BOUNDED TRACKING FOR NONMINIMUM PHASE NONLINEAR SYSTEMS WITH FAST ZERO DYNAMICS

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Bounded Tracking for Nonminimum Phase Nonlinear Systems with Fast Zero Dynamics

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Abstract

In this paper, we derive tracking control laws for nonminimum phase nonlinear systems with both fast and slow, possibly unstable, zero dynamics. The fast zero dynamics arise from a perturbation of a *nominal* system. These fast zeros can be problematic in that they may be in the right half plane and may cause large magnitude tracking control inputs. In this paper, we combine the ideas from some recent work of Hunt, Meyer, and Su with that of Devasia, Paden, and Chen on an asymptotic tracking procedure for nonminimum phase nonlinear systems. We give (somewhat subtle) conditions under which the tracking control input is bounded as the magnitude of the perturbation of the nominal system becomes zero. Explicit bounds on the control inputs are calculated for both SISO and MIMO systems using some interesting non-standard singular perturbation techniques. The method is applied to a suite of examples, including the simplified planar dynamics of VTOL and CTOL aircraft.

Keywords: Nonlinear control, zero dynamics, exact and asymptotic tracking, non-minimum phase, singular perturbation.

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1 Introduction

In this paper, we discuss tracking using bounded inputs for nonlinear nonminimum phase systems with "fast zero dynamics". While exact and asymptotic tracking for nonlinear minimum phase systems has now been well understood for some time (see [Isidori, 1989] for a comprehensive discussion), tracking for nonminimum phase nonlinear systems has been a tougher nut to crack. Early progress was made by the nonlinear regulator approach of Isidori-Byrnes [Isidori and Byrnes, 1990] which extended to the nonlinear case results of the Francis-Wonham regulator. Some drawbacks of this approach were the assumption of an exo-system and small domains of attraction [Teel, 1991]. A major advance in a general framework for tracking for nonminimum phase systems was made in a collection of papers by Devasia, Paden and Chen [Devasia *et al.*, 1996], [Devasia and Paden, 1994] in which they provide a *non-causal* exact tracking compensator for nonlinear (possibly multi-input multi-output) systems. In the time invariant case, their results use a clever technique of finding a bounded solution to a driven unstable nonlinear system over the time interval $-\infty < t < \infty$, provided that the "linear heart" of the nonlinear system is hyperbolic and the "residual nonlinearity" is Lipschitz continuous with small linear bounds. The extension to the time varying case is more subtle and needs some slowly time varying assumptions on the linearization due to Coppel. These results generalize the earlier results of Lanari and Wen [Lanari and Wen, 1991] for linear time invariant systems. Once the exact tracking compensator has been constructed the system is stabilized (through its linearization) about the trajectory to be exactly tracked.

In parallel with this work, we have been interested in the control of MIMO nonlinear systems where the decoupling matrix is close to being singular. We were heavily motivated in this regard by the flight control of a Vertical Take Off and Landing aircraft (VTOL) Harrier in [Hauser *et al.*, 1992b]. For systems such as this, the presence of the small control terms not only meant large control effort, but was symptomatic of "fast zero dynamics" which were possibly nonminimum phase. We explained this phenomenon in [Sastry *et al.*, 1989], [Isidori *et al.*, 1992] and [Godbole and Sastry, 1995] as a singular perturbation of the zero dynamics and discussed approximate methods for controlling the zero dynamics. A method for better approximating the exact control law is presented in [Barbot *et al.*, 1994]. Related work is presented in [Azam and Singh, 1994] and [Benvenuti *et al.*, 1993].

In new work on the problem of stable tracking for MIMO systems with fast zero dynamics, we attempted in [Tomlin *et al.*, 1995] to apply the Devasia-Paden-Chen techniques to a model of a Conventional Take Off and Landing (CTOL) aircraft and make comparisons with the other approximate techniques discussed above. The difficulty that we encountered was the presence of large magnitude control inputs in directly applying the Devasia-Paden-Chen scheme. More recently, Hunt, Meyer and Su in [Hunt and Meyer, 1995], [Meyer *et al.*, 1995a], and [Meyer *et al.*, 1995b] proposed an interesting variant to the application of the Devasia-Paden-Chen scheme by applying the method not to the given system, but to an "error system" obtained by comparing the given system to a nominal version of the system, which does not have the fast zero dynamics.

Our paper attempts to close the loop on this entire circle of ideas and to provide a reasonably complete[1] description of conditions under which bounded tracking control laws for nonlinear control systems with fast zero dynamics exist (in the limit that the perturbation of the system dynamics goes to zero). The paper considers a general class of both SISO and MIMO invertible (but not necessarily under static state feedback) nonlinear systems and as such is a generalization of the results in [Hunt and Meyer, 1995]. Unlike [Hunt and Meyer, 1995], we consider only systems which are affine in the inputs, yet this allows us to derive conditions under which bounded tracking may be proved and to work out the details of explicit bounds on the system inputs. What is striking about the current paper is the delicacy of the asymptotic calculations involving many interesting concepts from singular perturbations and differential equations. It is worthwhile to point out that while the two step procedure suggested in [Hunt and Meyer, 1995] is potentially useful from the standpoint of helping the numerical calculations, the analysis of whether the tracking input is bounded is intrinsic to the system rather than the process for calculating the control input. The results of [Barbot *et al.*, 1994] are relevant in that they provide a method for improving the accuracy of the approximate tracking scheme. We view our techniques as being relevant both for small ϵ perturbations (where they need to be compared with the control laws of [Barbot *et al.*, 1994]), as well as for moderate perturbations. Thus this paper appears to finish the program of finding control laws for a fairly general class of MIMO nonlinear systems. However, we are still concerned about the magnitude of signals that can be tracked using these methods (in other words, a resurgence of the concerns expressed by Teel about the Isidori-Byrnes regulator).

The outline of our paper is as follows. In Section 2, we consider SISO systems: we review the characterization of the fast zero dynamics and the Devasia-Paden-Chen scheme and show how these can be combined to produce bounded tracking control laws for the SISO case. Section 3 contains the MIMO case: there are two subcases to be treated based on whether the system has vector relative degree or not. Section 4 applies the theory first to some textbook style linear examples (for which our results are still interesting), and then to two

[1] We say *reasonably*, because we have some single time scale assumptions on the "fast" zero dynamics, which we would like to eventually remove.

MIMO flight control examples, planar models of VTOL and CTOL aircraft. Appendix 1 describes the computation of the steady state response of our systems using a "describing function" generalization of the Devasia-Paden-Chen method, when the output to be tracked (and hence the input to the system) is a bounded stationary signal.

2 Bounded Tracking for SISO systems

In this section, we will be concerned by a family of systems depending on a parameter ϵ, described by equations of the form

$$\begin{aligned} \dot{x} &= f(x,\epsilon) + g(x,\epsilon)u \\ y &= h(x,\epsilon) \end{aligned} \tag{1}$$

where $f(x,\epsilon)$ and the columns of $g(x,\epsilon)$ are smooth vector fields and $h(x,\epsilon)$ is a smooth function, defined in a neighborhood of $(x_0, 0)$ in $\mathbb{R}^n \times \mathbb{R}_+$. We will refer to the system of (1) with $\epsilon = 0$ as the *nominal* system and with $\epsilon \neq 0$ as the *perturbed* system. We will assume that $x = x_0$ is an equilibrium point for the nominal system, that is $f(x_0, 0) = 0$, and without loss of generality we will assume that $h(x_0, 0) = 0$.

2.1 Singularly Perturbed Zero and Driven Dynamics

In the two papers [Sastry *et al.*, 1989] and [Isidori *et al.*, 1992], it was shown that if the system (1) has relative degree $r(\epsilon) = r$ for $\epsilon \neq 0$, and relative degree $r(\epsilon) = r + d$ for $\epsilon = 0$, then there are *fast time scale zero dynamics* for the perturbed nonlinear system. This is in itself a rather surprising conclusion: we review one such result from these papers. As a consequence of the definition of relative degree we have that $r(\epsilon) = r$ and $r(0) = r + d$ implies that $\forall \epsilon \neq 0$

$$\begin{gathered} L_g h(x,\epsilon) = L_g L_f h(x,\epsilon) = \cdots = L_g L_f^{r-2} h(x,\epsilon) = 0 \quad \forall x \text{ near } x_0 \\ L_g L_f^{r-1} h(x_0,\epsilon) \neq 0 \end{gathered} \tag{2}$$

and for $\epsilon = 0$,

$$\begin{gathered} L_g h(x,0) = L_g L_f h(x,0) = \cdots = L_g L_f^{r+d-2} h(x,0) = 0 \quad \forall x \text{ near } x_0 \\ L_g L_f^{r+d-1} h(x_0,0) \neq 0 \end{gathered} \tag{3}$$

To keep the singularly perturbed zero dynamics from demonstrating *multiple time scale* behavior[2] we assume that for $0 \leq k \leq d$

$$L_g L_f^{r-1+k} h(x,\epsilon) = \epsilon^{d-k} \alpha_k(x,\epsilon) \tag{4}$$

where each $\alpha_k(x,\epsilon)$ is a smooth function of (x,ϵ) in a neighborhood of $(x_0, 0)$. The choice of $L_g L_f^{r-1} h(x,\epsilon) = O(\epsilon^d)$ rather than $O(\epsilon)$ is made to keep from having to use fractional powers of ϵ. What is critical about the assumption (4) is the decreasing powers of ϵ dependence as k increases from 0 to d.

