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TOWARDS A GEOMETRIC THEORY OF HYBRID SYSTEMS

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Towards a Geometric Theory of Hybrid Systems*

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Abstract

Given a deterministic, non-blocking hybrid system, we introduce the notion of its hybrid manifold (or hybrifold) with the associated hybrid flow on it. This enables us to study hybrid systems as (generally non-smooth) dynamical systems from a global geometric perspective. We introduce the notion of topological conjugacy of hybrid systems and locally classify Zeno states in dimension two. We show that the Zeno phenomenon is due to nonsmoothness of the hybrid flow and propose several ways of detecting and removing it. A stability result, capturing examples such as unstable + unstable = stable, and completely characterizing stable hybrid equilibria in dimension two, is proved in the last section.

1 Introduction

In this paper we present a unifying approach for treatment of hybrid systems. We define the notions of the *hybrid manifold* (or *hybrifold*) and *hybrid flow*, which enable us to study the hybrid system "in one piece", that is, as a single, generally non-smooth *dynamical system*. It is well known that even simple *smooth* dynamical systems can exibit a very complicated behavior which makes their global study very difficult using analytical methods. This is why developing qualitative (i.e. geometric and topological) techniques has been at the center of smooth dynamics, ever since Poincaré's foundational work at the end of the last century.

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Having established a reasonable framework for the geometric study of hybrid systems as dynamical systems, we focus particularly on the *Zeno phenomenon*, which does not occur in smooth dynamical systems. We study its causes, ways of removing it from the system, and classify it in dimension two. This classification is with respect to the notion of *topological conjugacy* borrowed from dynamical systems: two systems are conjugate if they are qualitatively the same.

The last section of the paper deals with stability of isolated hybrid equilibria. We prove a theorem which explains, among others, examples in which a *stable hybrid* equilibrium is composed of *unstable classical* equilibria.

2 Preliminaries

2.1 Definitions and examples

We start with the following, relatively standard definition of a hybrid system.

Definition 2.1 *An n-dimensional hybrid system is a 6-tuple*

$$\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R}),$$

where:

- $Q = \{1, \dots, k\}$ *is the collection of* (discrete) states *of* $\mathbf{H}$, *where* $k \geq 1$ *is an integer;*
- $E \subset Q \times Q$ *is the collection of* edges*;*
- $\mathcal{D} = \{D_i : i \in Q\}$ *is the collection of* domains *of* $\mathbf{H}$, *where* $D_i \subset \{i\} \times \mathbf{R}^n$ *for all* $i \in Q$*;*
- $\mathcal{X} = \{X_i : i \in Q\}$ *is the collection of vector fields such that* X_i *is Lipschitz on* D_i *for all* $i \in Q$*; we denote the local flow of* X_i *by* $\{\phi_t^i\}$*.*
- $\mathcal{G} = \{G(e) : e \in E\}$ *is the collection of* guards, *where for each* $e = (i, j) \in E$, $G(e) \subset D_i$*;*
- $\mathcal{R} = \{R_e : e \in E\}$ *is the collection of* resets, *where for each* $e = (i, j) \in E$, R_e *is a relation between elements of* $G(e)$ *and elements of* D_j, *i.e.* $R_e \subset G(e) \times D_j$*.*

A few remarks are in place here.

Remarks

(a) We choose to differ from the more standard terminology in which *domains* are called *invariants*. Since there is nothing dynamicaly invariant about these sets, we prefer to reserve the term for later, more appropriate use.

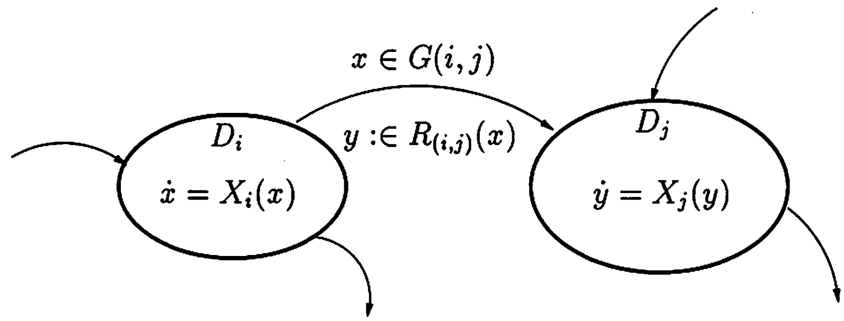

Figure 1: A hybrid system.

(b) Note that we do not consider the set of *initial states* as a separate item in the definition of a hybrid system. This is because we will restrict ourselves to studying only so called non-blocking systems in which all points can be initial conditions.

(c) The above definition clearly allows a hybrid system to be a very wild object. An important question is: what properties should the domains, guards, resets and vector fields in $\mathbf{H}$ satisfy to get a large enough class of hybrid systems about which something useful can be said. Soon we will deal with this question in detail and focus our attention on such a class of hybrid systems (which we will call regular).

(d) If a reset relation R_e is actually a map $G(e) \to D_j$, with $e = (i,j) \in E$, instead of $(x,y) \in R_e$ we write $y = R_e(x)$.

(e) Observe that domains D_i lie in distinct copies of $\mathbf{R}^n$. However, we will sometimes abuse the notation and consider the domains as subsets of a single copy of $\mathbf{R}^n$. We also set

$$D = \bigcup_{i \in Q} D_i,$$

and call this set the *total domain* of $\mathbf{H}$, and

$$G = \bigcup_{e \in E} G(e), \quad R = \bigcup_{e \in E} R_e(G(e)),$$

$$\overline{\mathcal{G}} = \{\overline{G(e)} : e \in E\}, \quad \overline{\mathcal{R}} = \{\overline{R_e(G(e))} : e \in E\}.$$

(f) To every hybrid system $\mathbf{H}$ we can associate its *graph*, $\Gamma(\mathbf{H})$, with elements of Q as vertices and E as the set of edges.

Given $\mathbf{H}$, the basic idea is that starting from a point in some domain D_i we flow according to X_i until (and if) we reach some guard $G(i,j)$, then switch via the reset $R_{(i,j)}$, continue flowing in D_j according to X_j and so on.

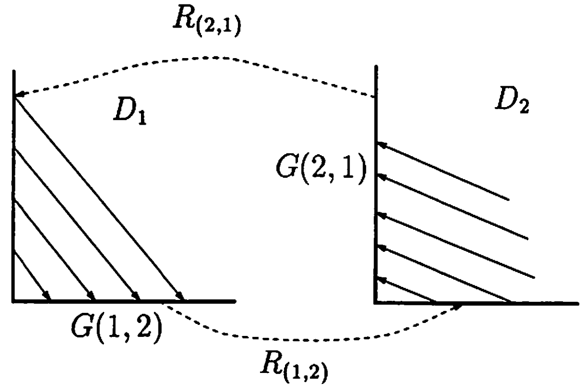

Figure 2: The water tank example.

Example 2.1 (Water Tank WT) Here $n = 2$, $k = 2$, $E = \{(1,2),(2,1)\}$,

$$D_1 = \{1\} \times C, \quad D_2 = \{2\} \times C,$$

where $C = [l_1, \infty) \times [l_2, \infty)$,

$$X_1 = (w - v_1, -v_2)^T, \quad X_2 = (-v_1, w - v_2)^T,$$

$$G(1,2) = \{(1, x_1, x_2) \in D_1 : x_2 = l_2\}, \quad G(2,1) = \{(2, x_1, x_2) \in D_2 : x_1 = l_1\},$$

and

$$R_{(1,2)}(1, x_1, l_2) = (2, x_1, , l_2), \quad R_{(2,1)}(2, l_1, x_2) = (1, l_1, x_2).$$

The interpretation is as follows (cf. Fig. 2). For $i \in Q$, x_i denotes the volume of water in tank i, v_i is the constant rate of flow of water out of tank i, and l_i is the desired volume of water in tank i. The constant rate of water flow into the system, dedicated exclusively to one tank at a time, is denoted by w. The control task is to keep the water volume above l_1 and l_2 (assuming the initial volumes are above l_1 and l_2 respectively) by a strategy that switches the inflow to the first tank whenever $x_1 = l_1$ and to the second tank whenever $x_2 = l_2$.

Example 2.2 (Bouncing Ball BB) This is a simplified model of an elastic ball that is bouncing and losing a fraction of its energy with each bounce. We denote by x_1 its altitude and by x_2 its vertical speed. Here $n = 2$, $k = 1$, $E = \{(1,1)\}$,

$$D_1 = \{(x_1, x_2) : x_1 \geq 0\}, \quad X_1(x_1, x_2) = (x_2, -g)^T,$$

$$G(1,1) = \{(0, x_2) : x_2 \leq 0\}, \quad R_{(1,1)}(0, x_2) = (0, -cx_2),$$

where g is the acceleration due to gravity and $0 < c < 1$ (cf. Fig. 3).

Example 2.3 (Bouncing m-Ball $BB(m)$) The only difference between this and the previous example is that we have m different domains in which the ball can bounce and after each

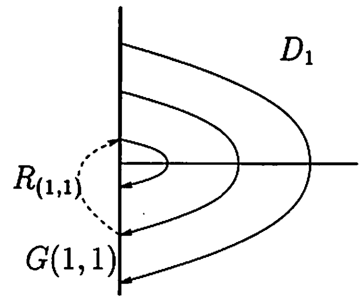

Figure 3: Bouncing ball.

bounce the ball switches to the next domain in a cyclic order. That is, $n = 2$, $k = m > 1$, $E = \{(1,2),(2,3),\dots,(m-1,m),(m,1)\}$, and for all $i \in Q$,

$$D_i = \{i\} \times \{(x_1,x_2) : x_1 \geq 0\}, \quad G(i,i+1) = \{i\} \times \{(0,x_2) : x_2 \leq 0\},$$

$$R_{(i,i+1)}(i,0,x_2) = (i+1,0,-cx_2),$$

where we conveniently identify $m+1 := 1$. Note that here the domains are just different copies of the closed right half-plane in $\mathbf{R}^2$.

