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ABSTRACT. A unified approach to decidability questions for verification algorithms of hybrid systems is obtained by the construction of a bisimulation. Bisimulations are finite state quotients whose reachability properties are equivalent to those of the original infinite state hybrid system. In this paper, we introduce the notion of *o-minimal hybrid systems*, which are initialized hybrid systems whose relevant sets and flows are definable in an o-minimal structure. We prove that o-minimal hybrid systems always admit finite bisimulations. We then present a list of o-minimal structures which captures most hybrid systems known to admit finite bisimulations as well as present new classes of hybrid systems with more complex dynamics for which finite bisimulations exist.

Keywords: Hybrid systems, bisimulations, model theory, o-minimality, decidability

1. INTRODUCTION

Hybrid systems consist of finite state machines interacting with differential equations. Various modeling formalisms, analysis, design and control methodologies, as well as applications, can be found in [3, 4, 5, 12, 13, 22]. The theory of formal verification is one of the main approaches for analyzing properties of hybrid systems. The system to be analyzed is first modeled as a hybrid automaton, and the desired property is expressed using a formula from some temporal logic. Then, model checking or deductive algorithms are used in order to guarantee that the system model indeed satisfies the desired property.

Verification algorithms are essentially reachability algorithms which check whether trajectories of the hybrid system can reach certain undesirable regions of the state space. Since hybrid systems have infinite state spaces, decidability of verification algorithms is very important. Decidability results for analyzing hybrid systems consider special finite state quotients of the original infinite state hybrid automaton called *bisimulations*. Bisimulations are reachability preserving quotient systems in the sense that checking a property on the quotient system is *equivalent* to checking the property on the original system. Showing that an infinite state hybrid automaton has a finite state bisimulation is the first step in proving that verification procedures are decidable. This approach has yielded several classes of decidable hybrid systems including timed automata, triangular timed automata, fixed-slope automata, periodic grid automata, and rectangular automata. The above decidability results as well as some undecidable classes are described in [1, 2, 14, 15] and the references therein. Computing finite bisimulations is clearly related to the problem of obtaining discrete abstractions of continuous systems which has been considered among others by [6, 11, 26] as well as [9].

The common approach to obtaining bisimulations has been to utilize an algorithm which refines an initial partition of the state space until it becomes compatible with the system dynamics and the property to be preserved. Using this approach, there are three main issues that must be resolved:

1. When does the algorithm terminate after a finite number of iterations?
2. When does the resulting partition consist of a finite number of equivalence classes?
3. Are all the steps of the algorithm constructive?

Resolving all three issues results in a decidable problem. Attacking the first two issues has been solved either by explicitly providing an equivalence relation which is checked to be a bisimulation (timed automata), or by transforming the problem to one for which a bisimulation is known to exist (multi-rate, rectangular automata). The third issue is typically tackled using quantifier elimination techniques from mathematical logic.

In this paper, we tackle the first two issues for a much wider class of hybrid systems. In order to solve them, we need to identify classes of sets and flows with finite, global intersection properties. This is provided by the concept of *o-minimal* theories in mathematical logic [24, 33, 32, 34, 35]. Using this concept, we introduce the notion of *o-minimal hybrid systems* which are initialized hybrid systems whose relevant sets (guards, resets, etc) and flows are definable in an o-minimal theory. We then prove that o-minimal hybrid systems always admit finite bisimulations. We also show using examples that relaxing the notion of o-minimality quickly leads into pathological situations. We then list various o-minimal theories and the corresponding hybrid systems that are definable in them. This list captures most hybrid systems known to admit finite bisimulations. Moreover, we present hybrid systems with much more complex dynamics which are definable in recently discovered o-minimal structures and thus also admit finite bisimulations. We also point out a new decidability result for a particular class of o-minimal hybrid systems.

In addition to generating more classes of hybrid systems with finite bisimulations, the importance of this paper can be summarized by the following:

1. The results presented provide a unified framework for decidability analysis of hybrid systems
2. Generation of more o-minimal theories immediately leads to new classes of o-minimal hybrid systems
3. Constructive results within o-minimal theories immediately lead to decidability results

By providing a purely model theoretic framework, we also extend the planar results of [19] and [20].

The outline of the paper is as follows: In Section 2 we review the notion of bisimulations of transitions systems. In Section 3 we define a general class of hybrid systems and describe the bisimulation algorithm as it applies to hybrid systems. Section 4 presents the notion of o-minimality from model theory which is used in Section 5 in order to define o-minimal hybrid systems and prove the main theorem. In Section 6, we list various classes of o-minimal hybrid systems. Section 7 contains conclusions and issues for further research.

2. Bisimulations of Transition Systems

We adopt here the terminology of [14] slightly modified for our purposes. A transition system $T = (Q, \Sigma, \rightarrow, Q_O, Q_F)$ consists of a (not necessarily finite) set Q of states, an alphabet Σ of events, a transition relation $\rightarrow \subseteq Q \times \Sigma \times Q$, a set $Q_O \subseteq Q$ of initial states, and a set $Q_F \subseteq Q$ of final states. A transition $(q_1, \sigma, q_2) \in \rightarrow$ is denoted as $q_1 \xrightarrow{\sigma} q_2$. The transition system is finite if the cardinality of Q is finite and it is infinite otherwise. A region is a subset $P \subseteq Q$. Given $\sigma \in \Sigma$ we define the predecessor $Pre_\sigma(P)$ of a region P as

$$Pre_\sigma(P) = \{q \in Q \mid \exists p \in P \text{ and } q \xrightarrow{\sigma} p\} \tag{2.1}$$

Given an equivalence relation $\sim \subseteq Q \times Q$ on the state space one can define a quotient transition system as follows. Let $Q/\sim$ denote the quotient space. For a region P we denote by $P/\sim$ the collection of all equivalence classes which intersect P. The transition relation $\rightarrow_\sim$ on the quotient space is defined as follows: for Q_1,$Q_2 \in Q/\sim$, $Q_1 \xrightarrow{\sigma}_\sim Q_2$ iff there exist $q_1 \in Q_1$ and $q_2 \in Q_2$ such that $q_1 \xrightarrow{\sigma} q_2$. The quotient transition system is then $T/\sim = (Q/\sim, \Sigma, \rightarrow_\sim, Q_0/\sim, Q_F/\sim)$.