As is standard in the literature, we will denote by $\xi \in \mathbb{R}^{r+d}$ the vector corresponding to the output and first $r+d-1$ derivatives of the system in (1), given by

$$\xi = \begin{pmatrix} h(x,\epsilon) \\ L_f h(x,\epsilon) \\ \vdots \\ L_f^{r-1} h(x,\epsilon) \\ \vdots \\ L_f^{r+d-1} h(x,\epsilon) \end{pmatrix} \tag{5}$$

where the first r coordinates correspond to the first r derivatives of the output, and the full set of $r+d$ coordinates, at $\epsilon = 0$, are the first $r+d$ derivatives of the output of the nominal system. It was shown in [Sastry *et al.*, 1989], [Isidori *et al.*, 1992] that for small ϵ we have the following "normal form" (in the sense of [Isidori, 1989]):

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= \xi_{r+1} + \epsilon^d \alpha_0(\xi,\eta,\epsilon) u \\ \dot{\xi}_{r+1} &= \xi_{r+2} + \epsilon^{d-1} \alpha_1(\xi,\eta,\epsilon) u \\ \dot{\xi}_{r+2} &= \xi_{r+3} + \epsilon^{d-2} \alpha_2(\xi,\eta,\epsilon) u \\ &\vdots \\ \dot{\xi}_{r+d} &= b(\xi,\eta,\epsilon) + a(\xi,\eta,\epsilon) u \\ \dot{\eta} &= q(\xi,\eta,\epsilon) \end{aligned} \tag{6}$$

Here, we have introduced the smooth functions a, b, and q; the details of how a and b depend on f, g, and h are discussed in [Sastry *et al.*, 1989], [Isidori *et al.*, 1992].

[2]This is an interesting case and though it is no different conceptually, the notation and the details of the assumptions needed are more involved.

Using the change of coordinates for the perturbed system given by

$$z_1 = \xi_{r+1}, \quad z_2 = \epsilon \xi_{r+2}, \quad \cdots \quad z_d = \epsilon^{d-1} \xi_{r+d} \tag{7}$$

it may be verified that the zero dynamics (corresponding to the output of the perturbed system being held identically to zero) have the form

$$\begin{aligned} \epsilon \dot{z}_1 = & \; -\tfrac{\alpha_1}{\alpha_0} z_1 + z_2 \\ \epsilon \dot{z}_2 = & \; -\tfrac{\alpha_2}{\alpha_0} z_1 + z_3 \\ \vdots & \\ \epsilon \dot{z}_d = & \; -\tfrac{a}{\alpha_0} z_1 + \epsilon^d b \\ \dot{\eta} = & \; q(z, \eta, \epsilon) \end{aligned} \tag{8}$$

Note that $\eta \in \mathbb{R}^{n-r-d}$, $z \in \mathbb{R}^d$. Also, we have abused notation for q from equation (6). Thus, the zero dynamics appear in singularly perturbed form, ie.

$$\begin{aligned} \epsilon \dot{z} = & \; r(z, \eta, \epsilon) \\ \dot{\eta} = & \; q(z, \eta, \epsilon) \end{aligned} \tag{9}$$

with $n - r - d$ slow states (η) and d fast states (z). This is now consistent with the zero dynamics for the system at $\epsilon = 0$ given by

$$\dot{\eta} = q(0, \eta, 0) \tag{10}$$

Thus, the presence of small terms in $L_g L_f^{r-1+k} h(x, \epsilon)$ for $0 \leq k \leq d$, causes the presence of singularly perturbed zero dynamics. The Jacobian matrix evaluated at $z = 0, \epsilon = 0$ of the fast zero subsystem is obtained to be

$$\begin{bmatrix} a_1(0,\eta,0) & 1 & 0 & \cdots & 0 \\ a_2(0,\eta,0) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{d-1}(0,\eta,0) & 0 & 0 & \cdots & 1 \\ a_d(0,\eta,0) & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{11}$$

Here $a_i = -\frac{\alpha_i}{\alpha_0}(\xi, \eta, \epsilon)$ for $1 \leq i < d$, and $a_d = -\frac{a}{\alpha_0}(\xi, \eta, \epsilon)$. It is clear that the perturbed system may be *nonminimum phase* either for positive ϵ, negative ϵ, or both positive and negative ϵ (according to whether the matrix in (11) has eigenvalues in $\mathbb{C}_-$, $\mathbb{C}_+$ or has indefinite inertia, respectively). If (10) has a stable equilibrium point at the origin (corresponding to the nominal system being minimum phase), but the origin of the system (8) is unstable, (corresponding to the perturbed system being nonminimum phase), we refer to these systems as *slightly nonminimum phase.*

We will need to generalize the preceding discussion of *zero dynamics* to the *driven dynamics* corresponding to the problem of tracking a desired output trajectory $y_D(t)$. If the output $y(t) \equiv y_D(t)$, it follows that, for the perturbed system,

$$\xi_D(t) = \begin{pmatrix} y_D(t) \\ \dot{y}_D(t) \\ \vdots \\ y_D^{(r-1)}(t) \end{pmatrix} \tag{12}$$

Then, the driven dynamics of the system are given by (6) with the choice of error coordinates

$$v_1 = \xi_{r+1} - y_D^{(r)}(t),\ v_2 = \epsilon(\xi_{r+1} - y_D^{(r+2)}(t)), \quad \cdots \quad v_d = \epsilon^{d-1}(\xi_{r+d} - y_D^{(r+d-1)}(t)) \tag{13}$$

and the input

$$u = \frac{-v_1}{\epsilon^d \alpha_0}$$

to be

$$\begin{aligned} \epsilon\dot{v}_1 &= a_1 v_1 + v_2 \\ \epsilon\dot{v}_2 &= a_2 v_1 + v_3 \\ &\vdots \\ \epsilon\dot{v}_d &= a_d v_1 + \epsilon^d(b - y_D^{(r+d)}(t)) \\ \dot{\eta} &= q(\xi, \eta, \epsilon) \end{aligned} \tag{14}$$

2.2 Two Step Procedure for Bounded Tracking

For systems of the form (1), difficulties with bounded tracking, that is the problem of finding bounded control laws for making $y(t)$ track a prescribed bounded trajectory $y_D(t)$ (with its first $r + d$ derivatives also bounded) may arise for two reasons:

1. **The nominal system may be nonminimum phase.** This means that the zero dynamics (10) of the nominal system are unstable.

2. **The presence of terms of $O(\epsilon^d)$ for $L_g L_f^{r-1} h(x, \epsilon)$ for the perturbed system.** This, in turn, may cause two different kinds of problems:

   (a) *The (exact) tracking control law given by (15) may become unbounded as* $\epsilon \to 0$.

   $$u = \frac{1}{L_g L_f^{r-1} h(x, \epsilon)}(y_D^{(r)} - L_f^r h(x, \epsilon)) \tag{15}$$

   (b) *The fast time scale zero dynamics of the perturbed system are likely to be nonminimum phase* as noted in the discussion following (11).

In this subsection, we combine some interesting new results of Devasia, Paden and Chen [Devasia *et al.*, 1996], [Devasia and Paden, 1994] on output tracking using bounded inputs for nonlinear systems with hyperbolic[3](but not necessarily minimum phase) zero dynamics, with a two step procedure suggested by Hunt, Meyer and Su in [Hunt and Meyer, 1995], [Meyer *et al.*, 1995a], and [Meyer *et al.*, 1995b] which we use to derive conditions for boundedness of the tracking control law (15). The algorithm proceeds in two steps:

1. **Step 1:** One finds a bounded input to cause the *nominal* system to track $y_D(t)$. If the nominal system is nonminimum phase, the algorithm of Devasia-Paden-Chen is applied, as follows. The nominal system with relative degree $r+d$ has driven dynamics given by

$$\dot{\eta} = q(\xi_D, \eta, 0) \tag{16}$$

with $\eta \in \mathbb{R}^{n-r-d}$ and ξ_D given by (12). The Devasia-Paden-Chen scheme (time invariant version) consists of defining a linear approximant to the smooth function q, usually

$$Q := \frac{\partial q}{\partial \eta}(\xi_{D_0}, \eta_0, 0)$$

and then under the hypothesis that Q is hyperbolic (i.e it has no eigenvalues on the $j\omega$ axis) and that the residual error defined by $r(\xi_D, \eta, 0) := q(\xi_D, \eta, 0) - Q\eta$ is Lipschitz continuous in both of its arguments, a condition referred to as *locally approximately linear*:

$$|r(\xi_1, \eta_1, 0) - r(|\xi_2, \eta_2, 0)| < K_1|\xi_1 - \xi_2| + K_2|\eta_1 - \eta_2|$$

with Lipschitz constants K_1, K_2 small enough, there exists for given bounded ξ_D a bounded solution $\eta(t)$ satisfying $\lim_{t\to\pm\infty} \eta(t) = 0$, which is obtained as the fixed point of the following integral equation:

$$\eta(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) r(\xi_D, \eta, 0) d\tau \tag{17}$$

Here $\Phi(t)$ is the Caratheodory solution of the matrix differential equation

$$\dot{X} = QX \quad X(\pm\infty) = 0 \quad X(0+) - X(0-) = I$$

Furthermore, one can find K_3 such that

$$|\eta(t)| \leq K_3 \sup_t |\xi_D(t)|$$

[3]More precisely, in the slowly time varying case, kinematically equivalent to uniformly hyperbolic

The strategy for solving the fixed point equation (17) is to use a Picard Lindelöf iteration scheme with any initial guess $\eta^0(t) : -\infty < t < \infty$,

$$\eta^{m+1}(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) r(\xi_D, \eta^m, 0) d\tau$$

The resulting controller is synthesized by using the bounded $\eta(t)$ to obtain

$$u(t) = \frac{y_D^{(r+d)}(t) - b(\xi_D(t), \eta(t), 0)}{a(\xi_D(t), \eta(t), 0)} \tag{18}$$

A drawback of this control law is that it is *non-causal*. One way this is remedied is to use a preview of a certain duration. Also, while the algorithm as stated is used for exact tracking, asymptotic tracking is achieved by stabilizing the linearization of the system (1). Thus, for small enough ξ_D and $(x(0) - x_0)$, bounded tracking is achieved using a non-causal input. A time varying version of the algorithm is given in [Devasia and Paden, 1994], which is very useful in applications such as in the linearization of equation (16) about $\xi_D(t)$ to produce a time varying matrix

$$Q(t) := \frac{\partial q}{\partial \eta}(\xi_D(t), \eta_0, 0)$$

If $Q(t)$ is slowly time varying, and is kinematically equivalent to a two-block diagonal matrix with an exponentially stable and an exponentially unstable state transition matrix (*uniformly hyperbolic*) the Picard iteration can be applied as before.

