TOWARDS CONTINUOUS ABSTRACTIONS OF DYNAMICAL AND CONTROL SYSTEMS

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Towards Continuous Abstractions of Dynamical and Control Systems*

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Abstract

The analysis and design of large scale systems is usually an extremely complex process. In order to reduce the complexity of the analysis, simplified models, called *abstractions*, which capture the system behavior of interest are obtained and analyzed. If the abstraction of system can be shown to satisfy certain properties of interest then so does the original complex plant. This can result in significant complexity reduction in the analysis of complex systems.

1 Introduction

Large scale systems such as intelligent highway systems [1] and air traffic control systems [2] result in systems of very high complexity. The analysis process for complex systems consists of proving that the designed system meets certain specifications. However, the analysis may be formidable due to the complexity and magnitude of the system.

For example, in air traffic control systems, aircraft are usually modeled by detailed differential equations which describe the behavior of engine dynamics, aerodynamics etc. The desired specification requires that any two aircraft do not collide with each other. Proving that the system indeed meets the specification may be prohibitingly complex due to the detailed modeled dynamics as well as the large scale of the system.

However, in the above example, it is clear that the specification is not interested in the details of aircraft operation, but only in the relative position of the aircraft. We can therefore reduce the complexity of the analysis by focusing only on dynamics which are of interest. This is performed by ignoring certain aspects of system behavior in a manner which is consistent with the behavior of the original system. This is essentially the idea behind system abstraction.

Webster's dictionary defines the word abstraction as *"the act or process of separating the inherent qualities or properties of something from the actual physical object or concept to which they belong"*. In system theory, the objects are usually dynamical or control systems, the properties are usually the behaviors of certain variables of interest and the act of separation is essentially the act of capturing all interesting behaviors. In summary, Webster's definition can be applied to define the *abstraction of a system to be another system which captures all system behavior of interest.*

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Once a system abstraction has been obtained, standard analysis methods are utilized on the abstracted models. For example, verification algorithms of hybrid systems [3, 4], which contain both discrete event dynamics at higher levels and continuous dynamics at lower levels, are based on abstracting continuous dynamics by rectangular differential inclusions [5, 6]. A similar methodology for reducing the verification complexity of discrete event systems can be found in [7].

In this paper, notions of system abstractions for dynamical and control systems are defined. Behaviors of interest are captured by *abstracting maps*. Abstracting maps are provided by the user depending on what information is of interest. Necessary and sufficient geometric conditions under which one system is an abstraction of another with respect to a given abstracting map are derived. Although abstractions of systems may capture all behaviors of interest, they might also allow evolutions which are not feasible by the original system. This is due to the information reduction which naturally occurs during the abstraction process and it is the price one has to pay in order to reduce complexity. System abstractions can therefore be ordered based on the "size" of redundant allowable system evolutions leading to a notion of *best abstraction*. Furthermore, we show that certain properties of interest, such as controllability, propagate from the original system to the abstracted system.

The structure of this paper is as follows: In Section 2 we review standard differential geometry which will be used throughout the paper. In Section 3 abstracting maps are introduced in order to define system behaviors of interest. A notion of an abstraction of a dynamical system is defined in Section 4 and we discuss when one vector field is an abstraction of another. Section 5 generalizes these notions for control systems while Section 6 discusses issues of further research.

2 Review of Differential Geometry

We first review some basic facts from differential geometry. The reader may wish to consult numerous books on the subject such as [8, 9, 10].

2.1 Differentiable Manifolds

Definition 1 (Manifolds) *A* manifold M *of dimension* n *is a metric space* [1] *which is locally homeomorphic to* R^n.

A simple example of a manifold, which is of great interest to us, is R^n itself. Other examples are the circle S^1, the sphere S^2 and the Euclidean groups $SO(3)$ and $SE(3)$.

A subset N of a manifold M which is itself a manifold is called a submanifold of M. Any open subset N of a manifold M is clearly a submanifold, since if M is locally homeomorphic to R^n then so is N. In particular, an open interval $I \subseteq R$ is also a manifold.

A coordinate chart on a manifold M is a pair (U, x) where U is an open set of M and x is a homeomorphism of U on an open set of R^n. The function x is also called a coordinate function and can also written as $(x_1, \dots, x_n)$ where $x_i : M \longrightarrow R$. If $p \in U$ then $x(p) = (x_1(p), \dots, x_n(p))$ is called the set of local coordinates in the chart (U, x).