Example 2.4 (Ball Bouncing on an N-step Staircase $BBS(N)$) Here a ball is bouncing on an N-step staircase. Assume that step $i = 1,\dots,N$ has width $w_i > 0$ and height $h_i > 0$, and define $\hat{w}_m = \sum_{i=1}^m w_i$ and $\hat{h}_m = \sum_{i=1}^m h_i$. Assume also that the ball loses a proportional amount of its vertical velocity (x_2) with each bounce and that the ball has constant horizontal speed (x_3). Denote by x_1 its vertical position. Then we have: $Q = \{1,\dots,N+1\}$, $E = \{(i,i) : 1 \leq i \leq N+1\} \cup \{(1,2),\dots,(N,N+1)\}$, and for $1 \leq i \leq N+1$:

$$D_i = \{i\} \times [\hat{h}_i,\infty) \times (-\infty,0] \times (-\infty,\hat{w}_i],$$

$$G(i,i) = \{(x_1,x_2,x_3) \in D_i : x_1 = \hat{h}_i\}, \quad R_{(i,i)}(i,x_1,x_2,x_3) = (i,x_1,-cx_2,x_3)$$

and $X_i(x_1,x_2,x_3) = (x_2,-g,v)^T$. Furthermore, for $1 \leq i \leq N$:

$$G(i,i+1) = \{(x_1,x_2,x_3) \in D_i : x_3 = \hat{w}_i\}, \quad R_{(i,i+1)}(i,\mathrm{x}) = (i+1,\mathrm{x}).$$

For more details see [JLSM].

Example 2.5 (Two Saddles $S2(\lambda)$) Here (see Fig. 4) $n = 2$, $k = 2$, $\lambda > 0$, $E = \{(1,2),(2,1)\}$, the domains are two copies of the square $S = [-1,1] \times [-1,1]$, i.e. for $i \in Q$,

$$D_i = \{i\} \times S, \quad X_1(x_1,x_2) = (\lambda x_1,-x_2)^T, \quad X_2(x_1,x_2) = (-x_1,\lambda x_2)^T,$$

$$G(1,2) = \text{union of the vertical sides of } D_1, \quad G(2,1) = \text{union of the horizontal sides of } D_2,$$

$$R_{(i,j)}(i,x) = (j,x), \text{ for all } (i,j) \in E.$$

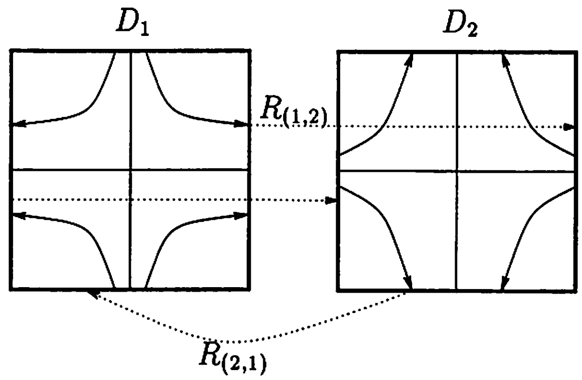

Figure 4: $S2(\lambda)$

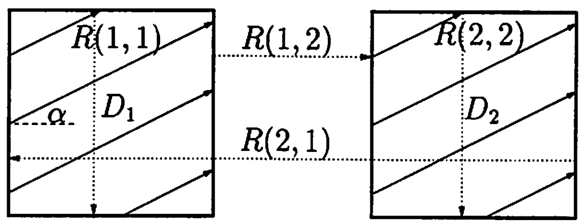

Figure 5: $T^2(\alpha)$.

Example 2.6 (Flow on the 2-torus $T^2(\alpha)$) We have $\alpha > 0$, $n = 2$, $k = 2$, $E = \{(1,2),(2,1)\}$, $D_i = \{i\} \times K$, where $K = [0,1] \times [0,1]$ is the unit square, $X_1 = X_2 = (1,\alpha)^T$ are constant vector fields,

$$G(i,i) = \{i\} \times S_{\text{upper}}, \quad G(i,j) = \{i\} \times S_{\text{right}},$$

$$R_{(i,i)}(i,x,1) = (i,x,0) \quad \text{and} \quad R_{(i,j)}(i,1,y) = (j,0,y),$$

where $i,j = 1,2$, $i \neq j$, $S_{\text{upper}} = [0,1] \times \{1\}$ and $S_{\text{right}} = \{1\} \times [0,1)$ denote the (closed) upper and (half-closed) right side of K. Note that $R_{(i,i)}(\{i\} \times S_{\text{upper}}) = \{i\} \times S_{\text{lower}}$ and $R_{(i,j)}(\{i\} \times S_{\text{right}}) = \{j\} \times S_{\text{left}}$, with the obvious meaning of S_{lower} and S_{left}.

If we proceed as is usually done in geometry and identify $\{i\} \times S_{\text{upper}}$ with $\{i\} \times S_{\text{lower}}$ via $R_{(i,i)}$ and $\{i\} \times S_{\text{right}}$ with $\{j\} \times S_{\text{left}}$ via $R_{(i,j)}$ (where $i,j = 1,2$, $i \neq j$), we obtain the standard 2-torus with a *smooth* flow with slope α on it. This is a baby-version of a construction we will later apply to more general hybrid systems.

Keeping in mind the examples above, we formally define the notion of an execution of a hybrid system.

Definition 2.2 *A (forward) hybrid time trajectory is a sequence (finite or infinite) $\tau = \{I_j\}_{j=0}^N$ of intervals such that $I_j = [\tau_j, \tau_j']$ for all $j \geq 0$ if the sequence is infinite; if N is finite, then $I_j = [\tau_j, \tau_j']$ for all $0 \leq j \leq N-1$ and I_N is either of the form $[\tau_N, \tau_N']$ or $[\tau_N, \tau_N')$. The sequences τ_j and τ_j' satisfy: $\tau_j \leq \tau_j' = \tau_{j+1}$, for all j.*

One thinks of τ_j's as time instants when discrete transitions (or switches) from one domain to another take place. If τ is a hybrid time trajectory, we will call N its *size* and denote it by $N(\tau)$. Also, we use $\langle\tau\rangle$ to denote the set $\{0,\ldots,N(\tau)\}$ if $N(\tau)$ is finite, and $\{0,1,2,\ldots\}$ if $N(\tau)$ is infinite.

We will say that τ is a *prefix* of an execution $\tau' = \{I'_j\}_{j=0}^{N'}$ if $N \leq N'$ (where the inequality is taken in the extened real number system), and for $0 \leq j < N$, we have $I_j = I'_j$; furthermore, if τ has finite size, then we must also have $I_N \subset I'_N$.

Definition 2.3 *An execution (or forward execution) of a hybrid system* $\mathbf{H}$ *is a triple* $\chi = (\tau, q, x)$*, where* τ *is a hybrid time trajectory,* $q : \langle\tau\rangle \to Q$ *is a map, and* $x = \{x_j : j \in \langle\tau\rangle\}$ *is a collection of* C^1 *maps such that* $x_j : I_j \to D_{q(j)}$ *and for all* $t \in I_j$,

$$\dot{x}_j(t) = X_{q(j)}(x_j(t)).$$

Furthermore, for all $j \in \langle\tau\rangle$*, we have*

$$(q(j), q(j+1)) \in E, \quad x_j(\tau'_j) \in G(q(j), q(j+1)),$$

and

$$(x_j(\tau'_j), x_{j+1}(\tau_{j+1})) \in R_{(q(j),q(j+1))}.$$

For an execution $\chi = (\tau, q, x)$, denote by $\tau_\infty(\chi)$ its *execution time*:

$$\tau_\infty(\chi) = \sum_{j=0}^{N(\tau)} (\tau'_j - \tau_j) = \lim_{j\to N(\tau)} \tau'_j - \tau_0.$$

Definition 2.4 *An execution* χ *is called:*

- *infinite, if* $N(\tau) = \infty$ *or* $\tau_\infty(\chi) = \infty$*;*
- *a Zeno execution if* $N(\tau) = \infty$ *and* $\tau_\infty(\chi) < \infty$*;*
- *maximal if it is not a strict prefix of any other execution of* $\mathbf{H}$.

The last statement means that there exists no other execution $\chi' = (\tau', q', x')$ such that τ is a strict prefix of τ' and $x = x'$ on τ (in the sense that $x_j = x'_j$ on I_j for all $j \in \langle\tau\rangle$).

Note that in Examples 2.1 (WT), 2.2 (BB) and 2.3 ($BB(m)$) every execution is Zeno. The same can be shown for Examples 2.4 ($BBS(N)$) if $0 < c < 1$ and 2.5 ($S2(\lambda)$) if $0 < \lambda < 1$. On the other hand, every execution in Example 2.6 ($T^2(\alpha)$) is infinite with infinite execution time.

We say that an execution $\chi = (\tau, q, x)$ starts at a point $p \in D$ if $p = x_0(\tau_0)$ and $\tau_0 = 0$. It *passes through* p if $p = x_j(t)$ for some $j \in \langle\tau\rangle$, $t \in I_j$, $t > \tau_0$.

Given $p \in D$, it is not difficult to see that there are many ways in which a hybrid system can *accept* several executions starting from or passing through p. For instance, this happens if at least one of the resets is a relation which is not a function.

Definition 2.5 *A hybrid system is called deterministic if for every $p \in D$ there exists at most one maximal execution starting from p. It is called non-blocking if for every $p \in D$ there is at least one infinite execution starting from p.*

Necessary and sufficient conditions for a hybrid system to be deterministic and non-blocking can be found in [LJSE]. Roughly speaking, resets have to be functions, guards have to be mutually disjoint and whenever a continuous trajectory of one of the vector fields in $\mathcal{X}$ is about to exit the domain in which it lies, it has to hit a guard.

2.2 Standing assumptions

From now on we will assume that every hybrid system $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$ in this paper satisfies the following assumptions.

(A1) $\mathbf{H}$ *is deterministic and non-blocking.*

This means that every point in D is the starting point of a unique infinite (and therefore maximal) execution of $\mathbf{H}$.

(A2) *Each domain D_i is a contractible n-dimensional smooth submanifold of $\mathbf{R}^n$, with piecewise smooth boundary. No two smooth components of the boundary meet at a zero angle.*

Recall that a space is contractible if it can be shrunk to a point (or more formally, if it is homotopically equivalent to a point). Note that this implies that the domains are connected. A manifold is called *piecewise smooth* if it is the union of finitely many smooth pieces. By smooth we will mean of class C^∞, unless specified otherwise.

The non-zero angle requirement eliminates, for instance, cusps in dimension two, but does not eliminate "corners". Thus for domains of a hybrid system we allow disks, half-spaces, rectangles, etc.

(A3) *Each guard is a piecewise smooth $(n-1)$-dimensional submanifold of the boundary of the corresponding domain. The boundary of each guard is piecewise smooth (or possibly empty).*

(A4) *Each reset is a piecewise smooth homeomorphism onto its image. The image of every reset lies in the boundary of the corresponding domain.*

Recall that a map f between (piecewise) smooth manifolds M and N is called *piecewise smooth* if M can be decomposed into finitely many pieces such that f is smooth on each one of them.

