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On Adaptive Synchronization and Control of Nonlinear Dynamical Systems

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Abstract

In this paper, we study the synchronization of two coupled nonlinear, in particular chaotic, systems which are not identical. We show how adaptive controllers can be used to adjust the parameters of the systems such that the two systems will synchronize. We use a Lyapunov function approach to prove a global result which shows that our choice of controllers will synchronize the two systems. We show how it is related to Huberman-Lumer adaptive control and the LMS adaptive algorithm.

We illustrate the applicability of this method using Chua's oscillators as the chaotic systems. We choose parameters for the two systems which are orders of magnitude apart to illustrate the effectiveness of the adaptive controllers. Finally, we discuss the role of adaptive synchronization in the context of secure and spread spectrum communication systems. In particular, we show how several signals can be encoded onto a single scalar chaotic carrier signal.

1 Introduction

Recently, there has been much work done in the area of chaotic synchronization [Fujisaka and Yamada, 1983; Pecora and Carroll, 1990; He and Vaidya, 1992; Wu and Chua, 1993; Wu and Chua, 1994; Wu and Chua, 1995a; Wu and Chua, 1995b; Wu *et al.*, 1995]. In most of the analysis done on two coupled chaotic systems, the two systems are assumed to be identical. In practical implementations this will not be the case, and some work has been done to address this problem using an adaptive approach [Sinha *et al.*, 1990; John and Amritkar, 1994; Celka, 1995; Chua *et al.*,

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1996]. This can be considered as a form of model following or model reference adaptive control where the plant system is driven in such a way as to track the dynamics of the model system. When the model system exhibits an unstable orbit, this can be interpreted as adaptive control to force the plant system dynamics to unstable orbits. On the other hand, such parameter mismatch between the two systems could be introduced intentionally as a way to encode information onto the chaotic carrier [Parlitz *et al.*, 1992; Dedieu *et al.*, 1993; Parlitz and Kocarev, 1995]. The adaptive controllers are then used to recover this encoded information.

The purpose of this paper is to further study the use of adaptive controllers to enforce the synchronization even though initially the two coupled systems are not identical. In particular, we propose a general theoretical framework in which to study this problem and present adaptive controllers based on a Lyapunov approach, similar to those used in model reference adaptive control.

In Section 2 we present the general framework and the main theorem which proves that the use of adaptive controllers will synchronize two coupled systems which are not identical initially. We discuss the relation to other proposed schemes for adaptive control and synchronization. In Section 3 we extend the result in Section 2 by allowing the coupling to be adaptive. This eliminates the need to estimate the dynamics of the two systems. In Section 4 we consider the special case of unidirectional coupling. In Section 5 we illustrate the results presented by means of two coupled Chua's oscillators. We show that the adaptive controllers can still synchronize the two systems and make the driven system's parameters converge towards the driving system's parameters even though the parameters of the two systems differ initially by orders of magnitude. Finally, in Section 6 we consider synchronization by means of adaptive controllers in the context of secure and spread spectrum communication systems. We show how several information signals can be encoded onto a single chaotic signal.

We use lowercase, bold uppercase and bold lowercase letters for scalars (or scalar-valued functions), matrices and vectors respectively. The transpose of a matrix $\mathbf{A}$ is denoted $\mathbf{A}^T$. The vector $\mathbf{0}$ denotes the zero vector. The identity matrix is written as $\mathbf{I}$. The integer n is usually used to denote the size of matrices and vectors.

2 Adaptive Synchronization of Two Coupled Systems

Our starting framework will be two systems coupled together, given by the following state equations:

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{m} \mathbf{A}_k \mathbf{f}_k(\mathbf{x}) + \mathbf{B}\mathbf{u} \quad \leftarrow \text{System 1} \tag{1}$$

$$\dot{\tilde{\mathbf{x}}}(t) = \sum_{k=1}^{m} \tilde{\mathbf{A}}_k \tilde{\mathbf{f}}_k(\tilde{\mathbf{x}}) + \tilde{\mathbf{B}}\tilde{\mathbf{u}} \quad \leftarrow \text{System 2} \tag{2}$$

where $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are the state vectors of System 1 and System 2 respectively. We assume that $\mathbf{f}_k$ and $\tilde{\mathbf{f}}_k$ are nonlinear functions which are continuous. The vectors $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are the inputs to the corresponding systems. The entries of the matrices $\mathbf{A}_k$ and $\tilde{\mathbf{A}}_k$ can be considered as *parameters* of the two systems respectively. Eqs. (1)-(2) can be considered as nonlinear systems which are linear in the parameters.

Note that the coupling *could* be bidirectional, i.e. System 1 influences System 2 and vice versa. Our emphasis will be on applications such as model-following, communication systems, and master-slave driving, where the coupling is unidirectional, i.e., one system influences the other but not the other way around. For example, if system 1 is the driving system and system 2 is the driven system, then $\mathbf{u}$ will not depend on $\tilde{\mathbf{x}}$ or $\tilde{\mathbf{u}}$.

We define the synchronization (or tracking) error to be:

$$\mathbf{r}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t) \tag{3}$$

Then we have:

$$\dot{\mathbf{r}} = \sum_{k=1}^{m} \left[\mathbf{A}_k \mathbf{f}_k(\mathbf{x}) - \tilde{\mathbf{A}}_k \tilde{\mathbf{f}}_k(\tilde{\mathbf{x}})\right] + \mathbf{B}\mathbf{u} - \tilde{\mathbf{B}}\tilde{\mathbf{u}} \tag{4}$$

This can be rewritten as

$$\dot{\mathbf{r}} = \sum_{k=1}^{m} \left[(\mathbf{A}_k - \tilde{\mathbf{A}}_k)\tilde{\mathbf{f}}_k(\tilde{\mathbf{x}}) + \mathbf{A}_k(\tilde{\mathbf{f}}_k(\mathbf{x}) - \tilde{\mathbf{f}}_k(\tilde{\mathbf{x}})) + \mathbf{A}_k(\mathbf{f}_k(\mathbf{x}) - \tilde{\mathbf{f}}_k(\mathbf{x}))\right] + \mathbf{B}\mathbf{u} - \tilde{\mathbf{B}}\tilde{\mathbf{u}} \tag{5}$$

Consider the following control laws for the 2 systems:

$$\begin{aligned} \mathbf{u} &= \mathbf{z} + \mathbf{g}(\mathbf{x}) - \tilde{\mathbf{g}}(\tilde{\mathbf{x}}) \\ \tilde{\mathbf{u}} &= \tilde{\mathbf{z}} + \tilde{\mathbf{h}}(\tilde{\mathbf{x}}) - \mathbf{h}(\mathbf{x}) \end{aligned} \tag{6}$$

where $\mathbf{z}$ and $\tilde{\mathbf{z}}$ are the external input signals that is fed into the two systems respectively (e.g. periodic forcing).

Substituting Eq.(6) into Eq.(5) gives:

$$\begin{aligned} \dot{\mathbf{r}} = & \textstyle\sum_{k=1}^{m} (\mathbf{A}_k - \tilde{\mathbf{A}}_k)\tilde{\mathbf{f}}_k(\tilde{\mathbf{x}}) \\ & +\textstyle\sum_{k=1}^{m} \mathbf{A}_k(\tilde{\mathbf{f}}_k(\mathbf{x}) - \tilde{\mathbf{f}}_k(\tilde{\mathbf{x}})) \\ & +\textstyle\sum_{k=1}^{m} \mathbf{A}_k(\mathbf{f}_k(\mathbf{x}) - \tilde{\mathbf{f}}_k(\mathbf{x})) \\ & +\mathbf{B}\mathbf{z} - \tilde{\mathbf{B}}\tilde{\mathbf{z}} + (\mathbf{B}\mathbf{g} + \tilde{\mathbf{B}}\mathbf{h})(\mathbf{x}) - (\mathbf{B}\tilde{\mathbf{g}} + \tilde{\mathbf{B}}\tilde{\mathbf{h}})(\tilde{\mathbf{x}}) \end{aligned} \tag{7}$$

Let us denote

$$\begin{aligned} \mathbf{p} &= \mathbf{B}\mathbf{g} + \tilde{\mathbf{B}}\mathbf{h} \\ \tilde{\mathbf{p}} &= \mathbf{B}\tilde{\mathbf{g}} + \tilde{\mathbf{B}}\tilde{\mathbf{h}} \end{aligned}$$

The goal is to change $\tilde{\mathbf{A}}_k$ adaptively such that $\mathbf{A}_k$ and $\tilde{\mathbf{A}}_k$ approach each other. Using a Lyapunov approach, we construct adaptive controllers for $\tilde{\mathbf{A}}_k$, given in the following theorem. We denote $\mathbf{A}_k = \{a_{ij}^k\}_{i,j=1}^{i,j=n}$ and $\tilde{\mathbf{A}}_k = \{\tilde{a}_{ij}^k\}_{i,j=1}^{i,j=n}$.

Theorem 1 *Assume that* $\mathbf{A}_k$ *is constant and*

$$\begin{aligned} \mathbf{Bz} &= \tilde{\mathbf{B}}\tilde{\mathbf{z}} && (8) \\ \mathbf{p} &= \tilde{\mathbf{p}} && (9) \\ \mathbf{f}_k &= \tilde{\mathbf{f}}_k && (10) \end{aligned}$$

for each k, *and there exists a real number* $\lambda > 0$ *and* $\mathbf{V}$ *symmetric positive definite such that*

$$\lambda \mathbf{I} + \sum_{k=1}^{m} (\mathbf{V}\mathbf{A}_k D\mathbf{f}_k(\mathbf{x})) + \mathbf{V} D\mathbf{p}(\mathbf{x}) \tag{11}$$

is negative definite for all $\mathbf{x}$, *where* $D\mathbf{f}_k(\mathbf{x})$ *and* $D\mathbf{p}(\mathbf{x})$ *are the Jacobian matrices of* $\mathbf{f}_k$ *and* $\mathbf{p}$ *at* $\mathbf{x}$ *respectively.*

Let $\mathbf{s} = (s_1, \ldots, s_n)^T = \mathbf{V}\mathbf{r}$. *If we use the adaptive controller for* $\tilde{\mathbf{A}}_k$ *given by:*

$$\dot{\tilde{a}}_{ij}^k = \tilde{\phi}^{i,j,k} s_i [\mathbf{f}_k(\tilde{\mathbf{x}})]_j \tag{12}$$

such that $\tilde{\phi}^{i,j,k} > 0$ *for all* i,j *and* k, *and the control laws specified by (6), then we can draw the following two conclusions:*

1. *The synchronization error* $\mathbf{r}(t)$ *and the parameter mismatch* $\mathbf{A}_k - \tilde{\mathbf{A}}_k(t)$ *can be made arbitrarily small for all time* $t \geq 0$, *if we choose the initial error* $\mathbf{r}(0)$ *and the initial parameter mismatch* $\mathbf{A}_k - \tilde{\mathbf{A}}_k(0)$ *to be sufficiently small.*

2. *The two systems in Eq. (1) and Eq. (2) are synchronized ,i.e.,*

$$\mathbf{r} \to \mathbf{0} \tag{13}$$

as $t \to \infty$. *Furthermore, if either* $\mathbf{x}$ *or* $\tilde{\mathbf{x}}$ *is bounded for all time* $t \geq 0$, *then we have*

$$\dot{\tilde{a}}_{ij}^k \to 0 \tag{14}$$

as $t \to \infty$.