Remark 1 The generalization of the computation of the steady state solution to (17) when the signals to be tracked are stationary is presented in Appendix 1.

Remark 2 Step 1 produces in (18) the input required for *exact* tracking of the *nominal system*, which, when applied to the perturbed system results in *approximate tracking*. If ϵ and the desired trajectory and its derivatives $(y_D, \dot{y}_D, \ldots, y_D^{(r+d)})$ are sufficiently small, then it may be proven that the states of the closed loop system will remain bounded and the tracking error will be $O(\epsilon)$. The proof requires an adaptation of the related proof in [Hauser *et al.*, 1992b] to systems with unstable zero dynamics, whose state response is calculated using the Devasia-Paden-Chen method. This method can also be adapted for approximate tracking in systems which are not regular but have a well defined "robust relative degree" [Hauser *et al.*, 1992a] provided that the "robust zero dynamics" are hyperbolic.

2. **Step 2:** Denote by $u^0(t)$ the input (18) required to produce exact tracking with bounded inputs for the nominal system and let the resultant state trajectory be given by $\xi^0(t), \eta^0(t)$. That is,

$$\begin{aligned} \dot{y}_D = \quad \dot{\xi}_1^0 &= \xi_2^0 \\ \ddot{y}_D = \quad \dot{\xi}_2^0 &= \xi_3^0 \\ \vdots \qquad\quad &\vdots \\ y_D^{(r)} = \quad \dot{\xi}_r^0 &= \xi_{r+1}^0 \\ y_D^{(r+1)} = \quad \dot{\xi}_{r+1}^0 &= \xi_{r+2}^0 \\ y_D^{(r+2)} = \quad \dot{\xi}_{r+2}^0 &= \xi_{r+3}^0 \\ \vdots \qquad\quad &\vdots \\ y_D^{(r+d)} = \quad \dot{\xi}_{r+d}^0 &= b(\xi^0, \eta^0, 0) + a(\xi^0, \eta^0, 0)u^0 \\ \dot{\eta}^0 &= q(\xi^0, \eta^0, 0) \end{aligned} \tag{19}$$

Now, define the input $u(t)$ for the perturbed system for exact tracking. The system equations are now given by the equations (6), repeated here:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= \xi_{r+1} + \epsilon^d \alpha_0(\xi, \eta, \epsilon)u \\ \dot{\xi}_{r+1} &= \xi_{r+2} + \epsilon^{d-1} \alpha_1(\xi, \eta, \epsilon)u \\ \dot{\xi}_{r+2} &= \xi_{r+3} + \epsilon^{d-2} \alpha_2(\xi, \eta, \epsilon)u \\ &\vdots \\ \dot{\xi}_{r+d} &= b(\xi, \eta, \epsilon) + a(\xi, \eta, \epsilon)u \\ \dot{\eta} &= q(\xi, \eta, \epsilon) \end{aligned}$$

Note that the first r coordinates of the perturbed system match those of the nominal system. To obtain the control $u(t)$ for the perturbed system, the expression for $\dot{\xi}_r$ from (6) is equated with that of $\dot{\xi}_r^0$ from (19). Also, define as before

$$\begin{aligned} v_i &= \epsilon^{i-1}(\xi_{r+i} - \xi_{r+i}^0) & i &= 1, \ldots, d \\ v_{i+d} &= \eta_i - \eta_i^0 & i &= 1, \ldots, n-r-d \end{aligned}$$

Subtracting equations (19) from equations (6) yields an algebraic equation for the control, namely:

$$v_1 = -\epsilon^d \alpha_0(\xi, \eta, \epsilon)u \tag{20}$$

and an error system

$$
\begin{aligned}
\epsilon\dot{v}_1 &= v_2 + a_1(\xi,\eta,\epsilon)v_1 \\
\epsilon\dot{v}_2 &= v_3 + a_2(\xi,\eta,\epsilon)v_1 \\
&\vdots \\
\epsilon\dot{v}_d &= a_d(\xi,\eta,\epsilon)v_1 + \epsilon^d(b(\xi,\eta,\epsilon) - b(\xi^0,\eta^0,0) - a(\xi^0,\eta^0,0)u^0) \\
\dot{v}_{d+1} &= q_1(\xi,\eta,\epsilon) - q_1(\xi^0,\eta^0,0) \\
&\vdots \\
\dot{v}_{n-r} &= q_{n-r-d}(\xi,\eta,\epsilon) - q_{n-r-d}(\xi^0,\eta^0,0)
\end{aligned}
\tag{21}
$$

One now applies the Devasia-Paden algorithm [Devasia and Paden, 1994] to the system of (21) to find the bounded control $u(t)$ for exact tracking. For the purpose of applying this algorithm, it is necessary to consider the linear approximant to the right hand side of (21). This is conveniently chosen to be

$$
\begin{bmatrix}
\begin{bmatrix}
a_1(\xi^0(t),\eta^0(t),0) & 1 & 0 & \cdots & 0 \\
a_2(\xi^0(t),\eta^0(t),0) & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
a_{d-1}(\xi^0(t),\eta^0(t),0) & 0 & 0 & \cdots & 1 \\
a_d(\xi^0(t),\eta^0(t),0) & 0 & 0 & \cdots & 0
\end{bmatrix} & 0 \\
\frac{\partial q(\xi^0(t),\eta^0(t),0)}{\partial \xi} & \frac{\partial q(\xi^0(t),\eta^0(t),0)}{\partial \eta}
\end{bmatrix}
\tag{22}
$$

The matrix in (22) is a time varying one. To apply the results of Devasia-Paden to this system we need to assume that it is *slowly varying* in time and *kinematically equivalent to a uniformly hyperbolic matrix.* One convenient way to do this is to *assume that the nominal trajectory y_D and its first $r+d$ derivatives are small enough and that the functions $a_i, 1 \leq i \leq d$ are Lipschitz continuous in their arguments.*

Thus for fixed $\epsilon > 0$, the control law u calculated in (20) is bounded. What is less clear is the magnitude of the control law as $\epsilon \to 0$. The following theorem gives conditions under which the control law remains bounded as $\epsilon \to 0$.

Theorem 1 (Bounded Tracking as $\epsilon \to 0$) *Assume that*

(a) the driven dynamics of the nominal system (16) is hyperbolic and slowly time varying;

(b) the error system (21) is hyperbolic, and each function $a_i(\xi,\eta,\epsilon)$, $\frac{\partial q(\xi,\eta,\epsilon)}{\partial \xi}$, $\frac{\partial q(\xi,\eta,\epsilon)}{\partial \eta}$ in the Jacobian (22) is smooth and slowly time varying; and, in addition,

(c) the functions $a_i(\xi,\eta,\epsilon)$ in the Jacobian (22) satisfy the following Lipschitz condition:

$$
\begin{aligned}
|a_i(\xi^1,\eta^1,\epsilon) - a_i(\xi^2,\eta^2,0)| \le L_{i,1}(\epsilon)|\xi^1_{r+1} - \xi^2_{r+1}| + \ldots + L_{i,d}(\epsilon)|\xi^1_{r+d} - \xi^2_{r+d}| \\
+L_{i,d+1}(\epsilon)|\eta^1_1 - \eta^2_1| + \ldots + L_{i,n-r}(\epsilon)|\eta^1_{n-r-d} - \eta^2_{n-r-d}|
\end{aligned}
\tag{23}
$$

where

$$
\begin{array}{lll}
L_{i,j}(\epsilon) = & o(\epsilon^{j+1}) & i = 1,\ldots,d, \quad j = 1,\ldots,d \\
L_{i,j}(\epsilon) = & o(\epsilon) = k\epsilon^{1+\alpha} & i = 1,\ldots,d, \quad j = d+1,\ldots,n-r
\end{array}
\tag{24}
$$

Under these assumptions, the input $u(t)$ required for exactly tracking a desired output signal $y_D(t)$ with bounded derivatives (all sufficiently small), is bounded as $\epsilon \to 0$.

Proof: By assumption, the system of equation (21) is hyperbolic. By assuming that y_D and its first $r+d$ derivatives are small enough, we can assume that the conditions of the Devasia-Paden method apply to this system. Thus, ξ^0, η^0, ξ, and η are bounded by Paden-Devasia construction, and there is a unique bounded $v \in \mathbb{R}^{n-r}$ satisfying $v(\pm\infty) = 0$. Suppose that $v_{d+1}, v_{d+2}, \cdots, v_{n-r}$ are bounded by M.