(A5) *Any sets in $\overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ (i.e. closures of guards and images of resets) can intersect only along their boundaries. Furthemore, if $p \in \overline{G} \cup \overline{R}$, then p can be of only one of the following*

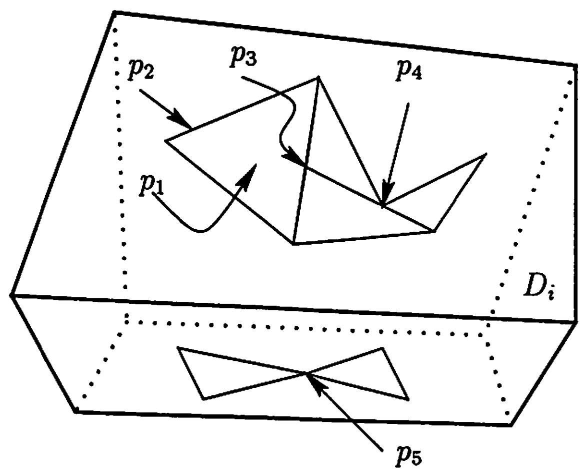

Figure 6: p_i is of Type (Roman) i ($1 \leq i \leq 4$).

four types (cf. Fig. 6):

Type I *:* $p \in \text{int}\, G \cup \text{int}\, R$;

Type II *:* $p \in \partial G \cup \partial R$ *and there exists exactly one set* $S \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ *which contains* p;

Type III *:* $p \in \partial G \cup \partial R$ *and there exist sets* $S_1, \ldots, S_l \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ $(l \geq 2)$ *such that* $p \in \partial S_1 \cap \ldots \cap \partial S_l$ *and some neighborhood of* p *in* $S_1 \cup \ldots \cup S_l$ *is homeomorphic to* $\mathbf{R}^{n-1}$;

Type IV *:* $p \in \partial G \cup \partial R$ *and there exist sets* $S_1, \ldots, S_l \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ $(l \geq 2)$ *such that* $p \in \partial S_1 \cap \ldots \cap \partial S_l$ *and some neighborhood of* p *in* $S_1 \cup \ldots \cup S_l$ *is homeomorphic to* $\mathbf{R}^{n-1}_+$.

Assumption (A5) ensures that intersections of guards and images of resets (that is, their closures) are sufficiantly nice. This in particular means that the configuration around p_5 in Fig. 6 is not allowed.

(A6) *For all* $e = (i, j) \in E$, X_i *points outside* D_i *along* $G(e)$, *and* X_j *is points inside* D_j *along* $\text{im}\, R_e$.

This means that if $p \in G(i,j)$, $q = R_{(i,j)}(p)$, then there exists $\epsilon > 0$ such that $\phi^i_{-t}(p) \in \text{int} D_i$ and $\phi^j_t(q) \in \text{int} D_j$, for all $0 < t < \epsilon$, where int denotes the interior of a set. In particular, we have that X_i is transverse to the smooth part of $G(e)$ and X_j is transverse to the smooth part of $\text{im}\, R_e$, the image of the map R_e.

(A7) *Each reset map* R_e *extends to a map* $\tilde{R}_e$ *defined on a neighborhood of* $\overline{G(e)}$ *(the closure of* $G(e)$*) in* D_i *such that* $\tilde{R}_e$ *is a piecewise smooth homeomorphism onto its image, which, in turn, is a neighborhood of* $\text{im}\, R_e$ *in* D_j. *Each vector field* X_i *can be smoothly extended to a neighborhood of* D_i *in* $\{i\} \times \mathbf{R}^n$.

The last one is a fairly technical assumption the need for which will become apparent later. Note that all the examples provided above satisfy this (as well as all other) assumptions. For instance, in Example 2.2 (BB), we can take $\tilde{R}_{(1,1)}(x_1, x_2) = (x_1, -cx_2)$.

Definition 2.6 *A hybrid system which satisfies assumptions* (A1) - (A7) *will be called regular.*

Given $\mathbf{H}$, define a map

$$\Phi^{\mathbf{H}} : \Omega_0 \to D,$$

(where $\Omega_0 \subset \mathbf{R} \times D$ will be specified later) as follows. Let $p \in D$ be arbitrary. Because of (A1), there exists a unique infinite execution $\chi(p) = (\tau, q, x)$ starting at p. For any $0 \leq t < \tau_\infty(\chi(p))$ there exist a unique $j \in Q$ such that $t \in [\tau_j, \tau_j')$. Then define

$$\Phi^{\mathbf{H}}(t, p) = x_j(t).$$

To define $\Phi^{\mathbf{H}}(t, p)$ for negative t, set

$$\Phi^{\mathbf{H}}(t, p) = \Phi^{\mathbf{H}'}(-t, p),$$

where $\mathbf{H}'$ is the *reverse* hybrid system $(Q', E', \mathcal{D}', \mathcal{X}', \mathcal{G}', \mathcal{R}')$ defined by:

- $Q' = Q$, $\mathcal{D}' = \mathcal{D}$, $X_i' = -X_i$ for all $i \in Q$;
- $(i, j) \in E'$ if and only if $(j, i) \in E$;
- for all $e = (i, j) \in E'$, $G'(e) = R_{(j,i)}(G(j, i))$ and $R_e' = R_e^{-1}$.

It can easily be checked that $\mathbf{H}'$ satisfies (A1) - (A7) if $\mathbf{H}$ does.

Let Ω_0 be the largest subset of $\mathbf{R} \times D$ on which $\Phi^{\mathbf{H}}$ is defined.

For instance, in Example 2.2, for any $p \neq \mathbf{0}$, $\Phi^{BB}(t, p) \to \mathbf{0}$, as $t \to \tau_\infty(\chi(p))$, where $\chi(p)$ is the unique infinite execution starting at p. Note, however, that $\chi(\mathbf{0})$ makes no time progress, i.e. $\tau_j = 0$ for all $j \geq 0$, but it involves infinitely many switches at the same, i.e. initial point, which happens to be fixed by the reset map.

Theorem 2.7 **(a)** *Ω_0 contains a neighborhood of $\{0\} \times \operatorname{int} D$ in $\mathbf{R} \times D$.*

(b) *For all $p \in D$, $\Phi^{\mathbf{H}}(0, p) = p$. Furthermore,*

$$\Phi^{\mathbf{H}}(t, \Phi^{\mathbf{H}}(s, p)) = \Phi^{\mathbf{H}}(t + s, p),$$

whenever both sides are defined.

PROOF: (a) If $p \in \operatorname{int} D_i$, then since X_i is Lipschitz on D_i, $t \mapsto \phi_t^i(p)$ is defined on a neighborhood of 0. Furthermore, there exists a neighborhood U of p in $\operatorname{int} D_i$ and $\epsilon > 0$ such that for each $p' \in U$, $t \mapsto \phi_t^i(p')$ is defined on $(-\epsilon, \epsilon)$. Thus $(-\epsilon, \epsilon) \times U$ is a neighborhood of $(0, p)$ in $\mathbf{R} \times \operatorname{int} D_i$. This proves (a).

(b) The first statement in (b) is clearly true. The second one follows from (A1). ∎

3 The hybrid manifold and hybrid flow

The basic idea in construction of the hybrid manifold from a hybrid system is simple: "glue" the closure of each guard to the image of the corresponding extended reset via the extended reset map. We make this more precise below and then show basic properties of the newly constructed object.

3.1 The hybrifold

Let $\mathbf{H}$ be a *regular* hybrid system. On D let $\sim$ be the equivalence relation generated by

$$p \sim \tilde{R}_e(p),$$

for all $e \in E$ and $p \in \overline{G(e)}$. Collapse each equivalence class to a point to obtain the quotient space

$$M_{\mathbf{H}} = D/\sim .$$

Definition 3.1 *We call $M_{\mathbf{H}}$ the hybrid manifold or hybrifold of* $\mathbf{H}$.[1]

Denote by π the natural projection $D \to M_{\mathbf{H}}$ which assigns to each p its equivalence class $p/\sim$. Put the *quotient topology* on $M_{\mathbf{H}}$. Recall that this is the smallest topology that makes π continuous, i.e. a set $V \subset M_{\mathbf{H}}$ is open if and only if $\pi^{-1}(V)$ is open in D.

Define the *hybrid flow* of $\mathbf{H}$,

$$\Psi^{\mathbf{H}} : \Omega \to M_{\mathbf{H}},$$

by

$$\Psi^{\mathbf{H}}(t, \pi(p)) = \pi\Phi^{\mathbf{H}}(t, p).$$

Here $\Omega = \{(t, \pi(p)) : (t, p) \in \Omega_0\}$. In other words, orbits of $\Psi^{\mathbf{H}}$ are obtained by projecting orbits of $\Phi^{\mathbf{H}}$ by π. By the $\Phi^{\mathbf{H}}$-orbit of p we mean the collection of points $\Phi^{\mathbf{H}}(t, p)$ for all possible t (i.e. all t such that $(t, p) \in \Omega_0$).

Let us run this construction on some of the examples listed above.

Example 3.1 (*WT* **continued)** Without loss we assume that $l_1 = l_2 = 0$. To obtain M_{WT} we have to identify the x_1-axis from D_1 with the same axis from D_2 via $R_{(1,2)}$ and similarly with the x_2-axis.

It is not difficult to see that M_{WT} is homeomorphic to $\mathbf{R}^2$ (see Fig. 7). However, M_{WT} has a singularity (or "corner") at $\mathbf{0} = \pi(1, 0, 0)$, i.e. π does not define a smooth structure on M_{WT}. Note that every execution starting at $x \neq \mathbf{0}$ converges to $\mathbf{0}$.

Example 3.2 (*BB* **continued)** Here we have to identify the negative part with the positive part of the x_2-axis. The resulting space M_{BB} is again homeomorphic to $\mathbf{R}^2$ (see Fig. 8), but π again does not define a smooth structure on it. As in the previous example, $\Psi^{BB}(t, x) \to \mathbf{0}$, as $t \to \tau_\infty(\chi(x))$, for all $x \neq \mathbf{0}$.

[1]The authors thank Renaud Dreyer for suggesting the term hybrifold. The term "manifold" will be justified by Theorem 3.2.