Proof: Construct the Lyapunov function V as follows:

$$V(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{A}_k, \tilde{\mathbf{A}}_k) = \frac{1}{2}(\mathbf{r}^T \mathbf{V} \mathbf{r}) + \frac{1}{2} \sum_{i,j,k} \frac{1}{\tilde{\phi}^{i,j,k}} (\tilde{a}_{ij}^k - a_{ij}^k)^2 \tag{15}$$

Given the assumptions, Eq. (7) can be simplified to:

$$\begin{aligned} \dot{\mathbf{r}} &= \textstyle\sum_{k=1}^m \mathbf{A}_k(\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\tilde{\mathbf{x}})) \\ &+ \textstyle\sum_{k=1}^m \left(\mathbf{A}_k - \tilde{\mathbf{A}}_k\right) \mathbf{f}_k(\tilde{\mathbf{x}}) + \mathbf{p}(\mathbf{x}) - \mathbf{p}(\tilde{\mathbf{x}}) \end{aligned} \tag{16}$$

Differentiating V along the trajectories, we get

$$\begin{aligned} \dot{V} &= \mathbf{r}^T \mathbf{V} \dot{\mathbf{r}} + \textstyle\sum_{i,j,k} \frac{1}{\tilde{\phi}^{i,j,k}} (\tilde{a}_{ij}^k - a_{ij}^k) \dot{\tilde{a}}_{ij}^k \\ &= \mathbf{r}^T \mathbf{V} \left(\textstyle\sum_{k=1}^m \mathbf{A}_k(\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\tilde{\mathbf{x}})) + \mathbf{p}(\mathbf{x}) - \mathbf{p}(\tilde{\mathbf{x}}) + \sum_{k=1}^m \left(\mathbf{A}_k - \tilde{\mathbf{A}}_k\right) \mathbf{f}_k(\tilde{\mathbf{x}}) \right) \\ &\quad + \textstyle\sum_{i,j,k} (\tilde{a}_{ij}^k - a_{ij}^k) s_i (\mathbf{f}_k(\tilde{\mathbf{x}}))_j \\ &= \mathbf{r}^T \mathbf{V} \left(\textstyle\sum_{k=1}^m \mathbf{A}_k(\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\tilde{\mathbf{x}})) + \mathbf{p}(\mathbf{x}) - \mathbf{p}(\tilde{\mathbf{x}}) \right) \leq -\lambda \mathbf{r}^T \mathbf{r} \end{aligned} \tag{17}$$

by the assumptions (see page 982 of [Wu and Chua, 1994]). The first conclusion follows from Theorem 1.2.1 of [Lakshmikantham and Liu, 1993]. Note that if $\tilde{\mathbf{A}}_k$ becomes unbounded, then $V \to +\infty$ which contradicts $\dot{V} \leq 0$. Thus for each k, $\tilde{\mathbf{A}}_k$ remains bounded for all time and the second conclusion follows from an application of Theorem 1.2.3 of [Lakshmikantham and Liu, 1993]. ■

Although the statement of the theorem dictates that all $\tilde{\phi}^{i,j,k} > 0$, it is clear that the theorem is still true when we set $\tilde{\phi}^{i,j,k} = 0$ for those $\{i, j, k\}$ such that $a^k_{i,j} = \tilde{a}^k_{ij}$. In other words, we only need to adapt the mismatched parameters. The condition of $\mathbf{x}$ or $\tilde{\mathbf{x}}$ being bounded in conclusion 2 can be removed if all the $\mathbf{f}_k$ have bounded range, e.g. if they are sigmoid functions.

Note that the theorem only implies that $\dot{\tilde{a}}^k_{ij} \to 0$. It doesn't say that $\tilde{a}^k_{ij}$ converges, and even if $\tilde{a}^k_{ij}$ converges it doesn't necessarily imply that $\tilde{a}^k_{ij} \to a^k_{ij}$ and it is unrealistic to expect that in general. First of all, it is possible that $\sum_k \mathbf{A}_k \mathbf{f}_k = \sum_k \tilde{\mathbf{A}}_k \mathbf{f}_k$ even when $\mathbf{A}_k \neq \tilde{\mathbf{A}}_k$. This suggests that in the design we should choose $\mathbf{f}_k$ such that this does not occur. Second of all, it is possible that several sets of parameters $\tilde{a}^k_{ij}$ generate the same trajectory in System 2 and $\tilde{a}^k_{ij}$ converge to only one such set. This is especially true if the trajectory is pretty boring and covers a small part of the phase space, e.g. an equilibrium point. One strategy is to run the adaptive controllers several times, but using different initial conditions to "map" out the dynamics of the systems. On the other hand, if for a given trajectory of the systems, there can correspond only one set of parameters which generates that trajectory, then the two sets of parameters $\tilde{a}^k_{ij}$ and a^k_{ij} might approach each other. We conjecture that this happens for chaotic systems, where the dynamics is very rich. In Section 5 we present examples which support this conjecture. In summary, in general we can only conclude that $\mathbf{r} \to \mathbf{0}$; it is possible that $\sum_k \tilde{\mathbf{A}}_k \mathbf{f}_k$ does not converge to $\sum_k \mathbf{A}_k \mathbf{f}_k$.

As for the choice of $\tilde{\phi}^{i,j,k}$, they should be chosen such that the terms in V (Eq. (15)) are all around the same order of magnitude, i.e. the large $a^k_{ij} - \tilde{a}^k_{ij}$ is, the larger $\phi^{i,j,k}$ should be. This requires an estimate of the range of the parameter mismatch and the synchronization error $\mathbf{r}$.

Note that Eq. (12) can be more compactly written as

$$\dot{\tilde{\mathbf{A}}}_k = \tilde{\Phi}_k \star \left(\mathbf{V}\mathbf{r}[\mathbf{f}_k(\tilde{\mathbf{x}})]^T\right)$$

where $\tilde{\Phi}_k = \{\tilde{\phi}^{i,j,k}\}_{i,j=1}^{i,j=n}$, and $\star$ is entry-wise multiplication of matrices.

When $k = 1$ and $\mathbf{V}$ is a diagonal matrix, Eq. (12) takes on the form of the well-known continuous-time LMS adaptive algorithm used widely in linear adaptive filters. In [Huberman and Lumer, 1990; John and Amritkar, 1994], the following adaptive controller is proposed for the system of equations $\dot{x}_i = f_i(\mathbf{x}, \mu_j)$, $\dot{\tilde{x}}_i = f_i(\tilde{\mathbf{x}}, \tilde{\mu}_j)$:

$$\dot{\tilde{\mu}}_j = G\left(x_i - \tilde{x}_i, \frac{df_i}{d\tilde{\mu}_j}\right)$$

where they chose

$$G\left(x_i - \tilde{x}_i, \frac{df_i}{d\tilde{\mu}_j}\right) = \delta(x_i - \tilde{x}_i)\mathrm{sgn}\left(\frac{df_i}{d\tilde{\mu}_j}\right)$$

for the computer simulations.

It is clear that by removing the *sgn* function, the resulting adaptive controllers correspond to Eq. (12) when $\mathbf{V}$ is diagonal, i.e. G becomes:

$$G\left(x_i - \tilde{x}_i, \frac{df_i}{d\tilde{\mu}_j}\right) = \delta(x_i - \tilde{x}_i)\left(\frac{df_i}{d\tilde{\mu}_j}\right) \tag{18}$$

Thus the results presented in this paper provide proofs for the Huberman-Lumer adaptive scheme when G is chosen as Eq. (18) and applied to Eqs. (1)-(2).

3 Choice of V, h, g, $\tilde{\mathbf{h}}$ and $\tilde{\mathbf{g}}$

Our control strategy should be to find $\mathbf{V}$, $\mathbf{g}$, $\mathbf{h}$, $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$ such that the matrix in Eq. (11) is negative definite.

Let us denote $\mathbf{r} = (r_1, \ldots, r_n)^T$. When $\mathbf{V}$ is the identity matrix or the diagonal matrix of the form $\mathbf{V} = \text{diag}\{v_1, \ldots, v_n\} > 0$, $s_i = v_i r_i$ and the adaptive controllers can be written as:

$$\dot{\tilde{a}}_{ij}^{k} = \tilde{\phi}_i^{i,j,k} r_i [\mathbf{f}_k(\tilde{\mathbf{x}})]_j \tag{19}$$

where $\tilde{\phi}_i^{i,j,k} = \tilde{\phi}^{i,j,k} v_i$. Thus in case where $\mathbf{V}$ is diagonal but unknown, we choose $\tilde{\phi}_i^{i,j,k}$ to be large and positive. This choice has to be made with care to ensure that the adaptive controllers will converge at proper rates, especially if the different state variables operate at different time scales.