Now focus attention on the fast time scale dynamics given by

$$
\begin{aligned}
\epsilon\dot{v}_1 = &\ v_2 + a_1(\xi,\eta,\epsilon)v_1 \\
\epsilon\dot{v}_2 = &\ v_3 + a_2(\xi,\eta,\epsilon)v_1 \\
\vdots & \\
\epsilon\dot{v}_d = &\ a_d(\xi,\eta,\epsilon)v_1 + \epsilon^d(b(\xi,\eta,\epsilon) - b(\xi^0,\eta^0,0) - a(\xi^0,\eta^0,0)u^0)
\end{aligned}
\tag{25}
$$

This is of the form of the linear time varying system

$$
\begin{aligned}
\epsilon\begin{bmatrix} \dot{v}_1 \\ \vdots \\ \dot{v}_d \end{bmatrix} = &\begin{bmatrix} a_1(\xi^0(t),\eta^0(t),0) & 1 & \cdots & 0 \\ \cdot & \cdot & \ddots & 1 \\ a_d(\xi^0(t),\eta^0(t),0) & 0 & \cdot & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} + \begin{bmatrix} a_1(\xi(t),\eta(t),\epsilon) - a_1(\xi^0(t),\eta^0(t),0) \\ \vdots \\ a_d(\xi(t),\eta(t),\epsilon) - a_d(\xi^0(t),\eta^0(t),0) \end{bmatrix} \\
+ &\ \epsilon^d \begin{bmatrix} 0 \\ \vdots \\ b(\xi,\eta,\epsilon) - b(\xi^0,\eta^0,0) - a(\xi^0,\eta^0,0)u^0 \end{bmatrix}
\end{aligned}
\tag{26}
$$

Let us denote

$$
A(t) := \begin{bmatrix} a_1(t) & 1 & \cdots & 0 \\ \cdot & \cdot & \ddots & 1 \\ a_d(t) & 0 & \cdot & 0 \end{bmatrix}
\tag{27}
$$

which by assumption is kinematically similar to

$$\left[\begin{array}{c|c} A_1(t) & 0 \\ \hline 0 & A_2(t) \end{array}\right] \tag{28}$$

where $A_1(t)$ corresponds to the exponentially (uniformly, asymptotically) anti-stable (i.e. stable as $t \to -\infty$) subsystem and $A_2(t)$ corresponds to the exponentially (uniformly, asymptotically) stable subsystem. Write the solution of (26) in integral form, with $\Phi_i(t)$ as the state transition matrix of $A_i(t)$:

$$\begin{aligned}
\left\|\begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}\right\| &\le \int_{-\infty}^{\infty} T(t) \left[\begin{array}{c|c} \Phi_1(t-\tau) & 0 \\ \hline 0 & \Phi_2(t-\tau) \end{array}\right] \left(\begin{bmatrix} L_{1,1}(\epsilon) \\ \vdots \\ L_{d,1}(\epsilon) \end{bmatrix} \left|\frac{v_1}{\epsilon}\right| + \ldots + \begin{bmatrix} L_{1,d}(\epsilon) \\ \vdots \\ L_{d,d}(\epsilon) \end{bmatrix} \left|\frac{v_d}{\epsilon^d}\right| \right) \\
&+ \int_{-\infty}^{\infty} T(t) \left[\begin{array}{c|c} \Phi_1(t-\tau) & 0 \\ \hline 0 & \Phi_2(t-\tau) \end{array}\right] M(n-r-d) \begin{bmatrix} \epsilon^\alpha \\ \vdots \\ \epsilon^\alpha \end{bmatrix} \\
&+ \epsilon^{d-1} \int_{-\infty}^{\infty} T(t) \left[\begin{array}{c|c} \Phi_1(t-\tau) & 0 \\ \hline 0 & \Phi_2(t-\tau) \end{array}\right] \begin{bmatrix} M_1 \\ \vdots \\ M_d \end{bmatrix} d\tau
\end{aligned} \tag{29}$$

Here the transformation $T(t)$ is a C^1 invertible matrix, $T(t)$ and $T^{-1}(t)$ transform $A(t)$ from the form (27) to the form (28) and are bounded, and the M_i denote the bounds on

$$T^{-1}(t) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ b(\xi,\eta,\epsilon) - b(\xi^0,\eta^0,0) - a(\xi^0,\eta^0,0)u^0 \end{bmatrix}$$

The bounds on the state transition matrices are

$$\|\Phi_1(t-\tau)\| \le k_1 e^{\lambda(t-\tau)/\epsilon} \quad \text{for} \quad \tau \ge t \tag{30}$$

$$\|\Phi_2(t-\tau)\| \le k_2 e^{-\mu(t-\tau)/\epsilon} \quad \text{for} \quad t \ge \tau \tag{31}$$

for $\lambda, \mu, k_1, k_2 > 0$. By assumption on the $L_{i,j}(\epsilon)$, $i = 1, \ldots, d, \quad j = 1, \ldots, d$, the first integral in (29) is $o(\epsilon)$. Thus,

$$(1 - o(\epsilon)) \left\|\begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}\right\| \le \epsilon^d \left\|\begin{bmatrix} M_1 \\ \vdots \\ M_d \end{bmatrix}\right\| \tag{32}$$

The term $(1 - o(\epsilon))$ is positive for small enough ϵ. Therefore, by integrating equation (29) we have that the bound on the components $v_i(t)$ is $O(\epsilon^d)$. This also establishes that for $1 \leq i \leq d$, the error coordinates are

$$\xi_{r+i} - \xi^0_{r+i} = O(\epsilon^{d-i+1})$$

In particular this establishes that the input given in equation (20):

$$u(t) = O(\epsilon^0)$$

This completes the proof. ■

An interesting by-product of the calculations in the proof that

$$\xi_{r+i} - \xi^0_{r+i} = O(\epsilon^{d-i+1})$$

is the verification that there is *no bounded peaking* in the dynamics (see for example, [Sussmann and Kokotović, 1991] for a description of this phenomenon). It is also important to realize that the control law may *not be a continuous function of* ϵ, that is there may be a discontinuity in the formula for the control law u at $\epsilon = 0$.

3 Bounded Tracking for MIMO Systems

To keep the notation in this section from becoming too complicated, we will consider two-input two-output systems of the form

$$\begin{aligned} \dot{x} &= f(x,\epsilon) + g_1(x,\epsilon)u_1 + g_2(x,\epsilon)u_2 \\ y_1 &= h_1(x,\epsilon) \\ y_2 &= h_2(x,\epsilon) \end{aligned} \quad (33)$$

We will need to distinguish, in what follows, between the following two cases:

1. Each perturbed system has vector relative degree at a point x_0, but as a function of ϵ the vector relative degree is not constant in a neighborhood of $\epsilon = 0$.

2. The perturbed system has vector relative degree, but the nominal system with $\epsilon = 0$ fails to have vector relative degree.

One could also consider as a third case, a scenario in which the perturbed system and nominal system fail to have vector relative degree, but they need different orders of dynamic extension for linearization. We will not consider this case here.

3.1 Singularly Perturbed Zero and Driven Dynamics

Case 1: Both the perturbed system and the nominal system have vector relative degree

Suppose that the system has vector relative degree $(r_1(\epsilon), r_2(\epsilon))$, with

$$\begin{aligned} r_1(\epsilon) &= s && \forall \epsilon \\ r_2(\epsilon) &= r && \forall \epsilon \neq 0 \\ r_2(0) &= r + d \end{aligned}$$

This implies that for the following matrix:

$$\begin{bmatrix} L_{g_1} L_f^i h_1(x,\epsilon) & L_{g_2} L_f^i h_1(x,\epsilon) \\ L_{g_1} L_f^j h_2(x,\epsilon) & L_{g_2} L_f^j h_2(x,\epsilon) \end{bmatrix} \tag{34}$$

1. The first row is identically zero for x near x_0 and all ϵ for $i < s-1$ and nonzero at $(x_0, 0)$ for $i = s-1$.
2. The second row is identically zero for x near x_0, when $\epsilon \neq 0$ for $j < r-1$, and is identically zero when $\epsilon = 0$ for $j < r+d-1$.
3. The matrix (34) is nonsingular at (x_0, ϵ) with $\epsilon \neq 0$, for $i = s-1, j = r-1$, and is nonsingular at $(x_0, 0)$ with $\epsilon = 0$ for $i = s-1, j = r+d-1$.

As in the SISO case, we will assume that there are only two time scales in the zero dynamics, by assuming that for $0 \leq k \leq d$,

$$\begin{bmatrix} L_{g_1} L_f^{s-1} h_1(x,\epsilon) & L_{g_2} L_f^{s-1} h_1(x,\epsilon) \\ L_{g_1} L_f^{r-1+k} h_2(x,\epsilon) & L_{g_2} L_f^{r-1+k} h_2(x,\epsilon) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^{d-k} \end{bmatrix} A_k(x,\epsilon) \tag{35}$$

where $A_k(x,\epsilon)$ is a matrix of smooth functions, and both $A_0(x_0, 0)$ and $A_d(x_0, 0)$ are nonsingular. By using elementary column operations to modify g_1, g_2, we can assume that $A_0(x,\epsilon) = I \in \mathbb{R}^{2\times 2}$. Since the first row of the left hand side of (35) does not change with k, we can assume $A_k(x,\epsilon)$ to be of the form

$$A_k(x,\epsilon) = \begin{bmatrix} 1 & 0 \\ \gamma_k(x,\epsilon) & \alpha_k(x,\epsilon) \end{bmatrix} \quad \text{with} \quad \alpha_d(x_0, 0) \neq 0 \tag{36}$$

We define the state variables

$$\begin{aligned} \xi_1^1 &= h_1(x,\epsilon), & \ldots \quad \xi_s^1 &= L_f^{s-1} h(x,\epsilon) \\ \xi_1^2 &= h_2(x,\epsilon), & \ldots \quad \xi_{r+d}^2 &= L_f^{r+d-1} h_2(x,\epsilon) \end{aligned}$$

Then, using variables $\eta \in \mathbb{R}^{n-s-r-d}$ to complete the basis, the "MIMO normal form" of the system is given by

$$\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1 \\
&\vdots \\
\dot{\xi}_s^1 &= b_1(\xi,\eta,\epsilon) + u_1 \\
\dot{\xi}_1^2 &= \xi_2^2 \\
&\vdots \\
\dot{\xi}_r^2 &= \xi_{r+1}^2 + \epsilon^d(\gamma_0(\xi,\eta,\epsilon)u_1 + \alpha_0(\xi,\eta,\epsilon)u_2) \\
\dot{\xi}_{r+1}^2 &= \xi_{r+2}^2 + \epsilon^{d-1}(\gamma_1(\xi,\eta,\epsilon)u_1 + \alpha_1(\xi,\eta,\epsilon)u_2) \\
&\vdots \\
\dot{\xi}_{r+d}^2 &= b_2(\xi,\eta,\epsilon) + \gamma_d(\xi,\eta,\epsilon)u_1 + \alpha_d(\xi,\eta,\epsilon)u_2 \\
\dot{\eta} &= q(\xi,\eta,\epsilon) + P(\xi,\eta,\epsilon)u
\end{aligned} \tag{37}$$