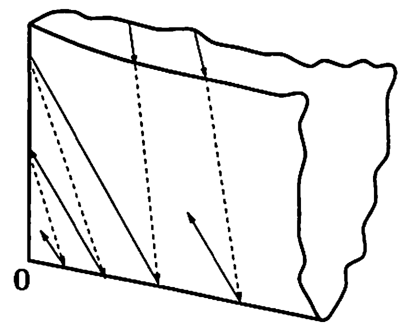

Figure 7: Hybrid manifold and on orbit of the hybrid flow for WT.

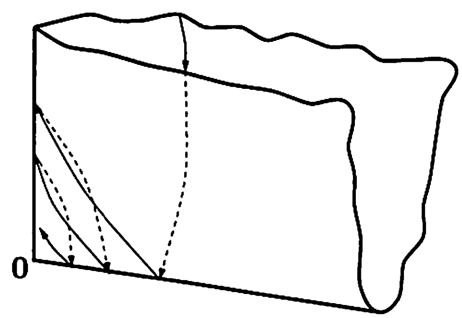

Figure 8: Hybrifold and an orbit of the hybrid flow for BB.

Example 3.3 **($BB(m)$ continued)** For simplicity assume $m = 2$. Then we see from Fig. 9 that $M_{BB(2)}$ is smooth and *diffeomorphic* to $\mathbf{R}^2$. However, the hybrid flow is not smooth.

Example 3.4 **($S2(\lambda)$ continued)** $M_{S2(\lambda)}$ is homeomorphic to the 2-sphere; it is not equipped with a smooth structure by π.

Example 3.5 **($T^2(\alpha)$ continued)** We already observed that $M_{T^2(\alpha)}$ is the standard 2-torus and $\Psi^{T^2(\alpha)}$ is a smooth linear flow on it. If α is rational, then every orbit is closed; if α is irrational, then every orbit is dense in T^2.

The following theorem establishes some basic properties of the hybrid manifold.

Theorem 3.2 **(a)** *$M_{\mathbf{H}}$ defined above is a topological n-manifold with boundary.*

(b) *Both $M_{\mathbf{H}}$ and its boundary are piecewise smooth.*

(c) *The restriction $\pi|_{\text{int}\,D} : \text{int}\, D \to \pi(\text{int}\, D)$ is a diffeomorphism.*

Recall that M is called a *topological n-manifold with boundary* if it is Hausdorff and every point in M has a neighborhood homeomorphic to either $\mathbf{R}^n$ or the closed upper half-space $\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) : x_n \geq 0\}$. Points having the latter property are said to be on the boundary ∂M, which is a topological $(n-1)$-manifold.

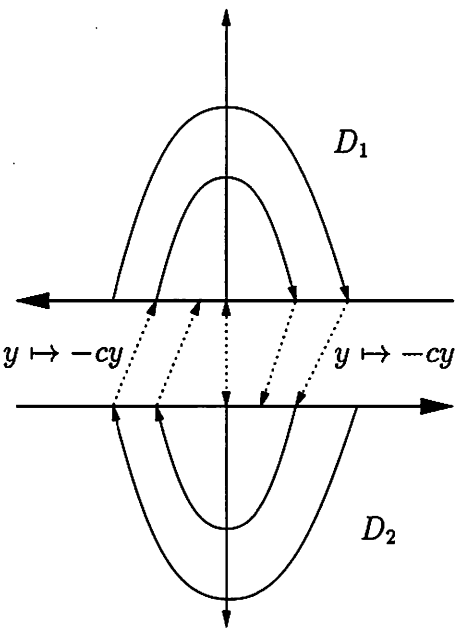

Figure 9: Hybrifold and an orbit of the hybrid flow for $BB(2)$.

PROOF OF THE THEOREM: (a) Recall that the quotient of a manifold M by an equivalence relation ρ is Hausdorff if and only if the graph of ρ is closed in $M \times M$. In our case, the graph of $\sim$ is

$$\{(p,p') : p \sim p'\} = \Delta_D \cup \bigcup_{e \in E} \{(p, \tilde{R}_e(p)) : p \in \overline{G(e)}\},$$

which is easily checked to be closed. Here $\Delta_D = \{(p,p) : p \in D\}$ is the diagonal of $D \times D$.

This shows why in the definition of $\sim$ we needed to identify p and $R_e(p)$ not only for $p \in G(e)$, but also for $p \in \partial G(e)$, since $G(e)$ may not be closed (in which case $M_{\mathbf{H}}$ would not necessarily be Hausdorff); this is why we had to use the maps $\tilde{R}_e$.

We sketch the rest of the proof of (a) when $n = 2$. In higher dimensions the proof is similar, but much more cumbersome to write.

Assume $x \in M_{\mathbf{H}}$. We need to show that there exists a neighborhood of x in $M_{\mathbf{H}}$ which is homeomorphic to $\mathbf{R}^2$ or $\mathbf{R}^2_+$.

Let $x = \pi(p)$, for some $p \in D$. If p is not identified with any other points, i.e. if it is not in $\overline{G} \cup \overline{R}$, then either $p \in \operatorname{int} D$ or $p \in \partial D - (\overline{G} \cup \overline{R})$. In the former case, p has a neighborhood V contained in the interior of a single domain, homeomorhic to $\mathbf{R}^2$; since π is 1-1 on V, $\pi(V)$ is homeomorphic to $\mathbf{R}^2$. In the latter case, p has a neighborhood W contained in a single domain, disjoint from $\overline{G} \cup \overline{R}$ and homeomorphic to $\mathbf{R}^2_+$. Since π is 1-1 on W (nothing in W gets glued to anything else), $\pi(W)$ is homeomorphic to $\mathbf{R}^2_+$. This completes the case when p is identified with no other points.

If $p \in \overline{G} \cup \overline{R}$, then according to (A5) we must consider the following four cases (cf. Fig. 10):

Case 1: p is of Type I. Then there exists a unique $e = (i,j) \in E$ such that either $p \in \operatorname{int} G(e)$ or $p \in \operatorname{int}(\operatorname{im} R_e)$. Without loss we can assume the former. Then p is identified with $p' = R_e(p)$; note that $\pi^{-1}(x) = \{p, p'\}$. There exist neighborhoods V (in D_i) and V' (in D_j) of p and p' respectively, homeomorphic to $\mathbf{R}^2_+$ and $\mathbf{R}^2_-$ (the closed lower half plane), respectively. If $U = \pi(V \cup V')$ and V and V' are small enough, it is not difficult to show that U is homeomorphic to $\mathbf{R}^2$. Thus $x \in \operatorname{int} M_{\mathbf{H}}$.

Case 2: p is of Type II. Then p is on the boundary of exactly one set $S \in \overline{G} \cup \overline{R}$. Without loss we may assume $S = G(e)$ for a unique $e = (i,j) \in E$. Then p gets identified with (a unique point) $p' = R_e(p) \in \partial(\operatorname{im} R_e)$; observe that $\pi^{-1}(x) = \{p, p'\}$. There exist neighborhoods V (in D_i) and V' (in D_j) of p and p' respectively, homeomorphic to $\mathbf{R}^2_+$ and $\mathbf{R}^2_-$ (the closed lower half plane), respectively. Note that only a proper subset of the boundary of V is identified with a proper part of the boundary of V'. So, if $U = \pi(V \cup V')$, and V and V' are small enough, it is not difficult to show that U is homeomorphic to $\mathbf{R}^2_+$. Thus $x \in \partial M_{\mathbf{H}}$.

Case 3: p is of Type III. Then there exist sets $S_1, \ldots, S_l \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ with $l \geq 2$ such that $p \in \partial S_1 \cap \ldots \cap \partial S_l$, and there exists a neighborhood W of p in $S_1 \cup \ldots \cup S_l$ homeomorphic to $\mathbf{R}$. Clearly, we must have $l = 2$.

The sets S_1 and S_2 are both contained in the same domain, say D_{i_1}. Assume that $\pi^{-1}(x) = \{p_1, \ldots, p_m\}$, where $p_1 = p$, and $p_j \in D_{i_j}$. Let $e_j = (i_j, i_{j+1})$ if $j = 1, \ldots, m-1$ and $e_m = (i_m, i_1)$. Without loss we may assume that

$$S_1 = \overline{\operatorname{im} R_{e_m}}, \quad S_2 = \overline{G(e_1)},$$

and

$$\overline{\operatorname{im} R_{e_{j-1}}} \cap \overline{G(e_j)} = \{p_j\},$$

for $j = 2, \ldots, m$. Each point p_j is also of Type III, so for each j there exists a neighborhood W_j of p_j in $\overline{\operatorname{im} R_{e_{j-1}}} \cup \overline{G(e_j)}$ (with $W_1 = W$) such that W_j is homeomorphic to $\mathbf{R}$. We can than find a neighborhood V_j of p_j in D_{i_j} such that $V_j \cap \partial D_{i_j} = W_j$ and V_j is homeomorphic to $\mathbf{R}^2_+$. Note that not all e_j's have to be in E, i.e. represent allowed discrete transitions. However, by (A5) only be the following two cases can occur:

- If $e_j \in E$ for all $j = 1, \ldots, m$, then $U = \pi(V_1 \cup \ldots \cup V_m)$ is a neighborhood of x homeomorphic to $\mathbf{R}^2$ and hence $x \in \operatorname{int} M_{\mathbf{H}}$.
- If one e_j is not in E, then $U = \pi(V_1 \cup \ldots \cup \widehat{V_j} \cup \ldots \cup V_m)$ is a neighborhood of x homeomorphic to $\mathbf{R}^2_+$. Thus $x \in \partial M_{\mathbf{H}}$. Here $\widehat{\ }$ denotes omission.

Case 4: p is of Type IV. Impossible if $n = 2$.