The difficulty here is that the adaptive controllers require knowledge of $\mathbf{V}$ while the choice of $\mathbf{g}$ and $\mathbf{h}$ requires the knowledge of $\mathbf{A}_k$ and $\mathbf{f}_k$ and $\mathbf{V}$. One way to overcome this difficulty is to change $\mathbf{g}$ and $\mathbf{h}$ adaptively as was done in [di Bernardo, 1995]. When $\sum_{k=1}^{m} \mathbf{A}_k \mathbf{f}_k$ has a bounded Jacobian matrix, it was shown in [Wu and Chua, 1994] that $\mathbf{V}$ can be chosen to be the identity matrix when $\mathbf{p}$ is chosen to be $\mathbf{p}(\mathbf{x}) = -\mathbf{K}^{max}\mathbf{x}$ for some positive definite diagonal matrix $\mathbf{K}^{max} = diag\{k_1^{max}, k_2^{max}, \cdots, k_n^{max}\} > 0$. Our goal is to adaptively change $\mathbf{p}$ to approach $\mathbf{K}^{max}$. With these modifications, Theorem 1 becomes:

Theorem 2 *Given the two systems*

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{m} \mathbf{A}_k \mathbf{f}_k(\mathbf{x}) + \mathbf{z} + \mathbf{K}(\tilde{\mathbf{x}} - \mathbf{x}) \quad \leftarrow \textit{System 1} \tag{20}$$

$$\dot{\tilde{\mathbf{x}}}(t) = \sum_{k=1}^{m} \tilde{\mathbf{A}}_k \mathbf{f}_k(\tilde{\mathbf{x}}) + \mathbf{z} + \tilde{\mathbf{K}}(\mathbf{x} - \tilde{\mathbf{x}}) \quad \leftarrow \textit{System 2} \tag{21}$$

The matrices $\mathbf{K}$ *and* $\tilde{\mathbf{K}}$ *are diagonal matrices of the form* $\mathbf{K} = diag(k_1, k_2, \cdots, k_n)$ *and* $\tilde{\mathbf{K}} = diag(\tilde{k}_1, \tilde{k}_2, \cdots, \tilde{k}_n)$. *Suppose that for each* k, $\mathbf{f}_k(\mathbf{x})$ *has a bounded Jacobian matrix for all* $\mathbf{x}$. *Suppose* $\mathbf{A}_k$ *and* $\tilde{\mathbf{A}}_k$ *are varied according to*

$$\begin{aligned} \dot{\tilde{a}}_{ij}^{k} &= \tilde{\phi}^{i,j,k} r_i [\mathbf{f}_k(\tilde{\mathbf{x}})]_j \\ \dot{a}_{ij}^{k} &= -\phi^{i,j,k} r_i [\mathbf{f}_k(\tilde{\mathbf{x}})]_j \end{aligned} \tag{22}$$

such that $\phi^{i,j,k} + \tilde{\phi}^{i,j,k} > 0$ *for each* i, j, *and* k. *Let the coefficients of* $\mathbf{K}$ *and* $\tilde{\mathbf{K}}$ *be varied as:*

$$\begin{aligned} \dot{k}_i &= \mu_i r_i^2 \\ \dot{\tilde{k}}_i &= \tilde{\mu}_i r_i^2 \end{aligned} \tag{23}$$

such that $\mu_i + \tilde{\mu}_i > 0$, *then the following statements are true:*

1. *If for each* i, j *and* k, $\phi^{i,j,k} = 0$, *then the synchronization error* $\mathbf{r}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ *and the parameter mismatch* $\mathbf{A}_k - \tilde{\mathbf{A}}_k(t)$ *can be made arbitrarily small for all time* $t \geq 0$ *by choosing the initial error* $\mathbf{r}(0)$ *and the initial parameter mismatch* $\mathbf{A}_k - \tilde{\mathbf{A}}_k(0)$ *to be sufficiently small and* $k_i(0) + \tilde{k}_i(0) > 0$ *to be sufficiently large.*

2. *If* $\mathbf{A}_k$ *is bounded for all* $t \geq 0$, *then* k_i *and* $\tilde{k}_i$ *converge to some finite values as* $t \to \infty$ *for all* i.

3. *One of two things can happen. Either* $\mathbf{A}_k$ *becomes unbounded as* $t \to \infty$ *for some* k, *or the two systems in Eq. (20) and Eq. (21) are synchronized ,i.e.,*

$$\mathbf{r} \rightarrow \mathbf{0} \tag{24}$$

as $t \to \infty$.

4. *If for each* i, j *and* k, $\phi^{i,j,k} = 0$, *then* $\mathbf{r} \to \mathbf{0}$, *as* $t \to \infty$.

5. *When* $\mathbf{r} \to \mathbf{0}$, *and either* $\mathbf{x}$ *or* $\tilde{\mathbf{x}}$ *is bounded for all time* $t \geq 0$, *we have*

$$\dot{\tilde{a}}_{ij}^k \rightarrow 0 \tag{25}$$

$$\dot{a}_{ij}^k \rightarrow 0 \tag{26}$$

as $t \to \infty$.

Note that in this case both $\tilde{\mathbf{A}}_k$ and $\mathbf{A}_k$ are time-varying.

Proof: The proof is the same as Theorem 1, except that we use the Lyapunov function

$$V(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{A}_k, \tilde{\mathbf{A}}_k, \mathbf{K}, \tilde{\mathbf{K}}) = \frac{1}{2}(\mathbf{r}^T\mathbf{r}) + \frac{1}{2}\sum_{i,j,k} \frac{1}{\phi^{i,j,k} + \tilde{\phi}^{i,j,k}}(\tilde{a}_{ij}^k - a_{ij}^k)^2 + \frac{1}{2}\sum_i \frac{1}{\mu_i + \tilde{\mu}_i}\left(k_i^{max} - k_i - \tilde{k}_i\right)^2$$

Taking the derivatives along trajectories, we get:

$$\begin{aligned} \dot{V} &= \mathbf{r}^T\dot{\mathbf{r}} + \textstyle\sum_{i,j,k} \frac{1}{\phi^{i,j,k} + \tilde{\phi}^{i,j,k}}(\tilde{a}_{ij}^k - a_{ij}^k)(\dot{\tilde{a}}_{ij}^k - \dot{a}_{ij}^k) - \sum_i \frac{1}{\mu_i + \tilde{\mu}_i}\left(k_i^{max} - k_i - \tilde{k}_i\right)\left(\dot{k}_i + \dot{\tilde{k}}_i\right) \\ &= \mathbf{r}^T\left(\textstyle\sum_{k=1}^m \mathbf{A}_k(\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\tilde{\mathbf{x}})) - (\mathbf{K} + \tilde{\mathbf{K}})\mathbf{r} + \sum_{k=1}^m\left(\mathbf{A}_k - \tilde{\mathbf{A}}_k\right)\mathbf{f}_k(\tilde{\mathbf{x}})\right) \\ &\quad + \textstyle\sum_{i,j,k}(\tilde{a}_{ij}^k - a_{ij}^k)s_i(\mathbf{f}_k(\tilde{\mathbf{x}}))_j + \sum_i(k_i + \tilde{k}_i - k_i^{max})r_i^2 \\ &= \mathbf{r}^T\left(\textstyle\sum_{k=1}^m \mathbf{A}_k(\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\tilde{\mathbf{x}})) - \mathbf{K}^{max}\mathbf{r}\right) \end{aligned} \tag{27}$$

If $\mathbf{K} + \tilde{\mathbf{K}}$ becomes unbounded as $t \to \infty$, then $V \to +\infty$ which contradicts $\dot{V} \leq 0$. Now we show that $\mathbf{K}$ is bounded for all time $t \geq 0$. Note that

$$\dot{k}_i = \mu_i r_i^2 = \frac{\mu_i}{\mu_i + \tilde{\mu}_i}(\mu_i + \tilde{\mu}_i)r_i^2 = \frac{\mu_i}{\mu_i + \tilde{\mu}_i}(\dot{k}_i + \dot{\tilde{k}}_i)$$

This implies that

$$k_i(t) = k_i(0) + \int_0^t \frac{\mu_i}{\mu_i + \tilde{\mu}_i}(\dot{k}_i + \dot{\tilde{k}}_i)d\tau = k_i(0) + \frac{\mu_i}{\mu_i + \tilde{\mu}_i}\left(k_i(t) + \tilde{k}_i(t) - k_i(0) - \tilde{k}_i(0)\right)$$

which is bounded for all $t \geq 0$ since $k_i(t) + \tilde{k}_i(t)$ is bounded for all $t \geq 0$. This implies that $\tilde{k}_i$ is also bounded for all t. Since k_i and $\tilde{k}_i$ are monotonic functions of time, this implies that k_i and $\tilde{k}_i$ converges to some finite values as $t \to \infty$. If $\mathbf{A}_k$ is bounded, then $\tilde{\mathbf{A}}_k$ is bounded since otherwise $V \to +\infty$, contradicting $\dot{V} \leq 0$. The rest of the proof is similar to Theorem 1. ∎

Note that we don't claim that $k_i + \tilde{k}_i \to k_i^{max}$ since in general there are many choices for $\mathbf{K}^{max}$. The condition expressed in (Eq. (10)) implies that the nonlinearities in the two systems are identical up to linear combinations. There are several reasons why this is not as restrictive as it first seems. First, for several chaotic circuits and systems in the literature that we have studied, there is only one scalar nonlinearity in the system. Thus in physical implementations the problem of matching the parameters should be concentrated on the single nonlinear element. Second, some of the $\mathbf{f}_k$ might depend on things such as circuit topology which we can assume to be identical in the two systems. Third, given enough basis functions, we can approximate any class of nonlinear functions by a linear combination of basis functions. Thus we can assume the nonlinearities in the two systems to consist of linear combinations of basis functions and adapt these coefficients. Some of these points will be illustrated in the examples using Chua's oscillator in Section 5.

4 Simplifications and Unidirectional Coupling

The framework proposed in Theorem 1 is very general and is perhaps overly complicated. However, some simplifications can be introduced if we are willing to sacrifice some generality. For example, we can assume that $\mathbf{B} = \tilde{\mathbf{B}}$, $\mathbf{g} = \tilde{\mathbf{g}}$ and $\mathbf{h} = \tilde{\mathbf{h}}$ in which case $\mathbf{p} = \tilde{\mathbf{p}}$ is satisfied. If we also assume that $\mathbf{z} = \tilde{\mathbf{z}}$ then $\mathbf{B}\mathbf{z} = \tilde{\mathbf{B}}\tilde{\mathbf{z}}$ is also satisfied. Let us also assume that $\mathbf{V}$ is a diagonal matrix.

