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Switching through Singularities*

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Abstract

Asymptotic tracking is studied for systems in which the relative degree is not well defined, meaning that the control law derived from exact input-output linearization has singularities in the state space. We propose a tracking control law which switches between approximate tracking [1] close to the singularities, and exact tracking away from the singularities, and we study the applicability of this law based on the behavior of the system's zero dynamics at the switching boundary. As in [1], the ball and beam example is used to motivate the study.

Keywords: Switching, nonlinear control, zero dynamics, exact and asymptotic tracking, nonminimum phase.

1 Introduction

The nonlinear control toolbox has built up a fair level of sophistication with the use of techniques for input-output and full state linearization, approximate linearization, and bounded tracking for nonminimum phase systems. One area in which results have been hard to come by is the tracking of *singular* or *non regular* nonlinear control systems, i.e. those that fail to have relative degree. While the problem of trying to track trajectories that go through singularities was begun by Hirschorn and Davis [2] who limited the class of outputs that could be tracked, the first set of practical schemes for approximate asymptotic tracking through singularities was given by Hauser, Sastry and Kokotović in [1], using an approximation technique. This in turn spurred the development of a result by Grizzle, Di Benedetto and Lamnabhi-Lagarrigue [3] which proved the necessity of "regularity" for asymptotically exact tracking.

In parallel with this activity has been the interest in using switching control laws for adaptive control ([4], [5]), for steering and stabilization ([6], [7]), as well as activity in *hybrid control systems* ([8], [9], [10]). In this paper, we combine ideas from switching along with our results on exact tracking of slightly non-minimum phase systems (developed in [11] building on the techniques and ideas of [12] and [13]) to describe our results in *approximate tracking for singular or non-regular nonlinear systems.*

The outline of our paper is as follows: in Section 2 we describe the ball and beam example of [1] with some added new insights about the zero dynamics of the singular system under perturbation and

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the subtleties of introducing switching across the singularity surface into the control law. In Section 3, we set up the problem formulation for general single-input single-output nonlinear systems, and prove approximate tracking for these systems when their zero dynamics are unstable. In Section 4 we discuss the nature of the zero dynamics of singular systems which we classify into three cases, and in Section 5 we give our current results about tracking using switched controllers.

For the three classes of systems that we obtain in Section 4, we give conditions for approximate tracking. The basic result is that *slow switching* is acceptable for the third case, where the equilibrium structure is preserved between the exact and approximate internal dynamics. In the first and second case (which is in fact a generalization of our ball and beam example), the change in equilibrium in the internal dynamics across the switching boundary is likely to destabilize the switched control law. Our results are not counterexamples to the Theorem of [3], since we use discontinuous control laws and ask for only asymptotic approximate tracking.

2 Motivating Example: the ball and beam

We describe in this section the proposed switching scheme applied to the ball and beam example. The insights gained by studying the zero dynamics as the system switches across the singularity surface motivate our general formulation in the next section.

Consider the following system, which describes the motion of a ball rolling along a rotating beam [1]. The state variables are

$$(x_1, x_2, x_3, x_4) = (r, \dot{r}, \theta, \dot{\theta})$$

where r is the distance of the ball from the pivot of beam, and θ is the angle of the beam. In the following, G is the acceleration due to gravity, and $B := M/(J_b/R^2 + M)$ where M and J_b are the mass and moment of inertia of the ball, and R is the radius of the ball. The input τ is the torque applied at the center of rotation: a preliminary change of coordinates in the input space, as in [1], is performed to define a new input variable u which appears linearly in the system equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= B(x_1 x_4^2 - G \sin x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \tag{1}$$

The output equation is $y = x_1 - r_0$, where r_0 is a given position of the ball along the beam. Thus

$$f(x) = \begin{bmatrix} x_2 \\ B(x_1 x_4^2 - G \sin x_3) \\ x_4 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad h(x) = x_1$$

The ball and beam system as modeled by (1) is non-regular, meaning that the relative degree of the system is not well defined globally over the state space. Indeed, $L_g L_f^2 h(x) = 2Bx_1x_4$, so that the feedback linearizing control is undefined when $x_1 x_4 = 0$. In [1], an approximate input-output linearization scheme is defined using the coordinates

$$\begin{aligned} \xi_1 &= x_1 - r_0 \\ \xi_2 &= x_2 \\ \xi_3 &= Bx_1x_4^2 - BG \sin x_3 \\ \xi_4 &= Bx_2x_4^2 - BGx_4 \cos x_3 \end{aligned} \tag{2}$$

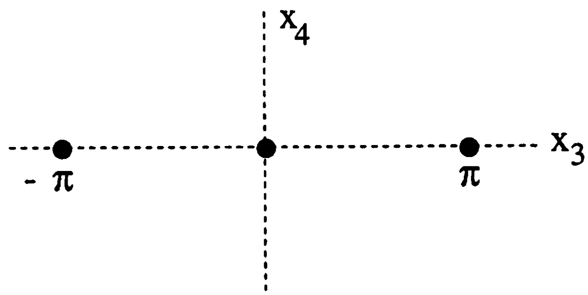

Figure 1: Zero Dynamics of the ball and beam using approximate tracking

The system dynamics in mixed x and ξ coordinates is

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 + 2Bx_1x_4u \\
\dot{\xi}_4 &= B(Bx_1x_4^2 - BG\sin x_3)x_4^2 + BGx_4^2\sin x_3 + (2Bx_2x_4 - BG\cos x_3)u
\end{aligned} \tag{3}$$

The approximately linearizing u is calculated by ignoring the coefficient of u in $\dot{\xi}_3$:

$$u = \frac{1}{2Bx_2x_4 - BG\cos x_3}[-B(Bx_1x_4^2 - BG\sin x_3)x_4^2 - BGx_4^2\sin x_3 + v] \tag{4}$$

for auxiliary input v. The zero dynamics are zero-dimensional, and consist of the equilibrium points (Figure 1):

$$x = \begin{bmatrix} r_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} r_0 \\ 0 \\ -\pi \\ 0 \end{bmatrix}, \quad \begin{bmatrix} r_0 \\ 0 \\ \pi \\ 0 \end{bmatrix}, \quad \text{with} \quad u_0 = 0.$$

which correspond to the beam being held in a stationary horizontal position.

This approximate scheme is exact when $2Bx_1x_4 = 0$, and approximates the exact scheme well when $|2Bx_1x_4|$ is small. However, results in [1] show that when $|2Bx_1x_4|$ becomes large (which is when the values of the desired trajectory and its derivatives become large) the approximate tracking scheme is unstable.