When doing operations on a manifold, we must ensure that our results are consistent regardless of the particular chart we use. We must therefore impose some conditions. Two charts (U, x) and (V, y)

[1]Or better replace metric space with Hausdorff and second countable topological space

with $U \cap V \neq \emptyset$, are called C^∞ compatible if the map

$$y \circ x^{-1} : x(U \cap V) \subset R^n \longrightarrow y(U \cap V) \subset R^n$$

is a C^∞ function. A C^∞ atlas on a manifold M is a collection of charts (U_α, x_α) with $\alpha \in A$ which are C^∞ compatible and such that the open sets U_α cover the manifold M, so $M = \bigcup_{\alpha \in A} U_\alpha$. An atlas is called maximal if it is not contained in any other atlas.

Definition 2 (Differentiable Manifolds) *A* differentiable *or* smooth manifold *is a manifold with a maximal, C^∞ atlas.*

Now that we have imposed this differential structure on our manifold M we can perform calculus on M. In particular let $f : M \longrightarrow R$ be a map. If (U, x) is a chart on M then the function

$$\hat{f} = f \circ x^{-1} : x(U) \subset R^n \longrightarrow R$$

is called the local representative of f in the chart (U, x). We therefore define the map f to be C^∞ or smooth if its local representative $\hat{f}$ is C^∞. Notice if f is C^∞ in one chart, then it is C^∞ in every chart since we required our charts to be C^∞ compatible and our atlas to be maximal. Hence our results are intrinsic to the manifold and do not depend on the particular chart we use. Similarly, if we have a map $f : M \longrightarrow N$, where M,N are differentiable manifolds, the local representation of f given a chart (U, x) of M and (V, y) of N is

$$\hat{f} = y \circ f \circ x^{-1}$$

which makes sense only if $f(U) \cap V \neq \emptyset$. Again f is a C^∞ map if $\hat{f}$ is a C^∞ map.

Let $f : M \longrightarrow N$ be a map between two manifolds. The map f is called a diffeomorphism if both f and f^{-1} are smooth. In this case, manifolds M and N are called diffeomorphic.

2.2 Tangent Spaces

Let p be a point on a manifold M. Let $C^\infty(p)$ denote the vector space of all smooth functions in a neighborhood of p. A tangent vector X_p at $p \in M$ is an operator from $C^\infty(p)$ to R which satisfies for $f, g \in C^\infty(p)$ and $a, b \in R$, the following properties,

1. Linearity $X_p(a \cdot f + b \cdot g) = a \cdot X_p(f) + b \cdot X_p(g)$
2. Derivation $X_p(f \cdot g) = f(p) \cdot X_p(g) + X_p(f) \cdot g(p)$

The set of all tangent vectors at $p \in M$ is called the tangent space of M at p and is denoted by T_pM. The tangent space T_pM becomes a vector space over R if for tangent vectors X_p, Y_p and real numbers c_1, c_2 we define

$$(c_1 \cdot X_p + c_2 \cdot Y_p)(f) = c_1 \cdot X_p(f) + c_2 \cdot Y_p(f)$$

for any smooth function f in the neighborhood of p. The collection of all tangent spaces of the manifold,

$$TM = \bigcup_{p \in M} T_pM$$

is called the tangent bundle. The tangent bundle has a naturally associated projection map $\pi : TM \longrightarrow M$ taking a tangent vector $X_p \in T_pM \subset TM$ to the point $p \in M$. The tangent space T_pM can then be thought of as $\pi^{-1}(p)$.

The tangent space can be thought of as a special case of a more general mathematical object called a fiber bundle. Loosely speaking, a fiber bundle can be thought of as gluing sets at each point of the manifold in a smooth way.

Definition 3 (Fiber Bundles [11]) *A* fiber bundle *is a five-tuple* $(B, M, \pi, U, \{O_i\}_{i\in I})$ *where* B, M, U *are smooth manifolds called the* total space, *the* base space *and the* standard fiber *respectively. The map* $\pi : B \longrightarrow M$ *is a surjective submersion and* $\{O_i\}_{i\in I}$ *is an open cover of* M *such that for every* $i \in I$ *there exists a diffeomorphism* $\Phi_i : \pi^{-1}(O_i) \longrightarrow O_i \times U$ *satisfying*

$$\pi_o \circ \Phi = \pi$$

where π_o *is the projection from* $O_i \times U$ *to* O_i. *The submanifold* $\pi^{-1}(p)$ *is called the* fiber *at* $p \in M$. *If all the fibers are vector spaces of constant dimension, then the fiber bundle is called a* vector bundle.