Given an equivalence relation $\sim$ on Q, we call a set a $\sim$-block if it is a union of equivalence classes. The equivalence relation $\sim$ is a *bisimulation* of T iff Q_O, Q_F are $\sim$-blocks and for all $\sigma \in \Sigma$ and all $\sim$-blocks P, the region $Pre_\sigma(P)$ is a $\sim$-block. In this case the systems T and $T/\sim$ are called *bisimilar*. We will also say that a partition is a bisimulation when its induced equivalence relation is a bisimulation. A bisimulation is called finite if it has a finite number of equivalence classes. Bisimulations are very important because bisimilar transition systems generate the same language [14]. Therefore, checking properties on the bisimilar transition system is equivalent to checking properties of the original transition system. This is very useful in reducing the complexity of various verification algorithms where Q is finite but very large. In addition, if T is infinite and $T/\sim$ is a finite bisimulation, then verification algorithms for infinite systems are guaranteed to terminate. Successful applications of this approach for hybrid systems include timed automata [2], initialized rectangular automata [25], and linear hybrid automata [14]. It should be noted that the notion of bisimulation is similar to the notion of dynamic consistency [8, 9, 23]. If $\sim$ is a bisimulation, it can be easily shown that if $p \sim q$ then

B1: $p \in Q_F$ iff $q \in Q_F$, and $p \in Q_O$ iff $q \in Q_O$
B2: if $p \xrightarrow{\sigma} p'$ then there exists q' such that $q \xrightarrow{\sigma} q'$ and $p' \sim q'$

Based on the above characterization, given a transition system T, the following algorithm computes increasingly finer partitions of the state space Q. If the algorithm terminates, then the resulting quotient transition system is a finite bisimulation. The state space $Q/\sim$ is called a bisimilarity quotient.

Algorithm 1: (Bisimulation Algorithm for Transition Systems)
Set: $Q/\sim = \{Q_O \cap Q_F, Q_O \setminus Q_F, Q_F \setminus Q_O, Q \setminus (Q_O \cup Q_F)\}$
while: $\exists\ P, P' \in Q/\sim$ and $\sigma \in \Sigma$ such that $\emptyset \neq P \cap Pre_\sigma(P') \neq P$
  set: $P_1 = P \cap Pre_\sigma(P')$, $P_2 = P \setminus Pre_\sigma(P')$

refine: $Q/\sim = (Q/\sim \backslash\{P\}) \cup \{P_1, P_2\}$
end while:

Notice that each time the partition $Q/\sim$ is refined, the transitions are updated to account for the newly subdivided sets. When checking specific properties, such as reachability to the set Q_F, one might simplify the algorithm by starting with a coarser partition, for example $\{Q_F, Q \backslash Q_F\}$. In general one should include in the initial partition all additional sets relevant to the verification problem of interest (such as safe or unsafe regions). The larger the initial class of sets the more difficult it is for the algorithm to terminate.

3. Bisimulations of Hybrid Systems

We focus on transition systems generated by the following class of hybrid systems.

Definition 3.1. A *hybrid system* is a tuple $H = (X, X_0, X_F, F, E, I, G, R)$ where

- $X = X_D \times X_C$ is the state space with $X_D = \{q_1, \ldots, q_n\}$ and X_C a manifold.
- $X_0 \subseteq X$ is the set of initial states
- $X_F \subseteq X$ is the set of final states
- $F : X \longrightarrow TX_C$ assigns to each discrete location $q \in X_D$ a vector field $F(q, \cdot)$
- $E \subseteq X_D \times X_D$ is the set of discrete transitions
- $I : X_D \longrightarrow 2^{X_C}$ assigns to each location a set $I(q) \subseteq X_C$ called the invariant.
- $G : E \longrightarrow X_D \times 2^{X_C}$ assigns to $e = (q_1, q_2) \in E$ a guard of the form $\{q_1\} \times U$, $U \subseteq I(q_1)$.
- $R : E \longrightarrow X_D \times 2^{X_C}$ assigns to $e = (q_1, q_2) \in E$ a reset of the form $\{q_2\} \times V$, $V \subseteq I(q_2)$.

Trajectories of the hybrid system H originate at any $(q, x) \in X_0$ and consist of either continuous evolutions or discrete jumps. Continuous trajectories keep the discrete part of the state constant, and the continuous part evolves according to the continuous flow $F(q, \cdot)$ as long as the flow remains inside the invariant set $I(q)$. If the flow exits $I(q)$, then a discrete transition is *forced.* If, during the continuous evolution, a state $(q, x) \in G(e)$ is reached for some $e \in E$, then discrete transition e is *enabled.* The hybrid system may then instantaneously jump from (q, x) to any $(q', x') \in R(e)$ and the system then evolves according to the flow $F(q', \cdot)$. Notice that even though the continuous evolution is deterministic, the discrete evolution may be nondeterministic. The discrete transitions allowed in our model are slightly more restrictive than those in initialized rectangular automata [1, 2, 25]. In rectangular automata, the continuous dynamics are decoupled and each component of the continuous part of the state may be either reset nondeterministically to an interval or remain the same. If, however, the dynamics of a particular component changes then the reset map cannot be the identity map on that component. In this paper, we restrict the reset maps in order to allow complex and fully coupled dynamics. However, one could use the techniques in this paper to deal with decoupled dynamics but more general reset maps. Finally, We assume that our hybrid system model is *non-blocking*, that is from every state either a continuous evolution or a discrete transition is possible.

Example 3.2. A typical hybrid system is shown in Figure 1. The state space is $\{Q1, Q2\} \times \mathbb{R}^2$. The initial states are of the form $\{Q1\} \times \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 1 < y < 2\}$. The discrete

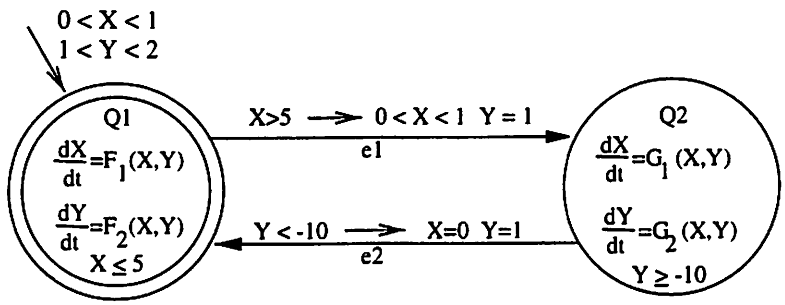

FIGURE 1. A typical hybrid automaton

dynamics consists of two transitions $e_1 = (Q1, Q2)$ and $e_2 = (Q2, Q1)$. Within location $Q1$, the continuous variables x and y evolve according to a differential equation as long as $(x, y) \in I(Q_1) = \{(x, y) \in \mathbb{R}^2 \mid x \leq 5\}$. Once $x > 5$, discrete transition e_1 is forced and x,y are nondeterministically reset to values in fixed sets. The system then flows according to the flow associated with $Q2$. The evolution from that point on is similar. We would like to find out whether the system will reach the set of final states $\{Q_2\} \times \{(x, y) \in \mathbb{R}^2 \mid x < -5\}$.