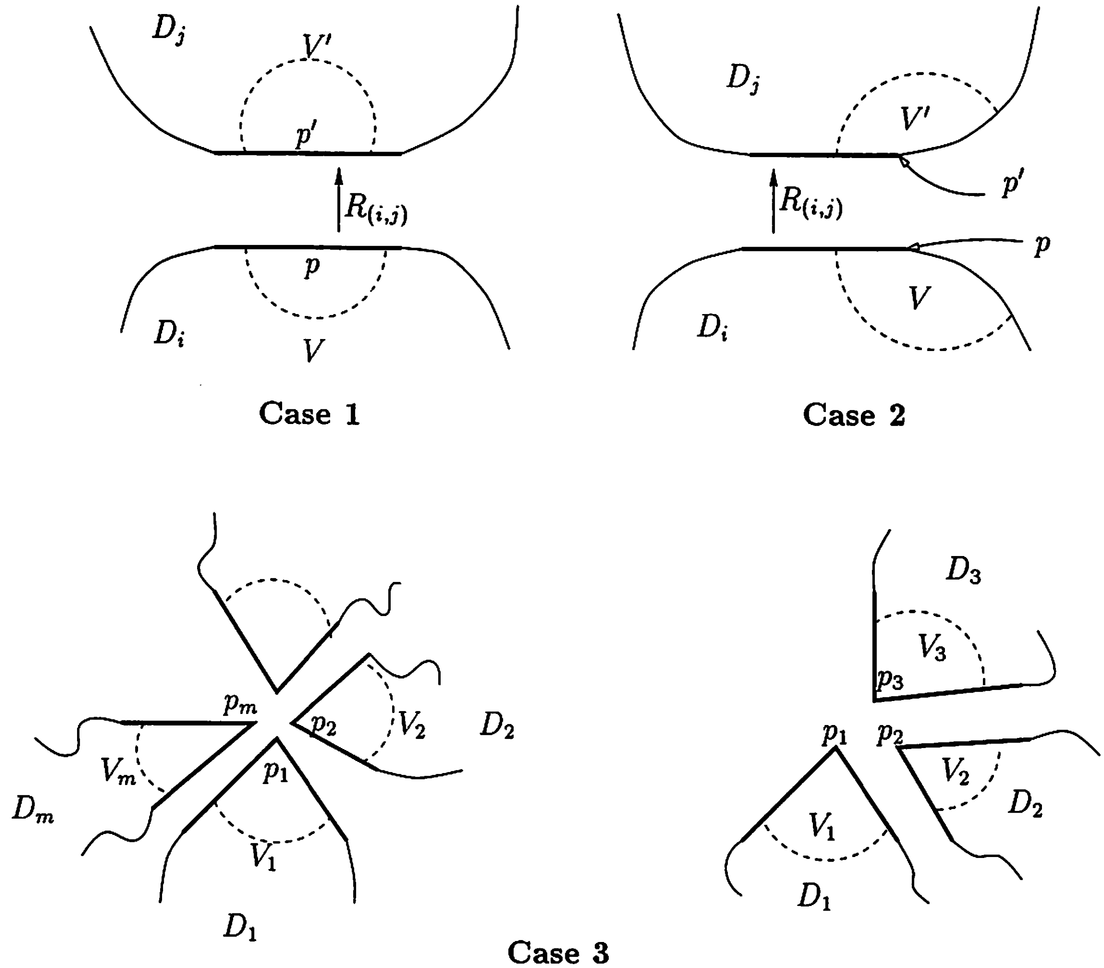

Figure 10: Proof of Theorem 3.2.

This completes the proof when $n = 2$. ∎

Remark. It needs to be pointed out that in dimensions greater than two a hybrid manifold can be a very complicated object, as one can realize by trying to imagine $M_{BBS(N)}$. However, it enables us to study the dynamics of a hybrid system on a *single* phase space. Advantages of this fact will become apparent soon.

If $c : [0,1] \to M_{\mathbf{H}}$ is a smooth curve contained in $\pi(\text{int } D_i)$ for some $i \in Q$, define its *arclength* by

$$\ell(c) = \int_0^1 \|T\pi^{-1}(\dot{c}(t))\| \, dt,$$

where $T\pi$ denotes the tangent map (i.e. the derivative) of π. This makes sense by part (c) of Theorem 3.2. If c is a piecewise smooth curve in $M_{\mathbf{H}}$, e.g. $c = \sum_j c_j$ with c_j smooth (and contained in the projection of the interior of a single domain), define

$$\ell(c) = \sum_j \ell(c_j).$$

Define a distance function on $M_{\mathbf{H}}$,

$$d_{\mathbf{H}} : M_{\mathbf{H}} \times M_{\mathbf{H}} \to \mathbf{R},$$

by: for $x, y \in M_{\mathbf{H}}$, let $d_{\mathbf{H}}(x, y)$ be the infimum of $\ell(c)$ where $c : [0,1] \to M_{\mathbf{H}}$ is an arbitrary piecewise smooth curve such that $c(0) = x$, $c(1) = y$. Then we have:

Theorem 3.3 *$(M_{\mathbf{H}}, d_{\mathbf{H}})$ is a metric space and π is a piecewise isometry. The topology induced by $d_{\mathbf{H}}$ coincides with the quotient topology on $M_{\mathbf{H}}$.*

PROOF: That $(M_{\mathbf{H}}, d_{\mathbf{H}})$ is a metric space is a standard fact from the theory of piecewise smooth manifolds. That π is a piecewise isometry follows from the definition of $d_{\mathbf{H}}$. Namely, for each $i \in Q$, $\pi|_{\text{int}\, D_i} : \text{int}\, D_i \to \pi(\text{int}\, D_i)$ is an isometry.

That $d_{\mathbf{H}}$-topology coincides with the quotient topology on $M_{\mathbf{H}}$ is also immediate by definition of $d_{\mathbf{H}}$. ∎

3.2 The hybrid flow

Next we establish some basic properties of the hybrid flow.

Let $\Psi := \Psi^{\mathbf{H}}$ be the hybrid flow of $\mathbf{H}$, as defined above. For each $t \in \mathbf{R}$ and $x \in M_{\mathbf{H}}$, let

$$M(t) = \{y \in M_{\mathbf{H}} : \Psi(t, y) \text{ is defined}\},$$

and

$$J(x) = \{s \in \mathbf{R} : \Psi(s, x) \text{ is defined}\}.$$

Observe that if $x = \pi(p)$, then $J(x) \cap [0, \infty) = [0, \tau_\infty(\chi(p)))$. Also, for $t > 0$, $M(t)$ contains all points $x = \pi(p)$ such that $\tau_\infty(\chi(p))) > t$. As usual, $\chi(p)$ denotes the unique execution of $\mathbf{H}$ starting at p.

If $M(t)$ is not empty, denote by $\Psi_t : M(t) \to M_{\mathbf{H}}$ the *time t map of* Ψ, defined by

$$\Psi_t(x) = \Psi(t, x).$$

Recall that a function (in particular, vector field) is said to be smooth on a *closed* set F if it is the restriction of a smooth function defined on a neighborhood of F. Then we have the following theorem.

Theorem 3.4 *Suppose each vector field X in $\mathcal{X}$ is smooth (in addition to being globally Lipschitz). Then:*

(a) *For each $x \in M_{\mathbf{H}}$ the map $t \mapsto \Psi_t(x)$ is continuous and, if $J(x)$ is not a single point, piecewise smooth on $J(x)$. More precisely, it is smooth except at (at most) countably many points in $J(x)$.*

(b) *Each map Ψ_t is injective.*

(c) *Whenever both sides are defined:*

$$\Psi_t^{\mathbf{H}}\Psi_s^{\mathbf{H}}(x) = \Psi_{t+s}^{\mathbf{H}}(x).$$

(d) *There is an open and dense subset of Ω on which Ψ is smooth.*

PROOF: (a) Let $x = \pi(p)$ for some $p \in D$. Let $\chi(p) = (\tau, q, x)$ be the unique execution of $\mathbf{H}$ starting at p (i.e. $\tau_0 = 0$ and $x_0(0) = p$). Recall that for positive t, $\Psi_t(x) = \pi\Phi^{\mathbf{H}}(t,p) = \pi x_j(t)$, if $t \in [\tau_j, \tau_j')$. Thus it is enough to check continuity of $t \mapsto \Psi_t(x)$ at $\tau_j' = \tau_{j+1}$, for $j \geq 0$. But recall that $t \mapsto \Phi^{\mathbf{H}}(t,p)$ is continuous from the right, with discontinuities of the first kind only at τ_j, $j \geq 0$. Since $x_j(\tau_j) \in G$ and

$$\pi(x_j(\tau_j')) = \pi(x_{j+1}(\tau_{j+1}))$$

for all $j \geq 0$, it follows that

$$\lim_{t \to \tau_j'-} \Psi_t(x) = \lim_{t \to \tau_j'-} \pi x_j(t) = \pi x_{j+1}(\tau_{j+1}) = \Psi_{\tau_j'}(x),$$

which shows that $t \mapsto \Psi_t(x)$ is continuous on $J(x) \cap [0, \infty)$.

Continuity of $t \mapsto \Psi_t(x)$ for negative t follows by observing that $\Psi_{-t}^{\mathbf{H}}(x) = \Psi_t^{\mathbf{H}'}(x)$ $(t > 0)$, where $\mathbf{H}'$ is the reverse hybrid system to $\mathbf{H}$.

The extreme case when $J(x) = \{0\}$ happens when $x = \pi(p)$ for some $p \in D$ such that:

$$p \in \overline{G(e_0)},\ p_1 = \tilde{R}_{e_0}(p) \in \overline{G(e_1)},\ p_2 = \tilde{R}_{e_1}(p_1) \in \overline{G(e_2)},\ \text{etc.},$$

for a sequence $e_0, e_1, e_2, \ldots$ in E. Then $t \mapsto \Psi_t(x)$ is trivially continuous.

Assume now that $J(x)$ is not a single point. With the notation as above, we have that for $\tau_j < t < \tau_j'$,

$$\frac{d}{dt}\Psi_t(x) = T\pi(X_{q(j)}(x_j(t))),$$

which proves that $t \mapsto \Psi_t(x)$ is piecewise smooth. Here $T\pi$ denotes the tangent (or derivative) map of π. It is defined at $x_j(t)$ because $x_j(t) \in \operatorname{int} D$ for $\tau_j < t < \tau_j'$ and π is smooth on $\operatorname{int} D$.

(b) Injectivity of Ψ_t follows directly from uniqueness of executions through any point.

(c) Follows from the analogous property of $\Phi^{\mathbf{H}}$.

(d) For a proof of this fact see [LJZS]. ∎

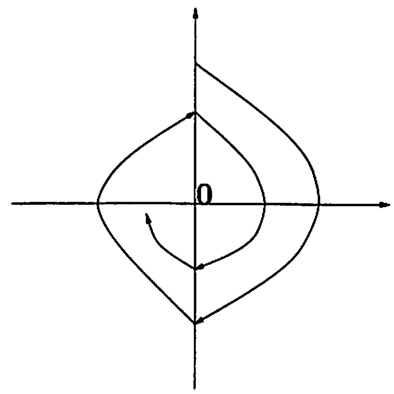

Figure 11: Flow in $\mathbf{R}^2$ topologically conjugate to Ψ^{WT}.