The tangent bundle is a vector bundle and the fiber at each point $p \in M$ is the tangent space T_pM. From Definition 3 it is clear that fiber bundles are locally diffeomorphic to $O_i \times U$. Therefore, fiber bundles are manifolds of dimension $n_M + n_U$ where n_M and n_U are the dimensions of M and U respectively. In particular, the tangent bundle TM has dimension $2n$.

Now let M be a manifold and let (U, x) be a chart containing the point p. In this chart we can associate the following tangent vectors

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

defined by

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial(f \circ x^{-1})}{\partial r_i}$$

for any smooth function $f \in C^\infty(p)$. The tangent space T_pM is an n-dimensional vector space and if (U, x) is a local chart around p then the tangent vectors

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

form a basis for T_pM. Therefore if X_p is a tangent vector at p then

$$X_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

where $a_1, \ldots, a_n$ are real numbers. From the above formula we can see that a tangent vector is an operator which simply takes the directional derivative of function in the direction of $[a_1, \ldots, a_n]$.

Now let M and N be smooth manifolds and $f : M \longrightarrow N$ be a smooth map. Let $p \in M$ and let $q = f(p) \in N$. We wish to push forward tangent vectors from T_pM to T_qN using the map f. The natural way to do this is by defining a map $f_* : T_pM \longrightarrow T_qN$ by

$$(f_*(X_p))(g) = X_p(g \circ f)$$

for smooth functions g in the neighborhood of q. One can easily check that $f_*(X_p)$ is a linear operator and a derivation and thus a tangent vector. The map $f_* : T_pM \longrightarrow T_{f(p)}N$ is called the push forward map of f. The push forward map $f_* : T_pM \longrightarrow T_{f(p)}N$ is a linear map. and furthermore if $f : M \longrightarrow N$ and $g : N \longrightarrow K$ then

$$(g \circ f)_* = g_* \circ f_*$$

which is essentially the chain rule.

2.3 Vector Fields

A vector field or dynamical system on a manifold M is a continuous map F which places at each point p of M a tangent vector from T_pM. Such functions are called sections of the tangent bundle TM since they satisfy $\pi \circ F = i$ where π is the projection from TM onto M and i is the identity on M. If F is of class C^∞ it is called a smooth section of TM.

Therefore since a vector field, F, places at each point p a tangent vector $F(p)$ we have that in the chart (U, x) the local expression for the vector field F is

$$F(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$$

We can easily see from the above that the vector field is C^∞ if and only if the scalar functions $a_i : M \longrightarrow R$ are C^∞.

Let $I \subseteq R$ be an open interval containing the origin. An integral curve of a vector field is a curve $c : I \longrightarrow M$ whose tangent at each point is identically equal to the vector field at that point. Therefore an integral curve satisfies for all $t \in I$,

$$c' = c_*(1) = X(c)$$

Finally, let $f : M \longrightarrow N$ be a smooth map between two manifolds and let X be a vector field on M. At every point $p \in M$ we can push forward $X(p)$ of the vector field to $T_{f(p)}N$. Then $f_*(X)$ is a well defined vector field only if f is a diffeomorphism or when f and X are such that $f_*(X_{p_1}) = f_*(X_{p_2})$ whenever $f(p_1) = f(p_2)$. In that case $Y = f_*(X)$ and X are called f-related. Equivalently, we have the following definition.

Definition 4 (f-related Vector Fields) *Let X and Y be vector fields on manifolds M and N respectively and $f : M \longrightarrow N$ be a smooth map. Then X and Y are* f-related *iff the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle Y} \\ TM & \xrightarrow{f_*} & TN \end{array} \qquad (1)$$

or otherwise iff $f_* \circ X = Y \circ f$.

3 Abstracting Maps

Let M be the state space of a system. In abstracting system dynamics, information about the state of the system which is not useful in the analysis process is usually ignored in order to produce a simplified model of reduced complexity. The state $p \in M$ is thus mapped to an abstracted state $q \in N$. It is clear that *complexity reduction requires that the dimension of N should be strictly lower than the dimension of M.*

For example, each state could be mapped to part of the state or to certain outputs of interest. What state information is relevant usually depends on the properties which need to be satisfied. The desired specification, however, could be quite different even for the same system since the functionality of the system may be different in various modes of system operation. It is therefore clear that it is very difficult to intrinsically obtain a system abstraction without any knowledge of the particular system functionality. System functionality determines what state information is of interest for analysis purposes. Given the functionality of the system, a notion of functionally equivalent states is obtained by defining an equivalence relation on the state space.

For example, given a dynamic model of some mechanical system one may be interested only in the configuration of the system. In this case, two states are functionally equivalent if the corresponding configurations are the same.