Then Theorem 1 can be simplified to:

Corollary 1 *Given the two systems*

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{m} \mathbf{A}_k \mathbf{f}_k(\mathbf{x}) + \mathbf{B}\left(\mathbf{z} + \mathbf{g}(\mathbf{x}) - \mathbf{g}(\tilde{\mathbf{x}})\right) \quad \leftarrow \textit{System 1} \tag{28}$$

$$\dot{\tilde{\mathbf{x}}}(t) = \sum_{k=1}^{m} \tilde{\mathbf{A}}_k \mathbf{f}_k(\tilde{\mathbf{x}}) + \mathbf{B}\left(\mathbf{z} + \mathbf{h}(\tilde{\mathbf{x}}) - \mathbf{h}(\mathbf{x})\right) \quad \leftarrow \textit{System 2} \tag{29}$$

where $\mathbf{r} = \mathbf{x} - \tilde{\mathbf{x}}$. *Suppose* $\mathbf{A}_k$ *is constant and* $\tilde{\mathbf{A}}_k$ *is varied according to*

$$\dot{\tilde{a}}_{ij}^{k} = \tilde{\phi}^{i,j,k} v_i r_i [\mathbf{f}_k(\tilde{\mathbf{x}})]_j \tag{30}$$

such that $\tilde{\phi}^{i,j,k} > 0$ *for each* i, j, *and* k, *and where* $\mathbf{V} = diag\{v_i, \ldots, v_n\} > 0$. *Suppose also that there exists* $\lambda > 0$ *such that*

$$\lambda \mathbf{I} + \sum_{k=1}^{m} \mathbf{V}\mathbf{A}_k D\mathbf{f}_k(\mathbf{x}) + \mathbf{V}\mathbf{B}(D\mathbf{g} + D\mathbf{h})(\mathbf{x})$$

is negative definite for all $\mathbf{x}$. *Then* $\mathbf{r} \to 0$, *as* $t \to \infty$. *If* $\mathbf{x}$ *or* $\tilde{\mathbf{x}}$ *is bounded for all time* $t \geq 0$, *then* $\dot{\tilde{a}}_{ij}^{k} \to 0$ *as* $t \to \infty$.

For the case of communication systems, master-slave driving or model following where we have unidirectional coupling, some additional constraints are introduced. Let System 1 be the master system (or transmitter) and System 2 be the slave system (or receiver). In this case we set $\mathbf{g} = \mathbf{0}$. Without loss of generality we set $\mathbf{B}$ to be the identity matrix and Corollary 1 reduces to

Corollary 2 *Given the two systems*

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{m} \mathbf{A}_k \mathbf{f}_k(\mathbf{x}) + \mathbf{z} \quad \leftarrow \textit{System 1} \tag{31}$$

$$\dot{\tilde{\mathbf{x}}}(t) = \sum_{k=1}^{m} \tilde{\mathbf{A}}_k \mathbf{f}_k(\tilde{\mathbf{x}}) + \mathbf{z} + \mathbf{h}(\tilde{\mathbf{x}}) - \mathbf{h}(\mathbf{x}) \quad \leftarrow \textit{System 2} \tag{32}$$

Let $\mathbf{V} = diag\{v_i, \ldots, v_n\} > 0$ *and* $\mathbf{r} = \mathbf{x} - \tilde{\mathbf{x}}$. *Suppose* $\mathbf{A}_k$ *is constant and* $\tilde{\mathbf{A}}_k$ *are varied according to*

$$\dot{\tilde{a}}_{ij}^{k} = \tilde{\phi}^{i,j,k} v_i r_i [\mathbf{f}_k(\tilde{\mathbf{x}})]_j$$

such that $\tilde{\phi}^{i,j,k} > 0$ *for all* i,j, *and* k. *Suppose also that there exists* $\lambda > 0$ *such that*

$$\lambda \mathbf{I} + \sum_{k=1}^{m} \mathbf{V}\mathbf{A}_k D\mathbf{f}_k(\mathbf{x}) + \mathbf{V} D\mathbf{h}(\mathbf{x}) \tag{33}$$

is negative definite for all $\mathbf{x}$. *Then* $\mathbf{r} \to 0$ *as* $t \to \infty$. *If* $\mathbf{x}$ *is bounded for all time* $t \geq 0$, *then* $\dot{\tilde{a}}_{ij}^{k} \to 0$ *as* $t \to \infty$.

Note that $\mathbf{h}$ in Corollary 2 can also be adaptively changed as in Theorem 2.

5 Examples Using Chua's Oscillator

The state equations of Chua's oscillator is given by

$$\begin{cases} \frac{dv_1}{dt} = \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] \\ \frac{dv_2}{dt} = \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} = \frac{1}{L}[-v_2 - R_0 i_3] \end{cases} \tag{34}$$

where

$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)(|v_1 + E| - |v_1 - E|) + I_b \tag{35}$$

There are many ways to decompose the state equations in the form of Eq. (1), and the choice depends on which parameters we want to adapt.

Example 1 Let us assume that two Chua's oscillators are connected in a master-slave configuration and the only mismatch is in the parameter C_1. The coupled system equations are given by:

$$\left\{\begin{array}{lcl} \frac{dv_1}{dt} &=& \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)] \\ \frac{dv_2}{dt} &=& \frac{1}{C_2}[G(v_1 - v_2) + i_3] \\ \frac{di_3}{dt} &=& \frac{1}{L}[-v_2 - R_0 i_3] \\ \frac{d\tilde{v}_1}{dt} &=& \frac{1}{\tilde{C}_1}[G(\tilde{v}_2 - \tilde{v}_1) - \tilde{f}(\tilde{v}_1) + G_c(v_1 - \tilde{v}_1)] \\ \frac{d\tilde{v}_2}{dt} &=& \frac{1}{\tilde{C}_2}[G(\tilde{v}_1 - \tilde{v}_2) + \tilde{i}_3] \\ \frac{d\tilde{i}_3}{dt} &=& \frac{1}{\tilde{L}}[-\tilde{v}_2 - \tilde{R}_0 \tilde{i}_3] \end{array}\right. \tag{36}$$

Let us assume that the nonlinearities are equal ($f = \tilde{f}$). The value of C_1 is the unknown parameter that the slave system wants to adapt to given only knowledge of v_1. Suppose that C_1, C_2, $R = \frac{1}{G}$, L, R_0 are positive.

Chua's oscillator (Eq. (34)) can be decomposed as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{f}(\mathbf{x})$ where

$$\mathbf{A} = \begin{pmatrix} \frac{1}{C_1} & & \\ & \frac{1}{C_2} & \\ & & \frac{1}{L} \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} G(x_2 - x_1) - f(x_1) \\ G(x_1 - x_2) + x_3 \\ -x_2 - R_0 x_3 \end{pmatrix}$$

In this case $\mathbf{V}$ can be chosen to be $\mathbf{A}^{-1}$ and $\mathbf{x} = (v_1, v_2, i_3)^T$. Note that v_1 and v_2 in this section are different from the v_i used in previous sections to denote the matrix $\mathbf{V}$.

Using Corollary 2, we construct an adapter for $\tilde{C}_1$ as follows:

$$\frac{d\left(\frac{1}{\tilde{C}_1}\right)}{dt} = \phi(v_1 - \tilde{v}_1)\tilde{i}_1 = \phi(v_1 - \tilde{v}_1)(G(\tilde{v}_2 - \tilde{v}_1) - f(\tilde{v}_1))$$

where $\tilde{i}_1$ is the current through capacitor $\tilde{C}_1$. This implies that

$$\frac{d\tilde{C}_1}{dt} = -\tilde{C}_1^2\phi(v_1 - \tilde{v}_1)(G(\tilde{v}_2 - \tilde{v}_1) - f(\tilde{v}_1))$$

We also want G_c to be adaptive, so by using a result similar to Theorem 2, we get

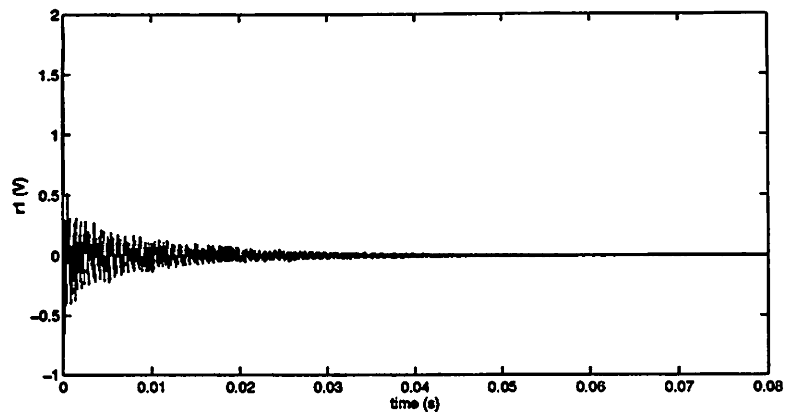
$$\frac{d\left(\frac{G_c}{\tilde{C}_1}\right)}{dt} = \mu(v_1 - \tilde{v}_1)^2$$

i.e.,

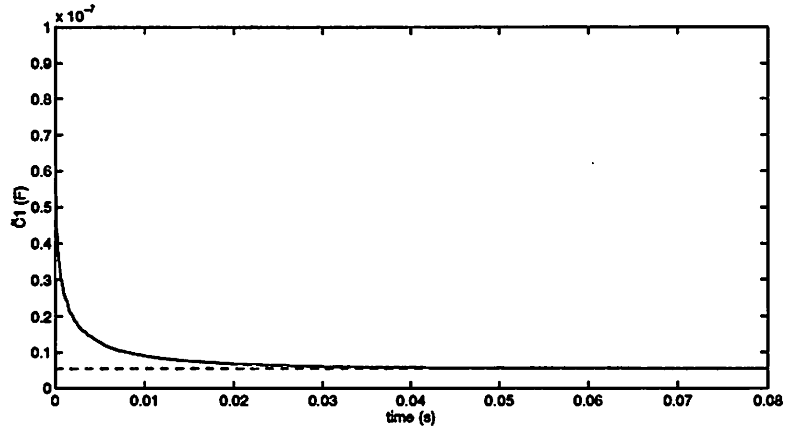
$$\frac{dG_c}{dt} = \tilde{C}_1\left(\mu(v_1 - \tilde{v}_1)^2 + \frac{G_c}{\tilde{C}_1^2}\frac{d\tilde{C}_1}{dt}\right) \tag{37}$$

The fixed parameters are set at $C_1 = 5.56nF$, $C_2 = 50nF$, $R = 1388\Omega$, $R_0 = 2\Omega$, $L = 7.14mH$, $E = 1V$, $G_a = -0.8mS$, $G_b = -0.5mS$, $I_b = 0A$.