Every hybrid system $H = (X, X_0, X_F, F, E, I, G, R)$ generates a transition system $T = (Q, \Sigma, \rightarrow, Q_O, Q_F)$ by setting $Q = X$, $Q_0 = X_0$, $Q_F = X_F$, $\Sigma = E \cup \{\tau\}$, and $\rightarrow = (\cup_{e \in E} \xrightarrow{e}) \cup \xrightarrow{\tau}$ where

Discrete Transitions: $(q, x) \xrightarrow{e} (q', x')$ for $e \in E$ iff $(q, x) \in G(e)$ and $(q', x') \in R(e)$
Continuous Transitions: $(q_1, x_1) \xrightarrow{\tau} (q_2, x_2)$ iff $q_1 = q_2$ and there exists $\delta \geq 0$ and a curve $x : [0, \delta] \longrightarrow M$ with $x(0) = x_1$, $x(\delta) = x_2$ and for all $t \in [0, \delta]$ it satisfies $x' = F(q_1, x(t))$ and $x(t) \in I(q_1)$.

The continuous τ transitions are time-abstract transitions, in the sense that the time it takes to reach one state from another is ignored. Having defined the continuous and discrete transitions $\xrightarrow{\tau}$ and $\xrightarrow{e}$ allows us to formally define $Pre_\tau(P)$ and $Pre_e(P)$ for $e \in E$ and any region $P \subseteq X$ using (2.1). Furthermore, the structure of the discrete transitions allowed in our hybrid system model result in

$$Pre_e(P) = \begin{cases} \emptyset & \text{if } P \cap R(e) = \emptyset \\ G(e) & \text{if } P \cap R(e) \neq \emptyset \end{cases} \tag{3.1}$$

for all discrete transitions $e \in E$ and regions P. Therefore, if the sets $R(e)$ and $G(e)$ are blocks of any partition of the state space, then no partition refinement is necessary in the bisimulation algorithm due to any discrete transitions $e \in E$. This fact, in a sense, decouples the continuous and discrete components of the hybrid system. In turn, this implies that the initial partition in the bisimulation algorithm should contain the invariants, guards and reset sets, in addition to the initial and final sets. This allows us to carry out the algorithm independently for each location.

More precisely, define for any region $P \subseteq X$ and $q \in X_D$ the set $P_q = \{x \in X_C : (q, x) \in P\}$. For each location $q \in X_D$ consider the finite collection of sets

$$\mathcal{A}_q = \{I(q), (X_0)_q, (X_F)_q\} \cup \{G(e)_q, R(e)_q : e \in E\} \tag{3.2}$$

which describes the initial and final states, guards, invariants and resets associated with location q. Let $\mathcal{S}_q$ be the coarsest partition of X_C compatible with the collection $\mathcal{A}_q$ (by compatible we mean that each set in $\mathcal{A}_q$ is a union of sets in $\mathcal{S}_q$). The (finite) partition $\mathcal{S}_q$ can be easily computed by successively finding the intersections between each of the sets in $\mathcal{A}_q$ and their complements. These collections $\mathcal{S}_q$ will be the starting partitions of the bisimulation algorithm.

Algorithm 2: (Bisimulation Algorithm for Hybrid Systems)
Set: $X/\sim \;= \bigcup_q \mathcal{S}_q$
for: $q \in X_D$
  while: $\exists\ P, P' \in \mathcal{S}_q$ such that $\emptyset \neq P \cap Pre_\tau(P') \neq P$
    Set: $P_1 = P \cap Pre_\tau(P')$; $P_2 = P \setminus Pre_\tau(P')$
    refine: $\mathcal{S}_q \;= (\mathcal{S}_q \setminus \{P\}) \cup \{P_1, P_2\}$
  end while:
end for:

The following example shows that, even in apparently simple situations, **Algorithm 2** does not terminate.

Example 3.3. Consider the hybrid system with only one discrete location and let F be the linear vector field $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathrm{x}$ on $\mathbb{R}^2$. Assume the partition of $\mathbb{R}^2$ consists of the following three sets (see Figure 2): $P_1 = \{(x, 0) : 0 \leq x \leq 4\}$, $P_2 = \{(x, 0) : -4 \leq x < 0\}$, $P_3 = \mathbb{R}^2 \setminus (P_1 \cup P_2)$. The integral curves of F are spirals moving away from the origin. The first iteration of

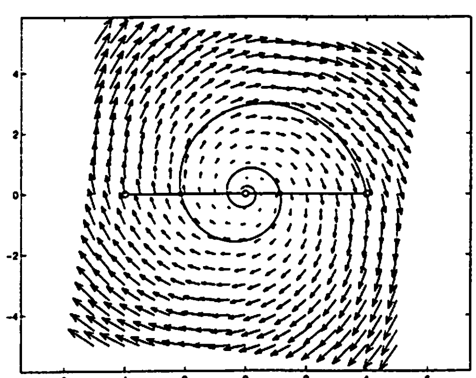

FIGURE 2. Algorithm 2 does not terminate

the algorithm partitions P_2 into $P_4 = P_2 \cap Pre_\tau(P_1) = \{(x, 0) : x_1 \leq x < 0\}$ and $P_2 \setminus Pre_\tau(P_1)$. Here $x_1 < 0$ is the x-coordinate of the first intersection point of the spiral through $(4, 0)$ with P_2. The second iteration subdivides P_1 into $P_5 = P_1 \cap Pre_\tau(P_4) = \{(x, 0) : 0 \leq x \leq x_2\}$ and

$P_1 \setminus Pre_\tau(P_4)$ where $x_2 > 0$ is the x-coordinate of the next point of intersection of the spiral with P_1. This process continues indefinitely since the spiral intersects P_1 in infinitely many points, and therefore the algorithm does not terminate.

From the above example it is clear that the critical problem one must investigate is how the flow of $F(q, \cdot)$ interacts with the sets S_q for a single location q. This requires that the trajectories of the vector field $F(q, \cdot)$ have "nice" intersection properties with such sets. Since the goal is to obtain finite partitions, it will become important that we restrict the study to classes of sets with global "finiteness" properties, for example, sets with finitely many connected components. In the next section, we identify such classes of sets and vector fields using the concept of o-minimal structures from model theory.

4. Model Theory

Model theory studies structures through properties of their definable sets (see [16, 29] for general background). The basic structures of interest for this paper are that of the real numbers, symbolized by $(\mathbb{R}, +, -, <, 0, 1)$, and its extensions. Every such structure L has an associated language $\mathcal{L}$ of formulas. The (first order) formulas over $\mathcal{L}$ are the well-formed logical expressions obtained by using logical connectives, quantifiers $\exists$ $\forall$ (quantification is allowed over the reals), real numbers as constants, the operation of addition, and the relations $<$ and $=$. All formulas will be interpreted over the real numbers. A *definable set* in the language $\mathcal{L}$ (or of the structure L) is a subset of $\mathbb{R}^n$ (for some n) of the form $\{(a_1, \ldots, a_n) \in \mathbb{R}^n : \Phi(a_1, \ldots, a_n)\}$, where $\Phi(x_1, \ldots, x_n)$ is a formula in $\mathcal{L}$ and $x_1, \ldots, x_n$ are free (i.e. not quantified) variables in Φ. A function f is definable if its graph is a definable set. The collection of definable sets is closed under Boolean operations and taking forward or inverse images under definable functions. While many of the concepts here apply to more general structures, in all that follows we consider only structures over the real numbers.

