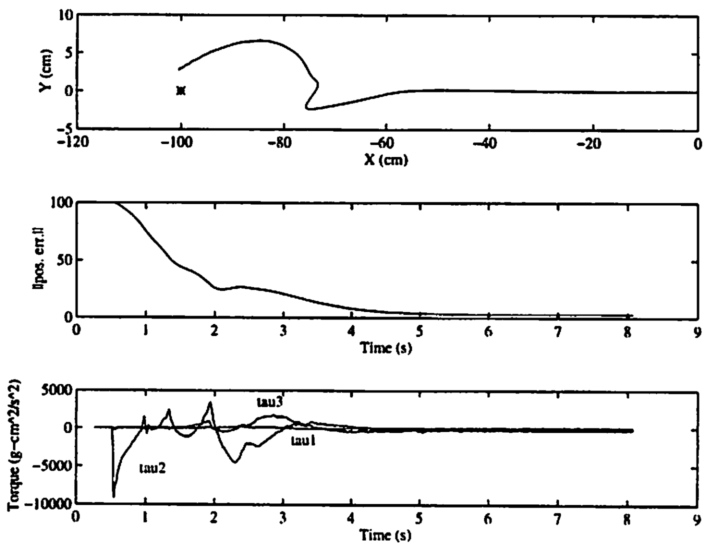

Figure 2: Controlling the Position of the Center of Mass

the two remaining axes are 10cm. The mass of the ellipsoid is 1.0Kg. The inertias are 10.0g cm^2 about the long axis and 25.0g cm^2 about the two remaining axes. The ellipsoid in the simulation is approximated by a polygon. The controller gains in the simulations are $k^p_{cm} = 1.0\frac{1}{s}$, $K^p_\omega = 25.0\frac{1}{s\ rad}$, and $k^p_q = 5.0\frac{1}{s}$.

4.1 Position Controller Results

In the first simulation, the ellipsoid is placed at the origin and commanded to reach $x_{cm} = -100.0\text{cm}, y_{cm} = 0.0\text{cm}$. The resulting motion of the ellipsoid is to initially roll in the direction of the long axis followed rotation about the long axis. The results of this simulation are shown in Figure 2. The first graph is the position in the XY plane. The asterisk is the desired center of mass position. The second graph is the norm of the position error as a function of time. The body torques are shown in the last graph. The ellipsoid does not reach the desired location, but the norm of the position error decreases to 2.85cm. The final position is $x_{cm} = -100.37\text{cm}, y_{cm} = 2.83\text{cm}$. The fact that the controller does not converge to the desired position may be due to the the controller torques being unable to overcome the effects of gravity and friction.

4.2 Orientation Controller Results

In this simulation, the ellipsoid is placed at the origin, and the position controller is activated to disturb the orientation. The position controller is then turned off and the orientation controller is turned on at approximately 1.8s. The simulation results are shown in Figure 3. The first graph is the norm of the error in the quaternion variables. The second graph is the torques over time. The orientation controller drives the error to a value close to zero. The norm of the error is 0.0220 at the end of the simulation. The error not converging to zero may be a result of the polygonal model, and the effects of gravity.

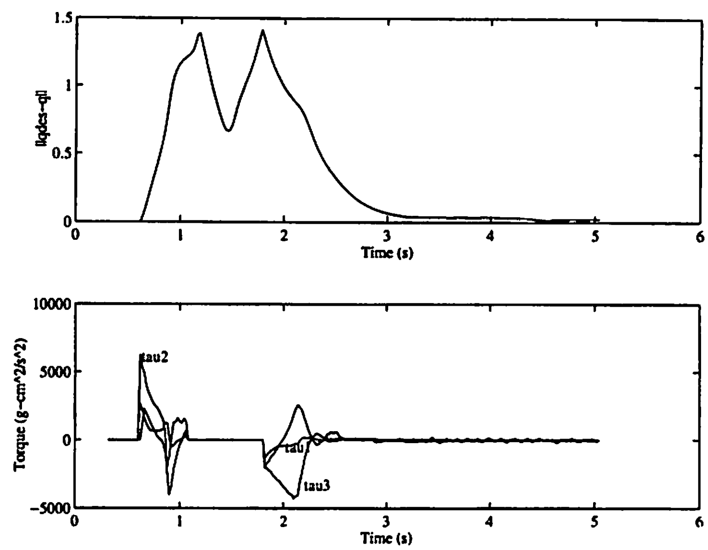

Figure 3: Controlling the Orientation

4.3 Position and Orientation Controller Results

In this simulation, the ellipsoid is commanded to $x_{cm} = 50.0$cm, $y_{cm} = -70.0$cm from the origin at the initial orientation. The simulation results are shown in Figure 4. The position in the plane is shown followed by the norm of the position error. The next graph is the norm of the orientation error. The last graph is the torques over time.

Initially, the position controller is switched on and the orientation controller is turned off. After the desired center of mass position is approximately reached, the position controller is turned off and the orientation controller is turned on. At this point, the orientation error converges close to zero but the position error increases. The orientation controller is turned off and the position controller is turned back on. The norm of the position error of the ellipsoid decreases again, but is greater than the value at 7s. The position controller is then turned off followed by the orientation controller turning on and then off. Finally, the position controller is turned on followed by the orientation controller turning on as well. The resulting norm in position and orientation error is between the maximum and minimum errors experienced during the switching. Using a switching controller came out of a discussion with John Lygeros.

5 Future Work and Conclusions

These relatively simple controllers produce reasonable results for position control and orientation control. The position controller effectively generates locomotion through internal torques. The effectiveness of the controllers are not proven and the simulations show that the position controller does not reach the final position in some cases. The position controller performs well even when the no-slip condition is not obeyed. Simulations have been performed on a slippery plane and the controller managed to bring the ellipsoid "close" to the desired position. The orientation controller seems to drive the error in quaternion space to zero for the orientation chosen. This orientation is

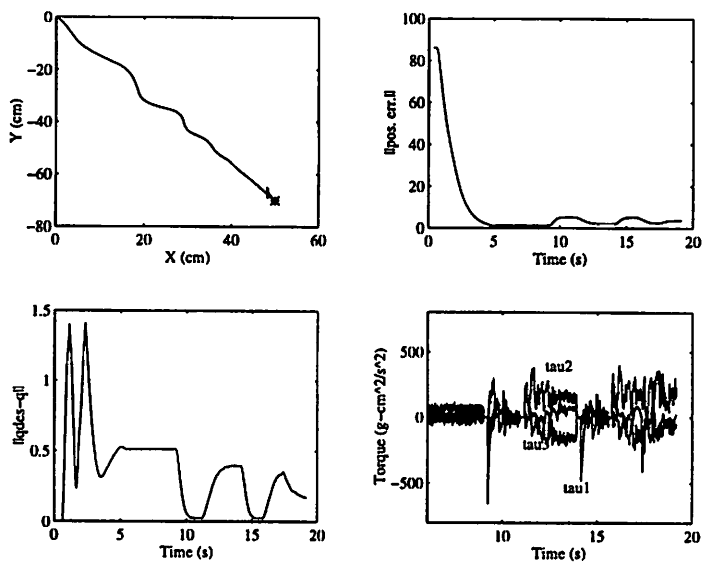

Figure 4: Attempting to Control Position and Orientation

stable with no torque. Other orientations which require finite torques at the desired location will produce finite errors. The switching controller seemed to bring the system to a region around the desired position and orientation but does not converge to the desired configuration.

The combination of these controllers does not accomplish the goal but is a first attempt. Future work involves designing better controllers and proving the effectiveness of the controllers. Another approach is to kinematically generate a path for the ellipsoid to follow from initial configuration to final configuration and design another controller to make the ellipsoid follow the trajectory. Yet another approach is to use the position controller to bring the ellipsoid close to the desired position and then to use a controller to drive the orientation to error as well as removing the small position error.

The control design and simulation method used for this example is a very efficient method of simulating the system and tuning the controller. The windows for the displayed ellipsoid and the window for controller parameters are shown in Figure 5 and 6. Gains can be changed quickly as the simulation is running and the results of the changes are seen almost instantly. Interactive controller design/simulation is a very effective tool for prototyping controlled mechanical systems.

References

[1] Brian Mirtich and John Canny. Impulse-based dynamic simulation. In *Workshop on Algorithmic Foundations of Robotics*, February 1994.

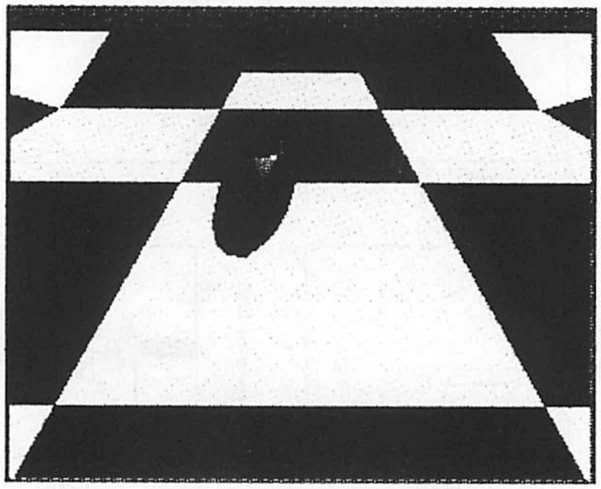

Figure 5: Ellipsoid Window

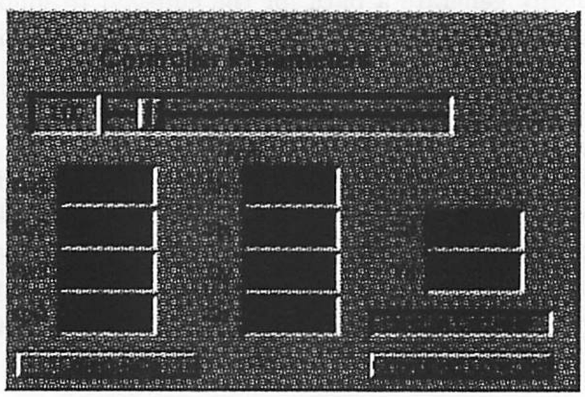

Figure 6: Control Parameter Window

Control and Steering of a Diver in the Plane

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Project Report
for
EECS 290B

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1 Introduction

Much work has been done on the control and steering of nonholonomic systems in the past few years, but the systems that have been discussed have generally been drift-free. Very little work exists concerning general systems with drift, although some specific cases have been addressed (for example, left-invariant systems on SO(3) and GL(n) with drift, [5], [2]). In the real world, however, systems with drift are common; bodies in free-fall with some initial angular momentum have drift. An example of this kind of problem is a human diver. After the diver has left the board, his angular momentum is conserved, but is generally not zero; the diver can change the drift velocity by changing his moment of inertia. The control goal for the system is to execute a certain maneuver, a forward one-and-a-half somersault pike, for example, and then enter the water in a fully extended, vertical position. Since the diver is falling while executing the maneuver, there is a predetermined length of time in which the controls can act. In this report, I will discuss a simplified, two-dimensional diver model and the associated control system.

2 Model

The two-dimensional diver model is shown in figure 1. The two-dimensional dynamical model used in simulation was developed from a fully three-dimensional model with 18 degrees of freedom in the joints, generously shared with us by Professor Jessica Hodgins at the Georgia Institute of Technology (see [7]). The two-dimensional diver model has 5 degrees of freedom: x, z, θ_1, θ_2, and θ_3. The x and z directions do not affect the control except to determine when the diver hits the water, so we will ignore them for now. This system is now an asymmetrical version of the planar skater discussed in [6]. There, it is demonstrated that the planar skater (when drift-free) can reorient its main axis by moving its two arms sinusoidally and out of phase with each other. This method of steering with sinusoids has also been used to steer cars with trailers and firetrucks [4],[1].