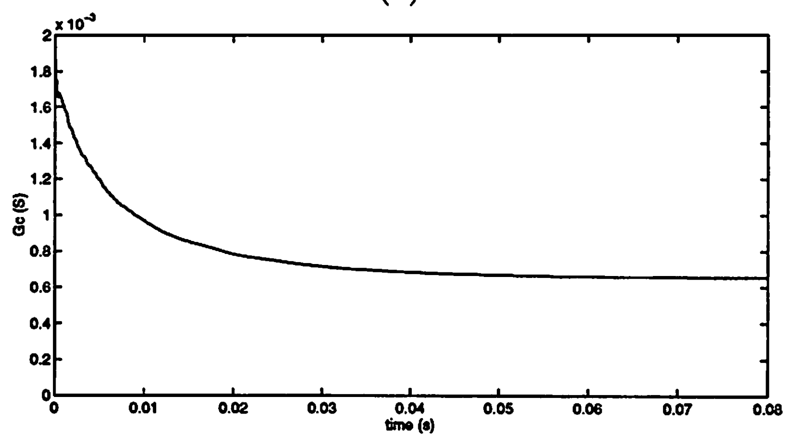
The initial $\tilde{C}_1(0)$ is set at $100nF$, which is about 18 times bigger than C_1 and the initial $G_c(0)$ is set at $0.08mS$, which is not big enough to synchronize the two Chua's oscillators. ϕ is set at 8×10^{14}, while $\mu = 4 \times 10^8$. The simulation results are shown in Figure 1. Figure 1a shows the synchronization error $r_1(t)$, while Figure 1b shows $\tilde{C}_1(t)$. We see that $\tilde{C}_1$ converges to the correct



(a)



(b)



(c)

Figure 1: Compensating for mismatch in C_1. (a) Synchronization error $r_1(t) = x_1(t) - \tilde{x}_1(t)$. (b) Capacitance $\tilde{C}_1(t)$ in System 2. The fixed capacitance C_1 in System 1 is shown with a dashed line. (c) The coupling conductance $G_c(t)$. $G_c(t)$ shoots up very quickly from $G_c(0) = 0.08mS$ then decreased to a constant value of $0.655mS$.

value of $C_1 = 5.56nF$, indicated by the dashed line. Similarly we see in Figure 1c that G_c has converged to a value close to $0.655mS$.

Example 2 Let us now apply the same technique to adapt the parameters G_a, G_b, I_b in $f(\cdot)$. In this case, we assume that C_1 and $\tilde{C}_1$ are matched but $f \neq \tilde{f}$. The state equations of the coupled system are the same as Eq. (36), where $\tilde{C}_1 = C_1$ and $\tilde{f}$ is given by

$$\begin{aligned} \tilde{f}(\tilde{v}_1) &= \tilde{G}_b\tilde{v}_1 + \tfrac{1}{2}(\tilde{G}_a - \tilde{G}_b)(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|) + \tilde{I}_b \\ &= \tfrac{1}{2}\tilde{G}_a(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|) + \tilde{G}_b\left(\tilde{v}_1 - \tfrac{1}{2}(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|)\right) + \tilde{I}_b \end{aligned} \tag{38}$$

We decompose the system (Eq. (34)) as $\dot{\mathbf{x}} = \sum_{i=1}^{4} \mathbf{A}_i \mathbf{f}_i(\mathbf{x})$ where

$$\mathbf{A}_1 = \begin{pmatrix} \frac{1}{C_1} & & \\ & \frac{1}{C_2} & \\ & & \frac{1}{L} \end{pmatrix}, \qquad \mathbf{f}_1(\mathbf{x}) = \begin{pmatrix} G(x_2 - x_1) \\ G(x_1 - x_2) + x_3 \\ -x_2 - R_0 x_3 \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} \frac{G_a}{C_1} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad \mathbf{f}_2(\mathbf{x}) = \begin{pmatrix} -\frac{1}{2}(|x_1 + E| - |x_1 - E|) \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} \frac{G_b}{C_1} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad \mathbf{f}_3(\mathbf{x}) = \begin{pmatrix} -\left(x_1 - \frac{1}{2}(|x_1 + E| - |x_1 - E|)\right) \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{A}_4 = \begin{pmatrix} \frac{I_b}{C_1} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad \mathbf{f}_4(\mathbf{x}) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

In this case, using Corollary 2, the adaptive controller for G_a and G_b, I_b are as follows:

$$\frac{d\left(\frac{\tilde{G}_a}{C_1}\right)}{dt} = \phi_2 r_1 [\mathbf{f}_2(\tilde{\mathbf{x}})]_1 = -\frac{1}{2}\phi_2(v_1 - \tilde{v}_1)(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|)$$

i.e.,

$$\frac{d\tilde{G}_a}{dt} = -\frac{1}{2}C_1\phi_2(v_1 - \tilde{v}_1)(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|)$$

Similarly,

$$\frac{d\tilde{G}_b}{dt} = C_1\phi_3 r_1[\mathbf{f}_3(\tilde{\mathbf{x}})]_1 = -C_1\phi_3(v_1 - \tilde{v}_1)(\tilde{v}_1 - \frac{1}{2}(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|))$$

and

$$\frac{d\tilde{I}_b}{dt} = C_1\phi_4 r_1[\mathbf{f}_4(\tilde{\mathbf{x}})]_1 = -C_1\phi_4(v_1 - \tilde{v}_1)$$

The controller for G_c is given by

$$\frac{dG_c}{dt} = C_1\mu(v_1 - \tilde{v}_1)^2$$

In Figure 2 we show the simulation results. The fixed parameters are as before, except that $I_b = 2\mu A$. The initial values for the parameters of $\tilde{f}$ are chosen to be $\tilde{G}_a(0) = -0.01mS$, $\tilde{G}_b(0) = -10mS$ and $\tilde{I}_b(0) = -0.01mA$, which differ by orders of magnitude from the corresponding values in System 1. $G_c(0) = 0.08mS$, while $\phi_2 = \phi_3 = \phi_4 = 5 \times 10^9$ and $\mu = 10^9$. We see in Fig. 2a that the synchronization error r_1 has decreased to zero. We also see in Fig. 2b-d that the values of $\tilde{G}_a$, $\tilde{G}_b$ and $\tilde{I}_b$ has converged to the corresponding values of G_a and G_b and I_b (indicated by the dashed lines) respectively. We show in Fig. 2e the value of G_c as a function of time.

Example 3 Let us now adapt all four parameters C_1, G_a and G_b and I_b. The state equations are the same as before (Eq. (36)), and we decompose the system (Eq. (34)) into $\dot{\mathbf{x}} = \sum_{i=1}^{4} \mathbf{A}_i \mathbf{f}_i(\mathbf{x})$ as before.

In this case, using Corollary 2, the adaptive controller for $\tilde{C}_1$ is as follows:

$$\frac{d\tilde{C}_1}{dt} = -\tilde{C}_1^2 \phi_1 (v_1 - \tilde{v}_1)(G(\tilde{v}_2 - \tilde{v}_1))$$

The adaptive controller for $\tilde{G}_a$ and $\tilde{G}_b$ are as follows:

$$\frac{d\left(\frac{\tilde{G}_a}{\tilde{C}_1}\right)}{dt} = -\frac{1}{2}\phi_2(v_1 - \tilde{v}_1)(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|)$$

which after some manipulation becomes

$$\begin{aligned} \frac{d\tilde{G}_a}{dt} &= \tilde{C}_1 \left(\frac{\tilde{G}_a}{\tilde{C}_1^2}\frac{d\tilde{C}_1}{dt} - \frac{1}{2}\phi_2(v_1 - \tilde{v}_1)(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|) \right) \\ &= -\tilde{C}_1(v_1 - \tilde{v}_1)\left(\tilde{G}_a G \phi_1 (\tilde{v}_2 - \tilde{v}_1) + \frac{1}{2}\phi_2(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|) \right) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{d\tilde{G}_b}{dt} &= \tilde{C}_1 \left(\frac{\tilde{G}_b}{\tilde{C}_1^2}\frac{d\tilde{C}_1}{dt} - \phi_3(v_1 - \tilde{v}_1)(\tilde{v}_1 - \frac{1}{2}(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|)) \right) \\ &= -\tilde{C}_1(v_1 - \tilde{v}_1)\left(\tilde{G}_b G \phi_1 (\tilde{v}_2 - \tilde{v}_1) + \phi_3 \left(\tilde{v}_1 - \frac{1}{2}(|\tilde{v}_1 + E| - |\tilde{v}_1 - E|) \right) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{d\tilde{I}_b}{dt} &= \tilde{C}_1 \left(\frac{\tilde{I}_b}{\tilde{C}_1^2}\frac{d\tilde{C}_1}{dt} - \phi_4(v_1 - \tilde{v}_1) \right) \\ &= -\tilde{C}_1(v_1 - \tilde{v}_1)\left(\tilde{I}_b G \phi_1 (\tilde{v}_2 - \tilde{v}_1) + \phi_4 \right) \end{aligned}$$

The controller for G_c is given in Eq. (37). In Figure 3 we show the simulation results. The fixed parameters are as before, except for $I_b = 1\mu A$. We choose $\tilde{C}_1(0) = 100nF$, $\tilde{G}_a(0) = -5mS$, $\tilde{G}_b(0) = -0.08mS$, $\tilde{I}_b(0) = -1mA$ and $G_c(0) = 0.08mS$. We choose $\phi_1 = 2 \times 10^{15}$, $\phi_2 = \phi_3 = \phi_4 = 5 \times 10^9$, $\mu = 1 \times 10^9$. We see in Fig. 3a that the synchronization error has decreased to zero.