Once a specific equivalence has been chosen, then the quotient space $M/\sim$ is the state space of the abstracted system. In order for the quotient space to have a manifold structure, the equivalence relation must be regular [9]. The surjective map $\alpha : M \longrightarrow M/\sim$ which sends each state $p \in M$ to its equivalence class $[p] \in M/\sim$ is called the identification map and is the map which sends each state to its abstracted state. In general, we have the following definition.

Definition 5 (Abstracting Maps) *Let M and N be given manifolds with $dim(N) < dim(M)$. A surjective map $\alpha : M \longrightarrow N$ from the state space M to the abstracted state space N is called an* abstracting map.

The identification map is an example of an abstracting map. In this paper, we will assume that M and N are manifolds and the abstracting maps to be smooth submersions.

4 Abstractions of Dynamical Systems

Once an abstracting map α has been given, then given a vector field X which governs the state evolution on M, then one is interested in obtaining the evolution of the abstracted dynamics. The evolution of a dynamical system is characterized by its integral curves. Let c be any integral curve of X. Then if we push forward the curve c by the abstracting map α we obtain that $\alpha \circ c$ describes the evolution of the abstracted dynamics on N. If we therefore want to abstract the vector field X on M by a vector field Y on N, then $\alpha \circ c$ should be an integral curve of Y. This motivates the following definition.

Definition 6 (Abstractions of Dynamical Systems) *Let X and Y be vector fields on M and N respectively and let $\alpha : M \longrightarrow N$ be a smooth abstracting map. Then vector field Y is* an abstraction of vector field X with respect to α *iff for every integral curve c of X, $\alpha \circ c$ is an integral curve of Y.*

Therefore if the curve c satisfies

$$c' = c_*(1) = X(c)$$

then it must also be true that

$$(\alpha \circ c)' = (\alpha \circ c)_*(1) = Y(\alpha \circ c)$$

From Definition 6 it is clear that a vector field Y may be an abstraction of some vector field X for some abstracting map α_1, but may not be for another abstracting map α_2.

The following theorem shows that the Definition 6 is equivalent to saying that the two vector fields are α-related.

Theorem 1 *Vector field Y on N is an abstraction of vector field X on M with respect to the map α if and only if X and Y are α-related.*

Proof: Let vector field Y on N be an abstraction with respect to α of vector field X on M. Then by Definition 6, for any integral curve c of X, $\alpha \circ c$ is an integral curve of Y. Thus

$$\begin{aligned}
&(\alpha \circ c)' = (\alpha \circ c)_*(1) = Y(\alpha \circ c) \Rightarrow \\
&\alpha_* \circ c_*(1) = Y \circ \alpha \circ c \Rightarrow \\
&\alpha_* \circ X(c) = Y \circ \alpha \circ c \Rightarrow \\
&\alpha_* \circ X \circ c = Y \circ \alpha \circ c \Rightarrow \\
&\alpha_* \circ X = Y \circ \alpha
\end{aligned}$$

But then, by Definition 4, X and Y are α-related. Conversely, let X and Y be α related. Then for any integral curve c of X,

$$\begin{aligned}
&\alpha_* \circ X = Y \circ \alpha \Rightarrow \\
&\alpha_* \circ X \circ c = Y \circ \alpha \circ c \Rightarrow \\
&\alpha_* \circ X(c) = Y(\alpha \circ c) \Rightarrow \\
&\alpha_* \circ c_*(1) = Y(\alpha \circ c) \Rightarrow \\
&(\alpha \circ c)_*(1) = Y(\alpha \circ c)
\end{aligned}$$

and thus $\alpha \circ c$ is an integral curve of Y. Therefore Y is an abstraction of vector field X with respect to α. □

Theorem 1 is important because it allows to check a condition on the vector fields rather than explicitly computing integral curves and verifying Definition 6. However, α-relatedness of two vector fields is a very restrictive condition which limits the cases where one dynamical system is an abstraction of another. Fortunately, this is not true for control systems.

5 Abstractions of Control Systems

The notions of Section 4 for dynamical systems will be extended to control systems. We first need to introduce some facts about control systems.

Definition 7 (Control Systems) *A* control system $S = (B, F)$ *consists of a fiber bundle $\pi : B \longrightarrow M$ called the control bundle and a smooth map $F : B \longrightarrow TM$ which is fiber preserving and hence satisfies*

$$\pi' \circ F = \pi$$

where $\pi' : TM \longrightarrow M$ is the tangent bundle projection.