4 Conjugacy of hybrid systems

In this section we discuss the following question: when are two hybrid systems qualitatively the same? For that purpose we borrow the notion of conjugacy from the theory of dynamical systems. Roughly speaking, two dynamical systems are conjugate if their phase portraits look qualitatively (or topologically) the same. Similarly, two hybrid systems are conjugate if their hybrid flows are conjugate. We now make this more precise.

Definition 4.1 *Two hybrid systems* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are said to be topologically conjugate (denoted by* $\mathbf{H_1} \approx \mathbf{H_2}$*) if there exists a homeomorphism* $h : M_{\mathbf{H1}} \to M_{\mathbf{H2}}$ *which sends orbits of* $\Psi^{\mathbf{H1}}$ *to orbits of* $\Psi^{\mathbf{H2}}$.

If $M_{\mathbf{H1}}$ *and* $M_{\mathbf{H2}}$ *happen to be smooth manifolds of class* C^r $(r \geq 1)$ *and* h *is a* C^r *diffeomorphism, then* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are said to be* C^r*-conjugate.*

As usual, by the *orbit* of a point x under a (local) flow $\{\phi_t\}$ we mean the set of points $\phi_t(x)$ for all t for which $\phi_t(x)$ is defined.

We usually think of h as a change of coordinates so that two hybrid systems are topologically conjugate if their hybrid flows are the same up to a continuous coordinate change.

Note that conjugacy does not necessarily preserve the time parameter t. If it does, it is called *equivalence.*

Example 4.1 WT is topologically conjugate to BB. This can be seen in the following way. Assume M_{WT} is embedded in $\mathbf{R}^3$ in such a way that its "origin" coincides with the point $(0,0,0)$ and M_{WT} lies entirely in the upper half space $\mathbf{R}^3_+$. Let P be the plane $x_3 = 0$ and let $h : M_{WT} \to P$ be the orthogonal projection. Then h is a homeomorphism which sends orbits of Ψ^{WT} to the orbits of the flow Φ in Fig. 11.

By smoothing Φ along the y-axis, we get that it is topologically conjugate to a (smooth) spiral sink at the origin, e.g., the flow of the linear vector field corresponding to the matrix

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Similarly, we obtain that Ψ^{BB} is topologically conjugate to the flow of A. Since conjugacy is an equivalence relation, we get that $WT \approx BB$, as claimed. We will see later that in dimension two this picture is typical.

Example 4.2 $T^2(1)$ is not conjugate to $T^2(\sqrt{2})$. Even though the hybrifold for both hybrid systems is the same (the 2-torus), every orbit of $T^2(1)$ is closed, while every orbit of $T^2(\sqrt{2})$ is dense in T^2. Since conjugacy always sends closed orbits to closed orbits, the statement above follows immediately.

Example 4.3 Suppose $\alpha_i < 0 < \beta_i$ for $i = 1, 2$, and $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Let $A_i = \text{diag}(\alpha_i, \beta_i)$. Then the flows of A_1 and A_2 both have a saddle at the origin of $\mathbf{R}^2$ and are topologically conjugate. However, they are *not* smoothly conjugate, because if they were, it is not difficult to check that their corresponding eigenvalues would be the same.

Ideally, one would like to be able to classify all hybrid systems up to topological conjugacy (smooth conjugacy being too strong a notion). Unfortunately, this attempt fails even for smooth dynamical systems on compact boundaryless manifolds of dimension greater than two, as can be seen in the standard dynamics literature (for instance, [PdM]). However, it turns out that it is possible to obtain a fairly detailed picture of the local behavior of 2-dimensional hybrid flows near a point called Zeno state. We will show this in Section 6. In the next section we investigate such points.

5 ω-limit sets and the Zeno phenomenon

It has to be pointed out that Zeno executions do not arise in physical systems and are a consequence of modeling over-abstraction. Therefore, one wishes to avoid such executions. However, from a mathematical viewpoint, the Zeno phenomenon poses several interesting questions: for instance, what is its topological cause? Is there a checkable criterion which guarantees the non-occurence of Zeno? How should the original system be modified to remove Zeno executions? In this section we show that, in short, the topological cause of Zenoness is a lack of smoothness in the hybrid flow and that the Zeno phenomenon can be removed by smoothing out the hybrifold and the hybrid flow on it.

Since we would like to study the long term behavior of executions of hybrid systems, we define the following notion (keeping the previously introduced notation).

Definition 5.1 *A point $y \in M_{\mathbf{H}}$ is called an ω-limit point of $x \in M_{\mathbf{H}}$ if*

$$y = \lim_{m \to \infty} \Psi^{\mathbf{H}}_{t_m}(x),$$

for some increasing sequence (t_m) in $J(x)$ such that $t_m \to \tau_\infty(x)$, as $m \to \infty$. The set of all ω-limit points of x is called the ω-limit set of x and is denoted by $\omega(x)$.

By $\tau_\infty(x)$ we denote the execution time of the unique execution of $\mathbf{H}$ starting from p, where $x = \pi(p)$; that is, $\tau_\infty(x) = \tau_\infty(\chi(p))$. It is easy to check that this is a well defined element of the extended real number system. In other words, ω-limit points for x are accumulation points of the orbit of x.

Suppose $x \in M_{\mathbf{H}}$ and denote by $E_\infty(x)$ the set of discrete transitions which occur infinitely many times in the execution starting from x. If $E_\infty(x)$ is empty, then the orbit of x eventually ends up in a single domain D_i (that is, its image under π in the hybrifold) in which case

$$\omega(x) \subset \pi(\overline{D_i}).$$

This means that every point $y \in \omega(x)$ is an accumulation point of the orbit of a single vector field, namely X_i. We will call such a point y, a *pure ω-limit point.*

If $E_\infty(x)$ is nonempty, then every ω-limit point for x is a result of both the continuous *and* discrete (i.e. hybrid) dynamics of $\mathbf{H}$ and will accordingly be called a *hybrid ω-limit point of x.*

Theorem 5.2 *For every $x \in M_{\mathbf{H}}$, $\omega(x)$ is invariant with respect to the hybrid flow. That is, if $y \in \omega(x)$, then*

$$\Psi_t^{\mathbf{H}}(y) \in \omega(x),$$

for all $t \in J(y)$.

PROOF: See [LJZS]. ∎

5.1 Properties of Zeno executions

A special case of an ω-limit point is a Zeno state:

Definition 5.3 *A point $z \in M_{\mathbf{H}}$ is called a Zeno state for x if $z \in \omega(x)$ and $\tau_\infty(x) < \infty$.*

We will also refer to points in $\pi^{-1}(z)$ as *Zeno states* in $\mathbf{H}$.

For example, the "origin" of M_{WT} (as well as M_{BB} and $M_{BB(2)}$) is a Zeno state for every pointf. Moreover, for each x, $\omega(x)$ contains only one Zeno state. We now show this is always the case.

Theorem 5.4 *If the execution starting from $x \in M_{\mathbf{H}}$ is Zeno, then $\omega(x)$ consists of exactly one Zeno state for x and*

$$\omega(x) \subset \bigcap_{e \in E_\infty(x)} \pi(\overline{G(e)}). \tag{1}$$

PROOF: Let $p \in \pi^{-1}(x)$ be arbitrary and, as before, let $\chi(p) = (\tau, q, x)$ be the unique execution starting from p. For $j \geq 0$, let $A_j = \pi(x_j(I_j))$. Then A_j is an arc in $\pi(D_{q(j)})$ of the $\Psi^{\mathbf{H}}$-orbit of x. Since X_i's are bounded along $\chi(p)$ (cf. [ZJL]) and $\chi(p)$ is Zeno, we have that

$$\sum_{j=0}^{\infty} |A_j| < \infty,$$

where $|A_j|$ is the length of A_j. Therefore, $A = \bigcup_j A_j$ is a bounded set and hence it has an accumulation point, z. Clearly, $z \in \omega(x)$ and z is a Zeno state for x.

Suppose there exists another accumulation point of A or equivalently, another Zeno state z' for x. Then we have:

$$z = \lim_{m \to \infty} \Psi^{\mathbf{H}}_{t_m}(x), \quad z' = \lim_{m \to \infty} \Psi^{\mathbf{H}}_{t'_m}(x),$$

for some increasing sequences (t_m) and (t'_m) in $J(x)$ such that $t_m, t'_m \to \tau_\infty(x)$, as $m \to \infty$. If

$$x_m = \Psi^{\mathbf{H}}_{t_m}(x) \in A_{j_m} \quad \text{and} \quad x'_m = \Psi^{\mathbf{H}}_{t'_m}(x) \in A_{j'_m},$$

for some $j_m, j'_m \to \infty$, we obtain:

$$d(x_m, x'_m) \leq \sum_{l=\min(j_m, j'_m)}^{\infty} |A_l| \to 0$$

as $m \to \infty$. Thus $z = z'$. This completes the first part of the proof.

To show (1), let $\omega(x) = \{z\}$, and let $e = (i,j) \in E_\infty(x)$. Then there exists a sequence (x_m) of points in D_i such that: x_m is on the forward orbit of x and $x_m \to z$ as $m \to \infty$. Thus: $z \in \pi(\overline{D_i})$. Similarly, $z \in \pi(\overline{D_j})$. But $\pi(\overline{D_i}) \cap \pi(\overline{D_j}) \subset \pi(\overline{G(e)})$, so $z \in \overline{\pi(G(e))}$. Since this holds for all $e \in E_\infty(x)$, the proof of (1) is complete. ∎

Note than in all the Zeno examples above none of the flows involved in creating the Zeno state has an equilibrium at the Zeno state. The following lemma shows that this is not a coincidence.

Lemma 5.5 *A Zeno state is not an equilibrium. More specifically, if $z \in M_{\mathbf{H}}$ is a Zeno state, then for every $p \in \pi^{-1}(z)$, if $p \in D_i$, then $X_i(p) \neq 0$.*

PROOF: Let z be a Zeno state for x. Consider the lift of the orbit of x by π to D and let us concentrate on its "trace" in a particular domain which it visits infinitely often.

More precisely, there exist $j \in Q$, $e = (i,j), e' = (j, i') \in E$ and $p_* \in D_j$ such that $\pi(p_*) = z$ and

$$p_* \in \overline{\operatorname{im} R_e} \cap \overline{G(e')}.$$

Furthermore, there exists a sequence (p_m) in $\operatorname{im} R_e$ converging to p_*, and a sequence (t_m) of positive numbers such that:

$$q_m = \phi^j_{t_m}(p_m) \in G(e') \quad \text{and} \quad \tau_\infty = \sum_{m=0}^{\infty} t_m < \infty.$$

Recall that $\{\phi_t^j\}$ is the (local) flow of the vector field $X := X_j$ on D_j.