We introduce a switching scheme which uses this approximate scheme in a neighborhood of the singularity, and switches to a tracking control based on exact input-output linearization outside of this neighborhood. To do this, partition the state space into the regions:

$$\begin{aligned}
M_0 &= \{x \in \mathbb{R}^4 : |x_1x_4| < \delta\} \\
M_- &= \{x \in \mathbb{R}^4 : x_1x_4 < -\delta\} \\
M_+ &= \{x \in \mathbb{R}^4 : x_1x_4 > \delta\}
\end{aligned}$$

In M_0, we use the control law (4). In the regions M_+ and M_-, we define a coordinate system:

$$\begin{aligned}
\xi_1 &= x_1 - r_0 \\
\xi_2 &= x_2 \\
\xi_3 &= Bx_1x_4^2 - BG\sin x_3 \\
\eta &= x_4
\end{aligned} \tag{5}$$

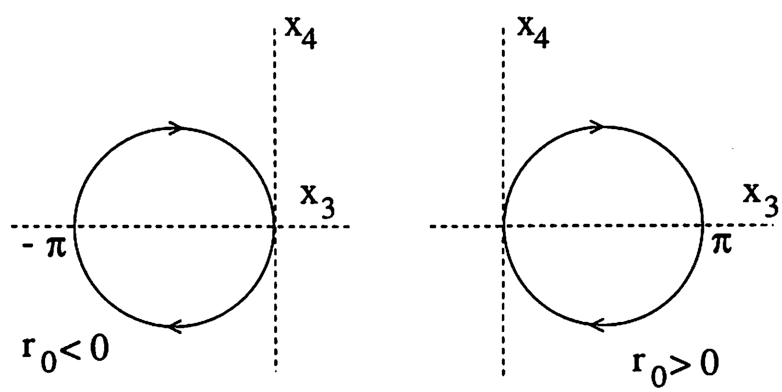

Figure 2: Zero Dynamics of the ball and beam using exact tracking

The nonlinear coordinate transformation defined by

$$\Phi : x \to (\xi, \eta)$$

is a local diffeomorphism away from $x_3 = \frac{\pi}{2}$. The system dynamics are

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= Bx_2x_4^2 - BGx_4 \cos x_3 + 2Bx_1x_4u \\ \dot{\eta} &= u\end{aligned} \tag{6}$$

The input-output linearizing u is therefore given by

$$u = \frac{1}{2Bx_1x_4}(-Bx_2x_4^2 + BGx_4 \cos x_3 + v) \tag{7}$$

for an auxiliary input v. The zero dynamics are obtained by setting ξ_1, ξ_2, ξ_3, and their derivatives to zero, for which the solutions are:

$$\dot{x}_4 = u_0, \quad \text{such that} \quad u_0 = 0 \quad \text{and} \quad (x_3, x_4) = (0,0) \text{ or } (\pi, 0)$$

and

$$\dot{x}_4 = u_0, \quad \text{such that} \quad u_0 = \frac{G \cos x_3}{2r_0} \quad \text{and} \quad r_0x_4^2 = G \sin x_3$$

The first solution above is trivial, since it corresponds to holding the ball and beam at the equilibrium position and applying zero input. The second solution is more interesting, since the zero dynamics evolve on the one-dimensional surface S_1, which causes the beam to do "cartwheels"! (See Figure 2) While these dynamics are bounded, they are such that tracking schemes using exact linearization are impossible.

To characterize the more robust features of the zero dynamics (such as the ω-limit sets) we consider a regular ϵ-perturbation of this system and study its zero dynamics as $\epsilon \to 0$:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= B(x_1x_4^2 - G \sin x_3) + \epsilon u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u\end{aligned} \tag{8}$$

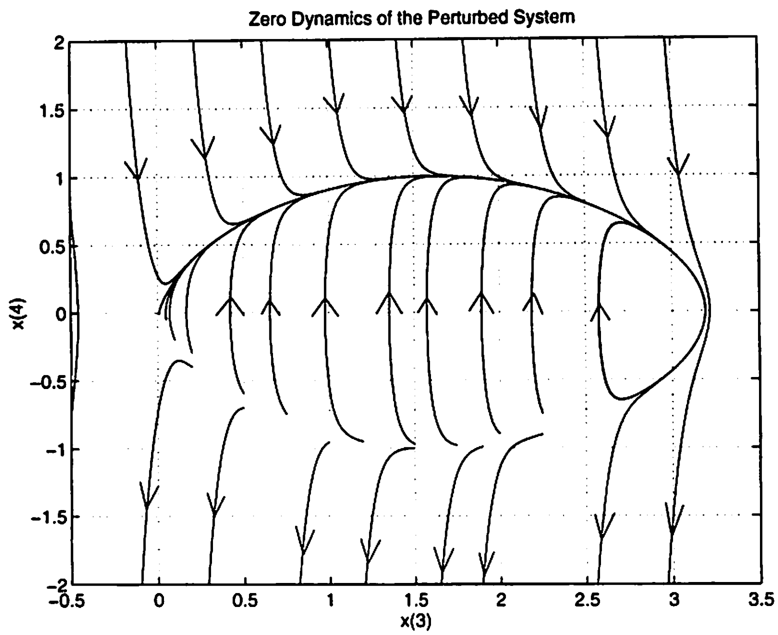

Figure 3: Zero Dynamics of the perturbed ball and beam

Physically, this perturbation reflects a coupling between the input torque and the velocity of the ball along the beam. This perturbed system has relative degree 2, with two dimensional zero dynamics ($x_1 = r_0, x_2 = 0$):

$$\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{G \sin x_3 - r_0 x_4^2}{\epsilon} \end{aligned} \tag{9}$$

which are illustrated in Figure 3 for $r_0 = 1, \epsilon = 0.1$. The zero dynamics have an unstable node at $(x_3, x_4) = (0,0)$ and a center equilibrium at $(x_3, x_4) = (\pi, 0)$. As $\epsilon \to 0$, the limit cycles around the center collapse into the equilibrium, and the center becomes an unstable node.

We would like the system to track a desired output trajectory $y_D(t)$ corresponding to a periodic motion of the ball along the beam. The physics of the ball and beam system insist that the state trajectory which implements $y_D(t)$ traverse the region in which $x_1 x_4 = 0$, since x_4 must necessarily change sign. We define a δ-region around $x_1 x_4 = 0$ by $\{x : |x_1 x_4| \leq \delta\}$ and in this region, the *approximate* tracking law of equation (4) may be used to stabilize the system to $y_D(t)$. The stabilizing control law may be calculated as

$$v = y_D^{(4)} - \alpha_4(\xi_4 - y_D^{(3)}) - \alpha_3(\xi_3 - y_D^{(2)}) - \alpha_2(\xi_2 - \dot{y}_D) - \alpha_1(\xi_1 - y_D)$$

where the α_i are selected so that $s^4 + \alpha_4 s^3 + \alpha_3 s^2 + \alpha_2 s + \alpha_1$ is Hurwitz. We expect the system to stabilize nicely in this region since the zero dynamics corresponds to a single equilibrium.