Essentially, the base manifold M of the control bundle is the state space and the fibers $\pi^{-1}(p)$ are the state dependent control spaces. In a local coordinate chart (V, x), the map F can be expressed as $\dot{x} = F(x, u)$ with $u \in U(x) = \pi^{-1}(x)$.

Definition 8 (Integral Curves of Control Systems) *A curve $c : I \longrightarrow M$ is called* an integral curve of the control system $S = (B, F)$ *if there exists a curve $c^B : I \longrightarrow B$ satisfying*

$$\pi \circ c^B = c$$
$$c' = c_*(1) = F(c^B)$$

Again in local coordinates, the above definition simply says that $x(t)$ is a solution to a control system if there exists an input $u \in U(x) = \pi^{-1}(x)$ satisfying $\dot{x} = F(x, u)$. We now define abstractions of control systems in a manner similar to dynamical systems.

Definition 9 (Abstractions of Control Systems) *Let $S_M = (B_M, F_M)$ with $\pi_M : B_M \longrightarrow M$ and $S_N = (B_N, F_N)$ with $\pi_N : B_N \longrightarrow N$ be two control systems. Let $\alpha : M \longrightarrow N$ be an abstracting map. Then control system S_N is* an abstraction of S_M with respect to abstracting map α *iff for every integral curve c_M of S_M, $\alpha \circ c_M$ is an integral curve of S_N.*

From Definition 9 it is clear that a control system S_N may be an abstraction of S_M for some abstracting map α_1 but may not be for another abstracting map α_2. Since the definition of an abstraction is at the level of integral curves, it is clearly difficult to conclude that one control system is an abstraction of another system by directly using Definition 9 since this would require integration of the system. One is therefore interested in easily checkable conditions under which one system is an abstraction of another. The following theorem, provides necessary and sufficient geometric conditions under which one control system is an abstraction of another system with respect to some abstracting map.

Theorem 2 (Necessary and Sufficient Conditions for Control System Abstractions) *Let $S_N = (B_N, F_N)$ and $S_M = (B_M, F_M)$ be two control systems and $\alpha : M \longrightarrow N$ be an abstracting map. Then S_N is an abstraction of S_M with respect to abstracting map α if and only if*

$$\alpha_* \circ F_M \circ \pi_M^{-1}(p) \subseteq F_N \circ \pi_N^{-1} \circ \alpha(p) \tag{2}$$

at every $p \in M$.

Proof: Before we proceed with the proof, we remark that condition (2) can be visualized using the following diagram,

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \pi_M^{-1}\downarrow & & \downarrow\pi_N^{-1} \\ B_M & & B_N \\ F_M\downarrow & & \downarrow F_N \\ TM & \xrightarrow{\alpha_*} & TN \end{array} \tag{3}$$

Then condition (2) states that in the above diagram the set of tangent vectors produced in the direction $(M \overset{\pi_M^{-1}}{\rightarrow} B_M \overset{F_M}{\rightarrow} TM \overset{\alpha_*}{\rightarrow} TN)$ is a subset of the tangent vectors produced in the direction $(M \overset{\alpha}{\rightarrow} N \overset{\pi_N^{-1}}{\rightarrow} B_N \overset{F_N}{\rightarrow} TN)$.

We begin the proof, by first showing that if $\alpha_* \circ F_M \circ \pi_M^{-1} \subseteq F_N \circ \pi_N^{-1} \circ \alpha$ at every point $p \in M$ then F_N is an abstraction of F_M. We will prove the contrapositive. Assume that F_N is not an abstraction of F_M. Then there exists an integral curve c_M of F_M such that $\alpha \circ c_M$ is not an integral curve of F_N. Therefore for all curves $c_N^B : I \longrightarrow B_N$ such that $\pi_N \circ c_N^B = \alpha \circ c_M$ we have that at some point $t^* \in I$

$$(\alpha \circ c_M)'(t^*) \neq F_N(c_N^B(t^*))$$

But since this is true for all curves c_N^B satisfying $\pi_N \circ c_N^B(t^*) = \alpha \circ c_M(t^*)$ and since π_N is a surjection we have

$$\begin{aligned}
&(\alpha \circ c_M)'(t^*) \notin F_N(\pi_N^{-1}(\alpha \circ c_M(t^*))) \Rightarrow \\
&(\alpha \circ c_M)'(t^*) \notin F_N \circ \pi_N^{-1} \circ \alpha \circ c_M(t^*) \Rightarrow \\
&\alpha_* \circ c_{M*}(t^*)(1) \notin F_N \circ \pi_N^{-1} \circ \alpha \circ c_M(t^*) \Rightarrow \\
&\alpha_* \circ F_M \circ c_M^B(t^*) \notin F_N \circ \pi_N^{-1} \circ \alpha \circ c_M(t^*)
\end{aligned} \tag{4}$$