Defining, in analogy to the SISO case,

$$z_1 = \xi_{r+1}^2, \; z_2 = \epsilon\xi_{r+2}^2, \quad \cdots \quad z_d = \epsilon^{d-1}\xi_{r+d}^2$$

it may be verified, using the controls

$$\begin{aligned}
u_1 &= -b_1 \\
u_2 &= \frac{\gamma_0}{\alpha_0}b_1 - \frac{1}{\epsilon^d}\frac{z_1}{\alpha_0}
\end{aligned}$$

that the zero dynamics are given by

$$\begin{aligned}
\epsilon\dot{z}_1 &= a_1z_1 + z_2 + \epsilon^d(-\gamma_1b_1 - a_1\gamma_0b_1) \\
\epsilon\dot{z}_2 &= a_2z_1 + z_3 + \epsilon^d(-\gamma_2b_1 - a_2\gamma_0b_1) \\
&\vdots \\
\epsilon\dot{z}_d &= a_dz_1 + \epsilon^d(b_2 - \gamma_db_1 - a_d\gamma_0b_1) \\
\dot{\eta} &= q(0,z_1,\tfrac{z_2}{\epsilon},\ldots,\eta,\epsilon) + P(0,z_1,\tfrac{z_2}{\epsilon},\ldots,\eta,\epsilon)\begin{bmatrix} -b_1 \\ \frac{\gamma_0}{\alpha_0}b_1 - \frac{z_1}{\epsilon^d\alpha_0} \end{bmatrix}
\end{aligned} \tag{38}$$

where the a_i are defined as before, with $a_d = -\frac{\alpha_d}{\alpha_0}$. The difference between the MIMO case and the SISO case is the presence of input terms in the right hand side of the η dynamics in (38). We will need to verify that no "bounded peaking" occurs in the dynamics of $\dot{\eta}$ in (38) when we estimate the magnitude of the variables z_i. Without this verification, it is possible that some of the η variables will also become fast time scale dynamics.

For the driven dynamics one considers, as in the SISO case, new "error coordinates" given by

$$v_1 = \xi_{r+1}^2 - y_{D2}^{(r)}(t), \; v_2 = \epsilon(\xi_{r+2}^2 - y_{D2}^{(r+1)}(t)), \quad \cdots \quad v_d = \epsilon^{d-1}(\xi_{r+d}^2 - y_{D2}^{(r+d-1)}(t)) \tag{39}$$

In these coordinates

$$
\begin{aligned}
\epsilon\dot{v}_1 =&\ a_1v_1 + v_2 + \epsilon^d(y^s_{D1} - b_1)(\gamma_1 + a_1\gamma_0) \\
\epsilon\dot{v}_2 =&\ a_2v_1 + v_3 + \epsilon^d(y^s_{D1} - b_1)(\gamma_2 + a_2\gamma_0) \\
&\vdots \\
\epsilon\dot{v}_d =&\ a_dv_1 + \epsilon^d(b_2 + (\gamma_d + a_d\gamma_0)(y^{(s)}_{D1} - b_1) - y^{(r+d)}_{D2}) \\
\dot{\eta} =&\ q(\xi_D, v_1, \tfrac{v_2}{\epsilon}, \ldots, \eta, \epsilon, t) + P(\xi_D, v_1, \tfrac{v_2}{\epsilon}, \ldots, \eta, \epsilon, t)\begin{bmatrix} y^{(s)}_{D1} - b_1 \\ -\frac{\gamma_0}{\alpha_0}(y^s_{D1} - b_1) - \frac{v_1}{\epsilon^d\gamma_0} \end{bmatrix}
\end{aligned} \tag{40}
$$

Case 2: The perturbed system has vector relative degree, but the nominal system does not have vector relative degree

We will suppose that the matrix in (34) satisfies

1. The first row is identically zero for x near x_0 and all ϵ for $i < s-1$ and nonzero at $(x_0, 0)$ for $i = s-1$.
2. The second row is identically zero for x near x_0, for all ϵ for $j < r-1$, and nonzero at $(x_0, 0)$ if $j = r-1$.
3. The matrix is non-singular at (x_0, ϵ) with $\epsilon \neq 0$, for $i = s-1, j = r-1$, but is singular at $(x_0, 0)$ with $\epsilon = 0$.

This implies that the system (33) has vector relative degree $(r_1(\epsilon), r_2(\epsilon)) = (s, r)$ for $\epsilon \neq 0$, but its relative degree cannot be defined at $\epsilon = 0$.

Under these assumptions and under dynamic extension of the system through u_1 and its first d derivatives, with $u_1^{(d)} = \overline{u_1}$, the system is decouplable. We will make the two time scale assumption, namely that the "decoupling matrix" at the kth step of the dynamic extension algorithm [Descusse and Moog, 1985] has the form

$$
\begin{bmatrix} \beta_k(x, u_1, \ldots, u_1^{(k-1)}, \epsilon) & \gamma_k(x, u_1, \ldots, u_1^{(k-1)}, \epsilon) \\ \delta_k(x, u_1, \ldots, u_1^{(k-1)}, \epsilon) & \alpha_k(x, u_1, \ldots, u_1^{(k-1)}, \epsilon) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^{d-k} \end{bmatrix} \tag{41}
$$

We denote the first matrix in (41) above $A_k(x, u_1, \ldots, u_1^{(k-1)}, \epsilon)$ for $0 \leq k \leq d$ and assume that $A_d(x, u_1, \ldots, u_1^{(k-1)}, \epsilon)$ is nonsingular. In fact, it may be verified that $\beta_k(x, u_1, \ldots, u_1^{(k-1)}, \epsilon) =$

$\beta_0(x,\epsilon)$ and $\delta_k(x,u_1,\ldots,u_1^{(k-1)},\epsilon)=\delta_0(x,\epsilon)$ for $0\le k\le d$. Now define

$$\begin{aligned}
\xi_1^1 &= h_1(x,\epsilon)\\
&\vdots\\
\xi_s^1 &= L_f^{s-1}h_1(x,\epsilon)\\
\xi_{s+1}^1 &= L_f^s h_1(x,\epsilon)+\beta_0(x,\epsilon)u_1\\
&\vdots\\
\xi_{s+d}^1 &= L_f^{s+d-1}h_1(x,\epsilon)+\phi_1(x,u_1,\dot u_1,\ldots,u_1^{(d-2)})+\beta_0(x,\epsilon)u_1^{(d-1)}\\
\xi_1^2 &= h_2(x,\epsilon)\\
&\vdots\\
\xi_r^2 &= L_f^{r-1}h_2(x,\epsilon)\\
\xi_{r+1}^2 &= L_f^r h_2(x,\epsilon)+\delta_0(x,\epsilon)u_1\\
&\vdots\\
\xi_{r+d}^2 &= L_f^{r+d-1}h_2(x,\epsilon)+\phi_2(x,u_1,\dot u_1,\ldots,u_1^{(d-2)})+\delta_0(x,\epsilon)u_1^{(d-1)}
\end{aligned}\tag{42}$$

Using $\eta\in\mathbb{R}^{n-r-s-d}$ to complete the coordinate transformation from $\mathbb{R}^n\times\mathbb{R}^d\to\mathbb{R}^n\times\mathbb{R}^d$, the MIMO normal form for Case 2 is given by

$$\begin{aligned}
\dot\xi_1^1 &= \xi_2^1\\
&\vdots\\
\dot\xi_s^1 &= \xi_{s+1}^1\\
\dot\xi_{s+1}^1 &= \xi_{s+2}^1+\epsilon^{d-1}\gamma_1(\xi,\eta,\epsilon)u_2\\
&\vdots\\
\dot\xi_{s+d}^1 &= b_1(\xi,\eta,\epsilon)+\beta_0(\xi,\eta,\epsilon)\overline{u_1}+\gamma_d(\xi,\eta,\epsilon)u_2\\
\dot\xi_1^2 &= \xi_2^2\\
&\vdots\\
\dot\xi_r^2 &= \xi_{r+1}^2+\epsilon^d\alpha_0(\xi,\eta,\epsilon)u_2\\
\dot\xi_{r+1}^2 &= \xi_{r+2}^2+\epsilon^{d-1}\alpha_1(\xi,\eta,\epsilon)u_2\\
&\vdots\\
\dot\xi_{r+d}^2 &= b_2(\xi,\eta,\epsilon)+\delta_0(\xi,\eta,\epsilon)\overline{u_1}+\alpha_d(\xi,\eta,\epsilon)u_2\\
\dot\eta &= q(\xi,\eta,\epsilon)+P(\xi,\eta,\epsilon)u
\end{aligned}\tag{43}$$

Note that the variables $\xi_{s+1}^1,\ldots,\xi_{s+d}^1$ are affinely related to $u_1,\dot u_1,\ldots,u_1^{(d-1)}$. Defining, as before, the "fast" zero dynamics variables as

$$z_1=\xi_{r+1}^2,\; z_2=\epsilon\xi_{r+2}^2,\quad\cdots\quad z_d=\epsilon^{d-1}\xi_{r+d}^2$$

and using the control inputs

$$
\begin{aligned}
u_1 = & \; -\frac{1}{\beta_0(x,\epsilon)} L_f^s h_1(x,\epsilon) \\
u_2 = & \; -\frac{1}{\epsilon^d \alpha_0(x,\epsilon)} (L_f^r h_2(x,\epsilon) - \frac{\delta_0(x,\epsilon)}{\beta_0(x,\epsilon)} L_f^s h_1(x,\epsilon))
\end{aligned}
$$

the zero dynamics are given by

$$
\begin{aligned}
\epsilon \dot{z}_1 = & \; a_1 z_1 + z_2 \\
\epsilon \dot{z}_2 = & \; a_2 z_1 + z_3 \\
& \vdots \\
\epsilon \dot{z}_d = & \; a_d z_1 + \epsilon^d (b_2 + \delta_0 \overline{u_1}) \\
\dot{\eta} = & \; q(0, z_1, \tfrac{z_2}{\epsilon}, \ldots, \eta, \epsilon) + P(0, z_1, \tfrac{z_2}{\epsilon}, \ldots, \eta, \epsilon) \begin{bmatrix} -\frac{1}{\beta_0(x,\epsilon)} L_f^s h_1(x,\epsilon) \\ -\frac{1}{\epsilon^d \alpha_0(x,\epsilon)} (L_f^r h_2(x,\epsilon) - \frac{\delta_0(x,\epsilon)}{\beta_0(x,\epsilon)} L_f^s h_1(x,\epsilon)) \end{bmatrix}
\end{aligned} \tag{44}
$$

The driven dynamics are derived accordingly.