Definition 4.1. The theory of $\mathcal{L}$ is *o-minimal* ("order minimal") if every definable subset of $\mathbb{R}$ is a finite union of points and intervals (possibly unbounded).

The class of o-minimal structures is quite rich. In [28] it was shown that the theory of the real numbers as a real closed field, $(\mathbb{R}, +, -, \times, <, 0, 1)$, admits elimination of quantifiers, which in turn implies it is o-minimal. Tarski was also interested in the extension of the theory of the real numbers by the exponential function, $\mathbb{R}_{\text{exp}} = (\mathbb{R}, +, -, \times, <, 0, 1, \exp)$ (i.e., there is an additional symbol in the language for the exponential function). While such theory does not admit elimination of quantifiers, it was shown in [34] that such theory is o-minimal. Another important extension is obtained as follows. Assume f is a real-analytic function in a neighborhood of the cube $[-1, 1]^n \subset \mathbb{R}^n$. Let $\hat{f} \colon \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in [-1, 1]^n \\ 0 & \text{otherwise} \end{cases}$$

We call such functions *restricted analytic functions.* The structure $\mathbb{R}_{\text{exp,an}} = (\mathbb{R}, +, -, \times, <, 0, 1, \exp, \{\hat{f}\})$ is then an extension of $\mathbb{R}_{\text{exp}}$ where there is a symbol for each restricted analytic function. In [32], it was shown that $\mathbb{R}_{\text{exp.an}}$ is also o-minimal. More recently it was shown in [27] that, so called Pfaffian extensions of o-minimal theories are also o-minimal.

The following table summarizes o-minimal structures (even very recent ones) along with some examples of sets and flows that are definable in these structures. We will examine the connection between these o-minimal extensions and different classes of hybrid system in Section 6.

Table 1 : O-minimal structures

Name	Theory	Definable Sets	Definable Flows
$\mathbb{R}_{\text{lin}}$	$(\mathbb{R}, +, -, <, 0, 1)$	Polyhedral sets	Linear flows
$\mathbb{R}_{\text{alg}}$	$(\mathbb{R}, +, -, \times, <, 0, 1)$	Semialgebraic sets	Polynomial flows
$\mathbb{R}_{\text{an}}$	$(\mathbb{R}, +, -, \times, <, 0, 1, \{\hat{f}\})$	Subanalytic sets	Polynomial flows
$\mathbb{R}_{\text{exp}}$	$(\mathbb{R}, +, -, \times, <, 0, 1, \exp)$	Semialgebraic sets	Exponential flows
$\mathbb{R}_{\text{exp,an}}$	$(\mathbb{R}, +, -, \times, <, 0, 1, \exp, \{\hat{f}\})$	Subanalytic sets	Exponential flows

Many geometric properties of the above structures can be found in [33] and the upcoming book [30]. We present below those properties of o-minimal structures that are used in the proof of the main theorem.

We assume given a structure L which is an extension of $(\mathbb{R}, +, -, <, 0, 1)$. Definability will refer to this structure.

Definition 4.2. We define a *cell* in $\mathbb{R}^n$ inductively as follows:

1. The cells in $\mathbb{R}$ are just the points $\{c\}$ with $c \in \mathbb{R}$ and the open intervals (a, b), $-\infty \leq a < b \leq +\infty$.
2. Let $C \subset \mathbb{R}^n$ be a cell and let $f, g : C \to \mathbb{R}$ be definable continuous functions such that $f < g$ on C. Then $(f, g) = \{(x, r) \in C \times \mathbb{R} : f(x) < r < g(x)\} \subseteq \mathbb{R}^{n+1}$, is a cell in $\mathbb{R}^{n+1}$. Also, for each definable function $f : C \to \mathbb{R}$, the graph of f and the sets $(-\infty, f) = \{(x, r) \in C \times \mathbb{R} : r < f(x)\}$, $(f, +\infty) = \{(x, r) \in C \times \mathbb{R} : f(x) < r\}$ and $C \times \mathbb{R}$ are cells in $\mathbb{R}^{n+1}$.

Theorem 4.3. *Assume L is an o-minimal structure which is an extension of* $(\mathbb{R}, +, -, <, 0, 1)$. *Then*

1. **(Cell Decomposition)** *Given any finite family $\{A_1, \ldots, A_l\}$ of definable subsets of $\mathbb{R}^n$ there exists a partition of $\mathbb{R}^n$ into cells so that each A_i is a union of such cells* [18].
2. *Any definable set has a finite number of connected components, each of which is a definable set. Moreover, if $A \subset \mathbb{R}^n \times \mathbb{R}$ is definable then there exists a positive integer N such that for each $x \in \mathbb{R}^n$ the number of connected components of $A_x = \{t \in \mathbb{R} : (x, t) \in A\}$ is less than N. (A consequence of cell decomposition.)*
3. *If A is definable and connected then it is arcwise connected. (A consequence of cell decomposition and the curve selection theorem in* [31].*)*

The above Cell Decomposition Theorem is used to provide the initial partition of **Algorithm 2**. It is also the first step in the proof of the main theorem.

5. O-Minimal Hybrid Systems

In this section we prove the main theorem and give specific examples of new classes of hybrid systems which admit a finite bisimulation. We also show how the existing results fit in this context.

Definition 5.1. A hybrid system $H = (X, X_0, X_F, F, E, I, G, R)$ is said to be o-minimal if

- $X_C = \mathbb{R}^n$
- for each $q \in X_D$ the flow of F_q is complete
- for each $q \in X_D$ the family of sets $\mathcal{A}_q = \{I(q), (X_0)_q, (X_F)_q\} \cup \{G(e)_q, R(e)_q : e \in E\}$ and the flow of F_q are definable in an o-minimal extension of $(\mathbb{R}, +, -, \times, <, 0, 1)$.

Theorem 5.2. *Every o-minimal hybrid system admits a finite bisimulation. In particular, the bisimulation algorithm, Algorithm 2, terminates for o-minimal hybrid systems.*

Proof. We assume given a fixed o-minimal extension $\overline{\mathcal{R}}$ of $\mathbb{R}(+, -, \cdot, <, 0, 1\}$, in which all relevant objects are definable. From now on, definable will mean definable in $\overline{\mathcal{R}}$. We start by applying the cell decomposition theorem on each family $\mathcal{A}_q$. As mentioned in Section 3, the special form of $Pre_e(P)$ allows us to construct the bisimulation quotient on each set $\{q\} \times X_C$ separately. Therefore, we assume given a finite partition $\mathcal{P}$ of $\mathbb{R}^n$ into definable sets and a vector field F whose flow is definable. Moreover, we will simply write Pre for Pre_τ.