The Lagrangian of the diver is given by:

$$
\begin{aligned}
L \;=\; & \frac{1}{2}(m_t + m_l + m_a)(\dot{x}^2 + \dot{z}^2) + \frac{1}{2}I_t\dot{\theta}_1^2 + \frac{1}{2}I_l(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}I_a(\dot{\theta}_1 + \dot{\theta}_3)^2 \\
& + \frac{1}{2}m_l[(j_{lz}^2 + j_{lx}^2)\dot{\theta}_1^2 + (l_z^2 + l_x^2)(\dot{\theta}_1 + \dot{\theta}_2)^2 \\
& \quad + 2((j_{lz}l_z + j_{lx}l_x)\cos\theta_2 + (j_{lz}l_x - j_{lx}l_z)\sin\theta_2)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)] \\
& + \frac{1}{2}m_a[(j_{az}^2 + j_{ax}^2)\dot{\theta}_1^2 + (a_z^2 + a_x^2)(\dot{\theta}_1 + \dot{\theta}_3)^2 \\
& \quad + 2((j_{az}a_z + j_{ax}a_x)\cos\theta_3 + (j_{az}a_x - j_{ax}a_z)\sin\theta_3)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_3)]
\end{aligned}
$$

$m_t = 51.7793$	$m_l = 27.6938$		$m_a = 9.1235$:
$I_t = 3.2900$	$I_l = 1.9921$		$I_a = 0.3226$
$j_{lx} = -0.0042$	$j_{lz} = -0.3754$	$j_{ax} = -0.0470$	$j_{az} = 0.1896$
$l_x = -0.0426$	$l_z = -0.2993$	$a_x = 0.0124$	$a_z = -0.2442$

where m_t, m_l, and m_a are the masses of the torso, legs, and arms respectively, I_t, I_l, and I_a are the inertias, j_{lx} and j_{lz} are the x and z components of the distance from the torso center of mass to the leg joint, j_{ax} and j_{az} are the components for the arm joint, l_x and l_z are the x and z components of the distance from the leg joint to the legs' center of mass, and a_x and a_z are the components for the distance from the arm joint to the arms' center of mass. Since

$$\frac{\partial L}{\partial \theta_1} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1}$$

we have $\frac{\partial L}{\partial \dot{\theta}_1} = \mu$, a constant. One thus finds that the single constraint for this system is

$$\begin{aligned} \mu &= [\alpha_1 + 2\beta\cos\theta_2 + 2\gamma\sin\theta_2 + 2\delta\cos\theta_3 + 2\epsilon\sin\theta_3]\,\dot{\theta}_1 \\ &\quad + [\alpha_2 + \beta\cos\theta_2 + \gamma\sin\theta_2]\,\dot{\theta}_2 + [\alpha_3 + \delta\cos\theta_3 + \epsilon\sin\theta_3]\,\dot{\theta}_3 \\ &=: \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \end{aligned}$$

with

$$\begin{aligned} \alpha_1 &= I_t + I_l + I_a + m_l(j_{lz}^2 + j_{lx}^2) + m_l(l_z^2 + l_x^2) + m_a(j_{az}^2 + j_{ax}^2) + m_a(a_z^2 + a_x^2) = 12.0394 \\ \alpha_2 &= I_l + m_l(l_z^2 + l_x^2) = 4.5230 \\ \alpha_3 &= I_a + m_a(a_z^2 + a_x^2) = 0.8681 \\ \beta &= m_l(j_{lz}l_z + j_{lx}l_x) = 3.1167 \\ \gamma &= m_l(j_{lz}l_x - j_{lx}l_z) = 0.4076 \\ \delta &= m_a(j_{az}a_z + j_{ax}a_x) = -0.4277 \\ \epsilon &= m_a(j_{az}a_x - j_{ax}a_z) = -0.0833 \end{aligned}$$

3 Control Task – Preliminary Analysis

Our initial goal is to control the diver in a forward $1\frac{1}{2}$ somersault pike. In three dimensions, combinations of somersaulting and twisting may be performed, but somersaults are the only dives that are possible for the two-dimensional model. Since there are three degrees of freedom and one constraint, there are two controls available. If we choose two linearly independent vectors g_1 and g_2 which annihilate the constraints and which allow us to control θ_2 and θ_3 directly, we can write

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} -\frac{b_2}{b_1} \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -\frac{b_3}{b_1} \\ 0 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} \frac{\mu}{b_1} \\ 0 \\ 0 \end{bmatrix} =: g_1(\theta)u_1 + g_2(\theta)u_2 + f(\theta)$$

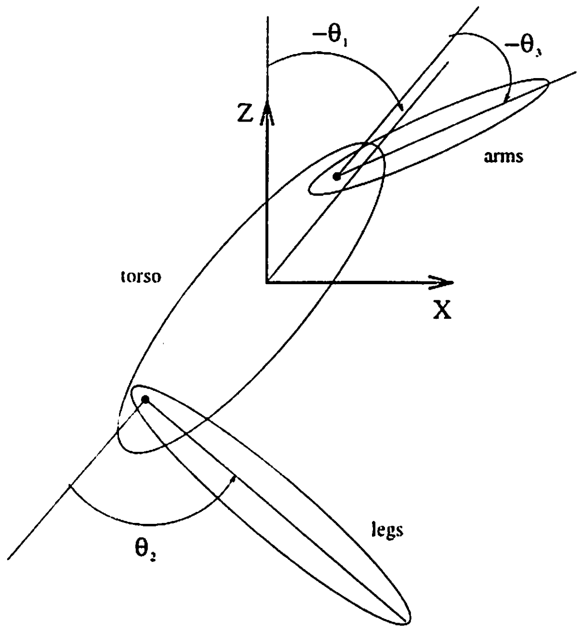

Figure 1: 2-D diver model.

In our diver model, b_1 can never equal zero. Note that the drift is a nonlinear function of (θ_2, θ_3). Since

$$[g_1, g_2] = \begin{bmatrix} \frac{1}{b_1^2}\left(b_3\frac{\partial b1}{\partial \theta_2} - b_2\frac{\partial b_1}{\partial \theta_3}\right) \\ 0 \\ 0 \end{bmatrix}$$

$\{g_1, g_2, [g_1, g_2]\}$ spans the space (except at (θ_2, θ_3) where the first entry in $[g_1, g_2]$ is zero), the system is locally controllable even without making use of the drift; the Lie bracket direction is the same as the drift direction. The first entry in $[g_1, g_2]$ is zero when $tan(\theta_2 - \theta_3) = \frac{\gamma\delta - \beta\epsilon}{\beta\delta + \gamma\epsilon}$; in our model, this happens at about $\theta_2 - \theta_3 = -3.57$ degrees.

We can, dually, look at the diving problem as a Pfaffian exterior differential system

$$I = \{\alpha\}$$

where $\alpha = b_1 d\theta_1 + b_2 d\theta_2 + b_3 d\theta_3 - \mu dt$. If we can convert this system into a standard form, we may be able to use sinusoidal or other steering methods that have been used for other systems. We have:

$$\begin{aligned} d\alpha &= db_1 \wedge d\theta_1 + db_2 \wedge d\theta_2 + db_3 \wedge d\theta_3 \\ &= \left(\frac{\partial b_2}{\partial \theta_1} - \frac{\partial b_1}{\partial \theta_2}\right) d\theta_1 \wedge d\theta_2 + \left(\frac{\partial b_3}{\partial \theta_1} - \frac{\partial b_1}{\partial \theta_3}\right) d\theta_1 \wedge d\theta_3 + \left(\frac{\partial b_3}{\partial \theta_2} - \frac{\partial b_2}{\partial \theta_3}\right) d\theta_2 \wedge d\theta_3 \\ &= 2(\beta \sin\theta_2 - \gamma\cos\theta_2) d\theta_1 \wedge d\theta_2 + 2(\delta\sin\theta_3 - \epsilon\cos\theta_3) d\theta_1 \wedge d\theta_3 \end{aligned}$$

$$\begin{aligned} d\alpha \wedge \alpha &= (2b_3(\beta\sin\theta_2 - \gamma\cos\theta_2) - 2b_2(\delta\sin\theta_3 - \epsilon\cos\theta_3))\, d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \\ &\quad -2\mu(\beta\sin\theta_2 - \gamma\cos\theta_2) d\theta_1 \wedge d\theta_2 \wedge dt - 2\mu(\delta\sin\theta_3 - \epsilon\cos\theta_3) d\theta_1 \wedge d\theta_3 \wedge dt \end{aligned}$$

Assuming $\mu \neq 0$, $d\alpha \wedge \alpha \neq 0$ except when $\tan\theta_2 = \frac{\gamma}{\beta}$ and $\tan\theta_3 = \frac{\epsilon}{\delta}$. (In our model, this happens at about $\theta_2 = 7.45$ degrees and $\theta_3 = 11.02$ degrees. Note that for these values, $\theta_2 - \theta_3 = -3.57$ degrees.) Also,

$$(d\alpha)^2 \wedge \alpha = 0,$$

so α has Pfaffian rank 1 (except at the points just described). In other words, the derived flag for this system looks like

$$\begin{aligned} I^{(0)} &= \{\alpha\} \\ I^{(1)} &= \{0\} \end{aligned}$$

as expected.

By Pfaff's theorem, we can find local coordinates (z_1, z_2, z_3) so that

$$\{\alpha\} = \{dz_3 - z_2 dz_1\}.$$

To do this, we need to find two functions f_1 and f_2 which satisfy

$$\begin{aligned} d\alpha \wedge \alpha \wedge df_1 &= 0 \\ \alpha \wedge df_1 &\neq 0 \\ \alpha \wedge df_1 \wedge df_2 &= 0 \\ df_1 \wedge df_2 &\neq 0 \end{aligned}$$

Then, by the theorem, $df_2 - g df_1 =: dz_3 - z_2 dz_1$. $\frac{df_1}{dt}$ and $\frac{dg}{dt}$ will become the directly controlled inputs of the system. $\frac{df_2}{dt}$ will be controlled indirectly, since the constraint $\alpha = 0$ requires $df_2 = g df_1$. If we write $df_1 = a_1 d\theta_1 + a_2 d\theta_2 + a_3 d\theta_3 + a_4 dt$, then the first equation becomes

$$\begin{aligned} d\alpha\wedge\alpha\wedge df_1 &= [2(\mu a_3 + b_3 a_4)(\beta\sin\theta_2 - \gamma\cos\theta_2) - 2(\mu a_2 + b_2 a_4)(\delta\sin\theta_3 - \epsilon\cos\theta_3)]\, d\theta_1\wedge d\theta_2\wedge d\theta_3\wedge dt \\ &= 0 \end{aligned}$$

The solution is not unique. It is notable that $f_1 = t$ is *not* a solution, so the system is not feedback linearizable. Simple time-scalings such as $f_1 = t + h(\theta_2)$ or $f_1 = t + h(\theta_1)$ also fail, as do $f_1 = \theta_2$ and $f_1 = \theta_3$. $f_1 = \theta_1$ will satisfy the equation, but since in the diver system only functions of θ_2 and θ_3 can be controlled directly, a system of coordinates with $\frac{d\theta_1}{dt}$ as an input is not useful. A more useful solution is

$$df_1 = (\beta\sin\theta_2 - \gamma\cos\theta_2)d\theta_2 + (\delta\sin\theta_3 - \epsilon\cos\theta_3)d\theta_3$$

Using this f_1 and writing $df_2 = c_1 d\theta_1 + c_2 d\theta_2 + c_3 d\theta_3 + c_4 dt$, we get

$$\begin{aligned} \alpha \wedge df_1 \wedge df_2 &= (\tilde{a}_2 b_1 c_3 - \tilde{a}_3 b_1 c_2 + \tilde{a}_3 b_2 c_1 - \tilde{a}_2 b_3 c_1) d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \\ &\quad + (\tilde{a}_2 \mu c_1 + \tilde{a}_2 b_1 c_4) d\theta_1 \wedge d\theta_2 \wedge dt + (\tilde{a}_3 \mu c_1 + \tilde{a}_3 b_1 c_4) d\theta_1 \wedge d\theta_3 \wedge dt \\ &\quad + (\tilde{a}_3 \mu c_2 - \tilde{a}_2 \mu c_3 + \tilde{a}_3 b_2 c_4 - \tilde{a}_2 b_3 c_4) d\theta_2 \wedge d\theta_3 \wedge dt \\ &= 0 \end{aligned}$$

where $\tilde{a}_2 = \beta \sin\theta_2 - \gamma\cos\theta_2$ and $\tilde{a}_3 = \delta\sin\theta_3 - \epsilon\cos\theta_3$. Continuing, one possible solution for df_2 is given by

$$df_2 = -b_1(\tilde{a}_2 b_3 + \tilde{a}_3 b_2)d\theta_1 + b_2(\tilde{a}_2 b_3 - \tilde{a}_3 b_2)d\theta_2 - b_3(\tilde{a}_2 b_3 - \tilde{a}_3 b_2)d\theta_3 + \mu(\tilde{a}_2 b_3 + \tilde{a}_3 b_2)dt$$