3.2 Two step Bounded Tracking

The two step bounded tracking procedure in the MIMO case proceeds exactly as in the SISO case. The first step is to find a bounded input for exact tracking of the nominal system, using, if necessary, the Devasia-Paden-Chen procedure. The second step is to apply to the perturbed system the Devasia-Paden procedure with slowly time varying linearization about the trajectory for the nominal system. Finally, the system is stabilized about the exact tracking input and trajectory using a linear time varying feedback law (about the linearization, see for example, [Cheng, 1979], [Walsh *et al.*, 1994]).

4 Examples

We illustrate the theory of the previous section with a comprehensive set of linear and nonlinear examples, for both single-input single-output and multi-input multi-output systems.

4.1 Textbook Linear Examples

4.1.1 Single-Input Single-Output Linear Systems

SISO Example 1

The first example we consider is the linear SISO system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 - \epsilon u \\ \dot{x}_3 &= u\end{aligned} \tag{45}$$

with output equation $y = x_1$. The transfer function of the system is given by

$$H(s) = \frac{1 - \epsilon s}{s^3}$$

It is clear that $(r, r + d) = (2, 3)$ and that the singularly perturbed zero dynamics are given by

$$\epsilon \dot{z}_1 = z_1, \quad \text{where} \quad z_1 = x_3 \tag{46}$$

In **Step 1** of the bounded tracking procedure, the bounded input required for the nominal system to track y_D is calculated as

$$u^0 = y_D^{(3)} \tag{47}$$

In **Step 2**, the control u is generated by solving the differential equation

$$\epsilon \dot{v}_1 = v_1 - \epsilon u^0, \tag{48}$$

where $v_1 = \epsilon u$, and $\dot{v}_1 = u - u^0$.

Proposition 2 (Bounded Input for (45)) *Suppose that the output $y = x_1$ of system (45) is to track a desired trajectory y_D. Then the control $u = v_1/\epsilon$ generated by the two step bounded tracking procedure, where v_1 is the solution to (48), is bounded as $\epsilon \to 0$.*

Proof: The proof follows from Theorem 1. In this simple example, we present the calculations explicitly. We solve equation (48) for v_1. Since the system described by (48) is anti-stable, we calculate its non-causal solution as

$$v_1 = \int_{-\infty}^{\infty} \Phi(t - \tau)(-u^0(\tau))d\tau \tag{49}$$

where $\Phi(t) = e^{t/\epsilon}\mathbf{1}(-t)$ and $\mathbf{1}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$. Restricting u^0 to be in L_∞ (ie. $\sup_t |u^0(t)| = M$)

$$\begin{aligned} v_1 &= -e^{t/\epsilon}\int_t^\infty e^{-\tau/\epsilon}u^0(\tau)d\tau && (50) \\ &\leq -\epsilon M && (51) \end{aligned}$$

Thus, v_1 is of $O(\epsilon)$, and $u = v_1/\epsilon$ is of $O(1)$. ■

SISO Example 2

The second SISO linear example we consider is

$$\begin{aligned} \dot{x}_1 &= x_2 + \epsilon x_3 + \epsilon^2 u \\ \dot{x}_2 &= x_3 + \epsilon u \\ \dot{x}_3 &= u \end{aligned} \tag{52}$$

with output $y = x_1$. The transfer function of the system is given by

$$H(s) = \frac{1 + 2\epsilon s + \epsilon^2 s^2}{s^3}$$

The relative degrees $(r, r+d) = (1,3)$ and the singularly perturbed zero dynamics are

$$\begin{aligned} \epsilon\dot{z}_1 &= -2z_1 + z_2 && (53) \\ \epsilon\dot{z}_2 &= -z_1 && (54) \end{aligned}$$

where $z_1 = x_2 + \epsilon x_3$ and $z_2 = \epsilon x_3$.

In **Step 1** of the bounded tracking procedure, the bounded input required for the nominal system to track y_D is calculated as

$$u^0 = y_D^{(3)} \tag{55}$$

In **Step 2**, the control law u for the perturbed system is calculated by solving the differential equation

$$\epsilon \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\epsilon^2 \end{bmatrix} u^0 \tag{56}$$

where

$$\begin{aligned} v_1 &= -\epsilon^2 u \\ v_2 &= \epsilon(x_3 - x_3^0) \end{aligned} \tag{57}$$

Proposition 3 (Bounded Input for (52)) *Suppose that the output $y = x_1$ of system (52) is to track a desired trajectory y_D. Then the control $u = -v_1/\epsilon^2$ generated by the two step bounded tracking procedure, where v_1 is the solution to (56), is bounded as $\epsilon \to 0$.*

Proof: The proof is again in integrating the differential equation (56) for v_1:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \int_{-\infty}^{t} \begin{bmatrix} -(t-\tau)e^{-(t-\tau)/\epsilon} \\ -(t-\tau)e^{-(t-\tau)/\epsilon} - \epsilon e^{-(t-\tau)/\epsilon} \end{bmatrix} u^0(\tau)d\tau \tag{58}$$

Restricting u^0 to be in L_∞ ($\sup_t |u^0(t)| = M$) then

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \leq \begin{bmatrix} -\epsilon^2 M \\ -2\epsilon^2 M \end{bmatrix} \tag{59}$$

Thus, v_1 is of $O(\epsilon^2)$, and $u = -v_1/\epsilon^2$ is of $O(1)$. ■

4.1.2 Multi-Input Multi-Output Linear Systems

MIMO Case 1

Consider the two-input two-output system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_5 - \epsilon^2 u_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \epsilon x_5 + u_1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2 \end{aligned} \tag{60}$$

with outputs $y_1 = x_3$ and $y_2 = x_1$, chosen so that the form of the equations conforms with the theoretical development of the previous section. The transfer function matrix of the system is given by

$$\begin{bmatrix} \frac{1}{s^2} & \frac{\epsilon}{s^4} \\ 0 & \frac{1-\epsilon^2 s^2}{s^4} \end{bmatrix}$$

The vector relative degree is well defined for both the perturbed and nominal system:

$$\begin{aligned} r_1(\epsilon) &\equiv s = 2 \\ r_2(\epsilon) &\equiv r = 2 \\ r_2(0) &\equiv r + d = 4 \end{aligned}$$

and the singularly perturbed zero dynamics are given by

$$\epsilon \dot{z}_1 = z_2 \tag{61}$$

$$\epsilon \dot{z}_2 = z_1 \tag{62}$$

where $z_1 = x_5$ and $z_2 = \epsilon x_6$.

The bounded inputs required for the nominal system to track y_{D1} and y_{D2} are calculated as:

$$u_1^0 = y_{D1}^{(2)} \quad (63)$$

$$u_2^0 = y_{D2}^{(4)} \quad (64)$$

The control law $u = [u_1 \; u_2]$ for the perturbed system is calculated by solving the differential equation

$$\epsilon \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\epsilon^2 \end{bmatrix} u_2^0 \quad (65)$$

where

$$\begin{aligned} v_1 &= \epsilon^2 u_2 \\ v_2 &= \epsilon(x_6 - x_6^0) \end{aligned} \quad (66)$$

and by solving the algebraic equation

$$u_1 = u_1^0 - \epsilon(v_1 + x_5^0) \quad (67)$$

The proof that the control law $u = v_1/\epsilon^2$ is bounded as $\epsilon \to 0$ follows from that of Theorem 1. Notice that since the system (65) is hyperbolic, we must calculate the causal solution of the stable subsystem (corresponding to eigenvalue -1), and the non-causal solution of the anti-stable subsystem (corresponding to eigenvalue 1).

MIMO Case 2

The second MIMO example describes a system in which the vector relative degree of the nominal system is not well defined:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_5 + u_1 - \epsilon^2 u_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \epsilon x_5 + u_1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2 \end{aligned} \quad (68)$$

with outputs $y_1 = x_3$ and $y_2 = x_1$. The transfer function matrix of the system is given by

$$\begin{bmatrix} \frac{1}{s^2} & \frac{\epsilon}{s^4} \\ \frac{1}{s^2} & \frac{1-\epsilon^2 s^2}{s^4} \end{bmatrix}$$

The vector relative degree is well defined for the perturbed system:

$$\begin{aligned} r_1(\epsilon) &\equiv s = 2 \\ r_2(\epsilon) &\equiv r = 2 \end{aligned}$$

At $\epsilon = 0$, the system (68) does not have vector relative degree. The singularly perturbed zero dynamics are given by

$$\epsilon \dot{z}_1 = z_2 \quad (69)$$

$$\epsilon \dot{z}_2 = z_1 - \epsilon z_1 \quad (70)$$

where $z_1 = x_5 + u_1$ and $z_2 = \epsilon(x_6 + \dot{u}_1)$.