In the region $\{x : |x_1 x_4| \geq \delta\}$, the *exact* tracking law of equation (7) may be used to stabilize the output of the system to $y_D(t)$. Again, the control law may be calculated as

$$v = y_D^{(3)} - \alpha_3(\xi_3 - y_D^{(2)}) - \alpha_2(\xi_2 - \dot{y}_D) - \alpha_1(\xi_1 - y_D)$$

where $s^3 + \alpha_3 s^2 + \alpha_2 s + \alpha_1$ is Hurwitz. Because the zero dynamics in this region is constrained to lie on S_1, we cannot expect such a nice behavior of the inverse dynamics of the system in this region.

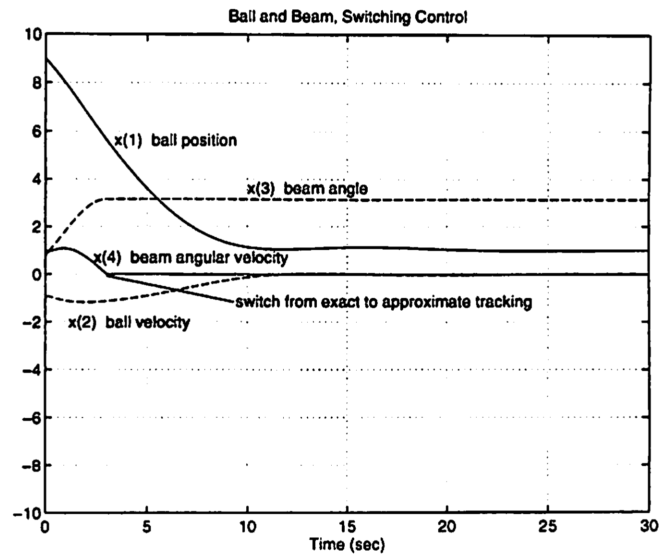

Figure 4: Regulating to $y_D(t) = 1$ using switched control

For example, if the desired trajectory is $y_D(t) \equiv x_1 - r_0 = 0$, the (x_3, x_4) dynamics traverses S_1 which corresponds to the beam rotating between $x_3 = 0$ and $x_3 = \pi$.

Figures 4, 5, and 6 display the results of regulating the ball and beam system to $y_D(t) = 1$ using the switched control scheme compared with using approximate linearization only. Figure 7 shows the results of tracking $y_D(t) = \sin(0.8t) + 3$ using switched control. Note that for regulation, the switched scheme causes the beam to flip upside down (Figure 4), since the zero dynamics of the system (Figure 5) follow S_1 until the switch to approximate tracking occurs. Compare this to the approximate linearization scheme in Figure 6 which, although it has larger overshoots and longer settling times, regulates the zero dynamics variables to zero. This problem is accentuated in the tracking results of Figure 7, where the beam continually flips upside down and right side up.

3 General Formulation

We strive for a nonlinear tracking control law which works well both close to singularities in the state space, as well as far away from these singularities. The ball and beam example provides inspiration. The exactly linearizing control law for the ball and beam system is not useful because of the intrinsic system property that the resulting one-dimensional zero dynamics are constrained to lie on S_1. But the same switching method applied to other systems in which the zero dynamics (either stable or unstable) are "better behaved" has the potential to be very successful in increasing the domains of attraction of the control law. For example, the planar flexible robot arm in [14] has a singularity at $\cos\theta_2 = \frac{2}{3}$, where θ_2 is the angle of the elbow. The exactly linearizing control produces two-dimensional zero dynamics which are unstable but hyperbolic, and so the nonlinear inversion technique of Devasia, Chen, and Paden [12] may be used away from the singularity to calculate the asymptotically stable solution to these dynamics.

In the general formulation, we will consider single-input single-output controllable nonlinear systems

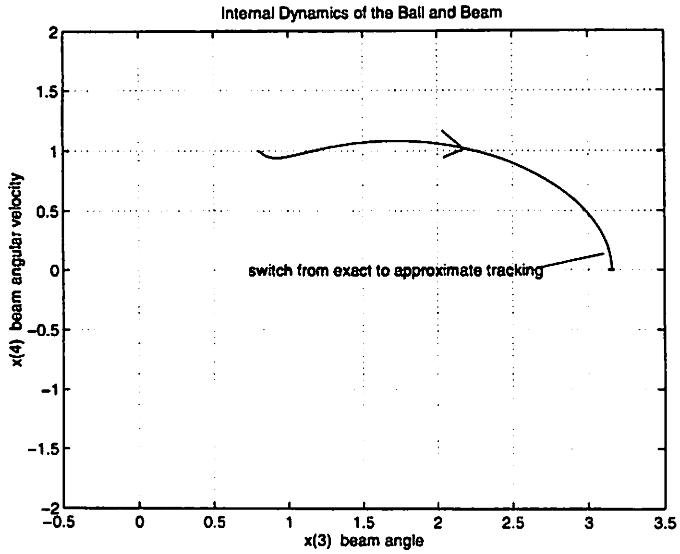

Figure 5: Regulating to $y_D(t) = 1$ using switched control, showing the zero dynamics