Requiring $\{\alpha\} = \{df_2 - g df_1\}$, we find

$$g = 2b_2 b_3$$

Now, since we have the constraint $df_2 - g df_1 = 0$, we can write

$$\begin{aligned}
df_1 &= (\beta\sin\theta_2 - \gamma\cos\theta_2)d\theta_2 + (\delta\sin\theta_3 - \epsilon\cos\theta_3)d\theta_3 = u_1 dt \\
dg &= 2b_3(\gamma\cos\theta_2 - \beta\sin\theta_2)d\theta_2 + 2b_2(\epsilon\cos\theta_3 - \delta\sin\theta_3)d\theta_3 = u_2 dt \\
df_2 &= g f_1 = g u_1 dt
\end{aligned}$$

4 Steering Methods

4.1 Sinusoids

If we can control u_1 and u_2, we can control f_2. Integrating the expression for df_2 above, we get

$$f_2 = (\tilde{a}_3 b_2 + \tilde{a}_2 b_3)(\mu t - b_1\theta_1) + h(\theta_2, \theta_3)$$

Thus by driving f_2, we can drive $\mu t - b_1\theta_1$, and thereby control θ_1, but not in arbitrary time, since we do not control t. To see if we can control u_1 and u_2, solve for $\frac{d\theta_2}{dt}$ and $\frac{d\theta_3}{dt}$:

$$\begin{aligned}
\frac{d\theta_2}{dt} &= \frac{b_2 u_1 + u_2}{2(b_3 - b_2)(\gamma\cos\theta_2 - \beta\sin\theta_2)} \\
\frac{d\theta_3}{dt} &= \frac{b_3 u_1 + u_2}{2(b_2 - b_3)(\epsilon\cos\theta_3 - \delta\sin\theta_3)}
\end{aligned}$$

We can provide arbitrary u_1 and u_2 except when $\tan\theta_2 = \frac{\gamma}{\beta}$, $\tan\theta_3 = \frac{\epsilon}{\delta}$, or $b_2 = b_3$ (which cannot happen with our diver model). Unfortunately, the first two conditions give values for θ_2 (7.45 degrees) and θ_3 (11.02 degrees) which fall within the desired ranges for these angles. If the diver is started with arms and legs at angles not too near these critical angles, applying $u_1 = A\sin\omega t$, $u_2 = B\cos\omega t$ can steer θ_1. As soon as A and B get too large, however, and try to push θ_2 or θ_3 toward the critical angles, driving u_1 and u_2 along the sinusoid becomes impossible.

This system was simulated using the SD/FAST equation generation and simulation package with the two-dimensional human model described above. Some sample trajectories are shown in figures 2- 3, and some images from one of the simulations are shown in figure 4. All of these simulations start with the same angular velocity about the center of mass. For simulations starting in different configurations, and therefore with different moments of inertia,

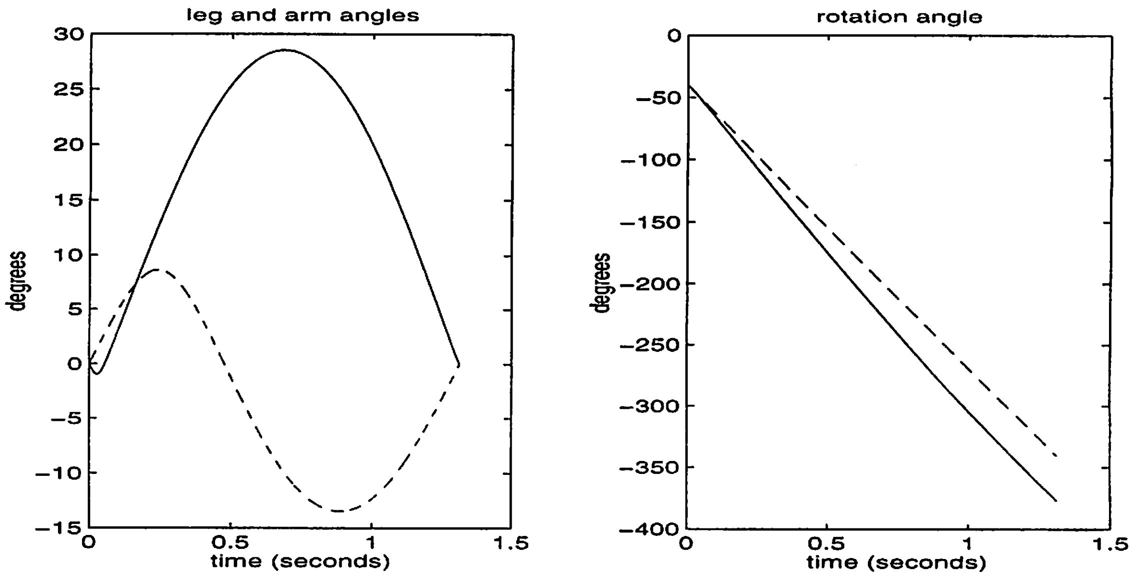

Figure 2: Simulation starting with $\theta_2 = 10$ degrees, $\theta_3 = -10$ degrees. A=2, B=-2, and ω was chosen so as to give one full period in the simulation time. The first plot shows the deviations of θ_2 (solid) and θ_3 (dashed) from their starting angles. The second plot shows θ_1 with these inputs (solid) and with no inputs (dashed).

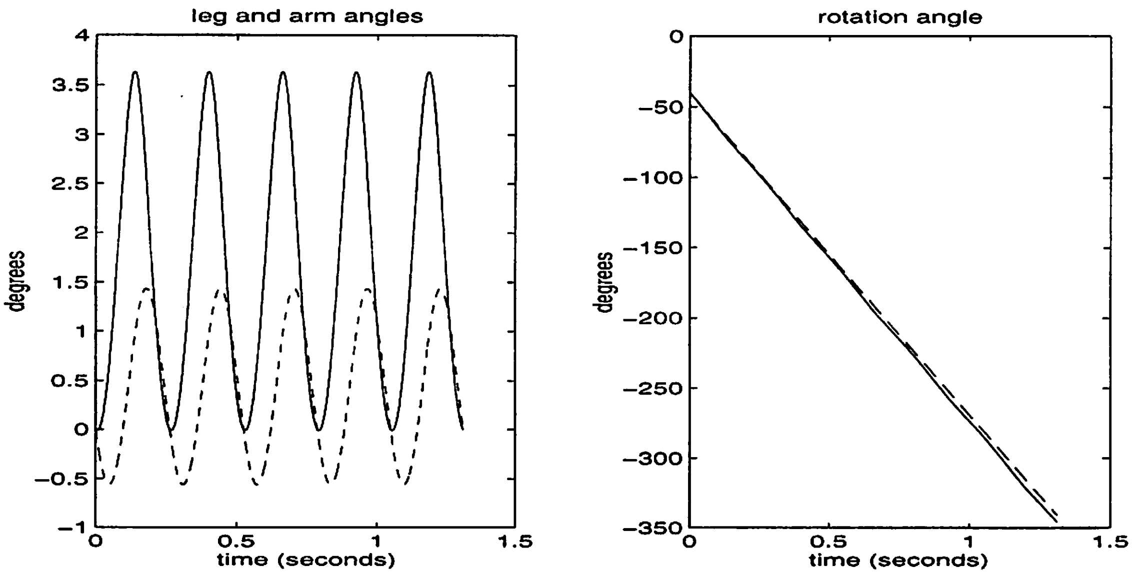

Figure 3: Simulation starting with $\theta_2 = 0$, $\theta_3 = 0$. A=-1.72, B=1.72, and ω was such that there are 5 full periods in the simulation time. The first plot shows θ_2 (solid) and θ_3 (dashed). The second plot shows θ_1 with these inputs (solid) and with no inputs (dashed).

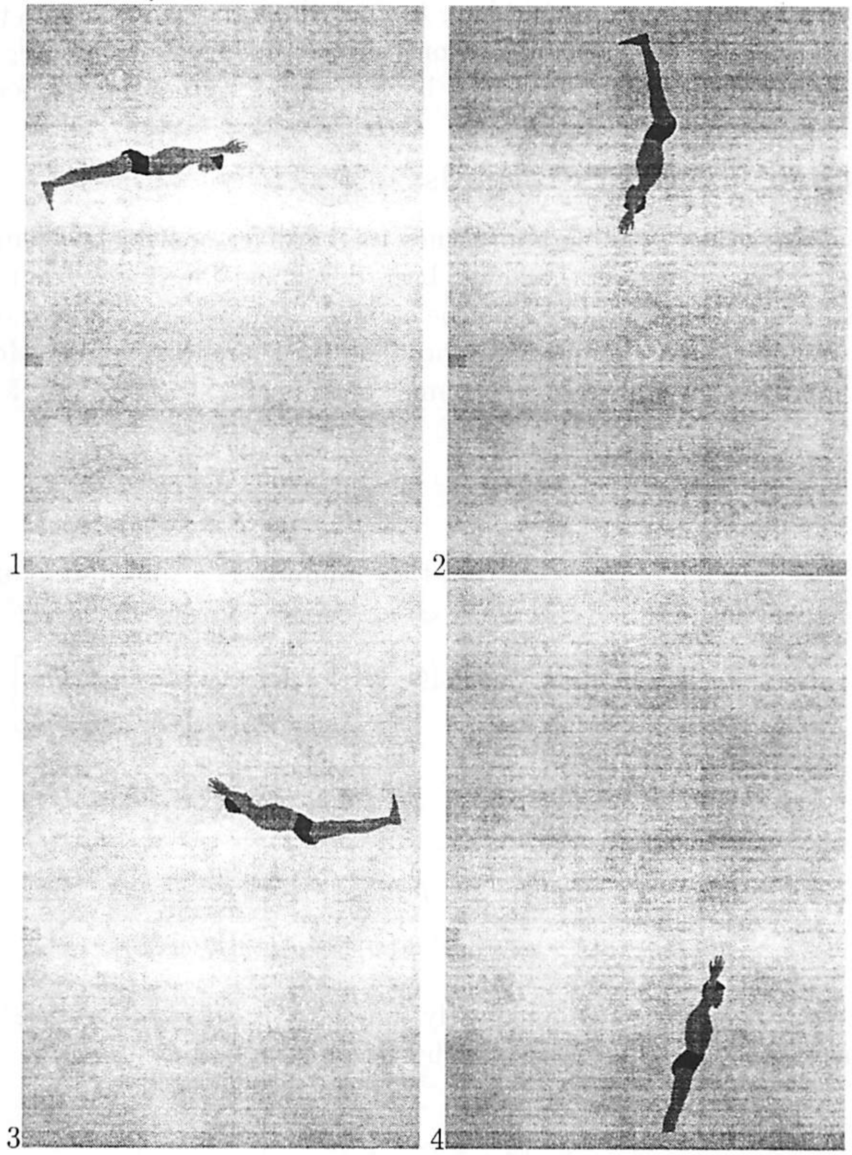

Figure 4: Frames from the simulation shown in figure 2. The graphical human model is from Viewpoint DataLabs.

this means they start with different angular momenta. Note that for a one-and-a-half somersault dive, the diver needs to rotate through exactly -540 degrees by the time he reaches the water.

These controls are promising, but, unfortunately, in the diver problem we want to start at $\theta_2 = \theta_3 = 0$, where only small amplitude controls are possible. In order to get a significant change in overall rotation, one would have to apply high frequency inputs (see figure 3), but this is not an attractive solution. Since α only drops rank at a few pairs (θ_2, θ_3), it is possible that with different choices of f_1 and f_2, a better control system may be found.