Let A_m be the arc tangent to X and connecting p_m and q_m (i.e. A_m is the X-orbit of p_m), and denote by $|A_m|$ its length. Finally, let $K \subset D_j$ be a compact set containing p_* and A_m for all $m \geq 0$. It exists, because $p_m \to p_*$, as $m \to \infty$.

The intuition is as follows: we start from p_0, flow by time t_0 to q_0 when we reach the guard $G(e')$ and are taken outside of D_j by a reset. We enter D_j again at p_1, flow by time t_1 until we reach q_1, etc.

To complete the proof, assume p_* *is* an equilibrium for X. Then $\phi_t^j(p_*) = p_*$ for all $t \in \mathbf{R}$. If $p \in A_m$, then $p = \phi_t^j(p_m)$ for some $0 \leq t \leq t_m$, so we have:

$$\begin{aligned} \|X(p)\| &= \|X(p) - X(p_*)\| \\ &\leq L\|p - p_*\| \\ &= L\|\phi_t^j(p_m) - \phi_t^j(p_*)\| \\ &\leq C\|p_m - p_*\|. \end{aligned}$$

where L is the Lipschitz constant of X and $C = L \max\{\|D\phi_t^j(q)\| : 0 \leq t \leq \tau_\infty,\ q \in K\} < \infty$. From this inequality we get that

$$\|X\|_{A_m} := \max_{p \in A_m} \|X(p)\| \leq C\|p_m - p_*\|,$$

for all $m \geq 0$.

Next observe that

$$|A_m| \leq t_m \, \|X\|_{A_m},$$

so $|A_m|/\|X\|_{A_m} \to 0$, as $m \to \infty$. However, by the non-zero angle requirement in (A2), there exists a constant $a > 0$ such that $|A_m| \geq a \, \|p_m - p_*\|$, for all $m \geq 0$ (cf. Fig. 12). Thus:

$$\frac{|A_m|}{\|X\|_{A_m}} \geq \frac{a}{C} > 0,$$

a contradiction. Therefore, $X(p_*) \neq 0$. ∎

Example 5.1 (equilibrium + cusp = Zeno) Consider the following one-domain hybrid system:

$$\begin{gathered} D = \{(x, y) \in \mathbf{R}^2 : y \geq 0, \quad -f(y) \leq x \leq f(y)\} \\ G = \{(-f(y), y) : y \geq 0\}, \quad R(-f(y), y) = (f(cy), cy), \\ X(x, y) = (-x - y, x - y)^T. \end{gathered}$$

Here $0 < c < 1$, $f : [0, \infty) \to [0, \infty)$ is a smooth function such that $f(0) = 0$ and for all $y \geq 0$,

$$f(y) \leq y^2.$$

In particular, $f'(0) = 0$, which means that D has a cusp at $\mathbf{0}$.

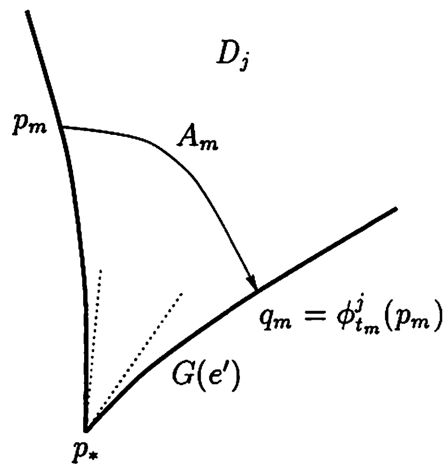

Figure 12: Lemma 5.5.

The vector field X has a spiral sink at the origin, and the time t map of its flow is the composition of the counterclockwise rotation by t (in radians) and contraction by e^{-t}.

Let p_0 be an arbitrary nonzero point on the right side, S, of D and let χ the execution starting from p_0. Let (p_m) (a sequence) be the intersection of χ with S; let $p_m = (f(y_m), y_m)$. Let t_m be the time it takes for the positive X-orbit of p_m to reach G. Then:

$$\|p_{m+1}\| = c\, e^{-t_m} \|p_m\|,$$

so

$$\|p_m\| = c^m \exp\left(-\sum_{i=0}^{m-1} t_i\right) \|p_0\|, \quad \text{and} \quad y_m \leq c^m\, y_0.$$

Let θ_m be the angle between the line $\mathbf{0}p_m$ and the positive y-axis and η_m the angle between the positive y-axis and the line $\mathbf{0}p'_m$, where p'_m is the intersection of the positive X-orbit of p_m and G. Then

$$t_m = \theta_m + \eta_m \leq 2\theta_m = 2 \arctan \frac{f(y_m)}{y_m} \leq 2y_m \leq 2c^m\, y_0.$$

Therefore, $\sum t_m$ converges and $\mathbf{0}$ is a Zeno state despite the fact that it is an equlibrium for X. This shows the importance of geometry of domains and assumption (A2).

Before we proceed, we need to remind the reader of the following *flow box theorem* for smooth flows. Namely, assume that X is a smooth vector field on an open set $U \subset \mathbf{R}^n$, $p \in U$ and $X(p) \neq 0$. Then there exists a neighborhood V of p in U (called a *flow box for X at p*) such that on V the flow of X i smoothly equivalent to the flow of the vector field $\frac{\partial}{\partial x_1}$, i.e. the flow

$$(t, x_1, \ldots, x_n) \mapsto (x_1 + t, x_2, \ldots, x_n).$$

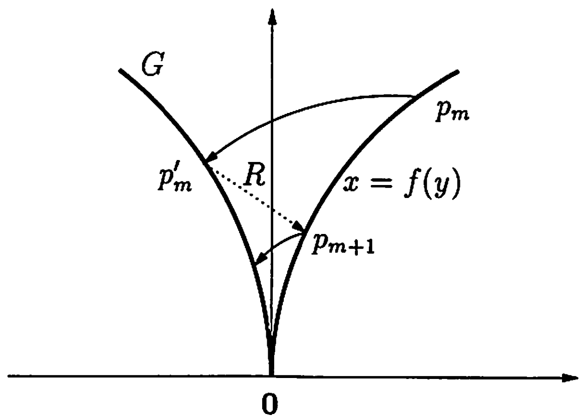

Figure 13: Zeno state at a cusp which is an equilibrium.

This means that in a neighborhood of any of it nonsingular points, the flow of a smooth vector field has a particularly simple form.

Now we can prove the following theorem.

Theorem 5.6 *Suppose* $\mathbf{H}$ *is a hybrid system such that its hybrid flow* $\Psi^{\mathbf{H}}$ *is smooth. (This in particular means that its hybrifold* $M_{\mathbf{H}}$ *is smooth.) Then* $\mathbf{H}$ *admits no Zeno executions or equivalently, there are no Zeno states in* $M_{\mathbf{H}}$.

PROOF. Assume the contrary and let $z \in M_{\mathbf{H}}$ be a Zeno state for some point x. Since the hybrid flow $\Psi^{\mathbf{H}}$ is smooth, it is generated by a smooth vector field on $M_{\mathbf{H}}$, which we denote by X. By Lemma 5.5, $X(z) \neq 0$. Therefore by the flow box theorem, $\Psi^{\mathbf{H}}$ is trivial in a neighborhood of z, which implies that $z = \Psi^{\mathbf{H}}_{\tau_\infty(x)}(x)$. But then $\Psi^{\mathbf{H}}_t(x)$ is clearly defined beyond the Zeno time $\tau_\infty(x)$, which is impossible. ∎

In general it may not be easy to check whether, given $\mathbf{H}$, the hybrifold $M_{\mathbf{H}}$ is smooth. Even if it were, non-smoothness of the hybrid flow may cause Zeno (cf. $BB(2)$). However, the following result provides an easily verifiable criterion for smoothness of $\Psi^{\mathbf{H}}$.

Theorem 5.7 *Suppose that* $M_{\mathbf{H}}$ *is smooth and for every* $e = (i,j) \in E$, X_i *and* X_j *are* $\tilde{R}_e$*-related on* $\overline{G(e)}$. *That is, for every* $p \in \overline{G(e)}$:

$$T\tilde{R}_e(X_i(p)) = X_j(\tilde{R}_e(p)). \tag{2}$$

Then the hybrid flow is smooth.

PROOF: Define a vector field Y on $M_{\mathbf{H}}$ as follows. If $x \in M_{\mathbf{H}}$, then $x = \pi(p)$ for some $p \in D_i$. Set

$$Y(x) = T\pi(X_i(p)).$$

We will show that Y is well defined.

If p is not in $\overline{G} \cup \overline{R}$ (i.e. $p/\sim$ is a single point), then there is no ambiguity in the definition of $Y(x)$.

If $p \in \overline{G} \cup \overline{R}$, then the ambiguity arises because p is identified with some p' via an extended reset map. Assume that $p \in \overline{G(e)}$ for some $e \in E$ and let $p' = \tilde{R}_e(p)$. Since $\pi \circ \tilde{R}_e = \pi$, (2) and the chain rule yield

$$\begin{aligned} T\pi(X_j(p')) &= T\pi\, T\tilde{R}_e(X_i(p)) \\ &= T(\pi \circ \tilde{R}_e)(X_i(p)) \\ &= T\pi(X_i(p)). \end{aligned}$$

Therefore, Y is well defined.

Next we show that Y is smooth. Let X the vector field on D which coincides with X_i on D_i for all $i \in Q$. Since D_i's are mutually disjoint and each X_i is smooth, X is smooth. The assumption that $M_{\mathbf{H}}$ is smooth means, in particular, that the projection π is smooth. By definition of Y, the vector fields X and Y are π-related. Therefore, since X and π are smooth, so is Y.

Smoothness of $\Psi^{\mathbf{H}}$ now follows directly from the fact that Y generates it, i.e.:

$$\left.\frac{d}{dt}\right|_0 \Psi_t^{\mathbf{H}}(x) = Y(x),$$

for all $x \in M_{\mathbf{H}}$. ∎

Example 5.2 Consider $BB(2)$. Here we have:

$$X_1(x_1, x_2) = (x_2, -g)^T = X_2, \quad \tilde{R}_{(i,j)}(i, x_1, x_2) = (j, x_1, -cx_2),$$

where $(i, j) = (1, 2)$ or $(2, 1)$. Therefore,

$$T\tilde{R}_{(1,2)}(X_1) = (x_2, cg)^T \neq X_2,$$

so the hybrid flow for $BB(2)$ is not smooth, as we already knew.