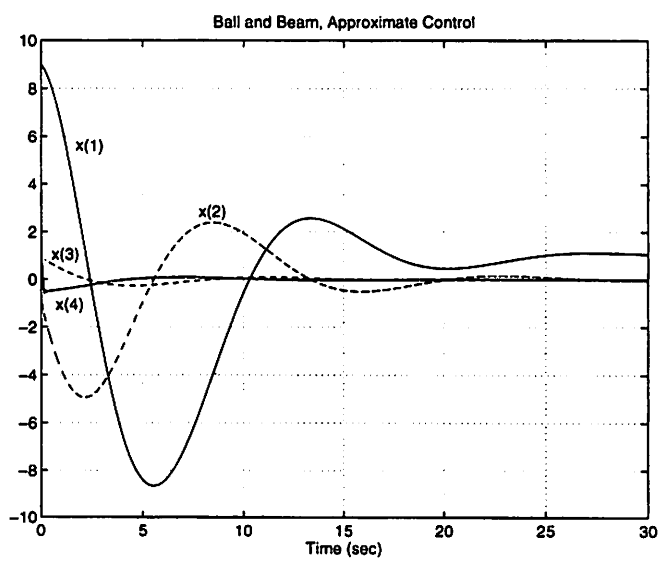

Figure 6: Regulating to $y_D(t) = 1$ using approximate control

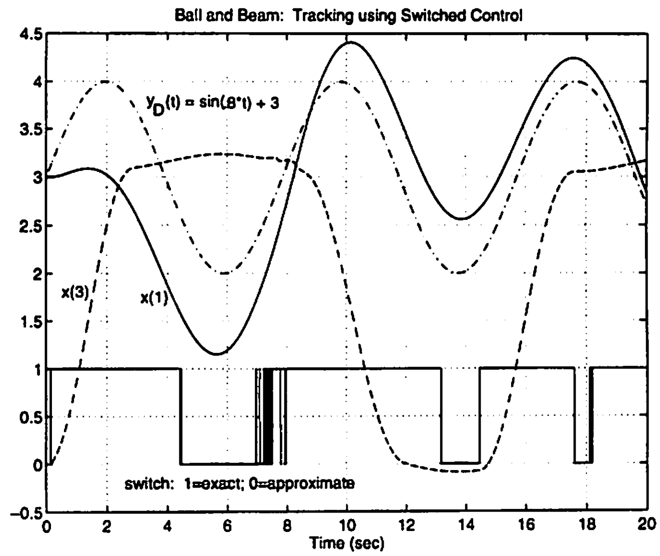

Figure 7: Tracking $y_D(t) = \sin(0.8t) + 3$, using switching control

of the form

$$\begin{aligned} \dot{x} = & \ f(x) + g(x)u \\ y = & \ h(x) \end{aligned} \tag{10}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $f(x)$ and the columns of $g(x)$ are analytic vector fields, and $h(x)$ is an analytic function. We assume that $x = x_0$ is an equilibrium point, that is $f(x_0) = 0$, and without loss of generality we assume that $h(x_0) = 0$.

In the following we need the definitions of *robust relative degree* and *uniformly higher order* [1]:

Definition 1 (Robust relative degree) *The nonlinear system (10) is said to have robust relative degree γ at x_0 if there exist smooth functions $\phi_i(x)$, $i = 1 \dots \gamma$ such that*

$$\begin{aligned} h(x) &= \phi_1(x) + \psi_0(x,u) \\ L_{f+gu}\phi_i(x) &= \phi_{i+1}(x) + \psi_i(x,u) \quad i = 1 \dots \gamma - 1 \\ L_{f+gu}\phi_\gamma(x) &= b(x) + a(x)u + \psi_\gamma(x,u) \end{aligned}$$

where the functions $\psi_i(x,u)$, $i = 0 \dots \gamma$ are sums of terms of $O(x)^2$, $O(x,u)$, or $O(u)^2$ (denoted $O(x,u)^2$).

Definition 2 (Uniformly higher order) *A function $\psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is said to be uniformly higher order on $U_\epsilon \times B_\sigma$ if for some $\sigma > 0$ there exists a monotone increasing function of ϵ, $K(\epsilon)$, such that*

$$|\psi(x,u)| \leq \epsilon K(\epsilon)[|x| + |u|] \forall x \in U_\epsilon, |u| \leq \sigma \tag{11}$$

Assume that the system (10) has robust relative degree γ. Adopting the notation of [1], we define new coordinates (ξ, η) by

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_\gamma \\ \eta_1 \\ \vdots \\ \eta_{n-\gamma} \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_\gamma(x) \\ \eta_1(x) \\ \vdots \\ \eta_{n-\gamma}(x) \end{bmatrix} \tag{12}$$

where the functions $\eta_i(x), i = 1, \ldots, n-\gamma$ are chosen so that $L_g \eta_i(x) = 0$. The system (10) is written in mixed coordinates as:

$$\begin{aligned} \dot{\xi}_1 = &\ \xi_2 + \psi_1(x,u) \\ \vdots & \\ \dot{\xi}_{\gamma-1} = &\ \xi_\gamma + \psi_{\gamma-1}(x,u) \\ \dot{\xi}_\gamma = &\ b(x) + a(x)u + \psi_\gamma(x,u) \\ \dot{\eta} = &\ q(\xi, \eta) \end{aligned} \tag{13}$$

with output $y = \xi_1 + \psi_0(x,u)$. The *approximate* system corresponding to (13) is

$$\begin{aligned} \dot{\xi}_1 = &\ \xi_2 \\ \vdots & \\ \dot{\xi}_{\gamma-1} = &\ \xi_\gamma \\ \dot{\xi}_\gamma = &\ b(x) + a(x)u \\ \dot{\eta} = &\ q(\xi, \eta) \end{aligned} \tag{14}$$

Since we are interested in tracking a desired trajectory $y_D(t)$, which is assumed to be at least γ times continuously differentiable, define

$$\xi_D(t) \equiv \begin{pmatrix} y_D(t) \\ \dot{y}_D(t) \\ \vdots \\ y_D^{(\gamma-1)}(t) \end{pmatrix} \tag{15}$$

We consider four possible scenarios for the system (13), which lead to four different nonlinear tracking control laws:

1. the exact system (13) has stable zero dynamics;
2. the exact system has unstable zero dynamics;
3. the approximate system (14) has stable zero dynamics;
4. the approximate system has unstable zero dynamics.

In the first scenario, feedback linearization results in exact tracking of $y_D(t)$ with stable zero dynamics. In the second, Devasia-Chen-Paden's nonlinear inversion technique [12] may be used to calculate the stable solution to the zero dynamics, and exact tracking results. For the third, bounded tracking and $O(\epsilon)$ tracking error are proven in [1] when the functions $\psi_i(x,u), i = 1,\ldots,\gamma-1$ are uniformly higher order in x and u, the zero dynamics of the approximate system ($\dot{\eta} = q(0,\eta)$) is exponentially stable, and the desired output and its derivatives $(y_D, \dot{y}_D, \ldots, y_D^{\gamma})$ are sufficiently small. In this section, we extend the result of [1] to prove bounded tracking for the fourth scenario, using the nonlinear inversion technique of [12].