4.2 Optimal Control Methods

There are several other possible approaches to the diver control problem. The system is not left invariant, so methods that have been developed for steering left-invariant control systems (see [5], [2], for example) cannot be applied. It may be possible, though, to make use of some optimal control techniques like those in [3]. There, Sastry and Montgomery derive coupled differential equations for the optimal controls of a system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

The controls are optimal in the sense that they minimize $\frac{1}{2}\int_0^T |u(t)|^2 dt$. For the $m = 2$ case, their result looks like:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & p^T[g_1, g_2] \\ p^T[g_2, g_1] & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -p^T[f, g_1] \\ -p^T[f, g_2] \end{bmatrix}$$

$$\begin{aligned} u_i &= -p^T g_i(x) \\ \dot{p} &= -\frac{\partial f}{\partial x}^T p + \frac{\partial g_1}{\partial x}^T p(p^T g_1(x)) + \frac{\partial g_2}{\partial x}^T p(p^T g_2(x)) \\ \dot{x} &= f(x) - g_1(x)(p^T g_1(x)) - g_2(x)(p^T g_2(x)) \end{aligned}$$

For the diving system, we get immediately that $\dot{p}_1 = 0$, so $p_1 = k$ (since f, g_1, and g_2 are not functions of θ_1). The Lie brackets with f become:

$$\begin{aligned} [f, g_1] &= \begin{bmatrix} \frac{\mu}{b_1^2}\frac{\partial b_1}{\partial \theta_2} \\ 0 \\ 0 \end{bmatrix} \\ [f, g_2] &= \begin{bmatrix} \frac{\mu}{b_1^2}\frac{\partial b_1}{\partial \theta_3} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$[g_1, g_2]$ also lies along the θ_1 axis, as shown above, so the equation for the controls becomes:

$$\begin{aligned}
\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} &= \begin{bmatrix} \frac{k}{b_1^2}\left(b_3 \frac{\partial b1}{\partial \theta_2} - b_2 \frac{\partial b_1}{\partial \theta_3}\right) u_2 \\ -\frac{k}{b_1^2}\left(b_3 \frac{\partial b1}{\partial \theta_2} - b_2 \frac{\partial b_1}{\partial \theta_3}\right) u_1 \end{bmatrix} - \begin{bmatrix} \frac{k\mu}{b_1^2} \frac{\partial b_1}{\partial \theta_2} \\ \frac{k\mu}{b_1^2} \frac{\partial b_1}{\partial \theta_3} \end{bmatrix} \\
&= \frac{k}{b_1^2} \begin{bmatrix} \frac{\partial b_1}{\partial \theta_2}(b_3 u_2 - \mu) - \frac{\partial b_1}{\partial \theta_3} b_2 u_2 \\ \frac{\partial b_1}{\partial \theta_3}(b_2 u_1 - \mu) - \frac{\partial b_1}{\partial \theta_2} b_3 u_1 \end{bmatrix}
\end{aligned}$$

with

$$\begin{aligned}
\dot{\theta} &= f(\theta) + g_1(\theta) u_1 + g_2(\theta) u_2 \\
\dot{p} &= \begin{bmatrix} 0 \\ \frac{\mu k}{b_1^2} \frac{\partial b_1}{\partial \theta_2} - \frac{k}{b_1^2}\left(-b_1 \frac{\partial b_2}{\partial \theta_2} + b_2 \frac{\partial b_1}{\partial \theta_2}\right) u_1 - \frac{k}{b_1^2}\left(b_3 \frac{\partial b_1}{\partial \theta_3}\right) u_2 \\ \frac{\mu k}{b_1^2} \frac{\partial b_1}{\partial \theta_3} - \frac{k}{b_1^2}\left(b_2 \frac{\partial b_1}{\partial \theta_2}\right) u_1 - \frac{k}{b_1^2}\left(-b_1 \frac{\partial b_3}{\partial \theta_3} + b_3 \frac{\partial b_1}{\partial \theta_3}\right) u_2 \end{bmatrix} \\
u_i &= -p^T g_i(\theta)
\end{aligned}$$

These equations would have to be solved numerically.

4.3 Adaptive or Learning Methods

One final approach to the diving problem is to design an adaptive controller which would search a restricted control space to find a good control. Even a very simple learning algorithm can, if given a suitable space to search, find a control law that will drive the diver through a $1\frac{1}{2}$ somersault pike. For example, the simulation shown in figures 5 and 6 used a control law chosen by an algorithm that searched among controls of the form

$$\begin{aligned}
u_1 &= \frac{A}{\sqrt{2\pi\sigma^2}} \left[e^{-\frac{(t-p_1)^2}{2\sigma^2}} - e^{-\frac{(t-p_2)^2}{2\sigma^2}} \right] \\
u_2 &= -\frac{A}{\sqrt{2\pi\sigma^2}} \left[e^{-\frac{(t-p_1)^2}{2\sigma^2}} - e^{-\frac{(t-p_2)^2}{2\sigma^2}} \right]
\end{aligned}$$

by varying the parameters A, σ^2, p_1, and $p_2 - p_1$ (within restricted ranges) around the best values it had found so far. The family of controls was chosen as a biologically plausible velocity profile. The controls were evaluated based on the error of the final rotation angle from -540 degrees and the final angles of the arms and legs away from zero. An initial control was found by taking the best of several random trials. The algorithm kept simple schemas characterizing the relationship (positive vs. negative) between the parameters and the final rotation angle. A schema was updated each time its parameter was changed. The schemas depended most heavily on recent data so that the algorithm could adapt to the very different, nonlinear regions of the control space. The memory of a schema decayed over several learning cycles, and some noise was added to the next parameter choice to avoid getting stuck at a parameter maximum or minimum, which would prevent the schema from adapting.

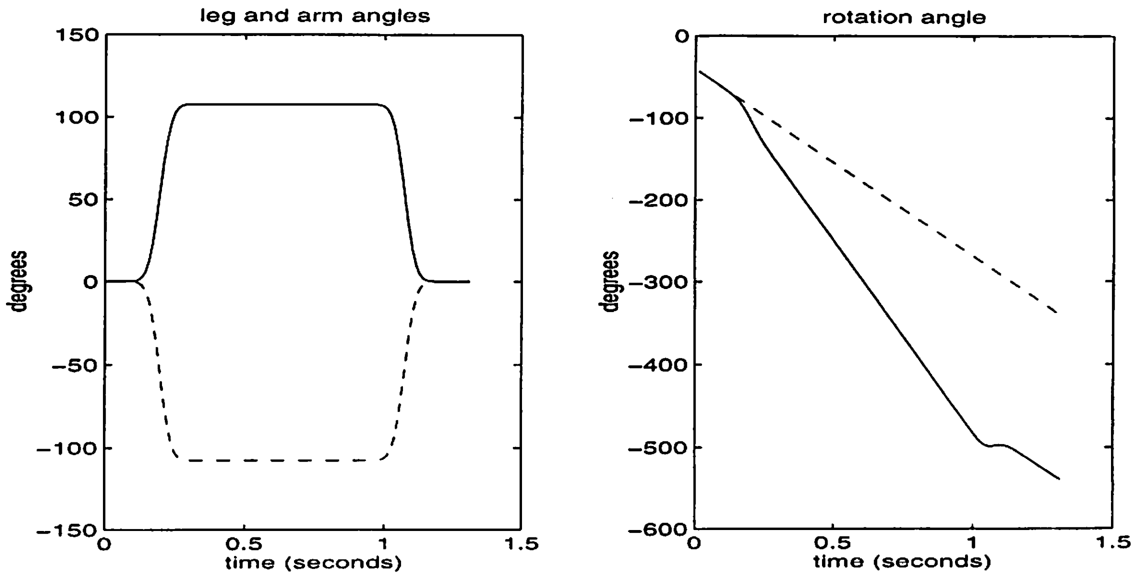

Figure 5: Simulation with a control with parameters chosen by learning algorithm. $A = 1.878723$. $\sigma^2 = .001017$. $p_1 = .195680$. and $p_2 - p_1 = .877551$. The first plot shows θ_2 (solid) and θ_3 (dashed). The second plot shows θ_1 with these inputs (solid) and with no inputs (dashed).

This learning approach can certainly be made more general, but any parameter-learning algorithm needs a family of controls among which to choose. For a real diver, the specified family of controls might be analogous to the instructions given by a teacher, with the parameter fine-tuning left for practice. More biologically-motivated learning approaches might give some insight into how an algorithm could learn the structure of the controls as well as fine-tune the parameters.

5 Future Work

Future work on this project will have two facets. I plan to try to improve on the sinusoidal control presented here, within the Pfaffian system framework, as well as to develop a more powerful, biologically-inspired learning algorithm that will require less prespecified information than the one described above. Developing controls for this kind of system has potential applications in dynamic animation and robotics, and may also lead to insights into biological control systems themselves.

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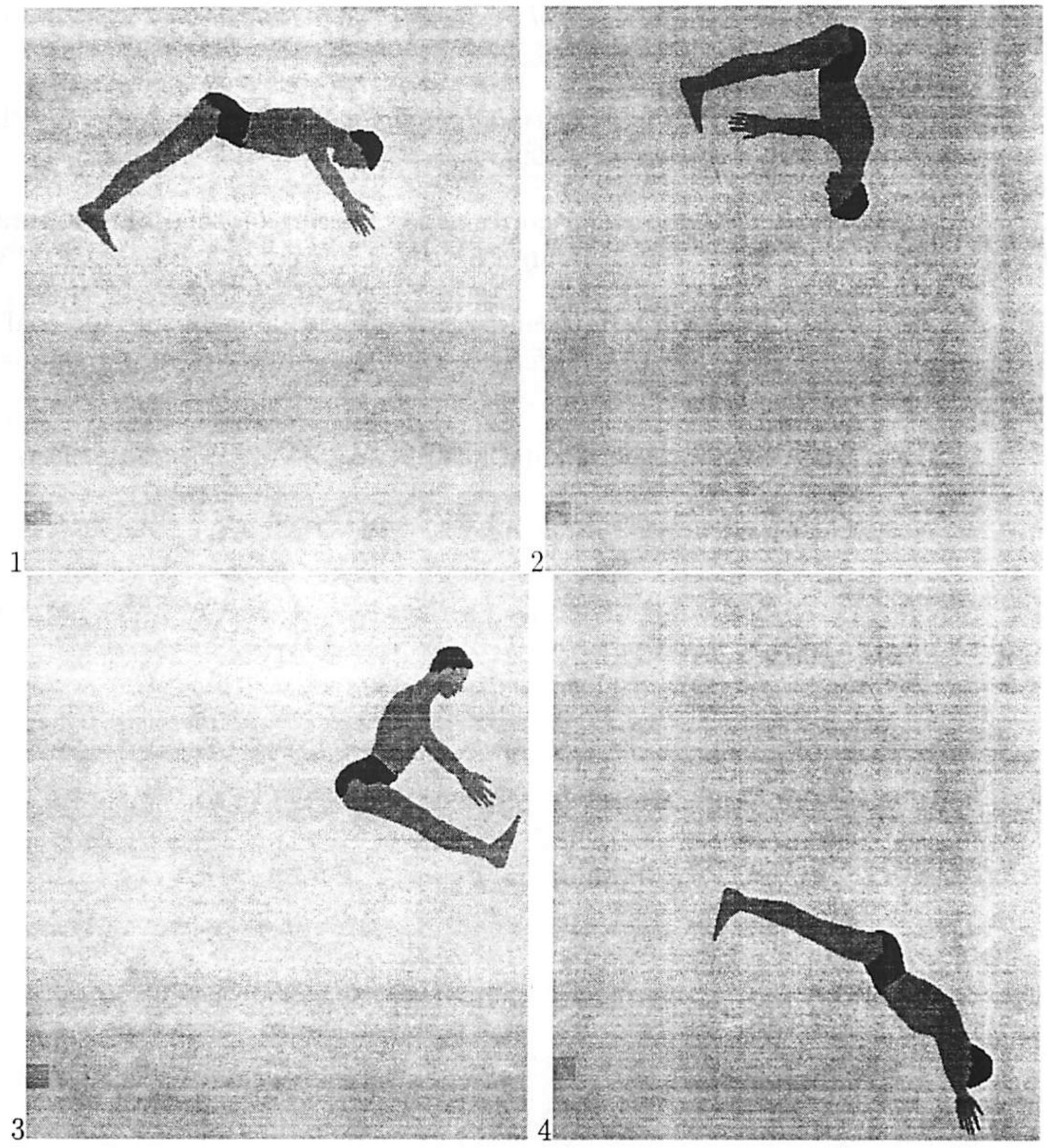

Figure 6: Frames from the simulation shown in figure 5. The graphical human model is from Viewpoint DataLabs.

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Registration of 3D shapes using least squares matching

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Abstract

The purpose of this paper is to investigate the applicability of the dynamical system $\dot{H} = [H,[H,N]]$, where H and N are symmetric $n \times n$ matrices and [A,B] = AB - BA. We have shown that this dynamical system can be used effectively in the registration of 3D maps from image correspondences. The idea is to recast the combinatorial optimization problem to that of function minimization. The method of steepest descent is used to generate an effective algorithm for the minimization. The minimization yields the information necessary to construct a transformation matrix T such that given two sets of points $\{x_i\}$ and $\{x_i'\}$ $x_i = Tx_i'$ $\forall i$ where $T \in SE(3)$ and $x_i, x_i' \in E^n$. Two matching problems are presented and applied to registration.