The outline of the proof is as follows. We first perform an initial finite refinement $\tilde{\mathcal{P}}$ of $\mathcal{P}$ which has the property that the intersection of any trajectory with each set has one connected component. Because of this property we can use a slight variation of the iterative step of the bisimulation algorithm to construct a finite partition $\mathcal{B}$ which is a further refinement, and satisfies the bisimulation condition, namely, that for any $B \in \mathcal{B}$, the set $Pre(B)$ is a finite union of set in $\mathcal{B}$. This guarantees that the bisimulation algorithm terminates.

We first notice that if $f : \mathbb{R} \to \mathbb{R}^n$ is continuous, periodic, and not constant, then f is not definable. Indeed, for such f there is $y \in \mathbb{R}^n$ such that the set $R = \{x \in \mathbb{R} : f(x) = y\}$ consists of an infinite number of isolated points. On the other hand, if f is definable, then so is R, but this contradicts o-minimality.

For each $x \in \mathbb{R}^n$, $\gamma_x(t)$ will denote the integral curve of F which passes through x at $t = 0$. That is, $\dot{\gamma}_x(t) = F(\gamma_x(t))$ and $\gamma_x(0) = x$. Therefore, $\Phi(x,t) = \gamma_x(t)$ denotes the flow of F and is definable by hypothesis. Combining this with the comment above we conclude that for each $x \in \mathbb{R}^n$, $\gamma_x(\cdot)$ is either constant or injective.

We will need the following lemma.

Lemma 5.3. *Let F and $\Phi(x,t)$ be as above, and let γ be an integral curve of F. Define $\Gamma = Im(\gamma) = \{\gamma(t) : t \in \mathbb{R}\}$. Let S be a definable set and C a connected component of $\Gamma \cap S$. If $t_0, t_1 \in \mathbb{R}$ are such that $\gamma(t_0), \gamma(t_1) \in C$, then $\gamma(t) \in C$ for all $t_0 \le t \le t_1$.*

Proof. Since C is definable and connected, it is also arcwise connected. Let $\beta : [0,1] \to C$ be continuous and such that $\beta(0) = \gamma(t_0)$ and $\beta(1) = \gamma(t_1)$. If γ is constant there is nothing to prove. We can then assume γ is injective and $F(\gamma(t)) \neq 0$ for any t. Therefore, the restriction of γ to any compact interval $[a, b]$ is a homeomorphism between $[a, b]$ and $\gamma([a, b])$. If $\beta([0,1]) \subseteq \gamma([a, b])$ then $\gamma^{-1} \circ \beta$ is continuous and so $\gamma^{-1} \circ \beta([0,1])$ is an interval containing t_0, t_1. Therefore, for all $t \in [t_0, t_1]$, $\gamma(t) \in \beta([0,1]) \subseteq C$ as desired.

Assume then that $\beta([0,1])$ is not contained in the image under γ of any finite interval. Hence there exist a sequence $\{t_n\}$ with $|t_n| \to \infty$ and $\gamma(t_n) \in \beta([0,1])$ for all n. By taking a subsequence if necessary we may assume that $\gamma(t_n) \to \tilde{x} \in \beta([0,1])$. Therefore, $\tilde{x} = \gamma(\tilde{t})$ for some $\tilde{t} \in \mathbb{R}$. We will show that this is a contradiction. In a (definable) neighborhood B of $\tilde{x}$ we can make a definable change of coordinates centered at $\tilde{x}$, so that in this coordinates $F \equiv \frac{\partial}{\partial x_1}$. In fact, after a translation and rotation (which are definable) we can assume that $\tilde{x} = 0$ and $F(0) = \frac{\partial}{\partial x_1}$. Then the desired change of coordinates is given by

$$(y_1, \dots, y_n) \longrightarrow \Phi((0, y_2, \dots, y_n), y_1)$$

Therefore, in that neighborhood all integral curves of F are of the form $\gamma(t) = (t, a_2, \dots, a_n)$ for some constant $a_2, \dots, a_n$. By restricting the neighborhood further we may assume it is of the form

$$B = \{(x_1, \dots, x_n) : \underline{a}_i \leq x_i \leq \bar{a}_i\}$$

The set $\Gamma \cap B$ is a union of at most countably many sets of the form $I_{a_2,\dots,a_n} = \{(t, a_2, \dots, a_n) : \underline{a}_1 \leq t \leq \bar{a}_1\}$ and so each such set is a connected component. By o-minimality, $\Gamma \cap B$ is a union of finitely many such sets. By shrinking the set B, if necessary, we may assume that

$$\Gamma \cap B = \{(t, 0, \dots, 0) : \underline{a}_1 \leq t \leq \bar{a}_1\}.$$

For n large enough we must have $\gamma(t_n) \in \Gamma \cap B$. Therefore, for such an n there exists t near $\tilde{t}$ such that $\gamma(t) = \gamma(t_n)$, which contradicts the injectivity of γ. This concludes the proof of the lemma. $\square$

We now continue with the proof of the main theorem. Given a set S, we define $H = \{(x, t) \in \mathbb{R}^{n+1} : \Phi(x, t) \in S\}$. If S is definable, then H is definable. Moreover, by o-minimality there exists $N_S \in \mathbb{N}$ such that the number of connected components of $H_x = \{t : (x, t) \in H\}$ is less than N for all $x \in \mathbb{R}^n$. This implies that if S is definable and Γ_x denotes the trajectory of F passing through x, then the number of connected components of $\Gamma_x \cap S$ is bounded above by a constant independent of x. We choose $N \in \mathbb{N}$ larger than the corresponding N_S for all sets $S \in \mathcal{P}$.

We begin the construction of the partition $\mathcal{B}$ by subdividing each set S in $\mathcal{P}$ as follows. Let

$$\begin{aligned}
S_0 &= \{x \in X : \forall t \geq 0 \ \ \gamma_x(t) \in S\} \\
S_1 &= \{x \in S \setminus S_0 : \forall t \geq 0 \ (\gamma_x(t) \notin S \setminus S_0 \Rightarrow \forall t' \geq t \ \gamma_x(t') \notin S \setminus S_0)\} \\
&\ \ \vdots \\
S_i &= \{x \in S \setminus (S_0 \cup \dots \cup S_{i-1}) : \\
&\qquad \forall t \geq 0 \ \ (\gamma_x(t) \notin S \setminus (S_0 \cup \dots \cup S_{i-1}) \Rightarrow \forall t' \geq t \ \gamma_x(t') \notin S \setminus (S_0 \cup \dots \cup S_{i-1}))\} \\
&\ \ \vdots
\end{aligned}$$

The set S_i is clearly definable for every i. For $i \geq 1$ the set S_i consists of those x for which γ_x leaves the set $S \setminus (S_0 \cup \cdots \cup S_{i-1})$ but never returns to it.