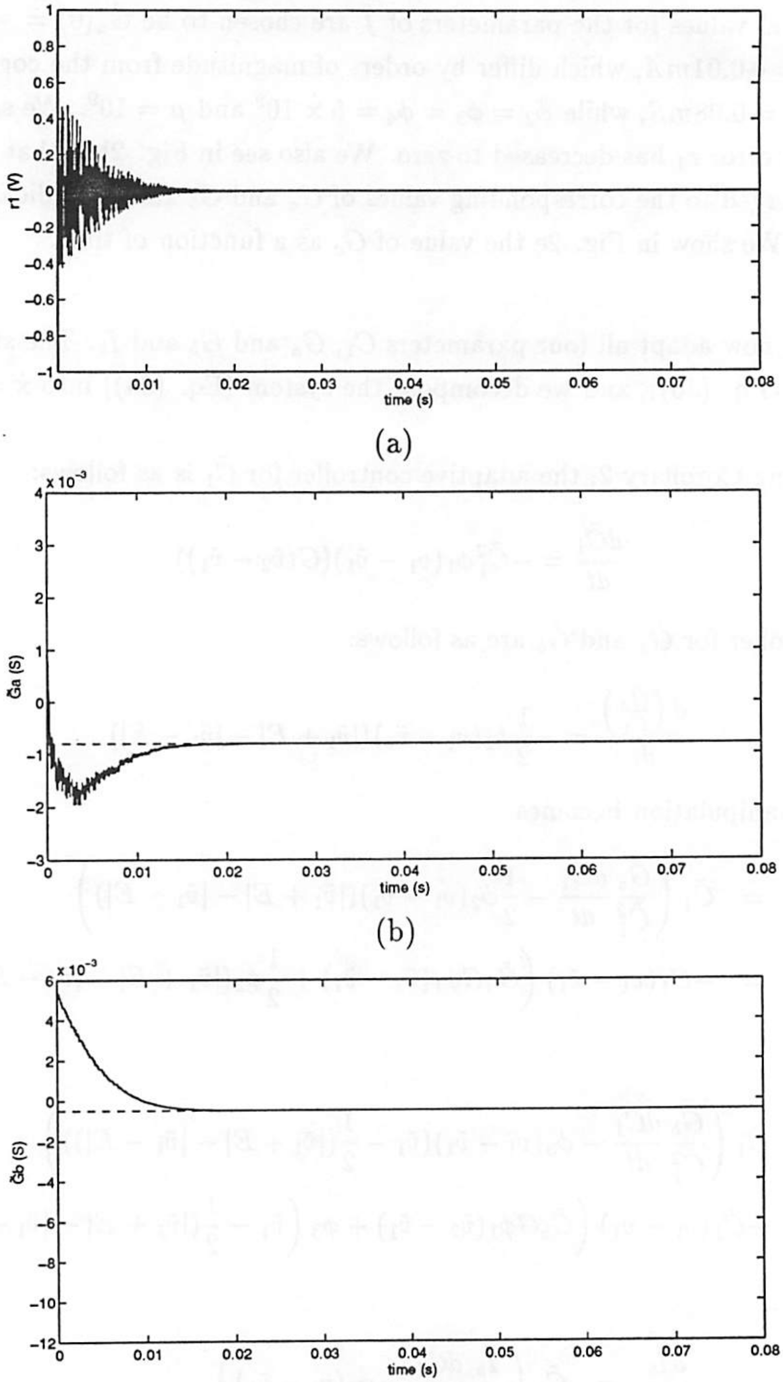
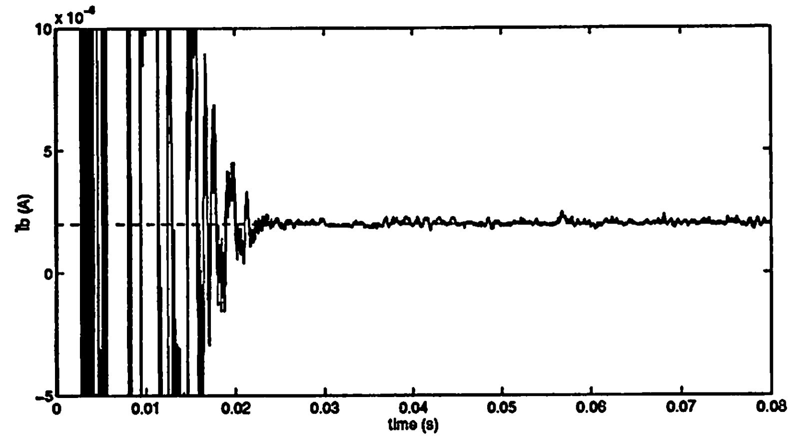
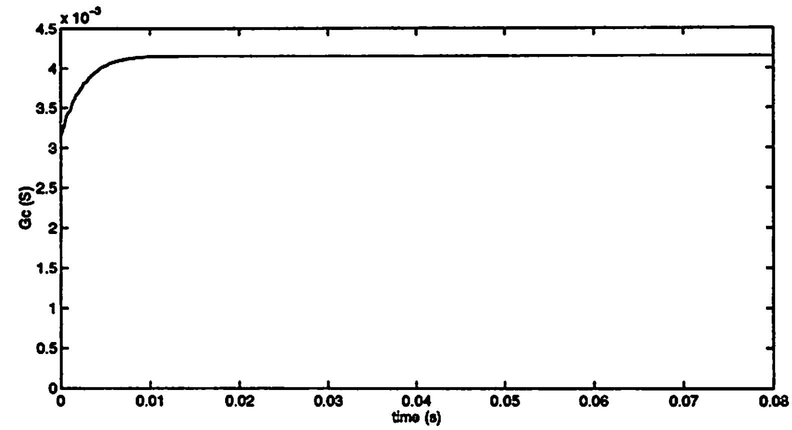


Figure 2: Compensating for mismatch in G_a, G_b and I_b. (a) Synchronization error $r_1(t) = x_1(t) - \tilde{x}_1(t)$. (b) Conductance $\tilde{G}_a(t)$ in System 2. The fixed conductance G_a in System 1 is shown by a dashed line. (c) Conductance $\tilde{G}_b(t)$ in System 2. The fixed conductance G_b in System 1 is shown with a dashed line.



(d)



(e)

Figure 2: (d) Current $\tilde{I}_b(t)$ in System 2. The fixed current I_b in System 1 is shown with a dashed line. (e) The coupling conductance $G_c(t)$.

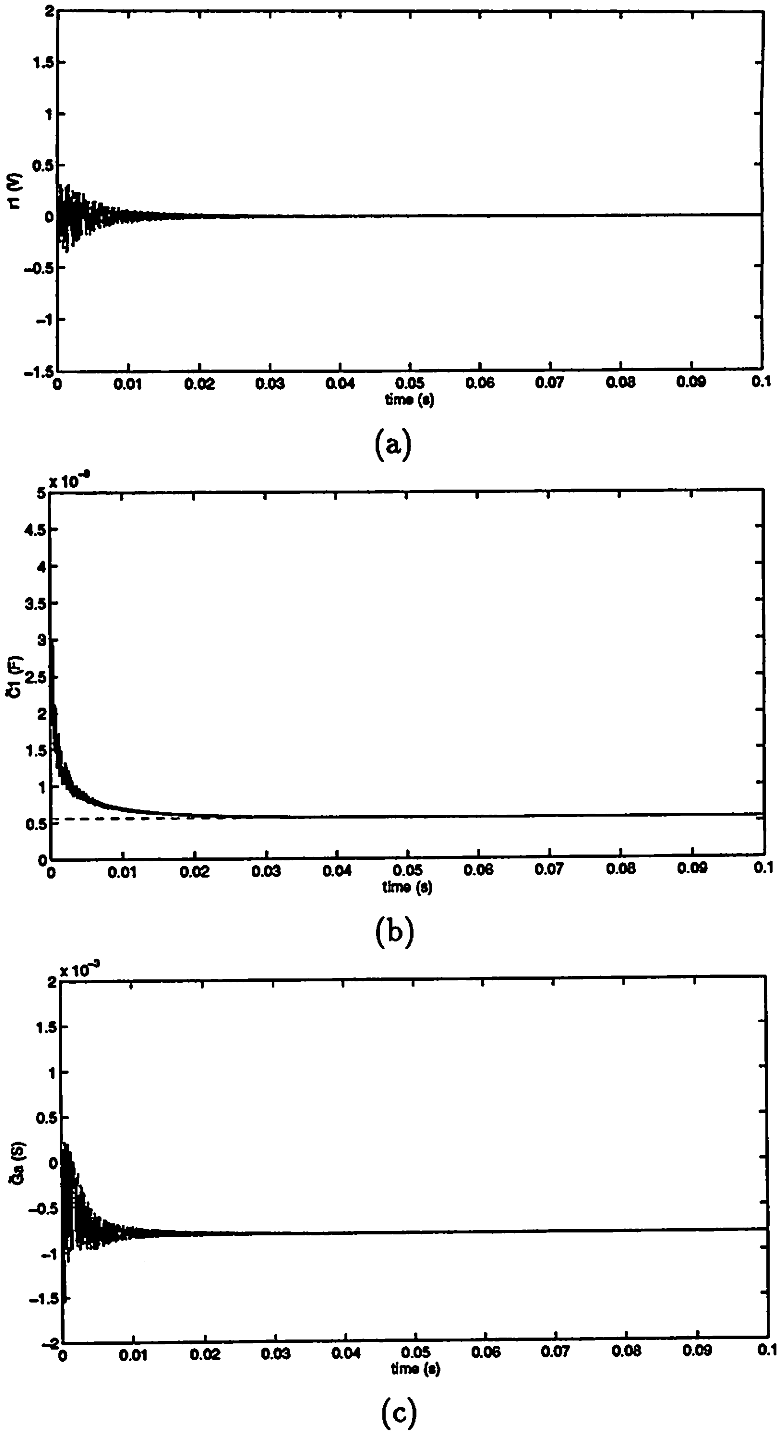
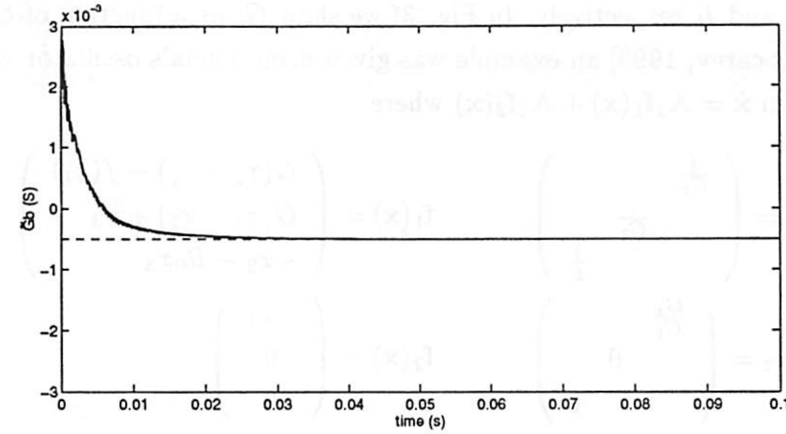
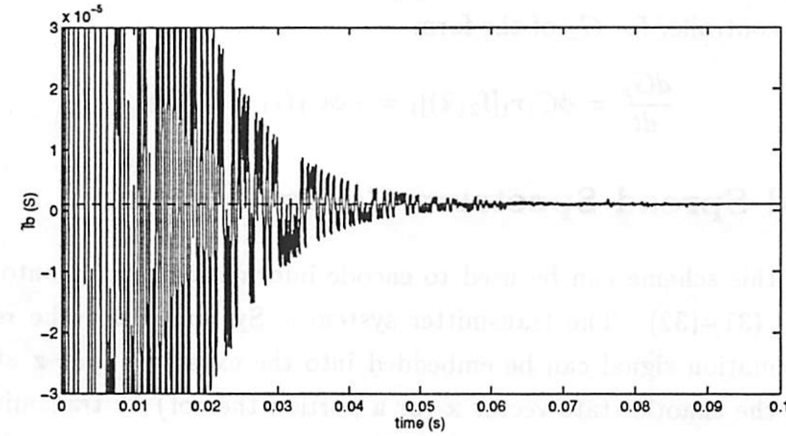