Keywords
Registration, Least squares matching, Steepest descent.

1 Introduction

In this paper, some results are presented involving a set of matrix differential equations:

$$\dot{H} = [H,[H,N]],\ H(0) = H_0 \qquad (1)$$

where $H, N \in R^{n \times n}$, N is a constant matrix and [A,B] = AB - BA denotes the matrix Lie bracket.

The following problems were proposed by Brockett [4]:

MATCHING PROBLEM A. Given a representation Φ of a Lie group acting on an inner product space (V,$<,>$) and given two ordered sets of points $\{x_1, x_2, ..., x_p\}$ and $\{y_1, y_2, ..., y_p\}$ in V, find $\phi \in \Phi$ such that

$$\eta = \sum_{i=1}^{p} < \phi(x_i) - y_i, \phi(x_i) - y_i >$$

is as small as possible.

By using a least squares matching criterion the above problem can be recast as one about minimizing the function $tr(\Theta^T Q \Theta N - 2M\Theta^T)$ where $\Theta \in SO(n)$ and Q, N and M are fixed $n \times n$ symmetric matrices. Equation (1) is equivalent to a certain gradient flow on the space of orthogonal matrices; that is, differential equations evolving in a Riemannian manifold and which are counterparts to the descent equation [5]

$$\dot{x} = \Delta\phi \qquad (2)$$

in Euclidean n-space, E^n. Let G be a subgroup of GL(n) and L_G the Lie algebra associated with G then the best $\Theta_0 \in G$ such that

$$tr(\Theta_0^T Q \Theta_0 N - M^T \Theta_0)L = 0$$

$\forall L \in L_G$ is the transformation which solves matching problem A. If we give GL(n), the set of $n \times n$ nonsingular matrices, the structure of a Riemannian manifold and investigate the gradient vector field associated with the function the descent equation can be used effectively for the minimization needed to find Θ_0.

The registration or geometric matching problem in computer vision involves finding the optimal rotation and translation, (R,t), that aligns, or registers two given sets of points. When an a *priori* solution of the correspondence problem is assumed it suffices to solve matching problem A in order to find the desired transformation $(R, t) = \phi \in \Phi$. We now present a more general problem.

MATCHING PROBLEM B. Given a representation Φ of a Lie group acting on an inner product space (V,$<,>$) and given two sets of points $\{x_1, x_2, ..., x_p\}$ and $\{y_1, y_2, ..., y_p\}$ in V, find $\phi \in \Phi$ and a permutation $\pi \in \Pi_p$, the set of all permutations of the integers 1,2,...,p, such that

$$\eta = \sum_{i=1}^{p} < \phi(x_{\pi(i)}) - y_i, \phi(x_{\pi(i)}) - y_i >$$

is as small as possible.

When complete a *priori* ignorance about the correspondence problem is assumed the solution of matching problem B has a variety of applications. For example, in pattern recognition features of objects need to be matched or registered with a data base before recognition of the object is possible. Note that problem A is a special case of problem B in the sense that $\pi(i) = i$ is assumed.

2 Least squares matching problem

We will show that the matching problem A is equivalent to one about minimizing $tr(\Theta^T Q \Theta N) - 2trM\Theta^T$. Consider the least squares criterion

$$\eta = \sum_{i=1}^{p} (\Theta x_i + b - y_i)^T Q (\Theta x_i + b - y_i)$$

where Θ is an element of a closed Lie group and Q is a symmetric positive definite $n \times n$ matrix. We can simplify notation by eliminating b. In homogeneous coordinates [3],

$$x_i := \begin{pmatrix} x_i \\ 1 \end{pmatrix}, \; y_i := \begin{pmatrix} y_i \\ 1 \end{pmatrix}$$

and

$$\Theta := \begin{pmatrix} \Theta & b \\ 0 & 1 \end{pmatrix}, \; Q := \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$\eta = \sum_{i=1}^{p} (\Theta x_i - y_i)^T Q (\Theta x_i - y_i)$$

By using properties of the trace, tr, we obtain

$$\eta = tr(\sum_{i=1}^{p} \Theta^T Q \Theta x_i x_i^T - Q(\Theta x_i y_i^T - y_i x_i^T \Theta^T) + y_i y_i^T),$$

Since trAB = trBA we have

$$\eta = tr(\Theta^T Q \Theta N - 2M\Theta^T) + \sum_{i=1}^{p} y_i^T Q y_i.$$

where

$$N = \sum_{i=1}^{p} x_i x_i^T, \; M = Q \sum_{i=1}^{p} x_i y_i^T.$$

Since y_i and x_i are given data, minimizing η is equivalent to maximizing the function $f(\Theta) = tr(M\Theta^T)$.

We will now show that matching problem B is equivalent to one about minimizing functions of the form

$$\eta = \sum_{i=1}^{n} \Theta^T Q_i \Theta N_i$$

where as before Θ is restricted to the orthogonal group.

Consider two sets of points $\{x_1, x_2, ..., x_p\}$ *and* $\{y_1, y_2, ..., y_p\} \in E^n$. We want to find a permutation π such that the least squares criterion

$$\eta = \sum_{i=1}^{p} ||x_{\pi(i)} - y_i||^2$$

is as small as possible. Since the elements of a diagonal matrix can be permuted with a permutation matrix Π, i.e.

$$\Pi^T diag(\alpha_1, \alpha_2, ..., \alpha_p)\Pi = diag(\alpha_{\pi(1)}, \alpha_{\pi(2)}, ..., \alpha_{\pi(n)}),$$

in the case n = 1 the above minimization problem is resolved by finding the permutation matrix maximizing

$$tr[\Pi^T diag(x_1, x_2, ..., x_n)\Pi diag(y_1, y_2, ..., y_n)].$$

which is true in view of

$$\sum_{i=1}^{p}(x_{\pi(i)} - y_i)^2 = \sum_{i=1}^{p}(x_{\pi(i)}^2 + y_i^2) - 2\sum_{i=1}^{p} x_{\pi(i)}y_i$$

Let $Q = diag(x_1, x_2, ..., x_n)$ and $N = diag(y_1, y_2, ..., y_n)$. Since $\Pi^T Q\Pi = diag(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ we have

$$\Omega(\Pi) = tr(Q^2 + N^2 - 2\Pi^T Q\Pi N)$$

Since the permutation matrices form a subgroup of SO(p) we could allow Π to be an arbitrary orthogonal matrix $\Theta \in SO(p)$ and obtain

$$\max_{\Pi} \Omega(\Pi) \leq \max_{\Theta} \Omega(\Theta)$$

In 1937 von Neumann [8] proved a theorem which shows that for Θ fixed the above inequality is in fact an equality. His theorem demonstrates a method by which the problem of rearranging a set of numbers defined on a finite symmetry group can be encoded as an optimization problem on SO(p). In order to solve the minimization problem we then set up a gradient flow system on SO(p).

For the general case $n \geq 1$ we denote the jth element of the ith vector x_i by x_{ij} and similarly for y_i. In addition, we define Q_j *and* N_j as

$$Q_j = diag(x_{1j}, x_{2j}, ..., x_{pj}),$$

$$N_j = diag(y_{1j}, y_{2j}, ..., y_{pj}),$$

To minimize η we must maximize the trace of a sum of terms, i.e.

$$\sum_{j=1}^{n} \Pi^T Q_j \Pi N_j.$$

This maximum value can only be increased by replacing Π by Θ and we have

$$\min_{\Pi} ||x_{\pi(i)} - y_i||^2 \leq \sum ||x_i||^2 + ||y_i||^2 - 2\max_{\Theta} \sum_{j=1}^{n} tr(\Theta^T Q_j \Theta N_j).$$

If for some π the magnitude of error between $x_{\pi(i)}$ and y_i is smaller than one-half the distance between y_i and its nearest neighbor, then the permutation solution

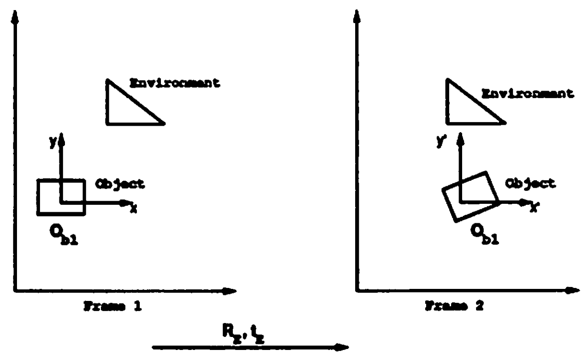

Figure 1: Geometric matching

is a local minimum. The following theorem [4] summarizes our results

Theorem 1. Let $x_1, x_2, ..., x_m$ and $y_1, y_2, ..., y_m$ be vectors in E^n. For i = 1,2,...,n let Q_i and N_i be m by m matrices defined as

$$Q_i = diag(x_{1i}, x_{2i}, ..., x_{mi}),$$

$$N_i = diag(y_{1i}, y_{2i}, ..., y_{mi}).$$

If the permutation π which minimizes

$$\eta = \sum_{i=1}^{m} \|x_{\pi(i)} - y_i\|^2$$

produces an approximate match which is sufficiently accurate so that for all i

$$\|x_{\pi(i)} - y_i\|^2 \leq \frac{1}{2} \min_j \|x_i - x_j\|^2,$$

then equating Θ to the corresponding permutation matrix Π yields a local minimum for

$$\eta(\Theta) = \sum_{i=1}^{n} tr(\Theta^T Q_i \Theta N_i).$$

Thus the best match is an attractor for the descent equation on the group of m by m orthogonal matrices

$$\dot{\Theta} = \sum_{i=1}^{n} \Theta Q_i \Theta^T N_i \Theta - N_i \Theta Q_i.$$

3 Registration

Registration or the geometric matching problem in computer vision involves establishing a correspondence between two given sets of features observed in two views. See Figure 1. There are two main applications [7]: object recognition and visual navigation. The problem in object recognition is to match data observed in a dynamic scene at different instants in order to recover object motions and to interpret the scene. The ability to deal with large motions is usually essential for applications to object recognition. In visual navigation, because the maximum velocity of an object is limited and the sample frequency is high, the motion between curves in successive frames is in general small or known within a reasonable precision. Given that the motion between two successive frames is small, a curve in the first frame is close to the corresponding curve in the second frame. By matching points on the curves in the first frame to their closest points on the curves in the second, a motion which brings the curves in the two frames closer can be found. See Figure 2. The rigidity assumption implies the conservation of local structure of objects, such as the angle and distance between two line segments, during their motion. This constraint provides a powerful tool when dealing with geometric matching [2].

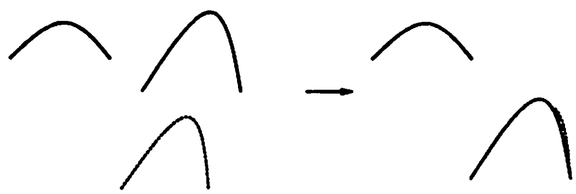

Figure 2: Closest point matching

4 Recognition and Locating 3D Shapes

There are a number of categories of representations for 3D shapes. Homogeneous representations deal with only one resolution and are simpler to use and build. We will use these representations to satisfy the requirements for solving the problem of recognizing and locating objects.

Our goal is to produce a list of matched model and scene primitives, such as points, lines and planar patches, which we refer to as the recognition task. In addition, we would like to locate the identified model in the workspace, i.e. compute the rigid displacement that takes the model onto the scene. In order to make the decomposition of the rigid displacement **T** unique, we assume that the axis of the rotation goes through the origin of coordinates and that the rotation is applied first.

The combinatorial complexity for the search strategy needed to reach our goal can be very high. However, by using the constraint of rigidity the size of the search space is reduced. Several techniques exist for producing such a search strategy. Some widely used examples are relaxation matching, Hough transform clustering and tree search. We have used a quaternion-based procedure for yielding the least squares rotation and translation.

4.1 Mathematical Representation of the Rigidity Constraint

We would like to apply the rigidity constraint to estimate the transformation between model and scene given a partial match. Thus given a set of pairings (M_i, S_i) where the M_i's and S_i's are primitives of the model and scene respectively, the problem is to compute the "best" transformation **T** that applies the model onto the scene. We find **T** by minimizing the sum of the distance between $\mathbf{T}(M_i)$ and S_i, i.e.

$$Min \sum_{i=1}^{p} D(\mathbf{T}(M_i), S_i).$$

The distance D and corresponding minimization problem depend on the type of primitives. In our case the primitives are points and hence D is the usual distance between the scene point and the transformed model point. We applied a least squares method and hence the problem is to minimize the function

$$V(R,t) = \sum_{i=1}^{p} ||Rp_i + t - x_i||^2 \qquad (3)$$

where p_i, x_i are points representing the scene and model respectively. R is constrained to be a rotation matrix and we have represented this rotation as a quaternion which is the simplest way of solving the problem.