Theorem 3 (Approximate Tracking for Unstable Zero Dynamics) *Consider the system (10) with normal form (13). Assume that in a neighborhood U_ϵ of x_0*

1. *the zero dynamics of the approximate system, $\dot{\eta} = q(0,\eta)$, is hyperbolic (its Jacobian $Q = \frac{\partial q}{\partial \eta}(0,\eta_0)$ has no eigenvalues on the $j\omega$-axis);*
2. *the error in Jacobian linearization of the corresponding driven dynamics, defined by $r(\xi,\eta) := q(\xi,\eta) - Q\eta$, is Lipschitz continuous in both of its arguments:*

$$|r(\xi_1,\eta_1) - r(\xi_2,\eta_2)| < k_1|\xi_1 - \xi_2| + k_2|\eta_1 - \eta_2|$$

   with Lipschitz constants k_1, k_2 small enough;

3. *the functions $\psi_i(x,u), i = 1,\ldots,\gamma-1$ are uniformly higher order; and*
4. *the desired trajectory and its derivatives are sufficiently small.*

Then, for $x \in U_\epsilon$, $u \in B_\sigma$ there exists a unique bounded solution to the driven dynamics, called η_D; the states of the closed loop system defined by (13) with control law

$$u(\xi,\eta,\xi_D,\eta_D) = \frac{1}{a(\xi,\eta)}[-b(\xi_D,\eta_D) + y_D^{(\gamma)}] + \frac{1}{a(\xi,\eta)}[f_1^T(\xi - \xi_D) + f_2^T(\eta - \eta_D)] \tag{16}$$

(where f_1 and f_2 are chosen to stabilize the closed loop system) remain bounded; and the tracking error is $O(\epsilon^2)$.

Proof: Assumptions 1 and 2 guarantee (see [12]) that there exists for given bounded ξ_D a bounded solution $\eta_D \in \mathbb{R}^{n-\gamma}$ satisfying $\lim_{t\to\pm\infty} \eta_D(t) = 0$, which is obtained as the fixed point of the following integral equation:

$$\eta_D(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) r(\xi_D,\eta_D) d\tau \tag{17}$$

in which $\Phi(t)$ is the Caratheodory solution of the matrix differential equation

$$\dot{X} = QX \quad X(\pm\infty) = 0 \quad X(0+) - X(0-) = I$$

Now, consider the control input (16) and define the error trajectory $e \in \mathbb{R}^n$ as

$$\begin{bmatrix} e_1 \\ \vdots \\ e_\gamma \\ e_{\gamma+1} \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_\gamma \\ \eta_1 \\ \vdots \\ \eta_{n-\gamma} \end{bmatrix} - \begin{bmatrix} y_D \\ \vdots \\ y_D^{(\gamma-1)} \\ \eta_{D,1} \\ \vdots \\ \eta_{D,n-\gamma} \end{bmatrix} \tag{18}$$

Then the closed loop system (13), (16) may be expressed as

$$
\begin{aligned}
\dot{e}_1 &= e_2 + \psi_1(\xi, \eta, u(\xi, \eta, \xi_D, \eta_D)) \\
&\vdots \\
\dot{e}_\gamma &= b(\xi, \eta) - b(\xi_D, \eta_D) + f(t)^T e + \psi_\gamma(\xi, \eta, u(\xi, \eta, \xi_D, \eta_D)) \\
\dot{e}_{\gamma+1} &= q_1(\xi, \eta) - q_1(\xi_D, \eta_D) \\
&\vdots \\
\dot{e}_n &= q_{n-\gamma}(\xi, \eta) - q_{n-\gamma}(\xi_D, \eta_D)
\end{aligned}
\tag{19}
$$

where $f(t) = [f_1(t)\ f_2(t)]$. Denote by e^γ the unit vector in $\mathbb{R}^n$ whose elements are all zero except for a 1 in the γ^{th} position. Jacobian linearization about the desired trajectory (ξ_D, η_D) results in

$$\dot{e} = A(t)e + f^T(t)\, e^\gamma e + \Gamma(\xi, \eta, \xi_D, \eta_D) \tag{20}$$

where

$$
A(t) = \left[\begin{array}{cccccc|ccc}
0 & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & & 0 & \cdots & 0 \\
\vdots & \vdots & 0 & \ddots & 0 & & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 & & 0 & \cdots & 0 \\
\hline
c_1 & c_2 & c_3 & \cdots & c_\gamma & & c_{\gamma+1} & \cdots & c_n \\
\hline
 & & \left.\frac{\partial q}{\partial \xi}\right|_{\xi_D, \eta_D} & & & & & \left.\frac{\partial q}{\partial \eta}\right|_{\xi_D, \eta_D} &
\end{array}\right]
\tag{21}
$$

and

$$(c_1 \cdots c_\gamma) = \left.\frac{\partial b}{\partial \xi}\right|_{\xi_D, \eta_D} \qquad (c_{\gamma+1} \cdots c_n) = \left.\frac{\partial b}{\partial \eta}\right|_{\xi_D, \eta_D}$$

$$
\begin{aligned}
\Gamma(\xi, \eta, \xi_D, \eta_D) &= \begin{bmatrix} \psi_1(\xi, \eta, \xi_D, \eta_D) \\ \vdots \\ \psi_\gamma(\xi, \eta, \xi_D, \eta_D) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \text{h.o.t}(e) \\ \text{h.o.t}(e) \\ \vdots \\ \text{h.o.t}(e) \end{bmatrix} \\
&\equiv \Psi(\xi, \eta, \xi_D, \eta_D) + \text{h.o.t}(e)
\end{aligned}
\tag{22}
$$

Here, h.o.t(e) means *higher order terms*, or terms which are of $O(e^2)$. By our assumption that (10) is controllable, f_1, f_2 may be chosen to make the closed loop system with dynamic matrix:

$$A_c(t) = A(t) + [f_1^T(t), f_2^T(t)]e^\gamma$$

exponentially stable. By choosing ξ_D small enough, η_D may be shown to be small enough using (17). Thus, using the uniformly higher order assumption (11) on $\Psi(\cdot)$, it follows that

$$|\xi_D, \eta_D| < \epsilon \Rightarrow |\Psi(\xi, \eta, \xi_D, \eta_D)| \leq \epsilon^2 K_1(\epsilon) + \epsilon K_2(\epsilon)|e| \tag{23}$$

Also, by definition

$$\text{h.o.t}(e) \leq \epsilon K_3(\epsilon)|e| \tag{24}$$

Thus, we will abuse notation and write (20) as

$$\begin{aligned} \dot{e} &= A_c(t)e + \epsilon[K_2(\epsilon) + K_3(\epsilon)]|e| + \epsilon^2\Gamma \\ &= \tilde{A}_c(t)e + \epsilon^2\Gamma \end{aligned} \tag{25}$$

Thus the error system is exponentially stable and is driven by an input of $O(\epsilon^2)$. Thus, the tracking error is $O(\epsilon^2)$. ■

In order to keep the notation as simple as possible, we concentrate on an example system in which the difference between the robust relative degree and the relative degree is 1. That is,

- the robust relative degree is γ, and the approximate zero dynamics have dimension $n-\gamma$;
- the relative degree is $\gamma-1$, and the exact zero dynamics have dimension $n-\gamma+1$.