Claim: $S_k = \emptyset$ for $k \geq N$.

To prove the claim it suffices to show that if $x \in S_i$ with $i \geq 1$, then $\Gamma_x \cap S$ has at least i connected components. To prove this we will use a couple of lemmas.

Lemma 5.4. *Let S and S_i, $i \geq 0$ be as above. Let I be an interval and $\gamma(\cdot)$ an integral curve of F such that $\gamma(I) \subseteq S$. If $\gamma(t_0) \in S_i$ for some $t_0 \in I$, then $\gamma(I) \subseteq S_i$.*

Proof. We proceed by induction. The statement is clearly true for S_0. Assume it holds for $i \leq k$. Let $\gamma(I) \subseteq S$, $t_0 \in I$ and $\gamma(t_0) \in S_{k+1}$. Then $\gamma(t_0) \in S \setminus (S_0 \cup \ldots \cup S_k)$. For any $t \in I$, if $\gamma(t) \in S_0 \cup \ldots \cup S_k$ then there is $j \leq k$ such that $\gamma(t) \in S_j$. By the inductive hypothesis, $\gamma(I) \subseteq S_j$, but this contradicts $\gamma(t_0) \notin S_j$. Therefore we have $\gamma(I) \subseteq S \setminus (S_0 \cup \ldots \cup S_k)$. Let $t \in I$ and $t' > t$ be such that $\gamma(t') \notin S \setminus (S_0 \cup \ldots \cup S_k)$. Then $t' \notin I$ and so $t' > t_0$. Since $\gamma(t_0) \in S_{k+1}$ we conclude that for any $t'' > t'$ we get $\gamma(t'') \notin S \setminus (S_0 \cup \ldots \cup S_k)$. This shows that $\gamma(t) \in S_{k+1}$. □

Lemma 5.5. *If $x \in S_i$ for $i \geq 2$ then there exist $t_1 > s_1 > t_2 > \cdots > s_{i-2} > t_{i-1} > s_{i-1} > 0$ such that $\gamma_x(s_j) \notin S$ and $\gamma_x(t_j) \in S_j$ for $j = 1, \ldots, i-1$.*

Proof. We proceed by induction. Let $x \in S_2$. Then $x \in S \setminus (S_0 \cup S_1) \subseteq S \setminus S_1$. Therefore there exist $t > s > 0$ such that $\gamma_x(s) \notin S \setminus S_0$ and $\gamma_x(t) \in S \setminus S_0$. We can not have $\gamma_x(s) \in S_0$ because then we would also have $\gamma_x(t) \in S_0$. Therefore $\gamma_x(s) \notin S$. We set $s_1 = s$. If $\gamma_x(t) \in S_1$ then we set $t_1 = t$. Otherwise, there exist $t' > s' > t$ such that $\gamma_x(s') \notin S \setminus S_0$ and $\gamma_x(t') \in S \setminus S_0$. Since $x \in S_2$, $\gamma_x(s) \notin S \setminus (S_0 \cup S_1)$, and $t' > s$ we must have $\gamma_x(t') \notin S \setminus (S_0 \cup S_1)$. Therefore $\gamma_x(t') \in S_1$ and we set $t_1 = t'$. This completes the proof for the case $i = 2$.

Assume now the conclusion holds for i and let $x \in S_{i+1}$. In particular, $x \in S \setminus S_i$, and there are $t > s > 0$ such that $\gamma_x(s) \notin S \setminus (S_0 \cup \ldots \cup S_{i-1})$ and $\gamma_x(t) \in S \setminus (S_0 \cup \ldots \cup S_{i-1})$. If $\gamma_x(s) \in S_j$ for some $j \leq i-1$ and $\gamma_x(\bar{s}) \in S$ for all $s \leq \bar{s} \leq t$, then Lemma 5.4 would imply that $\gamma_x(t) \in S_j$ which is not true. Therefore there exists $\bar{s}$, $s \leq \bar{s} < t$ such that $\gamma_x(\bar{s}) \notin S$. We set $s_i = \bar{s}$.

If $\gamma_x(t) \in S_i$ then we set $t_i = t$. Otherwise, there exist $t' > s' > t$ such that $\gamma_x(s') \notin S \setminus (S_0 \cup \ldots \cup S_{i-1})$ and $\gamma_x(t') \in S \setminus (S_0 \cup \ldots \cup S_{i-1})$. Since $x \in S_{i+1}$, $\gamma_x(\bar{s}) \notin S \setminus (S_0 \cup \ldots \cup S_i)$, and $t' > \bar{s}$ we must have $\gamma_x(t') \notin S \setminus (S_0 \cup \ldots \cup S_i)$. Therefore $\gamma_x(t') \in S_i$ and we set $t_i = t'$.

By the inductive hypothesis there exist $\tilde{t}_1 > \tilde{s}_1 > \cdots > \tilde{t}_{i-1} > \tilde{s}_{i-1} > 0$ such that $\gamma_{\gamma_x(t_i)}(\tilde{s}_j) \notin S_j$, $\gamma_{\gamma_x(t_i)}(\tilde{t}_j) \in S_j$, for $j = 1, \ldots, i-1$. Setting $s_j = \tilde{s}_j + t_i$, $t_j = \tilde{t}_j + t_i$ for $j = 1, \ldots, i-1$ we get the desired conclusion. □

The last lemma together with Lemma 5.3 proves that if $x \in S_i$ then $\Gamma_x \cap S$ has at least i connected components. This, in turn, proves the claim.

Notice that Lemma 5.3 also implies that if $x \in S_i$ then $\Gamma_x \cap S_i$ has exactly one connected component.

By carrying out the subdivision into the sets S_i for all $S \in \mathcal{P}$ we obtain a new finite partition $\widetilde{\mathcal{P}}$ of $\mathbb{R}^n$ with the property

(P) For each $S \in \widetilde{\mathcal{P}}$, and each trajectory γ of F such that $\gamma(t_0), \gamma(t_1) \in S$ we have $\gamma(t) \in S$ for all t with $t_0 \leq t \leq t_1$. In particular, for each $x \in S$, the set $\Gamma_x \cap S$ has exactly one connected component.

We will denote by $\rho = \rho(\widetilde{\mathcal{P}})$ the number of sets in $\widetilde{\mathcal{P}}$ and write $\widetilde{\mathcal{P}} = \{S_i : i = 1, \dots, \rho\}$.