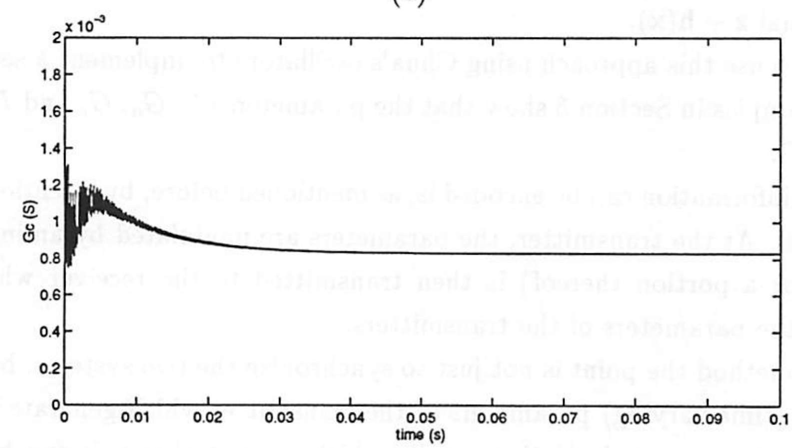
Figure 3: Compensating for mismatch in C_1, G_a, G_b and I_b. (a) Synchronization error $r_1(t) = x_1(t) - \tilde{x}_1(t)$. (b) Capacitance $\tilde{C}_1(t)$ in System 2. The fixed capacitance C_1 in System 1 is shown by a dashed line. (c) Conductance $\tilde{G}_a(t)$ in System 2. The fixed conductance G_a in System 1 is shown by a dashed line.



(d)



(e)



(f)

Figure 3: (d) Conductance $\tilde{G}_b(t)$ in System 2. The fixed conductance G_b in System 1 is shown with a dashed line. (e) Current $\tilde{I}_b(t)$ in System 2. The fixed current I_b in System 1 is shown with a dashed line. (f) The coupling conductance $G_c(t)$.

We also see in Fig. 3b-e that the values of $\tilde{C}_1$, $\tilde{G}_a$ and $\tilde{G}_b$, $\tilde{I}_b$ has converged to the corresponding values of C_1, G_a, G_b and I_b respectively. In Fig. 3f we show G_c as a function of time.

In [Parlitz and Kocarev, 1995] an example was given using Chua's oscillator where the decomposition is of the form $\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{f}_1(\mathbf{x}) + \mathbf{A}_2\mathbf{f}_2(\mathbf{x})$ where

$$\mathbf{A}_1 = \begin{pmatrix} \frac{1}{C_1} & & \\ & \frac{1}{C_2} & \\ & & \frac{1}{L} \end{pmatrix}, \qquad \mathbf{f}_1(\mathbf{x}) = \begin{pmatrix} G(x_2 - x_1) - f(x_1) \\ G(x_1 - x_2) + x_3 \\ -x_2 - R_0 x_3 \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} \frac{G_x}{C_1} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad \mathbf{f}_2(\mathbf{x}) = \begin{pmatrix} -x_1 \\ 0 \\ 0 \end{pmatrix}$$

and G_x is the only parameter to adapt.

This results in a controller for G_x of the form

$$\frac{dG_x}{dt} = \phi C_1 r_1 [\mathbf{f}_2(\tilde{\mathbf{x}})]_1 = -\phi C_1 (v_1 - \tilde{v}_1)\tilde{v}_1$$

6 Secure and Spread Spectrum Communication

There are two ways this scheme can be used to encode information signals onto chaotic carriers. First, consider Eqs. (31)-(32). The transmitter system is System 1 and the receiver system is System 2. The information signal can be embedded into the external input $\mathbf{z}$ at the transmitter and sent along with the chaotic state vector $\mathbf{x}$ (or a portion thereof) by transmitting $\mathbf{z} + \mathbf{h}(\mathbf{x})$ to the receiver. When synchronization is achieved, $\mathbf{h}(\tilde{\mathbf{x}})$ approaches $\mathbf{h}(\mathbf{x})$ and $\mathbf{z}$ can be recovered from the transmitted signal $\mathbf{z} + \mathbf{h}(\mathbf{x})$.

When we want to use this approach using Chua's oscillators to implement a secure communication system, the examples in Section 5 show that the parameters C_1, G_a, G_b and I_b are not suitable to be used as "keys".

The second way information can be encoded is, as mentioned before, by intentionally introducing parameter mismatch. At the transmitter, the parameters are modulated by an information signal. The state vector (or a portion thereof) is then transmitted to the receiver where the adaptive controllers recover the parameters of the transmitters.

For this second method the point is not just to synchronize the two systems, but also to use the receiver to find the (time-varying) parameters of the transmitter which generated the transmitted waveform. We show here some simulation results which suggest that this can be done when the parameters are sufficiently slowly varying.

Example 4 The setup is the same as Example 2, except that the parameters G_a, G_b and I_b in System 1 are modulated by slowly varying information signals. In particular, we set G_a, G_b and I_c in the transmitter (System 1) to be sinusoids of the form $G_a(t) = -0.8 \times 10^{-3}(1 - 0.01\sin(300t))$,

$G_b(t) = -0.5 \times 10^{-3}(1 - 0.01\sin(150t))$, $I_b(t) = 7 \times 10^{-6}\sin(75t)$. We set $\tilde{G}_a(0) = -1mS$, $\tilde{G}_b(0) = -1mS$, $\tilde{I}_b(0) = 10\mu A$, $G_c(0) = 0.08mS$.

The simulation results are shown in Fig. 4. The transmitted signal $v_1(t)$ is shown in Fig. 4a. The three sinusoidal information signals can be recovered from $\tilde{G}_a(t)$, $\tilde{G}_b(t)$ and $\tilde{I}_b(t)$ as shown in Fig. 4b-d. In Fig. 4e we show the coupling conductance $G_c(t)$.

It is easy to see how by decomposing the nonlinearity $f(\cdot)$ into many "orthogonal" basis functions, it is possible to transmit an arbitrary number of signals on a single scalar signal $v_1(t)$. Orthogonality here means the following. Let L be the range of v_1 of the attractor (in our case $v_1(t) \in L \approx [-2,2]$, see Fig. 4a). The function f should be decomposed as $f(v_1) = \sum_i a_i f_i(v_1)$ such that if $\sum_i a_i f_i(v_1) = \sum_i \tilde{a}_i f_i(v_1)$ for all $v_1 \in L$, then $a_i = \tilde{a}_i$ for all i. The decomposition in Eq. (38) where $\tilde{G}_a$, $\tilde{G}_b$, and $\tilde{I}_b$ are the coefficients is of this nature.

Of course, both these methods of transmitting information can be used simultaneously in the same system.

7 Conclusions

In this paper we have shown how adaptive controllers can be used to adapt the parameters in two coupled systems such that the two systems are synchronized. We conjecture that the parameters of the 2 systems converge towards each other if they exhibit chaotic dynamics. This provides a theoretical basis to study the well known Huberman-Lumer adaptive control scheme. The results given are global and we give examples where the parameters of the two systems differing by orders of magnitude approach each other to maintain synchronization. We also illustrate their use in transmitting several signals onto a single chaotic carrier signal.

Acknowledgements

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References

P. Celka, [1995], "Synchronization of chaotic systems through parameter adaptation,", In *IEEE International Symposium of Circuits and Systems Proceedings*, 692–695.

L. O. Chua, T. Yang, G.-Q. Zhong, and C. W. Wu, [1996], "Adaptive synchronization of chua's oscillators," *International Journal of Bifurcation and Chaos*, **6**(1).