Under this assumption, system (13) becomes

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{\gamma-1} &= \underbrace{L_f^{\gamma-1}h(x)}_{\xi_\gamma} + \underbrace{L_gL_f^{\gamma-2}h(x)u}_{\psi_{\gamma-1}(x,u)} \\ \dot{\xi}_\gamma &= \underbrace{L_f^{\gamma}h(x)}_{b(x)} + \underbrace{L_gL_f^{\gamma-1}h(x)}_{a(x)}\,u \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{26}$$

We also assume that $L_gL_f^{\gamma-2}h(x_0) = 0$, meaning that the singular surface contains x_0.

We have shown in this section that for the system (26) with approximate control law (16), if the dynamics $\dot{\eta} = q(0,\eta)$ is unstable but hyperbolic, the states of the system remain bounded and the tracking error is small if the desired trajectory and its derivatives $(y_D, \dot{y}_D, \ldots, y_D^{(\gamma)})$ remain small. However, we would like to track trajectories in which the desired output and its derivatives may be large, meaning that even though the state of the system is forced *through* the singular surface of $L_gL_f^{\gamma-2}h(x)$, it is also forced *far away* from this singular surface. Far away from this singular surface, $|L_gL_f^{\gamma-2}h(x)|$ is large, and the approximate control law fails. We propose a switching scheme which uses the control law derived from the approximate input-output linearization method of [1] close to the singular surface, and switches to a control law derived from exact input-output linearization far away from the singular surface. The remainder of this paper is devoted to deriving conditions under which such a switching scheme is possible. We approach the problem by classifying the behavior of the internal dynamics of systems of the form (26) as the state switches through the singular surface $L_gL_f^{\gamma-2}h(x) = 0$.

4 Classification of the Internal Dynamics

Consider the system (26). The internal dynamics of the system are the $(n-\gamma+1)$-dimensional "exact internal dynamics" in the region in which $L_gL_f^{\gamma-2}h(x) \neq 0$:

$$\begin{aligned} \dot{\xi}_\gamma &= L_f^{\gamma}h(x) + L_gL_f^{\gamma-1}h(x)u^{\mathrm{ex}}(x) \\ \dot{\eta} &= q(\xi,\eta) \end{aligned} \tag{27}$$

and the $(n-\gamma)$-dimensional "approximate internal dynamics" when $L_g L_f^{\gamma-2} h(x) = 0$:

$$\dot{\eta} = q(\xi, \eta) \tag{28}$$

Let us analyze the exact and approximate zero and driven dynamics as the state switches through x_0.

Exact Internal Dynamics

Assuming that $L_g L_f^{\gamma-2} h(x) \neq 0$, consider the *exact* control law $u_0^{\text{ex}}(x)$ required to hold the output and its derivatives identically at zero:

$$L_f^{\gamma-1} h(x) + L_g L_f^{\gamma-2} h(x) u_0^{\text{ex}}(x) = 0 \tag{29}$$

As the state trajectory approaches x_0, this control law becomes

$$u_0^{\text{ex}}(x_0) = \lim_{x \to x_0} -\frac{L_f^{\gamma-1} h(x)}{L_g L_f^{\gamma-2} h(x)} \tag{30}$$

Note that for the class of *singular systems* that we are considering both the numerator $L_f^{\gamma-1} h(x_0)$ and denominator $L_g L_f^{\gamma-2} h(x_0)$ of this limit vanish, since $f(x_0) = 0$, and $x_0 \in M_0$. But because the numerator and denominator are both analytic functions, we can rewrite (29) as

$$(x - x_0)^r q_1(x) + (x - x_0)^s q_2(x) u_0^{\text{ex}}(x) = 0 \tag{31}$$

where $q_1(x_0) \neq 0$, $q_2(x_0) \neq 0$, and $r > 0$ (again since $f(x_0) = 0$).

For the driven dynamics, if the desired output is $y_D(t)$, the required input $u^{\text{ex}}(t)$ may be calculated from

$$(x - x_0)^r q_1(x) + (x - x_0)^s q_2(x) u^{\text{ex}}(x) = y_D^{\gamma-1}(t) \tag{32}$$

We consider three cases:

1. **No Zero Dynamics:** $s > r$, $u_0^{\text{ex}}(x_0) = \infty$.

   In this case, there does not exist an input $u_0^{\text{ex}}(x_0)$ to hold the output and its derivatives at zero.

   We may still study the nonlinear zeros of the driven system, which are the eigenvalues of the Jacobian linearization about x_0 of the driven dynamics. Substituting $u^{\text{ex}}(x)$ into the linearized equation for $\dot{\xi}_\gamma$ and forcing the system towards x_0 results in the term:

$$L_g L_f^{\gamma-1} h(x) \cdot \left(\frac{-q_1(x)}{(x - x_0)^{s-r} q_2(x)} + \frac{y_D^{\gamma-1}}{(x - x_0)^s q_2(x)} \right) \Bigg|_{x \to x_0}$$

   As x goes through x_0, the denominators of these terms may change sign, depending on s and r. This corresponds to an eigenvalue of the linearized driven dynamics either moving to $-\infty$ in the left half plane and returning at ∞ in the right half plane, or moving to ∞ in the right half plane and returning at $-\infty$ in the left half plane. This suggests that the internal dynamics may become unstable as the system switches through x_0.

2. **Change in Equilibrium of the Zero Dynamics:** $s = r$, $u_0^{\text{ex}}(x_0) = -\frac{q_1(x_0)}{q_2(x_0)}$.

   The ball and beam is an example of a system of this class. In this case, the zero dynamics exist and are the dynamics of

$$\dot{x} = f(x) + g(x)u_0^{\text{ex}}(x)|_{M=\{x:h(x)=L_f h(x)=\ldots=L_f^{\gamma-2}h(x)=0\}} \tag{33}$$

   Since $u_0^{\text{ex}}(x_0) \neq 0$, x_0 is *not* an equilibrium of (33). This is interesting, since it means that a feedback linearizing control may trigger "higher order" zero dynamics in the switching from an approximate to an exact control law. This is evident in the ball and beam example.

   In this case the linearized driven dynamics include the term

$$L_g L_f^{\gamma-1} h(x) \cdot \left(\frac{-q_1(x)}{q_2(x)} + \frac{y_D^{\gamma-1}}{(x-x_0)^s q_2(x)}\right)\Bigg|_{x\to x_0}$$

   which, depending on $y_D(t)$, may cause a stable eigenvalue to become unstable as the system switches through x_0.