for some curve $c_M^B : I \longrightarrow B_M$ such that $\pi_M \circ c_M^B = c_M$. But then $c_M^B(t^*) \in \pi_M^{-1}(c_M(t^*)) = \pi_M^{-1} \circ c_M(t^*)$. Therefore, there exists a tangent vector $Y_{\alpha(c_M(t^*))} \in T_{\alpha(c_M(t^*))}N$, namely

$$Y_{\alpha(c_M(t^*))} = \alpha_* \circ F_M \circ c_M^B(t^*)$$

such that

$$Y_{\alpha(c_M(t^*))} \in \alpha_* \circ F_M \circ \pi_M^{-1} \circ c_M(t^*)$$

since $c_M^B(t^*) \in \pi_M^{-1} \circ c_M(t^*)$ but

$$Y_{\alpha(c_M(t^*))} \notin F_N \circ \pi_N^{-1} \circ \alpha \circ c_M(t^*)$$

by condition (4). But then we have that at $c_M(t^*) \in M$,

$$\alpha_* \circ F_M \circ \pi_M^{-1}(c_M(t^*)) \not\subseteq F_N \circ \pi_N^{-1} \circ \alpha(c_M(t^*)) \tag{5}$$

Conversely, we now prove that if F_N is an abstraction of F_M then $\alpha_* \circ F_M \circ \pi_M^{-1} \subseteq F_N \circ \pi_N^{-1} \circ \alpha$. We will use contradiction. Assume that F_N is an abstraction of F_M but at some point $p \in M$ we have $\alpha_* \circ F_M \circ \pi_M^{-1}(p) \not\subseteq F_N \circ \pi_N^{-1} \circ \alpha(p)$. Then there exists tangent vector $Y_{\alpha(p)} \in T_{\alpha(p)}N$ such that

$$\begin{aligned}
Y_{\alpha(p)} &\in \alpha_* \circ F_M \circ \pi_M^{-1}(p) && (6)\\
Y_{\alpha(p)} &\notin F_N \circ \pi_N^{-1} \circ \alpha(p) && (7)
\end{aligned}$$

Since $Y_{\alpha(p)} \in \alpha_* \circ F_M \circ \pi_M^{-1}(p)$, we can write $Y_{\alpha(p)} = \alpha_*(X_p)$ for some (not necessarily unique) tangent vector $X_p \in F_M \circ \pi_M^{-1}(p)$. But since $X_p \in F_M \circ \pi_M^{-1}(p)$ then there exists an integral curve $c_M : I \longrightarrow M$ such that at some $t^* \in I$ we have

$$\begin{aligned}
c_M(t^*) &= p && (8)\\
c_M'(t^*) &= X_p && (9)
\end{aligned}$$

To see that such a curve exists assume that such an integral curve does not exist. But then for all curves c_M satisfying (8,9) and for all curves $c_M^B : I \longrightarrow B_M$ such that $\pi_M \circ c_M^B = c_M$ we have that

$$c_M'(t^*) \neq F_M(c_M^B) \Rightarrow X_p \neq F_M(c_M^B) \tag{10}$$

But since this is true for all such c_M^B we obtain

$$X_p \notin F_M(\pi_M^{-1}(c_M(t^*)))$$

which is clearly a contradiction. Therefore, an integral curve satisfying (8,9) always exists.

We know that F_N is an abstraction of F_M. Therefore by definition, for every integral curve c_M of F_M, $\alpha \circ c_M$ must be an integral curve of F_N. Let c_M be the integral curve satisfying (8,9). Then it must be true that

$$(\alpha \circ c_M)' = F_N(c_N^B)$$

for some $c_N^B : I \longrightarrow B_N$ such that $\pi_N \circ c_N^B = \alpha \circ c_M$. But at $t^* \in I$ we have that

$$(\alpha \circ c_M)'(t^*) = \alpha_* \circ c_{M*}(t^*)(1) = \alpha_*(X_p) = Y_{\alpha(p)}$$