We introduce two functions, I and C, acting on pairs of sets, defined by

$$\begin{aligned} I(A,B) &= A \cap Pre(B) \\ C(A,B) &= A \setminus Pre(B) \end{aligned}$$

It is clear that if A and B are definable, then $I(A,B)$ and $C(A,B)$ are definable. Notice also that for each A, B the sets $I(A,B), C(A,B)$ form a partition of A.

For each i, $1 \leq i \leq \rho$ consider all the partitions of S_i defined by

$$I(S_i, Q(S_{j_1}, Q(S_{j_2}, \dots, Q(S_{j_{k-1}}, S_{j_k}) \dots))) \tag{5.1}$$

$$C(S_i, Q(S_{j_1}, Q(S_{j_2}, \dots, Q(S_{j_{k-1}}, S_{j_k}) \dots))) \tag{5.2}$$

where Q is either I or C and $1 \leq j_l \leq \rho$ for $l = 1, \dots, k$. This is a finite collection of partitions. We let $\mathcal{B}$ denote the coarsest partition of $\mathbb{R}^n$ compatible with all such partitions.

Claim: $\mathcal{B}$ is a bisimulation.

The intuitive basis for this proof is the fact that the partitions constructed so far are done "along the flow of F." That is, two sets in $\mathcal{B}$ which are subsets of the same set in $\widetilde{\mathcal{P}}$ can not be connected by a trajectory of F.

To prove the claim first notice that the sets in $\mathcal{B}$ are (finite) intersections of sets of the form (5.1) or (5.2). Notice also that by construction $\mathcal{B}$ is a refinement of $\mathcal{P}$.

To check the bisimulation property let $B \in \mathcal{B}$, $B \subseteq S \in \widetilde{\mathcal{P}}$, be written as

$$B = \bigcap_{l=1}^{m} P_l$$

where each P_l is of the form (5.1) or (5.2). We want to prove first that

$$Pre(B) = \bigcap_{l=1}^{m} Pre(P_l). \tag{5.3}$$

The inclusion $Pre(B) \subseteq \cap_{l=1}^{m} Pre(P_l)$ is straightforward. For the other one let $x \in \cap_{l=1}^{m} Pre(P_l)$. For each l there exists $t_l \geq 0$ such that $\gamma_x(t_l) \in P_l$. Each set P_l is of the form $I(S_i, A_l)$ or $C(S_i, A_l)$ for some A_l's. Hence, $\gamma_x(t_l) \in S_i$ for all l. We now want to show that indeed $\gamma_x(t_l) \in B$ for all t_l. Consider the following property of a set A.

(Q) for any trajectory γ of F, if $\gamma(s_0) \in A \subseteq S \in \widetilde{\mathcal{P}}$, then for all s with $\gamma(s) \in S$, $\gamma(s) \in A$.

We show that if a set A has Property **(Q)**, then so do $I(S', A)$ and $C(S', A)$ for any $S' \in \widetilde{\mathcal{P}}$. Let $\gamma(s_0) \in I(S', A) \subseteq S'$. Then $\gamma(s_0) \in S'$ and there exists $t \geq s_0$ such that $\gamma(t) \in A$. If $\gamma(t) \in S'$, then we have $S = S'$ since both belong to $\widetilde{\mathcal{P}}$. By **(Q)** $\gamma(s) \in A \subseteq Pre(A)$ for all s such that $\gamma(s) \in S'$. Therefore $\gamma(s) \in I(S', A)$ for all such s. On the other hand, if $\gamma(t) \notin S'$, then $A \cap S' \subseteq S \cap S' = \emptyset$. Let $\gamma(s) \in S'$. By Property (P) applied to S' we get that $s \leq t$. But then $\gamma(s) \in Pre(A) \cap S'$ as desired. The proof for $C(S', A)$ is analogous.

Proceeding by induction it is easy to show that the sets P_l have Property **(Q)** and this completes the proof of (5.3).

Notice also, that $Pre(A \cup B) = Pre(A) \cup Pre(B)$ for all sets A, B.

To complete the proof that $\mathcal{B}$ is a bisimulation we only need to show that for each l, and each set $S \in \widetilde{\mathcal{P}}$, the set $S \cap Pre(P_l)$ is a union of sets in $\mathcal{B}$. The set $S \cap Pre(P_l) = I(S, P_l)$ is of the form (5.1) with $k \leq \rho + 1$. If $k < \rho + 1$ we already know that $I(S, P_l)$ is a union of sets in $\mathcal{B}$. We only need to consider the case $k = \rho + 1$.

There are two possibilities for $I(S, P_l)$:

1. there are two occurrences of C in $I(S, P_l)$,
2. there are $\rho + 1$ occurrences of I in $I(S, P_l)$, and therefore, at least one $S_i \in \widetilde{\mathcal{P}}$ is repeated as an argument of I.

In case 1 the following two formulas, and boolean algebra, show how to rewrite $I(S, P_l)$ either with fewer terms or using only I.

$$C(S_3, C(S_2, S_1)) = C(S_3, S_2) \cup I(S_3, I(S_2, S_1)) \tag{5.4}$$

$$C(S_3, I(S_2, S_1)) = C(S_3, S_2) \cup I(S_3, C(S_2, S_1)) \tag{5.5}$$

Both formulas can be proved with arguments similar to the ones above, relying on Property (P).

Finally, in case 2 we can show, again using (P) that $I(S, A) = \emptyset$. This concludes the proof that $\mathcal{B}$ is a bisimulation. □

Notice that in the proof we used multiplication only to find a suitable rotation to "straighten out" the flow of F_q. In the structure $(\mathbb{R}, +, -, <, 0, 1)$ where multiplication is not defined, the only definable flows are already complete and consist of straight lines. This leads to the following corollary.

Corollary 5.6. *Consider the hybrid system H with $X_C = \mathbb{R}^n$ and for each $q \in X_D$ the collection of sets $\mathcal{A}_q$ and the vector field F_q are definable in $(\mathbb{R}, +, -, <, 0, 1)$. Then the bisimulation algorithm terminates.*

In the next section we list various classes of o-minimal hybrid systems.

6. Classes of O-minimal Hybrid Systems

In this section, we apply Theorem 5.2 to several special classes of o-minimal hybrid systems. For each o-minimal structure of Table 1, we provide examples of definable, o-minimal hybrid systems. While it is clearly possible to identify other special cases, the ones described below cover most known results and several natural extensions.

6.1. $\mathbb{R}_{\text{lin}} = (\mathbb{R}, +, -, <, 0, 1)$. The definable sets in this theory capture polyhedral sets whereas the definable flows capture linear flows. Therefore the corresponding o-minimal hybrid system captures the standard initialized model of timed automata. In addition, it captures all other hybrid system models that can be transformed to timed automata such as multirate automata and rectangular automata.