3. **No qualitative change in the Zero Dynamics:** $s < r$, $u_0^{\text{ex}}(x_0) = 0$.

   In this case the zero dynamics exist and have an equilibrium point at x_0.

   The driven dynamics include the term

$$L_g L_f^{\gamma-1} h(x) \cdot \left(\frac{y_D^{\gamma-1}}{(x-x_0)^s q_2(x)}\right)\Bigg|_{x\to x_0}$$

   which again, depending on $y_D(t)$, may cause the driven dynamics to become unstable on one side of the singular surface.

Approximate Internal Dynamics

Now consider the *approximate* control law $u_0^{\text{app}}(x)$ required to hold the approximate system's output and its derivatives at zero:

$$L_f^{\gamma} h(x) + L_g L_f^{\gamma-1} h(x) u_0^{\text{app}}(x) = 0 \tag{34}$$

so that

$$u_0^{\text{app}}(x_0) = \lim_{x\to x_0} -\frac{L_f^{\gamma} h(x)}{L_g L_f^{\gamma-1} h(x)} \tag{35}$$

We can rewrite (34) as

$$(x-x_0)^{r'} q_1'(x) + q_2'(x) u_0^{\text{app}}(x) = 0 \tag{36}$$

where $q_1'(x_0) \neq 0$, $q_2'(x_0) \neq 0$, and $r' > 0$ (note that $L_g L_f^{\gamma-1} h(x_0) \neq 0$ since the robust relative degree is γ). The *only* solution to this is $u_0^{\text{app}}(x_0) = 0$, so that x_0 is always an equilibrium of the approximate zero dynamics.

For the approximate driven dynamics, if the desired output is $y_D(t)$, the required input $u^{\text{app}}(t)$ may be calculated from

$$(x-x_0)^{r'} q_1'(x) + q_2'(x) u^{\text{app}}(x) = y_D^{\gamma}(t) \tag{37}$$

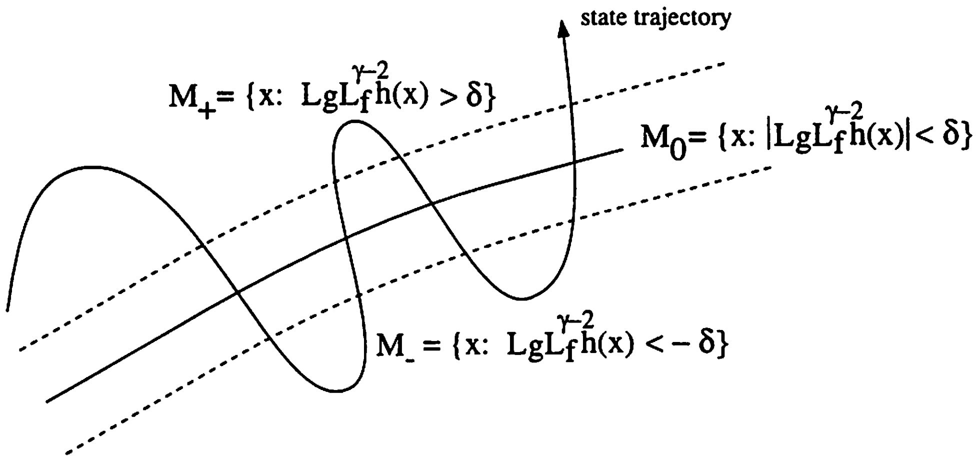

Figure 8: Partition of the state space into M_0, M_-, and M_+

Thus, $u^{\rm app}(x_0)$ is always finite.

The analysis in this section indicates that a control scheme which switches between exact and approximate control laws calculated from (32) and (37) is not likely to work for cases 1 and 2 above, when there is a change in equilibrium between the exact and approximate internal dynamics. As we saw in Section 2, the switching control in the ball and beam example triggered higher order zero dynamics (a limit cycle with no equilibria) resulting in cyclic oscillations of the beam from 0 to π and back. This seems to be a prototype behavior for the general case: in the tracking problem, while the driven dynamics would remain bounded, they may not go to zero as $(y_D, \dot{y}_D, \ldots)$ goes to zero. Our conclusions are that for this class of singular systems, it is best to not use a switching control law, but to use the approximate control law (39) for small enough values of $y_D(\cdot)$ and its derivatives. Thus, for this class of nonlinear systems there are intrinsic limitations to the magnitude of the inputs and their derivatives that can be approximately tracked.

In the next section, we prove approximate tracking for "slow switching", for case 3 above.

5 Tracking $y_D(t)$

Consider the system (26). Partition the state space as in the ball and beam example, illustrated in Figure 8:

$$\begin{aligned} M_0 &= \{x \in \mathbb{R}^n : |L_g L_f^{\gamma-2} h(x)| < \delta\} \\ M_- &= \{x \in \mathbb{R}^n : L_g L_f^{\gamma-2} h(x) < -\delta\} \\ M_+ &= \{x \in \mathbb{R}^n : L_g L_f^{\gamma-2} h(x) > \delta\} \end{aligned} \tag{38}$$

As in Section 3, we define in analogy to the control (16):

1. an approximate control law

$$u^{\rm app} = \frac{1}{L_g L_f^{\gamma-1} h(x^{\rm app})}[-L_f^\gamma h(x_D^{\rm app}) + y_D^{(\gamma)} + f^{{\rm app}^T} e^{\rm app}] \tag{39}$$

in the region M_0, where x_D^{app} is obtained from $\xi_D^{\text{app}} = (y_D, \dots, y_D^{(\gamma-1)})$ and

$$\eta_D^{\text{app}}(t) = \int_{-T}^{T} \Phi^{\text{app}}(t-\tau) r^{\text{app}}(\xi_D^{\text{app}}, \eta_D^{\text{app}}) d\tau; \tag{40}$$

2. and an exact control law

$$u^{\text{ex}} = \frac{1}{L_g L_f^{(\gamma-2)} h(x^{\text{ex}})} [-L_f^{(\gamma-1)} h(x_D^{\text{ex}}) + y_D^{(\gamma-1)} + f^{\text{ex}^T} e^{\text{ex}}] \tag{41}$$

in the region M_- and M_+, where x_D^{ex} is obtained from $\xi_D^{\text{ex}} = (y_D, \dots, y_D^{(\gamma-2)})$ and

$$\eta_D^{\text{ex}}(t) = \int_{-T}^{T} \Phi^{\text{ex}}(t-\tau) r^{\text{ex}}(\xi_D^{\text{ex}}, \eta_D^{\text{ex}}) d\tau \tag{42}$$

For both cases, the choice of T is discussed below.