But by condition (7), $Y_{\alpha(p)} \notin F_N \circ \pi_N^{-1} \circ \alpha(p)$ and therefore for all curves c_N^B satisfying $\pi_N \circ c_N^B = \alpha \circ c_M$ we get

$$(\alpha \circ c_M)'(t^*) = Y_{\alpha(p)} \notin F_N(c_N^B(t^*))$$

But then $\alpha \circ c_M$ is not an integral curve of F_N which is a contradiction since we assumed that F_N is an abstraction of F_M with respect to the abstracting map α. Therefore, at all points $p \in M$ we must have $\alpha_* \circ F_M \circ \pi_M^{-1} \subseteq F_N \circ \pi_N^{-1} \circ \alpha$. This completes the proof. □

Theorem 2 is the analogue of Theorem 1 for control systems. However, unlike Theorem 1 which required the α-relatedness of two vector fields, Theorem 2 does not require the commutativity of diagram 3. This is actually quite fortunate since, as the following corollaries of Theorem 2 show, *every* control and dynamical system is abstractable by another control system.

Corollary 1 (Abstractable Control Systems) *Every control system $S_M = (B_M, F_M)$ is abstractable by a control system S_N with respect to any abstracting map $\alpha : M \longrightarrow N$.*

Proof: Simply let $B_N = TN$ and $F_N : TN \longrightarrow TN$ equal the identity. Then condition (2) is trivially satisfied. Thus $S_N = (B_N, F_N)$ is an abstraction of S_M. □

As a subcollorary of Corollary 1 we have.

Corollary 2 (Abstractable Dynamical Systems) *Every dynamical system on M is abstractable by a control system with respect to any abstracting map $\alpha : M \longrightarrow N$.*

Proof: Every vector field X can be thought of a trivial control system $S_M = (B_M, F_M)$ where $B_M = M \times \{0\}$ and F_M is equal to $X \circ \pi$. Then Corollary 1 applies. □

In local coordinates, Corollaries 1 and 2 simply state the fact that the behavior of any system can be abstracted by a differential inclusion $\dot{x} \in R^n$ where x are the local coordinates of interest and n is the dimension of manifold N. However, such an abstraction may not be useful in proving properties. Therefore, it is clear that there is a notion of order among abstractions of a given system.

If one considers fiber subbundles Δ of the tangent bundle TN where at each $q \in N$,

$$\Delta(q) = F_N \circ \pi_N^{-1}(q) \subseteq T_q N \tag{11}$$

for a control system $S_N = (B_N, F_N)$ then Theorem 2 essentially allows us to think of abstractions of a given system $S_M = (B_M, F_M)$ as subbundles $\Delta \subseteq TN$ that satisfy at each point $p \in M$,

$$\alpha_* \circ F_M \circ \pi_M^{-1}(p) \subseteq \Delta(\alpha(p)) \tag{12}$$

and therefore capture all possible tangent directions in which the abstracted dynamics may evolve. Note that Δ is not needed to be a distribution or to have any vector space structure.

It is clear from (11,12) that if Δ is an abstraction of a control system S_M then so is any superset of Δ, say $\bar{\Delta}$ and thus $\bar{\Delta}$ is also an abstraction. But if $\Delta \subset \bar{\Delta}$ then a straightforward application of Theorem 2 shows that $\bar{\Delta}$ is an abstraction of Δ with respect to the identity map $i : N \longrightarrow N$. Therefore, any integral curve of Δ is also an integral curve of $\bar{\Delta}$ but not vice versa. But since Δ has captured all evolutions of S_M which are of interest, $\bar{\Delta}$ can only contain more redundant evolutions which are not feasible by S_M. It is therefore clear that Δ is a more desirable abstraction than $\bar{\Delta}$. This raises a notion of order among abstractions.

Let $S_M = (B_M, F_M)$ be a control system and an abstracting map $\alpha : M \longrightarrow N$ be given. Let control systems $S_{N_1} = (B_{N_1}, F_{N_1})$ and $S_{N_2} = (B_{N_2}, F_{N_2})$ be abstractions of S_M with respect to α. Define at each $q \in N$,

$$\Delta_1(q) = F_{N_1} \circ \pi_{N_1}^{-1}(q) \subseteq T_qN$$
$$\Delta_2(q) = F_{N_2} \circ \pi_{N_2}^{-1}(q) \subseteq T_qN$$

Then we say that S_{N_1} is a better abstraction than S_{N_2}, denoted $S_{N_1} \preceq S_{N_2}$ iff at each point $p \in M$ we have

$$\Delta_1(\alpha(p)) \subseteq \Delta_2(\alpha(p)) \tag{13}$$

It is clear that $\preceq$ is a partial order among abstractions since the order is essentially set inclusion at each fiber. The following Theorem shows that the resulting lattice has a diamond-like structure since there is a unique minimal and maximal element.