It is a well known fact that this theory is not only o-minimal but also decidable. Therefore, the definable o-minimal hybrid systems do not only admit finite bisimulations but there is also an effective procedure to compute them. This immediately leads to decidability results for o-minimal hybrid systems defined in $\mathbb{R}_{\text{lin}}$. In particular, it captures initialized (in the sense defined in Section 3) timed automata [2], where all relevant sets are conjuctions of predicates of the form $x \diamond c$ with $\diamond$ being one of $>, \geq, \leq, <, =$, and $c \in \mathbb{Q}$, and flows are of the form $\dot{x} = 1$. We also capture initialized versions of multirate automata [1], and rectangular automata [15, 25]. Rectangular automata also allow for identity reset maps as long as the dynamics from one location to another remain the same. If we restrict the dynamics to be decoupled, then more general reset maps can be allowed.

6.2. $\mathbb{R}_{\text{alg}} = (\mathbb{R}, +, -, \times, <, 0, 1)$. It was shown in [28] that $\mathbb{R}_{\text{alg}}$ (without parameters) is decidable. In fact, the decision procedure consisted of two parts: first an algorithm for eliminating quantifiers, and second an algorithm for deciding quantifier free formulas. Because of these results, the definable sets in $\mathbb{R}_{\text{alg}}$ (with parameters) are the *semialgebraic sets*, which are defined as Boolean combinations of sets of the form $\{x : p(x) < 0\}$ and $\{x : p(x) = 0\}$ where $p(x)$ is a polynomial. The definable flows in this theory are polynomial. Therefore, the o-minimal hybrid systems corresponding to this theory are hybrid systems H where all sets all semialgebraic and all flows all polynomial. Moreover, if all polynomials involved in the description of the hybrid system have rational coefficients, we obtain a new class of decidable hybrid systems.

The o-minimality of this structure can also be used to show the existence of finite bisimulations in special cases when the flow is not definable. This was illustrated in [19] for the case of planar hybrid systems whose vector fields admit definable Hamiltonians. This captures the decidability result of [10].

6.3. $\mathbb{R}_{\text{an}} = (\mathbb{R}, +, -, \times, <, 0, 1, \{\hat{f}\})$. In order to describe the definable sets in this theory, we need the notions of *semianalytic* and *subanalytic sets*. We provide below an informal definition of these notions. For precise definitions and properties the reader is referred to [7]. We say that a bounded subset S of $\mathbb{R}^n$ is semianalytic in $\mathbb{R}^n$ if for every $x \in \mathbb{R}^n$ there exists a neighborhood U of x such that $U \cap S$ is a boolean combination of sets of the form

$\{x : f(x) < 0\}$ and $\{x : f(x) = 0\}$ where f is an analytic function on U. Roughly speaking, a local description of a semianalytic set is analogous to that of a semialgebraic set with analytic functions replacing polynomials. A bounded subset S of $\mathbb{R}^n$ is subanalytic in $\mathbb{R}^n$, if it is the image of a relatively compact semianalytic set T under an analytic map (defined on $\bar{T}$). The bounded subanalytic sets in $\mathbb{R}^n$ are definable in this theory.

Even though polynomial flows are definable in this theory, since the functions $\hat{f}$ are zero outside a compact set, they cannot be used to define complete flows. However, the *Pre* operator corresponding to some periodic flows may still be definable. Consider for example, a hybrid system H whose vector fields are diagonalizable linear vector fields with purely imaginary eigenvalues and all relevant sets are definable in $\mathbb{R}_{\mathbf{an}}$. Since the restriction of sin on $[-\pi, \pi]$ is definable, the *Pre* operator corresponding to F is definable. This leads to the following theorem which generalizes the planar result in [19].

Theorem 6.1. *Let H be a hybrid system for which all relevant sets are subanalytic and all vector fields are diagonalizable linear vector fields with purely imaginary eigenvalues. Then H admits a finite bisimulation.*

6.4. $\mathbb{R}_{\mathbf{exp}} = (\mathbb{R}, +, -, \times, <, 0, 1, \mathbf{exp})$. The main difference between $\mathbb{R}_{\mathbf{exp}}$ and the previous theories, besides enriching the class of definable sets, is the fact that the symbol exp represents a globally defined function. This allows new classes of definable flows. In particular, the flows of linear vector fields with real eigenvalues are definable. The following theorem is then a special case of Theorem 5.2.

Theorem 6.2. *Let H be a hybrid system for which all relevant sets are semialgebraic and all vector fields are linear with real eigenvalues. Then H admits a finite bisimulation.*

It is not known if the theory of $\mathbb{R}_{\mathbf{exp}}$ is decidable, although in [21] it was shown that it would be a consequence of Schanuel's conjecture in number theory.

6.5. $\mathbb{R}_{\mathbf{exp,an}} = (\mathbb{R}, +, -, \times, <, 0, 1, \mathbf{exp}, \{\hat{f}\})$. This theory extends both $\mathbb{R}_{\mathbf{an}}$ and $\mathbb{R}_{\mathbf{exp}}$. We can therefore combine the Theorems 6.1 and Theorems 6.2 to obtain the following result.

Theorem 6.3. *Let H be a hybrid system for which all relevant sets are subanalytic and all vector fields are of one of the following two forms:*

- *linear vector fields with real eigenvalues*
- *diagonalizable linear vector fields with purely imaginary eigenvalues*

Then H admits a finite bisimulation.

The above theorem extends the planar results in [19] to $\mathbb{R}^n$. Note that relaxations of Theorem 6.3 would allow spiraling, linear vector fields which are not definable in $\mathbb{R}_{\mathbf{exp,an}}$. As was shown by Example 3.3, such systems, in general, do not admit finite bisimulations.

6.6. Other Extensions. It is shown in [27] that extensions of o-minimal theories by Pfaffian functions are also o-minimal. Informally, the sequence of analytic functions $G_1, \ldots, G_l : U \longrightarrow \mathbb{R}^n$ form a Pfaffian chain on U if there exist polynomials p_{ij} such that for all $x \in U$

$$\frac{\partial G_i}{\partial x_j}(x) = p_{ij}(x, G_1(x), \ldots, G_i(x))$$

A function is called Pfaffian if it is the last function of some Pfaffian chain (see [17] for more precise definitions). While this theory provides new globally defined functions, there are no easily described classes of vector fields whose flows are definable in it. The search for such classes is a topic for current research.

7. Conclusions

In this paper, we presented a unified framework for tackling decidability questions of hybrid systems. We introduced the notion of o-minimal hybrid systems as initialized hybrid systems whose relevant sets and flows are definable in an o-minimal theory. We showed that all o-minimal hybrid systems admit finite bisimulations. Various examples from recently discovered o-minimal theories were presented. The examples capture most of the known decidable classes of hybrid systems. In addition, they extend the class of hybrid systems which admit finite bisimulations by enriching the class of relevant sets and incorporating more complex dynamics at each discrete location.

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