Consider the normal form (26) with

$$u = u^{\text{ex}}(x) = \frac{-(x-x_0)^r q_1(x)}{(x-x_0)^s q_2(x)} + \frac{y_D^{\gamma-1}}{(x-x_0)^s q_2(x)}$$

calculated from (31). As was shown in Section 4 the internal dynamics (27) may be nonminimum phase either in M_- or in M_+ or both. If so, the tracking control law that is used in (41) would have to be calculated using the technique of Devasia, Paden and Chen [12]. The main drawback of the computations involved in the Devasia et al. scheme is that it involves computation of the "steady state" response as discussed in [11]. To address this, we will *assume* that the rate of switching is slow compared to the zero dynamics of the exact and approximate system. We formalize this assumption in the following definition:

Definition 4 (Slow Switching) *Let $\{t_i\}_{i \in I}$ be the sequence of switching times indexed by the set I. The switching between control laws (39) and (41) is called slow if*

$$\inf_{i \in I} \{t_{i+1} - t_i\} > 2T \tag{43}$$

where T is such that

$$\tilde{\eta}_D(t) = \int_{-T}^{T} \Phi(t-\tau) r(\xi_D, \tilde{\eta}_D) d\tau \tag{44}$$

is an ϵ-approximation to the infinite time computation of (17), repeated here:

$$\eta_D(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) r(\xi_D, \eta_D) d\tau$$

i.e. that

$$\sup_{t \in [-T,T]} |\tilde{\eta}_D(t) - \eta_D(t)| < \epsilon \tag{45}$$

In this definition, $\eta_D(t)$ could be either $\eta_D^{\text{app}}(t)$ (40) or $\eta_D^{\text{ex}}(t)$ (42). ϵ will be determined in the proof of the theorem. We now state the approximate tracking theorem using switching for the case in which the system satisfies the assumptions of slow switching.

Theorem 5 (Approximate Tracking Using Switching Control) *Consider the system (10) with normal form (26), and with the control law switching between (39) and (41) at the surfaces*

$$\{x \in \mathbb{R}^n : L_g L_f^{\gamma-2} h(x) = \pm\delta\}$$

Assume that conditions $1-3$ *of Theorem 3 hold for both the approximate internal dynamics and the exact internal dynamics, thus the state* x_D *in control laws (39), (41) is computed by applying the Devasia-Paden-Chen scheme to the approximate internal dynamics dynamics of (28) and exact internal dynamics of (27) respectively.*

Then, if the desired trajectory $y_D(\cdot)$ *is such that the conditions of slow switching are satisfied, for small enough* ϵ*, the switching control law results in asymptotic approximate tracking with output error less than or equal to* $K\delta$ *with all state variables bounded.*

Sketch of proof: We prove that for the conditions stated above, the error in approximating $\eta_D(t)$ (17) with the truncated version $\tilde{\eta}_D(t)$ (44) is small.

Let λ be the smallest eigenvalue of Φ^{app} in (40) and Φ^{ex} in (42).

Subtracting equation (44) from (17) and assuming that t is fixed and lies in the interval $[-\frac{T}{2}, \frac{T}{2}]$:

$$\begin{aligned}
|\eta_D - \tilde{\eta}_D| &\leq |\int_T^\infty \Phi(t-\tau) r(\xi_D, \eta_D) d\tau| + |\int_{-\infty}^{-T} \Phi(t-\tau) r(\xi_D, \eta_D) d\tau| \\
&+ |\int_{-T}^{T} \Phi(t-\tau)[r(\xi_D, \eta_D) - r(\xi_D, \tilde{\eta}_D)] d\tau| && (46) \\
&\leq M_1(T) + \int_{-T}^{T} |\Phi(t-\tau)| K_2 |\eta_D - \tilde{\eta}_D| && (47)
\end{aligned}$$

Since the zero dynamics are hyperbolic,

$$\begin{aligned}
|\Phi(t-\tau)| &\leq K_3 e^{-\lambda(t-\tau)} \text{ for } t \geq \tau \\
|\Phi(t-\tau)| &\leq K_3 e^{\lambda(t-\tau)} \text{ for } t \leq \tau
\end{aligned}$$

then it follows from the Bellman-Gronwall lemma that

$$|\eta_D - \tilde{\eta}_D| \leq M_1(T) e^{-M_2(T)} \tag{48}$$

with both M_1 and M_2 decreasing functions of T. Thus, if T is chosen large enough (as per the definition of slow switching) then the error introduced by switching in the computation of the desired trajectory is small. Now, continuing the analysis along the lines of [1], we can establish that $|y(t) - y_D(t)| \leq K_3\delta$ for t large enough. ∎

Though none of the cases discussed above are explicitly mentioned here, the conditions for slow switching are unlikely to be met in cases 1 and 2, because the change in equilibrium across the switching boundary would cause larger tracking errors and thus require a longer time T to compute the stable solution to the internal dynamics. For case 3, provided that the frequency of the signal being tracked is not too high and the amplitude is small enough, the condition of slow switching is likely to be met. This presents a limitation to the size of the signal being tracked and its derivatives.

Remark 1 The linear stabilizing terms f^{app^T} and f^{ex^T} should be such that the closed loop system $A_c(t)$ has eigenvalues which are greater than the slowest eigenvalues of the zero dynamics.

Remark 2 Theorem 5 clearly holds if the number of switches is finite provided that the switches are placed far enough apart.

Remark 3 In light of the discussion before Theorem 5 about the need for "steady state" calculations in the Devasia-Paden-Chen scheme, we feel that the foregoing switching control is not a good alternative to approximate linearization if the switching is fast.

Remark 4 Even though the last step of the proof of Theorem 5 uses the method of [1], it is important to note that the theorem is different and may yield better asymptotic tracking because of the switching.

6 Conclusions

The ball and beam example is used to motivate the study of a tracking control law which switches between an approximate tracking law close to singularities in the state space, and an exact tracking law away from these singularities. We study a general nonlinear system (10) which has relative degree $\gamma - 1$ and robust relative degree γ. We first prove that if the system has unstable zero dynamics, then approximate tracking can be achieved using the nonlinear inversion technique of [12] to calculate the stable solution to the zero dynamics. We then classify the zero dynamics of the general system into three cases, in which the first two involve a change in equilibrium of the zero dynamics across the switching boundary. We prove approximate tracking for the third case, in which there is no change in equilibrium, using the assumption that switching must be *slow*. For the first two cases, we advise that a tracking control law based on approximate tracking alone should be used.

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