Theorem 3 (Structure of Abstractions) *The partial order $\preceq$ has a unique maximal and minimal element.*

Proof: It is easy to see that the unique maximal abstraction is given by $\bar{S} = (TN, i)$ where i is the identity map from TN to TN.

From condition (12) it is clear that it is clear that if

$$\Delta_1(q) = F_{N_1} \circ \pi_{N_1}^{-1}(q) \subseteq T_qN$$
$$\Delta_2(q) = F_{N_2} \circ \pi_{N_2}^{-1}(q) \subseteq T_qN$$

are abstractions of a control system S_M with respect to α then so is the control system $\Delta = \Delta_1 \cap \Delta_2$ where the intersection of the two bundles is defined at each fiber. It is therefore clear that the unique minimal element of $\preceq$ is given by the intersection of all abstractions of S_M. But the intersection of all abstractions can be seen from condition (12) to be the subbundle that satisfies

$$\underline{\Delta}(\alpha(p)) = \alpha_* \circ F_M \circ \pi_M^{-1}(p) \tag{14}$$

for every $p \in M$. □

Therefore the best abstraction results in diagram (3) being commutative. Since the notion of f-related vector fields was defined as vector fields satisfying commutative diagram (1), we extend this notion for control systems by requiring that the control systems S_M and S_N make diagram (3) commutative.

Definition 10 (f-related Control Systems) *Let $S_M = (B_M, F_M)$ with $\pi_M : B_M \longrightarrow M$ and $S_N = (B_N, F_N)$ with $\pi_N : B_N \longrightarrow N$ be two control systems and $f : M \longrightarrow N$ be a smooth map. Then control systems S_N and S_M are called f-*related *iff the following diagram*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \pi_M^{-1}\big\downarrow & & \big\downarrow\pi_N^{-1} \\ B_M & & B_N \\ F_M\big\downarrow & & \big\downarrow F_N \\ TM & \xrightarrow{\alpha_*} & TN \end{array} \tag{15}$$

is commutative or equivalently iff at every point $p \in M$ we have $f_* \circ F_M \circ \pi_M^{-1}(p) = F_N \circ \pi_N^{-1} \circ f(p)$.

Therefore, Definition 10 and Theorem 3 simply state that the best abstraction of a control system S_N is α-related to the original control system S_M.

Once a system abstraction has been obtained, it is useful to propagate properties of interest from the original system to the abstracted system. For control systems, one of those properties is controllability.

Definition 11 (Controllability) *Let $S = (B, F)$ be a control system. Then S is called controllable iff given any two points $p_1, p_2 \in M$, there exists an integral curve c such that for some $t_1, t_2 \in I$ we have $c(t_1) = p_1$ and $c(t_2) = p_2$.*

Theorem 4 (Controllable Abstractions) *Let control system $S_N = (B_N, F_N)$ be an abstraction of $S_M = (B_M, F_M)$ with respect to some abstracting map α. If S_M is controllable then S_N is controllable.*

Proof: Let q_1 and q_2 be any two points on N. Then let $p_1 \in \alpha^{-1}(q_1)$ and $p_2 \in \alpha^{-1}(q_2)$ be any two points on M. Since F_M on B_M is controllable then there exists an integral curve c_M such that $c_M(t_1) = p_1$ and $c_M(t_2) = p_2$. The curve $\alpha \circ c_M$ satisfies $\alpha \circ c_M(t_1) = q_1$ and $\alpha \circ c_M(t_2) = q_2$. But since F_N is an abstraction of F_M, then $\alpha \circ c_M$ is an integral curve of F_N on B_N. Therefore, the abstracted system is controllable. □

6 Issues for Further Research

In this paper, some preliminary results on abstracting dynamical and control systems have been discussed. A notion of system abstraction has been defined and necessary and sufficient conditions under which one system is an abstraction of another have been obtained. Furthermore, a notion of order among abstractions was introduced by ordering the conservativeness of the given abstractions. Finally, desirable system properties were found to propagate from original models to abstracted models.

Issues for further research include, on the theoretical front, obtaining better, less conservative abstractions. Better abstractions can be obtained by propagating to the abstracted models qualitative aspects, such as stability, of the evolution of the ignored system dynamics. Furthermore, approximate abstractions are also of interest. Approximate abstractions approximate integral curves of a given system with some guaranteed margin of error. If the margin of error is acceptable then analysis of the approximate system will be sufficient. Furthermore, propagating additional properties from original models to abstracted models is useful in both system analysis and design.

Finally, the developments presented in this paper will be applied to various applications of interest. Particular applications of interest include aircraft and automobile models.

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