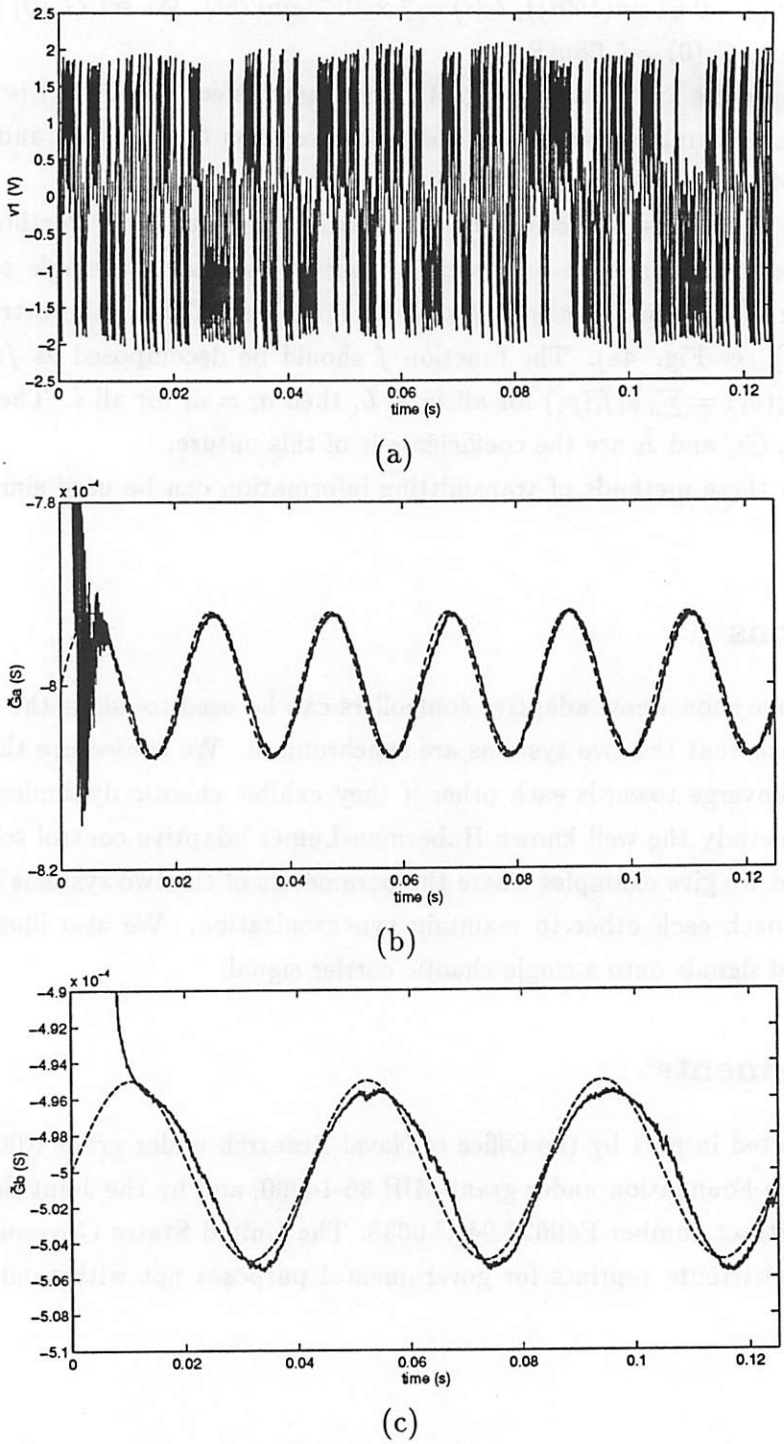
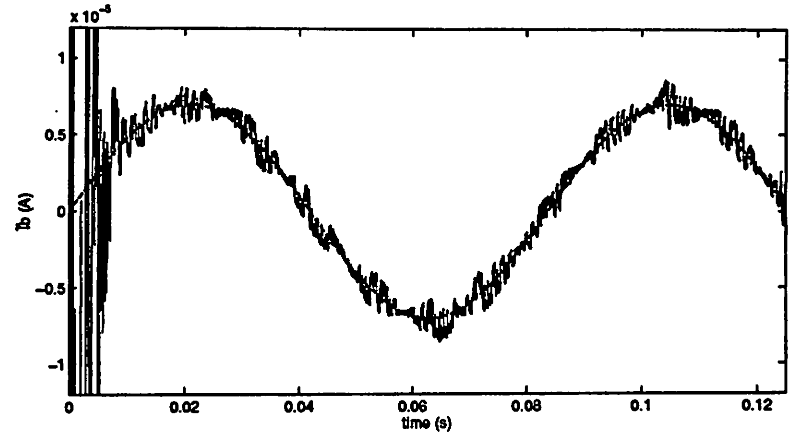
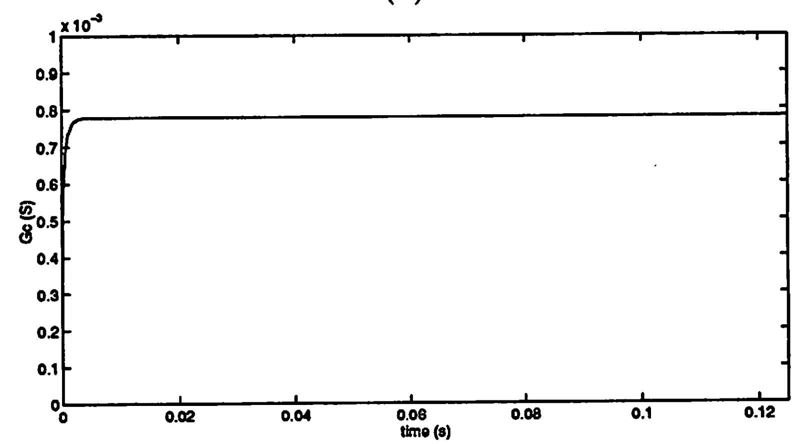


Figure 4: Transmitting several signals by modulating G_a, G_b, and I_b. (a) The transmitted signal $v_1(t)$. (b) Conductance $\tilde{G}_a(t)$ in System 2. The conductance G_a (shown as a dashed curve) in System 1 is a sinusoid of the form $-0.8 \times 10^{-3}(1 - 0.01\sin(300t))$. (c) Conductance $\tilde{G}_b(t)$ in System 2. The conductance G_b (shown as a dashed curve) in System 1 is a sinusoid of the form $-0.5 \times 10^{-3}(1 - 0.01\sin(150t))$.



(d)



(e)

Figure 4: (d) Current $\tilde{I}_b(t)$ in System 2. The current I_b (shown in a dashed curve) in System 1 is a sinusoid of the form $7 \times 10^{-6} \sin(75t)$. (e) The coupling conductance $G_c(t)$.

H. Dedieu, M. P. Kennedy, and M. Hasler, [1993], "Chaos shift keying: Modulation and demodulation of a chaotic carrier using self-synchronizing Chua's circuits," *IEEE Transactions on Circuits and Systems–II: Analog and Digital Signal Processing*, **40**(10), 634–642.

M. di Bernardo, [1995], "An adaptive approach to the control and synchronization of continuous-time chaotic systems,", Technical report.

H. Fujisaka and T. Yamada, [1983], "Stability theory of synchronized motion in coupled-oscillator systems," *Progress of Theoretical Physics*, **69**(1), 32–47.

R. He and P. G. Vaidya, [1992], "Analysis and synthesis of synchronous periodic and chaotic systems," *Physical Review A*, **46**(12), 7387–7392.

B. A. Huberman and E. Lumer, [1990], "Dynamics of adaptive systems," *IEEE Transactions on Circuits and Systems*, **37**(4), 547–550.

J. K. John and R. E. Amritkar, [1994], "Synchronization by feedback and adaptive control," *International Journal of Bifurcation and Chaos*, **4**(6), 1687–1695.

V. Lakshmikantham and X. Z. Liu, [1993], *Stability Analysis In Terms of Two Measures*. World Scientific, Singapore.

U. Parlitz and L. Kocarev, [1995], "Multichannel communication using auto-synchronization,", Technical report.

U. Parlitz, L. O. Chua, L. Kocarev, K. S. Halle, and A. Shang, [1992], "Transmission of digital signals by chaotic synchronization," *International Journal of Bifurcation and Chaos*, **2**(4), 973–977.

L. M. Pecora and T. L. Carroll, [1990], "Synchronization in chaotic systems," *Physical Review Letters*, **64**(8), 821–824.

S. Sinha, R. Ramaswamy, and J. S. Rao, [1990], "Adaptive control in nonlinear dynamics," *Physica D*, **43**, 118–128.

C. W. Wu and L. O. Chua, [1993], "A simple way to synchronize chaotic systems with applications to secure communication systems," *International Journal of Bifurcation and Chaos*, **3**(6), 1619–1627.

C. W. Wu and L. O. Chua, [1994], "A unified framework for synchronization and control of dynamical systems," *International Journal of Bifurcation and Chaos*, **4**(4), 979–998.

C. W. Wu and L. O. Chua, [1995a], "Synchronization in an array of linearly coupled dynamical systems," *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, **42**(8), 430–447.

C. W. Wu and L. O. Chua, [1995b], "Application of graph theory to the synchronization in an array of coupled nonlinear oscillators," *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, **42**(8), 494–497.

C. W. Wu, G. Q. Zhong, and L. O. Chua, [1995], "Synchronizing nonautonomous chaotic systems without phase-locking," *Journal of Circuits, Systems, and Computers*. To appear.

Figure Captions

Figure 1 Compensating for mismatch in C_1.

(a) Synchronization error $r_1(t) = x_1(t) - \tilde{x}_1(t)$.

(b) Capacitance $\tilde{C}_1(t)$ in System 2. The fixed capacitance C_1 in System 1 is shown with a dashed line.

(c) The coupling conductance $G_c(t)$. $G_c(t)$ shoots up very quickly from $G_c(0) = 0.08mS$ then decreased to a constant value of $0.655mS$.

Figure 2 Compensating for mismatch in G_a, G_b and I_b.

(a) Synchronization error $r_1(t) = x_1(t) - \tilde{x}_1(t)$.

(b) Conductance $\tilde{G}_a(t)$ in System 2. The fixed conductance G_a in System 1 is shown by a dashed line.

(c) Conductance $\tilde{G}_b(t)$ in System 2. The fixed conductance G_b in System 1 is shown with a dashed line.

(d) Current $\tilde{I}_b(t)$ in System 2. The fixed current I_b in System 1 is shown with a dashed line.

(e) The coupling conductance $G_c(t)$.

Figure 3 Compensating for mismatch in C_1, G_a, G_b and I_b.

(a) Synchronization error $r_1(t) = x_1(t) - \tilde{x}_1(t)$.

(b) Capacitance $\tilde{C}_1(t)$ in System 2. The fixed capacitance C_1 in System 1 is shown by a dashed line.

(c) Conductance $\tilde{G}_a(t)$ in System 2. The fixed conductance G_a in System 1 is shown by a dashed line.

(d) Conductance $\tilde{G}_b(t)$ in System 2. The fixed conductance G_b in System 1 is shown with a dashed line.

(e) Current $\tilde{I}_b(t)$ in System 2. The fixed current I_b in System 1 is shown with a dashed line.

(f) The coupling conductance $G_c(t)$.

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