The bounded inputs required for the nominal system to track y_{D1} and y_{D2} are calculated as:

$$\ddot{u}_1^0 = y_{D1}^{(4)} \quad (71)$$

$$u_2^0 = y_{D2}^{(4)} - y_{D1}^{(4)} \quad (72)$$

The control law $u = [u_1\ u_2]$ for the perturbed system is calculated by solving the differential equation

$$\epsilon \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-\epsilon & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\epsilon^2 \end{bmatrix} u_2^0 \quad (73)$$

where

$$\begin{aligned} v_1 &= \epsilon^2(u_2 - u_2^0) \\ v_2 &= \epsilon(x_6 - x_6^0 + \dot{u}_1 - \dot{u}_1^0) \end{aligned} \quad (74)$$

and by solving the algebraic equation

$$u_1 = u_1^0 - \frac{\epsilon}{1-\epsilon}(v_1 + x_5^0) \quad (75)$$

The control u generated by solving equations (73), (74), and (75) is bounded as $\epsilon \to 0$. As in the previous cases, the proof follows from Theorem 1.

4.2 Flight Control for VTOL/CTOL Aircraft

We may apply the theory of the previous sections to simple planar aircraft models. The development of section 3.1 using the zero dynamics algorithm was presented in its most general form so as to enable it to be applied to general MIMO systems. Our examples are motivated by our study of flight control for vertical take off and landing (VTOL) and conventional take off and landing (CTOL) aircraft, in [Hauser *et al.*, 1992b] and [Tomlin *et al.*, 1995]. These are two-input two-output systems in which the nominal systems have no zero dynamics.

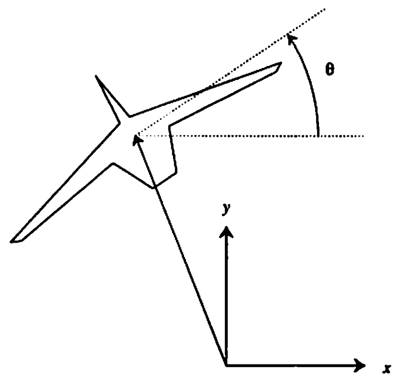

Figure 1: The planar vertical takeoff and landing (PVTOL) aircraft (figure courtesy of J. Hauser)

PVTOL Aircraft

We consider a model of a planar vertical takeoff and landing (PVTOL) aircraft, an example being the YAV-8B Harrier of the McDonnell Douglas Corporation. The simplified PVTOL equations, corresponding to the aircraft in the hover mode, derived in [Hauser *et al.*, 1992b], are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_5 u_1 + \epsilon^2 \cos x_5 u_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \cos x_5 u_1 + \epsilon^2 \sin x_5 u_2 - 1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2 \end{aligned} \tag{76}$$

where $x_1 = x$, $x_3 = y$, $x_5 = \theta$, and x, y, and θ are as illustrated in Figure 1. Note that we have used ϵ^2 in the equations instead of the standard ϵ. We choose the standard outputs $y_1 = x_1$, $y_2 = x_3$.

The vector relative degree is well defined for the perturbed system:

$$\begin{aligned} r_1(\epsilon) &\equiv s = 2 \\ r_2(\epsilon) &\equiv r = 2 \end{aligned}$$

At $\epsilon = 0$, the system (76) does not have vector relative degree. The zero dynamics manifold of the unperturbed system is trivial, and the two time scales assumption is satisfied since the decoupling matrices are

$$\begin{bmatrix} -\sin x_5 & \cos x_5 \\ \cos x_5 & \sin x_5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix}, \begin{bmatrix} -\sin x_5 & 0 \\ \cos x_5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}, \begin{bmatrix} -\sin x_5 & -\cos x_5 u_1 \\ \cos x_5 & -\sin x_5 u_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \epsilon^0 \end{bmatrix}$$

with $\det A_2(x, u_1, \dot{u}_1) = u_1 \neq 0$.

Rather than follow the procedure of section 3.1, we proceed more informally for simplicity. The singularly perturbed zero dynamics are given by

$$\begin{aligned} \epsilon \dot{z}_1 &= z_2 \\ \epsilon \dot{z}_2 &= \sin z_1 \end{aligned} \tag{77}$$

where $z_1 = x_5$ and $z_2 = \epsilon x_6$.

The bounded inputs required for the nominal system to track y_{D1} and y_{D2} are calculated as:

$$\ddot{u}_1^0 = x_6^2 u_1^0 - \sin x_5^0 y_{D1}^{(4)} + \cos x_5^0 y_{D2}^{(4)} \tag{78}$$

$$u_2^0 = -2x_6^0 \dot{u}_1^0 / u_1^0 - y_{D1}^{(4)} \cos x_5^0 / u_1^0 - y_{D2}^{(4)} \sin x_5^0 / u_1^0 \tag{79}$$

The control law $u = [u_1\ u_2]$ for the perturbed system is calculated by solving the differential equations

$$\begin{aligned} \epsilon \dot{v}_1 &= v_2 \\ \epsilon \dot{v}_2 &= u_1^0 \sin v_1 - \epsilon^2 u_2^0 \end{aligned} \tag{80}$$

where

$$\begin{aligned} \sin v_1 &= \epsilon^2 u_2 / u_1^0 \\ v_2 &= \epsilon(x_6 - x_6^0) \end{aligned} \tag{81}$$

and by solving the algebraic equation

$$u_1 \sin(v_1 + x_5^0) = u_1^0 \sin x_5^0 + \epsilon^2 \cos(v_1 + x_5^0) u_2 \tag{82}$$

The control

$$u_2 = u_1^0 \sin v_1 / \epsilon^2 \tag{83}$$

$$u_1 = \frac{u_1^0 \sin x_5^0 + \epsilon^2 \cos(v_1 + x_5^0) u_2}{\sin(v_1 + x_5^0)} \tag{84}$$

is bounded as $\epsilon \to 0$. The proof does not follow directly from Theorem 1, because the form of (80) is not in the standard form of section 3.1. We maintain the appealing simplicity of (80) and use the following proposition, the proof of which is available from the authors.

Proposition 4 *Consider the nonlinear differential equation (85)*

$$\dot{x} = Ax + \psi(x) + \epsilon^2 v \tag{85}$$

in which $A \in \mathbb{R}^{n \times n}$ *is hyperbolic and* ψ *belongs to a class of functions called Lip(r) in [Hale, 1980]:*

$$\sup_{|x| \leq r} |\psi(x)| \leq L(r)|x|$$

and L *is a continuous, nondecreasing, nonnegative function on* $[0, \infty)$ *with* $L(0) = 0$*. Then if* ϵ *is small enough the unique bounded solution of (85) is of* $O(\epsilon^2)$*.*

CTOL Aircraft

The second aircraft model we consider is the planar conventional take off and landing (PCTOL) aircraft introduced in [Tomlin *et al.*, 1995]. The simplified PCTOL equations are

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (-D + u_1)\cos x_5 - (L - \epsilon^2 u_2)\sin x_5 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= (-D + u_1)\sin x_5 + (L - \epsilon^2 u_2)\cos x_5 - 1 \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= u_2
\end{aligned} \tag{86}$$

where $x_1 = x$, $x_3 = y$, $x_5 = \theta$. Unlike the example of the hovering VTOL, we now have aerodynamic forces: L and D, the aerodynamic lift and drag forces given by

$$L = a_L(x_2^2 + x_4^2)(1 + c\alpha) \tag{87}$$

$$D = a_D(x_2^2 + x_4^2)(1 + b(1 + c\alpha)^2) \tag{88}$$

and α is the angle of attack

$$\alpha = x_5 - \tan^{-1}(x_4/x_2) \tag{89}$$

The coordinates are illustrated in Figure 2. The angle of attack α is assumed to be zero for these calculations. The outputs are $y_1 = x_1$, $y_2 = x_3$.

The vector relative degree is well defined for the perturbed system:

$$\begin{aligned}
r_1(\epsilon) &\equiv s = 2 \\
r_2(\epsilon) &\equiv r = 2
\end{aligned}$$

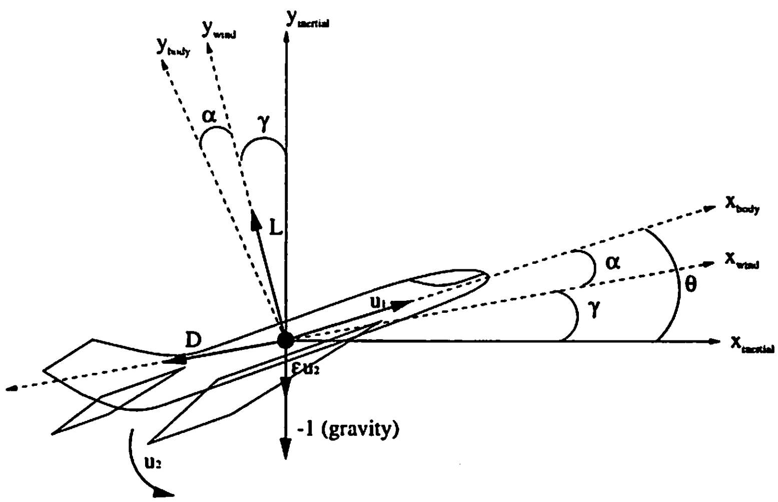

Figure 2: The planar vertical takeoff and landing (PCTOL) aircraft

At $\epsilon = 0$, the system (86) does not have vector relative degree. The zero dynamics manifold of the unperturbed system is trivial, and the two time scales assumption is satisfied.