Definition of quaternions Quaternions are elements of a vector space endowed with multiplication. A quaternion q can be considered as being a 4-dimensional vector (α, γ) where $\alpha = q_0 \in R_+$ and $\gamma = [q_1\ q_2\ q_3]^T \in R^3$. A multiplication is defined over the set of quaternions as

$$(\alpha, \gamma) * (\alpha', \gamma') = (\alpha\alpha' - \gamma^T\gamma', \alpha\gamma' + \alpha'\gamma + \gamma \wedge \gamma')$$

where $\wedge$ is the cross product.

The conjugate and magnitude of a quaternion q are defined as follows:

$$\overline{q} = (\alpha, -\gamma),$$

$$|q|^2 = q * \overline{q} = (\alpha^2 + ||\gamma||^2, 0) = (||q||^2, 0)$$

Note that R^3 is a subspace of Q due to the identification v = (0,v). Similarly, the magnitude is an extension of the euclidean magnitude and is "multiplicative":

$$|q * q'|^2 = |q|^2 * |q'|^2$$

A rotation R of axis v and angle θ can be represented by two quaternions q = (α, γ) and $-q$ with $|q| = 1$. The application of the rotation is translated into a quaternion product by the relation

$$Ru = q * u * \overline{q},$$

where the vectors and quaternions are identified. In addition, the 3×3 rotation matrix generated by a unit rotation quaternion, $q = [q_0\ q_1\ q_2\ q_3]^T$, is given below:

$$R = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

The mapping between the rotations and the quaternions is defined by

$$\alpha = cos(\theta/2) \ \ and \ \ \gamma = sin(\theta/2)\mathbf{v}.$$

Hence the relation defining the minimization problem can be translated into a minimization in the space of quaternions with the new constraint $|q| = 1$.

4.2 Corresponding Point Set Registration

We next review a procedure for yielding the least squares rotation and translation. Since reflections are not desired we use a quaternion-based algorithm instead of a singular value decomposition (SVD) method in three dimensions. However, the SVD approach, based on the cross-covariance matrix of two point distributions, does, generalize easily to n dimensions and would be the method of choice for n > 3 in any n-dimensional applications.

Let $q_R = [q_0\ q_1\ q_2\ q_3]^T$ be a unit rotation quaternion and $q_T = [q_4\ q_5\ q_6]^T$ be a translation vector. The complete registration state vector $\vec{q}$ is denoted $\vec{q} = (q_R\ \ q_T)^T$. Let $P = \{p_i\}$ be a measured data point set to be aligned with a model point set $X = \{x_i\}$, where $N_x = N_p$ and where each point p_i corresponds to the point x_i with the same index. The following approach can be found in a number of computer vision papers. Using the mean square the minimization problem in eq. (3) can be rewritten in quaternion notation as [6]

$$V(q_R, q_T) = \frac{1}{N_p}\sum_{i=1}^{N_p} \|R(q_R)p_i + q_T - x_i\|^2.$$

The "center of mass" μ_p of the measured point set P and the center of mass μ_x for the model point set X are given by

$$\mu_p = \frac{1}{N_p}\sum_{i=1}^{N_p} p_i \quad and \quad \mu_x = \frac{1}{N_x}\sum_{i=1}^{N_x} x_i.$$

The cross-covariance matrix $\sum_{px}$ of the sets P and X is given by

$$\textstyle\sum_{px} = \frac{1}{N_p}\sum_{i=1}^{N_p}[(p_i - \mu_p)(x_i - \mu_x)^T] = \frac{1}{N_p}\sum_{i=1}^{N_p}[p_i x_i^T] - \mu_p \mu_x^T.$$

Next the cyclic components of the anti-symmetric matrix $A_{ij} = (\sum_{px} - \sum_{px}^T)_{ij}$ are used to form the column vector $\Delta = [A_{23}\ A_{31}\ A_{12}]^T$. This vector is then used to form the symmetric 4×4 matrix $Q(\sum_{px})$

$$Q(\textstyle\sum_{px}) = \begin{pmatrix} tr(\sum_{px}) & \Delta^T \\ \Delta & \sum_{px} + \sum_{px}^T - tr(\sum_{px})I_3 \end{pmatrix}$$

where I_3 is the 3×3 identity matrix. The unit eigenvector $q_R = [q_0\ q_1\ q_2\ q_3]^T$ corresponding to the maximum eigenvalue of the matrix $Q(\sum_{px})$ is selected as the optimal rotation. The 3×3 rotation matrix, $R(q_R)$, is computed using the equation

$$R(q_R) = 2r(q_R)r(q_R)^T - q_R^T q_R I_3 + 2q_0 Q^x(q_R)$$

where $r(q_R) = [q_1\ q_2\ q_3]^T$ and $Q^x(q_R)$ is given by

$$Q^x(q_R) = \begin{pmatrix} q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0. \end{pmatrix}$$

The optimal translation vector is given by

$$q_T = \mu_x - R(q_R)\mu_p. \tag{4}$$

This least squares quaternion operation is $O(N_p)$ and is denoted as and similarly for $ad_f^{i-1}g_2$. By the assumed involutivity of $\hat{\Delta}_{i-1}$ the last two terms are already in Δ_{i-1}. Therefore we can write:

$$\Delta_i = \Delta_{i-1} + \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ ad_f^{i-1}g_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ ad_f^{i-1}g_2 \end{bmatrix} \right\}$$

The conditions of Theorem 4 requires that $\hat{I}^{(i)}$ be integrable. This is equivalent to $\hat{\Delta}_i$ being involutive. As before the only pairs that can cause trouble are the ones not involving v_1 and v_2. Hence the condition is equivalent to $G_{i-1} = \{ad_f^k g_j : 0 \leq k \leq i-1, j = 1,2\}$ being involutive.

By induction, for any i, the condition of Theorem 4 for the the i^{th} iteration of the derived flag holds if and only if distribution G_{i-1} is involutive, i.e. if and only if condition (3) of Theorem 1 holds. In addition note that the dimension of G_i keeps increasing by at least one, until, for some M, $G_{M-1} = G_M$. The involutivity assumption on G_{M-1} prevents any further increase after this stage is reached. Note that the dimension of G_i is bounded above by n. Hence, as the dimension of G_i increases by at least one at each step until it saturates, $M \leq n$. Note that the Δ_M will have dimension three greater than the dimension of G_{M-1}. Moreover $\Delta_M = \Delta_{M+1}$ and therefore $I^{(M)} = I^{(M+1)}$, i.e. the derived flag stops shrinking after M steps. The remaining condition of Theorem 4, namely that there exists N such that $I^{(N)} = \{0\}$ is clearly equivalent to saying that $I^{(M)} = \{0\}$, or equivalently that Δ_M has dimension $n+3$. This in turn is equivalent to the dimension of G_{M-1} being n. As $M \leq n$ this is equivalent to the dimension of G_{n-1} being equal to n, i.e. condition (2) of Theorem 1. The remaining condition of Theorem 1, namely that the dimension of G_i is constant for all $0 \leq i \leq n-1$ is taken care of by the implicit assumption that the dimension of all co-distributions in the derived flag is constant. □

A few more comments can be made on the relation between the vector field and Pfaffian approaches to feedback linearization:

- The coordinate transformation considered in the forms context corresponds to both state feedback and coordinate transformation in the vector field notation. The reason is that the state space $\mathbb{R}^{n+m+1}$ in the forms context does not discriminate between states, inputs and time, hence a coordinate transformation will make the inputs in the original coordinates functions of the state in the original coordinates and possibly time. However it can be shown [7] that time does not enter into the transformation at all, that is a time invariant state feedback and coordinate change can always be found. In addition the coordinate transformation can also be chosen to be both time and input.

- Theorems 3 and 4 are **not** equivalent to 1 in their general form. The reason is that they allow π to be any integrable one-form and not just dt. Therefore we expect more systems to match the conditions of 3 and 4 than those of 1. It should be noted however that a choice of π other than dt implies, rescaling of time as a function of the state. Even though this effect is very useful for the case of driftless systems (where the role of time is effectively played by an input) solutions for $\pi \neq dt$ are probably not very helpful for linearizing control systems with drift.

- Theorems 3 and 4 are capable of dealing with the more general case of systems of the form 10 (or equivalently 1). The corresponding conditions for the vector field case have not been thoroughly investigated. Some progress in this direction can be made by following the steps of Proposition 1, but the resulting conditions do not seem to be terribly informative.

- The equivalence between Theorems 1 and 4 implies that some of the integrability conditions of 4 may be redundant. In fact the only steps of the flag construction that we need to worry about are the ones where a tower terminates.

- Theorem 5 is a very interesting alternative to Theorems 3 and 4 as it provides a way of determining if a Pfaffian system can be brought into Goursat normal form just by looking at the annihilating distributions, without having to determine a one-form π or an appropriate basis. Unfortunately a generalization to multi-input systems (or more precisely to the extended Goursat normal form) is not easy to formulate. It should be noted that the conditions on the filtrations are very much like involutivity conditions. It may be interesting to try to relate these conditions to the conditions of Theorem 4 (the connection to the conditions of Theorem 3 is provided in [6]) and see if a formulation for the extended problem can be constructed in this way.

3 Linearization by Dynamic State Feedback

3.1 Problem Statement

Following (more or less) the notation of [8] the problem of exact linearization by dynamic state feedback and coordinate transformation can be stated as follows:

Problem 3 (Dynamic Feedback Linearization Problem)[2]
Given a control system of the form 1 find, if possible, a dynamic feedback compensator:

$$\begin{aligned}\dot{w} &= a(x,w) + B(x,w)v \\ u &= \alpha(x,w) + \beta(x,w)v\end{aligned}$$

where $w \in \mathbf{R}^q, v \in Re^{m'}$ *and an extended state space diffeomorphism* $z = \Phi(x,w), z \in \mathbf{R}^{n'} = \mathbf{R}^{n+q}$ *such that the resulting system is linear and controllable (without loss of generality in Brunovsky form).*

It should be noted here that the problem statement above requires that both the system and the controller dynamics be rendered linear and controllable. This condition is relaxed in the problem statement considered in where only the system dynamics are required to be linear and controllable. This relaxed condition implies that a separate analysis (primarily for stability) is carried out for the controller dynamics.

An interesting special case of the general Dynamic Feedback Linearization Problem is the following:

[2]Similar to problem statement 1 all conditions may be restricted to a neighborhood U of an equilibrium point x^o.

Problem 4 *(Feedback Linearization by Dynamic Extension)*
Given a control system of the form 1 find, if possible, a dynamic feedback compensator of the form:

$$\dot{w}_i^j = w_{i+1}^j \quad 1 \le i \le \mu_j - 1, \mu_j > 1 \tag{12}$$

$$\dot{w}_{\mu_j}^j = \alpha_j(x,w) + \sum_{l=1}^{m'} \beta_{j,l}(x,w) v_l(t) \quad 1 \le j \le m, \mu_j > 0 \tag{13}$$

$$u_j = w_1^j \quad 1 \le j \le m, \mu_j > 0 \tag{14}$$

$$u_j = \alpha_j(x,w) + \sum_{l=1}^{m'} \beta_{j,l}(x,w) v_l(t) \quad 1 \le j \le m, \mu_j = 0 \tag{15}$$

for some integers $\mu_j \ge 0$ where $m' \ge m$ and $\beta(x,w)$ has full rank m in a neighborhood an equilibrium point in $\mathbb{R}^{n+\mu}$, $\mu = \sum_{i=1}^m \mu_j$.

In other words the added dynamics simply involve adding strings of integrators in front of some of the input channels and defining the new inputs to be linear combinations of the resulting derivatives of the old inputs:

$$\begin{pmatrix} u_1^{(\mu_1)} \\ \cdots \\ u_m^{(\mu_m)} \end{pmatrix} = \alpha(x,w) + \beta(x,w) \begin{pmatrix} v_1 \\ \cdots \\ v_{m'} \end{pmatrix} \tag{16}$$

3.2 The Vector Field Approach

As for the static feedback case the results using the vector field approach are restricted to systems of the form 2. The problem of linearization by general dynamic state feedback and coordinate transformation is still largely open. Even for the special case of dynamic extension no necessary and sufficient conditions exist. In [8] the following results are proved:

Theorem 6 *If system 2 is locally dynamic feedback linearizable, then its Jacobian linearization at the origin is completely controllable.*

Theorem 7 *If for a set of integers $\{\mu_1, \ldots, \mu_m\}$, $0 \le \mu_1 \le \ldots \le \mu_m$, $\mu = \sum_{i=1}^m \mu_j$, the distributions, up to input reordering,*

$$\begin{aligned} \Delta_0 &= span\{g_k : \mu_k = 0\} \\ \Delta_{i+1} &= \Delta_i + ad_f \Delta_i + span\{g_k : \mu_k = i+1\} \quad i \ge 0 \end{aligned}$$

are such that in a neighborhood of the origin in $\mathbb{R}^n$:

1. *Δ_i is of constant rank for $0 \le i \le n + \mu_m - 1$*
2. *Δ_i is involutive for $0 \le i \le n + \mu_m - 1$*
3. *$rank \Delta_{n+\mu_m-1} = n$*
4. *$[g_j, \Delta_i] \subset \Delta_{i+1}$ for all $j, 1 \le j \le m$ such that $\mu_j \ge 1$ and all $i, 0 \le i \le n + \mu_m - 1$*

then the system is locally dynamic feedback linearizable by dynamic extension and a local diffeomorphism on a neighborhood of the extended state space $\mathbb{R}^{n+\mu}$.

The necessary condition of Theorem 6 is shown to be not sufficient by means of a counter example.

3.3 The Prolongation Approach

Problem 4 can also be approached in the framework of Pfaffian systems by means of prolongations by differentiation.

Definition 3 *Let* I *be a Pfaffian system of co-dimension* $m+1$ *in* $\mathbb{R}^{n+m+1}$ *with coordinates* t, u, x *where* dt *is an independence condition and* $\{dt, du_1, \ldots, du_m\}$ *forms a complement. Let* $\{\mu_1, \ldots, \mu_m\}$ *be a set of non-negative integers,* $\mu = \sum_{i=1}^m \mu_j$. *The system* I *augmented by the* μ *one forms:*

$$\begin{array}{lll} du_1 - w_1^1 dt, & \ldots, & dw_1^{\mu_1-1} - w_1^{\mu_1} dt, \\ du_2 - w_2^1 dt, & \ldots, & dw_2^{\mu_1-1} - w_2^{\mu_1} dt, \\ \ldots & \ldots & \ldots \\ du_1 - w_m^1 dt, & \ldots, & dw_m^{\mu_1-1} - w_m^{\mu_1} dt \end{array}$$

is called a **prolongation by differentiation** *of* I. *The augmented system is defined on* $\mathbb{R}^{n+m+\mu+1}$.

Prolongation by differentiation is a special case of the Cartan prolongation. Utilizing this definition the following theorem can be stated:

Theorem 8 *Consider the Pfaffian system* $I = \{\alpha^1, \ldots, \alpha^n\}$ *on* $\mathbb{R}^{n+m+1}$ *with independence condition* dz^0 *and complement* $\{dz^0, du_1, \ldots, du_m\}$. *If there exists a list of non-negative integers* $\{\mu_1, \ldots, \mu_m\}, \mu = \sum_{i=1}^m \mu_j$ *such that the prolonged system:*

$$\begin{array}{rll} J & = \{ & \alpha^1, \ldots, \alpha^n \\ & & du_1 - w_1^1 dt, \ldots, dw_1^{\mu_1-1} - w_1^{\mu_1} dt, \\ & & du_2 - w_2^1 dt, \ldots, dw_2^{\mu_1-1} - w_2^{\mu_1} dt, \\ & & \ldots, \ldots, \ldots \\ & & du_1 - w_m^1 dt, \ldots, dw_m^{\mu_1-1} - w_m^{\mu_1} dt\} \end{array}$$

satisfies the condition that $\{J^{(k)}, dz^0\}$ *is integrable for all* k, *then* I *can be transformed to extended Goursat Normal Form using prolongation by differentiation.*

For the relevant theorems and definitions the reader is refered to [5]

3.4 The Infinitesimal Brunovsky Form

An altogether different approach to dynamic feedback linearization is presented in [9]. It revolves around an alternative flag construction that can be used to derive a special normal form, the Infinitesimal Brunovsky Form. An interesting fact about the whole construction is that **any** accessible nonlinear system can be brought into this form.

Consider the system 2 and let $\mathcal{K}$ denote the field of meromorphic functions of $x, u, \dot{u}, \ldots$, where the dot stands for the usual time differentiation. Let $\mathcal{E}$ denote the $\mathcal{K}$ vector space of one forms, spanned by $\{dx_1, \ldots, dx_n, du_1, \ldots, du_m, d\dot{u}_1, \ldots, d\dot{u}_m, \ldots\} = \{dx, du, d\dot{u}, \ldots\}$. Define the time derivative of $\omega = \sum_j \alpha_j dv_j \in \mathcal{E}$ by

$$\begin{aligned}
\dot{\omega} &= \sum_j (\dot{\alpha}_j dv_j + \alpha_j d\dot{v}_j) \\
\dot{\alpha}_j &= \frac{\partial \alpha_j}{\partial x}(f(x) + g(x)u) + \sum_{j \geq 0} \frac{\partial \alpha_j}{\partial u^{(j)}} u^{(j+1)} \\
d\dot{x}_i &= \sum_{k=1}^{n} (\frac{\partial f_i}{\partial x_k} + \sum_{j=1}^{m} \frac{\partial g_{ij}}{\partial x_k} u_j) dx_k + \sum_{k=1}^{m} g_{ij} du_j
\end{aligned}$$

The **relative degree** of a one form ω is defined as the smallest integer r such that $\omega^{(r)} \notin \text{span}_{\mathcal{K}}\{dx\}$. If such an integer does not exist define $r = \infty$.

Consider a flag defined iteratively by:

$$H_0 = \text{span}_{\mathcal{K}}\{dx, du\} \tag{17}$$

$$H_k = \{\omega \in H_{k-1} : \dot{\omega} \in H_{k-1}\} \quad k > 0 \tag{18}$$

Clearly, $\mathcal{E} \supset H_0 \supset H_1 = \text{span}_{\mathcal{K}}\{dx\} \supset H_2 \supset \ldots$. Moreover, as the dimension of H_1 is finite (n), the flag will stop decreasing after a finite number of steps, i.e. there exists $k^* > 0$ such that $H_{k^*+1} = H_{k^*+2} = \ldots = H_\infty$. The reason this flag is important is highlighted by the following theorems:

Proposition 2 *The following statements are equivalent:*

1. *The system satisfies the strong accessibility rank condition*
2. *Any non zero form has finite relative degree*
3. $H_\infty = \{0\}$

Theorem 9 *Suppose $H_\infty = \{0\}$. There exist a list of integers $\{r_1, \ldots, r_m\}$, invariant under regular static state feedback, and m one forms $\omega_1, \ldots, \omega_m$ with relative degrees $r_1, \ldots, r_m$ such that:*

1. $span_{\mathcal{K}}\{\omega_i^{(j)}, 1 \leq i \leq m, 0 \leq j \leq r_i - 1\} = span_{\mathcal{K}}\{dx\}$
2. $span_{\mathcal{K}}\{\omega_i^{(j)}, 1 \leq i \leq m, 0 \leq j \leq r_i\} = span_{\mathcal{K}}\{dx, du\}$
3. *The forms $\{\omega_i^{(j)}, 1 \leq i \leq m, j \geq 0\}$ are linearly independent. In particular, $\sum_{i=1}^{m} r_i = n$.*

An equivalent form of the last result is the following:

Corollary 1 *Suppose $H_\infty = \{0\}$. Then the basis $\{\omega_{i,j}, 1 \le i \le m, 0 \le j \le r_i\}$ of $span_{\mathcal{K}}\{dx\}$ defined by $\omega_{i,j} = \omega_i^{(j-1)}$ yields:*

$$\begin{aligned} \dot{\omega}_{i,1} &= \omega_{i,2} \\ &\dots \\ \dot{\omega}_{i,r_i-1} &= \omega_{i,r_i} \\ \dot{\omega}_{i,r_i} &= \sum_{j=1}^{n} a_{i,j} dx_j + \sum_{j=1}^{m} b_{i,j} du_j \end{aligned}$$

where $a_{i,j}, b_{i,j} \in \mathcal{K}$ and the matrix $[b_{i,j}]$ has an inverse in the ring of $m \times m$ matrices with entries in $\mathcal{K}$.

The last representation, called the **Infinitesimal Brunovsky form**, highlights the similarity of this construction with the regular Brunovsky form: the two forms are identical, with scalar quantities replaced by one forms. Using this normal form the following Theorems can be proved:

Theorem 10 *The system is linearizable by static state feedback if and only if $H_\infty = \{0\}$ and for all $k = 1, \dots, k^*$, H_k is integrable.*

Theorem 11 *Suppose $H_\infty = \{0\}$ and let $\Omega = (\omega_1, \dots, \omega_m)^T$. There exists a system of linearizing outputs $y = h(x, u, \dots, u^{(\nu-1)}) \in \mathbb{R}^m$ if and only if there exist an invertible polynomial operator $P \in \mathcal{K}^{m \times m}\left[\frac{d}{dt}\right]$ such that $d(P\Omega) = 0$.*

3.5 Connection between the approaches

A first observation is that, a system that is not linearizable by state feedback may be linearizable by dynamic feedback only if it has more than one input. In other words:

Theorem 12 *The following statements are equivalent:*

1. *System 2 with $m = 1$ is static feedback linearizable*
2. *System 2 with $m = 1$ is dynamic feedback linearizable*

This fact is relatively easy to illustrate in the case where the only dynamic feedback of interest is dynamic extension:

Proposition 3 *A single input system of the form 2 is feedback linearizable by dynamic extension if and only if it is static feedback linearizable.*

Proof: The proof can be carried out using any of the above approaches. One step of dynamic extension (integrator) suffices to illustrate the point. Consider the Pfaffian system $I^{(0)} = \{dx_1 - f_1(x,u)dt, \dots, dx_n - f_n(x,u)dt\}$ in $\mathbb{R}^{n+2}$. Add a prolongation to obtain the augmented system $\hat{I}^{(0)} = \{dx_1 - f_1(x,u)dt, \dots, dx_n - f_n(x,u)dt, du - vdt\}$ of co-dimension 2 in $\mathbb{R}^{n+3}$ The first step of the derived flag of the augmented system yields $\hat{I}^{(1)} = \{dx_1 - f_1(x,u)dt, \dots, dx_n - f_n(x,u)dt\}$ as $d(du - vdt) = -dv \wedge dt \not\equiv 0 \bmod \hat{I}^{(0)}$ whereas $d(dx_i - f_i(x,u)dt)$ only contain terms in $dx_i \wedge dt$ and $du \wedge dt$ which are both 0 mod $\hat{I}^{(0)}$. Note that $\hat{I}^{(1)}$ (and consequently $\hat{I}^{(i)}$ for all i) is independent of v. Moreover, $\hat{I}^{(i)} = I^{(i-1)}$, while $\{\hat{I}^{(i)}, dt\} = \{I^{(0)}, du - vdt, dt\} = \{I^{(0)}, du, dt\}$. Therefore, the extended system satisfies the conditions of Theorem 4 for $\pi = dt$ if and only if the original system does. □

The proof for the general dynamic feedback case can be found in [8] for the vector field formalism and in [9] for the infinitesimal Brunovsky form formalism. It should be noted here that the second proof is extremely simple whereas the vector field proof is rather complicated.

A simple calculation, similar to the one in Proposition 3 can be used to show the following:

Proposition 4 *Consider system 1. An extended system obtained by adding the same number k of integrators in front of each input is linearizable by static state feedback if and only if the original system is.*

The same is probably true about feedback linearization by dynamic extension as well.