The singularly perturbed zero dynamics are given by

$$\epsilon \dot{z}_1 = z_2 \quad (90)$$

$$\epsilon \dot{z}_2 = -\cos z_1 \quad (91)$$

where $z_1 = x_5$ and $z_2 = \epsilon x_6$. Note that we have again used a non-standard form for simplicity.

The bounded inputs required for the nominal system to track y_{D1} and y_{D2} are calculated as in the PVTOL case by dynamic extension. The calculations are more involved because of the presence of the lift and drag and are available from the authors. As before, they result in bounded values for $\ddot{u}_1^0$ and u_2^0.

The control law $u = [u_1\ u_2]$ for the perturbed system is calculated by solving the differential equations

$$\epsilon \dot{v}_1 = v_2 \quad (92)$$

$$\epsilon \dot{v}_2 = u_1^0 \sin v_1 - D^0 \sin v_1 - L^0 \cos v_1 + L - \epsilon^2 u_2^0 \quad (93)$$

where

$$\begin{aligned} \sin v_1 &= \tfrac{1}{u_1^0}(\epsilon^2 u_2 - L + D^0 \sin v_1 + L^0 \cos v_1) \\ v_2 &= \epsilon(x_6 - x_6^0) \end{aligned} \quad (94)$$

and L^0, D^0 correspond to the aerodynamic lift and drag at $\epsilon = 0$.

The control

$$u_2 = \frac{1}{\epsilon^2}(u_1^0 \sin v_1 + L - D^0 \sin v_1 - L^0 \cos v_1) \quad (95)$$

$$u_1 = \frac{(u_1^0 - D^0)\cos x_5^0 - L^0 \sin x_5^0 + D\cos(v_1 + x_5^0) + (L - \epsilon^2 u_2)\sin(v_1 + x_5^0)}{\cos(v_1 + x_5^0)} \qquad (96)$$

is bounded as $\epsilon \to 0$. The proof follows from Proposition 4.

5 Conclusions

This work presents a method for tracking systems with singularly perturbed zero dynamics. We combined recent results in exact tracking by Devasia, Paden, and Chen, and Hunt, Meyer, and Su, with a general framework for describing nonlinear nonminimum phase MIMO systems with singularly perturbed zero dynamics. Using this framework, we prove boundedness of the control inputs required for exact tracking. We showed, using planar dynamic models of VTOL and CTOL aircraft, that this method may be successfully applied to the slightly nonminimum phase systems characteristic of flight control.

Appendix 1 Computation of the "Steady State" Solution

Consider the general system

$$\dot{x} = Ax + f(x, u) \qquad (97)$$

in which $x \in \mathbb{R}^n$ represents the driven internal dynamics of the system, $A \in \mathbb{R}^{n \times n}$ represents the (hyperbolic) linearization of these dynamics, and $f(x, u)$ is the residual error in the linearization. We assume, as before, that $f(0, 0) = 0$ and $f(x, u)$ satisfies the Lipschitz condition

$$|f(x, u) - f(y, v)| \leq k_1|x - y| + k_2|u - v| \qquad (98)$$

Devasia-Paden-Chen solve for the "steady state" response of the system (97) on $(-\infty, \infty)$, given $u(t) : -\infty < t < \infty$. We will assume that the state variables x have been partitioned into $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$, such that

$$A = \operatorname{diag}(A_1 \quad A_2)$$

with the eigenvalues of A_1 in $\mathbb{C}_-^0$ and those of A_2 in $\mathbb{C}_+^0$. By way of notation, we will refer to the first n_1 components of f as f_1 and the remaining n_2 components as f_2. Define the Heavyside state transition function of A on $(-\infty, \infty)$ by

$$\Phi(t) = \operatorname{diag}(e^{A_1 t} 1(t) \quad -e^{A_2 t} 1(-t)) \qquad (99)$$

Here $1(t)$ is the unit step function defined to be 1 for $t \geq 0$, and 0 for $t < 0$. By using the variation of constants formula, we have

$$\begin{aligned} x_1(t) = &\; e^{A_1(t-\sigma)}x_1(\sigma) + \textstyle\int_\sigma^t e^{A_1(t-\tau)} f_1(x(\tau), u(\tau))d\tau \\ x_2(t) = &\; e^{A_2(t-\sigma)}x_2(\sigma) + \textstyle\int_\sigma^t e^{A_2(t-\tau)} f_2(x(\tau), u(\tau))d\tau \end{aligned} \tag{100}$$

By setting $\sigma = -\infty$ in the first of these equations and $\sigma = \infty$ in the second, the contributions due to the initial conditions for *bounded solutions* of (97) disappear [Hunt and Meyer, 1995]. It may be verified that a bounded solution to (97) on $(-\infty, \infty)$, must satisfy the integral equation

$$x(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) f(x(\tau), u(\tau)) d\tau \tag{101}$$

Denoting by T the integral operator given by

$$T(x(.))(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) f(x(\tau), u(\tau)) d\tau$$

It may also be verified that for given $u(.) \in L_\infty$, $T : L_\infty \to L_\infty$ [Hunt and Meyer, 1995], and that for given $u(.) \in L_\infty \cap L_1$, $T : L_\infty \cap L_1 \to L_\infty \cap L_1$ [Devasia *et al.*, 1996]. Further if the L_1 norm of Φ is M, then

$$|T(x) - T(y)|_\infty \leq M k_1 |x - y|_\infty$$

The same estimate holds for the L_1 norm of $T(x) - T(y)$ as well. Thus, when $Mk_1 < 1$, the map T is a contraction map, and the solution to (101) exists, is unique, and may be found by the Picard Lindelöf iteration scheme

$$x_{n+1}(.) = T(x_n(.))$$

The fixed point of the map T is the so-called "steady state response" of the system (97). A local version of this result can also be proven when the condition of (98) holds only for x, y in a ball of radius r and u, v in a ball of radius s. This version of the result requires that $|u(.)|_\infty$ be small enough. Also, in the instance that $u(.) \in L_1 \cap L_\infty$, it may be verified that $x(t) \to 0$ as $t \to \pm\infty$. In particular this is the case when $u(.)$ has compact support.

Periodic Inputs

It is of interest to investigate the steady state solution of (97) given by (101) in the special case that $u(.)$ is periodic with period T. In the instance that A is hyperbolic, it follows from a basic perturbation theorem ([Wiggins, 1990], page 111) that for $|u(.)|_\infty$ small enough, there is a unique periodic steady state solution of (97) which inherits the stability properties of A. In this section, we show how the solution of (101) can be obtained using some techniques reminiscent of describing function techniques. These have also been suggested in somewhat different form by Meyer, Hunt and Su [Meyer *et al.*, 1995b] as "Fourier techniques".

Sinusoidal Inputs Consider first the case that $u(t) = u_1 e^{j\omega t}$. Since the steady state solution of (101) is periodic, it may be represented by its Fourier series

$$x_{ss}(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega t} \tag{102}$$

with $x_k \in \mathbb{C}$. Consider also the approximation of the steady state solution with the Fourier series truncated to $2N+1$ terms

$$x_{ap-ss} = \sum_{k=-N}^{N} x_k e^{jk\omega t} \tag{103}$$

Using (102) in the expression for $f(x,u)$, we get that

$$f(x_{ss}(t), u(t)) = \sum_{k=-\infty}^{\infty} f_k(\ldots, x_{-1}, x_0, x_1, \ldots, u_1) e^{jk\omega t} \tag{104}$$

Using this in (101) and integrating explicitly yields the following (infinite) set of equations for the x_k:

$$x_k = \mathrm{diag}(\quad (jk\omega I - A_1)^{-1} \quad (jk\omega I - A_2)^{-1}) f_k \tag{105}$$

Note that the f_k are functions of the $x_k, -\infty < k < \infty$. This set of equations may be solved approximately by truncating the steady state solution as in (103) to give $2N+1$ equations of the form (105) in $2N+1$ unknowns $x_k, -N \leq k \leq N$. The neglect of the higher order terms can be justified using the same degree theoretic arguments as are used by Bergen and Franks [Bergen and Franks, 1971] to justify the describing function method. Since $(jk\omega I - A_1)^{-1}, (jk\omega I - A_2)^{-1}$ are "low pass", the conditions are roughly that N is chosen large enough for a given $f(x,u)$ to make the neglected terms x_k small enough.

Arbitrary Periodic Inputs When $u(.)$ is merely periodic rather than sinusoidal, the development of the equation (105) goes through verbatim with the extra condition that formula (104) is derived using the explicit form of $u(.)$. One truncates the Fourier series of the response as before to obtain an approximate solution. There is no need to truncate the Fourier series of the input (or, for that matter, to derive it!).

Stationary Inputs

The foregoing discussion may be generalized to stationary inputs, in the sense that the input u has a well defined autocovariance ([Sastry and Bodson, 1989]) given by the following limit (independent of t):

$$R_u(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} u(\sigma) u^T(\sigma + \tau) d\sigma \tag{106}$$

The autocovariance function of u is a positive semidefinite matrix belonging to $\mathbb{R}^{n_i \times n_i}$ with Fourier transform $S_u(\omega)$, the power spectral density. Using the techniques of [Sastry and Bodson, 1989] it may be proved that when $u(.)$ is stationary, then so is $x(.)$. The equation for the power spectral density of x (the analog of (105)) is

$$S_x(\omega) = \text{diag}[(-jk\omega I - A_1)^{-1}(-jk\omega I - A_2)^{-1}]S_f(\omega)[(-jk\omega I - A_1)^{-1}(-jk\omega I - A_2)^{-1}] \quad (107)$$

The formula above is not explicit since $S_f(\omega)$ is also a function of $S_x(\omega)$ (and $S_u(\omega)$). Some functional approximations of $S_x(\omega)$ (such as the truncation of the previous subsection are then needed to solve equation (107)). We omit these details for brevity.

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