Concerning the relation between the dynamic extension results 7 and 8 the following statement is probably true:

Conjecture 1 *There exist integers satisfying the conditions of Theorem 7 if and only if there exist integers satisfying the conditions of Theorem 8 for $\pi = dt$.*

One way to prove this statement would be by making use of the fact that the proof of Theorem 7 relies on the fact that the given conditions on the distributions (involving vector fields in $\mathbb{R}^n$) are equivalent to the conditions of Theorem 1 for distributions involving the vector fields in the extended space $\mathbb{R}^{n+\mu}$. Proposition 1 can then be used to show that these conditions on $\mathbb{R}^{n+\mu}$ are equivalent to the conditions of Theorem 8. A direct proof of this conjecture will probably involve rather cumbersome calculations, but should be possible.

The relation between the standard Pfaffian system approach and the infinitesimal Brunovsky form is a bit more obscure. The following can be said:

Proposition 5 *If the system is linearizable by static state feedback (equivalently the conditions of Theorems 3 and 4 hold for $dz^0 = dt$) then the two flag constructions are the same, mod dt, i.e.:*

$$H_k = I^{(k-1)} \text{ mod } dt$$

Proof: As both flag constructions are intrinsic we can assume, without loss of generality, that the system is already in the canonical coordinates of the Goursat normal form. Then:

$$\begin{aligned} I^{(0)} &= \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \ldots, s_j; j = 1, \ldots, m\} \\ H_1 &= \{dz_i^j : i = 1, \ldots s_j; j = 1, \ldots, m\} \end{aligned}$$

Recall that, in the context of system 1 (equivalently 10), z^0 plays the role of time (hence $dz^0 = dt$) and $z_{s_j+1}^j$ plays the role of u_j, $j = 1, \ldots, m$. Observe that the above co-distributions are identical if the terms in dz^0 are dropped from $I^{(0)}$.

The next iteration of the two flags yields:

$$\begin{aligned} I^{(1)} &= \{\alpha \in I^{(0)} : d\alpha \equiv 0 \bmod I^{(0)}\} \\ &= \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \ldots, s_j - 1; j = 1, \ldots, m\} \\ H_2 &= \{\omega \in H_1 : \dot{\omega} \in H_1\} \\ &= \{dz_i^j : i = 1, \ldots, s_j - 1; j = 1, \ldots, m\} \end{aligned}$$

Note that $d(dz_{s_j}^j - z_{s_j+1}^j dz^0) = -dz_{s_j+1}^j \wedge dz^0$ which is not equal to 0 mod $I^{(0)}$ as, when wedged with all the forms in $I^{(0)}$ will produce $\pm \bigwedge_{j=1,i=1}^{m,s_j} dz_i^j \wedge dz_{s_j+1}^j \wedge dz^0 \neq 0$. Similarly $d\dot{z}_{s_j}^j = dz_{s_j+1} = du_j \notin \text{span}_{\mathcal{K}}\{dz_i^j : i = 1, \ldots, s_j; j = 1, \ldots, m\}$. Again the two constructions are the same if the terms in dz^0 are dropped from $I^{(1)}$.
In general, for the k^{th} step assume that:

$$\begin{aligned} I^{(k-1)} &= \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \ldots, s_j - k + 1; j = 1, \ldots, m\} \\ H_k &= \{dz_i^j : i = 1, \ldots, s_j - k + 1; j = 1, \ldots, m\} \end{aligned}$$

Then:

$$\begin{aligned} I^{(k)} &= \{\alpha \in I^{(k-1)} : d\alpha \equiv 0 \bmod I^{(k-1)}\} \\ &= \{dz_i^j - z_{i+1}^j dz^0 : i = 1, \ldots, s_j - k; j = 1, \ldots, m\} \\ H_{k+1} &= \{\omega \in H_k : \dot{\omega} \in H_k\} \\ &= \{dz_i^j : i = 1, \ldots, s_j - k; j = 1, \ldots, m\} \end{aligned}$$

Note again that $d(dz_{s_j-k+1}^j - z_{s_j-k+2}^j dz^0) = -dz_{s_j-k+2}^j \wedge dz^0$ which will not be zero when wedged with all the one forms spanning $I^{(k-1)}$. Similarly, $d\dot{z}_{s_j-k+1}^j = dz_{s_j-k+2} \notin H_k$. Yet again the two co-distributions are identical if the terms in dz^0 are dropped from $I^{(k)}$. □

In view of Theorem 10 the following is also likely to be true.

Conjecture 2 *The two flag constructions are related by:*

$$H_k = I^{(k-1)} \text{ mod } dt$$

only if the system is linearizable by static state feedback.

Unfortunately, the proof of this statement eludes me at the moment. Here are some examples that illustrate the point:

Example: Ball and Beam (give or take a sine)
This is an example of a single input system that is not linearizable by static (and hence dynamic) state feedback. The equations of the dynamics read:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 x_4^2 - x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned}$$

The flag associated with the infinitesimal Brunovsky form is:

$$\begin{aligned} H_0 &= \{dx_1, dx_2, dx_3, dx_4, du\} \\ H_1 &= \{dx_1, dx_2, dx_3, dx_4\} \\ H_2 &= \{dx_1, dx_2, dx_3\} \\ H_3 &= \{dx_1, dx_2 - 2x_1x_4dx_3\} \\ H_4 &= \{(1 + 2x_2x_4 + 2x_1u)dx_1 + 2x_1x_4(dx_2 - 2x_1x_4dx_3)\} \\ H_5 &= \{0\} \end{aligned}$$

Note that, if we let $\omega = (1 + 2x_2x_4 + 2x_1u)dx_1 + 2x_1x_4(dx_2 - 2x_1x_4dx_3)$, $d\omega \wedge \omega \not\equiv 0$ (as it will contain, among other things, the term $x_1^2x_4du \wedge dx_1 \wedge dx_2$), therefore $H_4 = \{\omega\}$ is not integrable. Hence, according to Theorem 10 and Theorem 12, the system will not be linearizable by any of the techniques considered here, as expected.

The derived flag construction for the same system leads to:

$$\begin{aligned} I^{(0)} &= \{dx_1 - x_2dt, dx_2 - (x_1x_4^2 - x_3)dt, dx_3 - x_4dt, dx_4 - udt\} \\ I^{(1)} &= \{dx_1 - x_2dt, dx_2 - (x_1x_4^2 - x_3)dt, dx_3 - x_4dt\} \\ I^{(2)} &= \{dx_1 - x_2dt, dx_2 - 2x_1x_4dx_3 + (x_1x_4^2 + x_3)dt\} \\ I^{(3)} &= \{0\} \end{aligned}$$

Note that the two flags are identical (neglecting the dt terms) until the fourth step where the dimension of the derived flag drops by two. According to Theorem 4 (and 5 this implies that the system is not linearizable by static state feedback. My guess is that the reason for this is the fact that at this step the input u appears explicitly as a coefficient in H_4. Indeed it turns out that, if in the derived flag calculation we substitute $dx_4 = udt$ in this last step (going from $I^{(2)}$ to $I^{(3)}$) the augmented flag becomes:

$$\begin{aligned} \hat{I}^{(3)} &= \{(1 + 2x_2x_4 + 2x_1u)(dx_1 - x_2dt) + 2x_1x_4(dx_2 - 2x_1x_4dx_3 + (x_1x_4^2 + x_3)dt))\} \\ \hat{I}^{(4)} &= \{0\} \end{aligned}$$

which again is identical to the last two steps of the H_k flag. Loosely speaking H_4 "detects" that the system is not feedback linearizable with static feedback when it reaches H_4 and attempts some form of brute force dynamic extension.

Example: Harrier (give or take some parasitic effects)
This is an example of a two input system that is not linearizable by static state feedback, but is linearizable by dynamic extension. The dynamics of the system read:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_5u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \cos x_5u_1 - 1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2 \end{aligned}$$

It is easy to show that the system is not feedback linearizable by static state feedback. However, if two integrators are added in front of input u_1 the resulting eight state, two input system is feedback linearizable.

The flag associated with the infinitesimal Brunovsky form is:

$$\begin{aligned} H_0 &= \{dx_1, dx_2, dx_3, dx_4, dx_5, dx_6, du_1, du_2\} \\ H_1 &= \{dx_1, dx_2, dx_3, dx_4, dx_5, dx_6\} \\ H_2 &= \{dx_1, dx_3, dx_5, \cos x_5 dx_2 + \sin x_5 dx_4\} \\ H_3 &= \{\cos x_5 dx_1 + \sin x_5 dx_3, (\sin x_5) x_6 dx_1 - (\cos x_5) x_6 dx_3 + (\cos x_5 dx_2 + \sin x_5 dx_4)\} \\ H_4 &= \{0\} \end{aligned}$$

Letting $\omega_1 = \cos x_5 dx_1 + \sin x_5 dx_3$ and $\omega_2 = (\sin x_5)x_6 dx_1 - (\cos x_5)x_6 dx_3 + (\cos x_5 dx_2 + \sin x_5 dx_4)$ it is easy to show that $d\omega_1 \wedge \omega_1 \wedge \omega_2 \not\equiv 0$ (as it contains terms in $dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5$ among other things). Therefore, $H_3 = \text{span}_{\mathcal{K}}\{\omega_1, \omega_2\}$ is not integrable and hence, according to Theorem 10 the system is not linearizable by static state feedback. Even though it is known that the system is linearizable by dynamic extension, there seems to be no easy way of determining the form of the invertible operator P of Theorem 11.

The derived flag on the other hand has the form:

$$\begin{aligned} I^{(0)} &= \{dx_1 - x_2 dt, dx_2 + \sin x_5 u_1 dt, dx_3 - x_4 dt, dx_4 - (\cos x_5 u_1 - 1)dt, dx_5 - x_6 dt, dx_6 - u_2 dt\} \\ I^{(1)} &= \{dx_1 - x_2 dt, dx_3 - x_4 dt, dx_5 - x_6 dt, \cos x_5 dx_2 + \sin x_5 dx_4 + \sin x_5 dt\} \end{aligned}$$

The calculation involved in the next step of the derived flag are rather complicated. However, the pair of one forms we would expect to find in $I^{(2)}$ because of the structure of H_3, namely $\cos x_5(dx_1 - x_2 dt) + \sin x_5(dx_3 - x_4 dt)$ and $\sin x_5 x_6(dx_1 - x_2 dt) - \cos x_5 x_6(dx_3 - x_4 dt) + (\cos x_5 dx_2 + \sin x_5 dx_4 + \sin x_5 dt)$, do not satisfy the necessary conditions. It seems that the two flags diverge at this point. An interesting observation is that, if we define outputs $y_1 = x_1$ and $y_2 = x_3$ (the position of the plane), and attempt to input-output linearize the system this is exactly the step where input u_1 shows up (without u_2) and we can conclude that the linearization will fail. It would be interesting to try to relate this observation with the maximal linearizable subsystem [10] and hence give a bit more substance to this observation.

Some more loose comments on the various approaches presented above:

- The infinitesimal Brunovsky form approach presented above is the only one of the three that can tackle the more general problem 3.

4 Concluding Remarks & Topics of Interest

A number of different approaches that can be used to linearize a nonlinear system by state feedback and coordinate transformation were presented. It was shown that they all produce comparable results in most cases, even though some are better suited to tackle certain problems than others. These techniques represent significant progress for all the problems posed here. There is still a lot to look strive for however. For example an extension of the vector field conditions for bringing the system to extended Goursat normal form may be very useful and may provide insight into many hard problems in the area of exterior differential systems.

Another direction that deserves further attention is linearization by dynamic state feedback. It should be noted that, even for the case of dynamic extension, all the results provide just sufficient conditions for the problem to be solvable. In particular all the theorem statements start with an assumption of the form "if there exist ..." ("integers such that ..." or "invertible operator ..."), but provide no insight on if those integers or operators exist and how to determine them. Moreover there are no bounds (upper or lower) on the number of dynamic extension steps required or the degree of the operators. All of these are properties that depend only on the system specification, therefore it should be possible to answer the above questions given just the system equations. It should also be noted that most of the literature is concerned with dynamic extension and non-singular input transformations. Of the theorems presented here only 11 claims to address the general dynamic state feedback case. Singular input transformations are briefly discussed in [5] and compared with the corresponding results using the extended Goursat normal form and prolongations. Both these topics merit further attention.

Finally it should be noted that the conditions for feedback linearization are "closed", i.e. they essentially hold for a set of "measure zero" in the "space" of dynamical systems. It is therefore useful to know what, if anything, can be done about systems that don't satisfy these conditions, as most systems encountered in practice fall in this category. This problem was first addressed in [11] and then, more formally, in [12]. A different approach, related more to input-output linearization is taken in [13] and [8]. It would be interesting to compare the two approaches, and hopefully determine classes of systems that are better suited for one or the other.

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