Example 5.3 It is not difficult to check that in case of $T^2(\alpha)$, (2) is satisfied for every $\alpha > 0$. Thus $T^2(\alpha)$ does not admit Zeno, as was already shown above.

Corollary 5.8 *If* $\mathbf{H}$ *is a hybrid system satifying condition (2), then* $\mathbf{H}$ *accepts no Zeno executions.*

Next we discuss two ways of removing Zeno from a hybrid system. They are: smoothing and suspension.

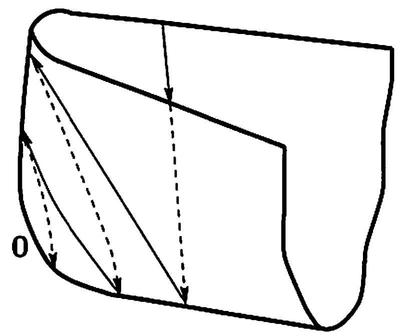

Figure 14: Smoothed water tank M_{WT}^{smooth}.

5.2 Removal of Zeno

Suppose that $\mathbf{H}$ is a regular hybrid system and that $z \in M_{\mathbf{H}}$ is a Zeno state. We have seen that $M_{\mathbf{H}}$ in a certain sense has a singularity at z. Consider the following ways of removing such singularities.

Smoothing. Suppose that $M_{\mathbf{H}}$ can be equipped with a smooth structure which induces the same topology as the original one and denote the smoothed hybrifold by $M_{\mathbf{H}}^{smooth}$ (cf. Fig. 14). Note that $M_{\mathbf{H}}$ and $M_{\mathbf{H}}^{smooth}$ are homeomorphic. It is not guaranteed that the hybrid flow $\Psi^{\mathbf{H}}$ will be smooth on $M_{\mathbf{H}}^{smooth}$. If, however, $\Psi^{\mathbf{H}}$ *is* smooth with respect to the differentiable structure on $M_{\mathbf{H}}^{smooth}$, then Theorem 5.6 implies that there are no Zeno states in $M_{\mathbf{H}}^{smooth}$. We say that we have removed Zeno by *smoothing.*

Hybrid suspension. [2] The basic idea is to "interpolate" executions between guards and images of corresponding resets, i.e. to make "instantaneous" discrete transitions given by reset maps "last" some time ϵ. The constructions goes as follows. Let $\epsilon > 0$ be arbitrary and assume $e = (i,j) \in E$. Instead of gluing $\overline{G(e)}$ to $\overline{\operatorname{im} R_e}$ via $\tilde{R}_e$, first enlarge the domain D_i by

$$D_i^\epsilon = D_i \cup (\overline{G(e)} \times [0,\epsilon]),$$

and then identify

$$(p,\epsilon) \sim \tilde{R}_e(p),$$

for every $p \in \overline{G(e)}$. Denote the space obtained by this identification for all $e \in E$ by $S^\epsilon M_{\mathbf{H}}$ and by π^ϵ the quotient (i.e. identification) map. On each $\overline{G(e)} \times [0,\epsilon]$, consider the trivial "vertical" flow: $(p,s,t) \mapsto (p,s+t)$ ($p \in \overline{G(e)}, 0 \le s \le \epsilon, t \in \mathbf{R}$). Denote by $S^\epsilon\Psi^{\mathbf{H}}$ the flow on $S^\epsilon M_{\mathbf{H}}$ obtained by projecting via π^ϵ this flow (for each $e \in E$) as well as $\Phi^{\mathbf{H}}$. We will call $S^\epsilon M_{\mathbf{H}}$ the *ϵ-suspended hybrid manifold* and $S^\epsilon\Psi^{\mathbf{H}}$ the associated *ϵ-suspended hybrid flow* (see Fig. 15). (This construction resembles the standard suspension of a map; cf. e.g. [PdM].)

It is immediate by construction that for ever $\epsilon > 0$, $S^\epsilon\Psi^{\mathbf{H}}$ accepts no Zeno-type executions.

[2] We thank Morris W. Hirsch for suggesting this idea in a recent conversation.

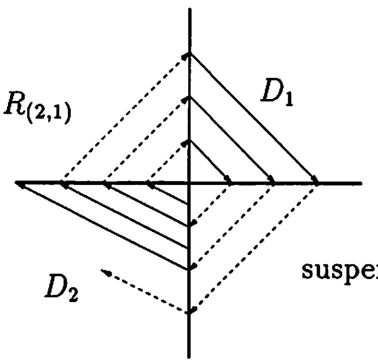

Figure 15: ϵ-suspended water tank $S^\epsilon M_{WT}$.

6 Classification of Zeno states in dimension two

We show that in dimension two, every hybrid flow near its Zeno state locally looks like the hybrid flow of WT near $\mathbf{0}$.

Theorem 6.1 *Let* $\mathbf{H}$ *be a 2-dimensional hybrid system and suppose that* $z \in M_{\mathbf{H}}$ *is a Zeno state. Then there is a neighborhood* U *of* z *in* $M_{\mathbf{H}}$ *and a neighborhood* V *of* $\mathbf{0}$ *in* M_{WT} *such that* $\Psi^{\mathbf{H}}|_U$ *is topologically conjugate to* $\Psi^{WT}|_V$.

PROOF: Let z be the Zeno state for some $x_0 \in M_{\mathbf{H}}$, and denote by χ_0 the execution starting from x_0. Assume

$$\pi^{-1}(z) = \{z_1, \ldots, z_l\},$$

and $z_j \in D_j$ (if this is not the case, reorder the domains). We can, without loss, assume that the execution $\tilde{\chi}_0 = \pi^{-1}(\chi_0)$ visits $D_1, D_2, \ldots, D_l, D_1, D_2, \ldots$ respectively.

Denote

$$A_j = \operatorname{im} R_{(j-1,j)}, \quad B_j = G(j, j+1),$$

so that $z_j \in \overline{A_j} \cap \overline{B_j}$ $(1 \le j \le l)$. (Here we identify 0 with l and $l+1$ with 1, i.e. $R_{(0,1)} = R_{(l,1)}$ and $G(l, l+1) = G(l, 1)$.)

1. We claim that for each j there exists V_j, a neighborhood of z_j in D_j such that every execution starting in $A_j \cap V_j - \{z_j\}$ reaches $B_j \cap V_j - \{z_j\}$.

To prove this, let V_j be the region in D_j bounded by A_j, B_j and a single arc of $\tilde{\chi}_0$, and let $p \in A_j \cap V_j - \{z_j\}$ be an arbitrary point. The execution $\chi(p)$ cannot intersect $\tilde{\chi}_0$, so it must reach $\overline{B_j} \cap V_j$. If it passes through z_j before it reaches $B_j \cap V_j - \{z_j\}$, then *every* execution starting from a point in $\overline{A_j}$ between p and z_j must pass through z_j. But this is impossible since, according to Lemma 5.5, z_j is not an equilibrium of X_j (here we also used the fact that X_j points inside D_j along A_j). Therefore, $\chi(p)$ reaches $B_j \cap V_j - \{z_j\}$.

Let $V = \cup V_j$ and $U = \pi(V)$.

2. We now investigate the only two possibilities:

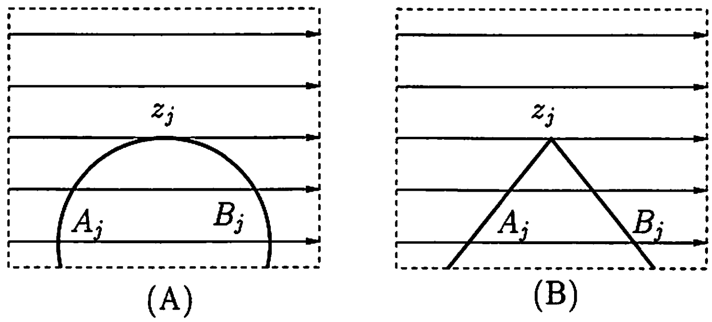

Figure 16: Local picture around z_j.

- **Case 1**: ∂D_j is smooth at z_j. Then it follows immediately from (A6) that X_j is *tangent* to ∂D_j at z_j. Therefore, by **1.**, in a smooth flow box around z_j, the local picture is as in part (A) of Fig. 16. (Recall that X_j extends smoothly to a neighborhood of D_j.)

- **Case 2**: ∂D_j is not smooth at z_j. Because of **1.**, it is not difficult to see that the local picture around z_j looks like part (B) of Fig. 16.

3. In fact, in both cases, up to a continuous change of coordinates, the local picture around z_j looks like part (B) of Fig. 16, but with A_j and B_j straight line segments. To construct a topological conjugacy between $\Psi^{\mathbf{H}}$ near z_j and Ψ^{WT} near $\mathbf{0}$, subdivide D_1 of WT into $l-1$ subdomains $D'_1, \ldots, D'_{l-1}$ by $l-2$ rays from the origin. Let $D'_l = D_2$. Define a hybrid system WT_l by: the domains are $D'_1, \ldots, D'_l$, the vector fields X'_j are the restrictions of the vector fields of WT to the corresponding new domains, and the resets are identity maps.

It is easily seen that X_j on V_j is topologically conjugate to X'_j on D'_j. Call the conjugating homeomorphism h_j. Glue the h_j's together to obtain a homeomorphism h between $\Psi^{\mathbf{H}}$ on U and Ψ^{WT_l} on a neighborhood of $\mathbf{0}$. Since Ψ^{WT_l} is clearly conjugate to Ψ^{WT}, the proof of the theorem is complete. ∎

7 Stability of Hybrid Equilibria

Recall that if ϕ_t is a local flow generated by a smooth vector field X on some set U (in $\mathbf{R}^n$ or any manifold), then $p \in U$ is an *equilibrium* for X (equivalently: for ϕ_t) if $X(p) = 0$ (equivalently: if $\phi_t(p) = p$ for all $t \in \mathbf{R}$). In case of a hybrid system there is usually more than one vector field at play, and even in the case when there is only one, resets are involved in generating the hybrid dynamics. Taking this into account we define a hybrid equilibrium as follows.

Definition 7.1 *Let* $\mathbf{H}$ *be a hybrid system. A point* $x \in M_{\mathbf{H}}$ *is called an (hybrid) equilibrium for the hybrid flow* $\Psi^{\mathbf{H}}$ *if* $\Psi^{\mathbf{H}}(t, x) = x$ *for all* $t \in J(x)$.

Equivalently, $x \in M_{\mathbf{H}}$ is a hybrid equilibrium if the hybrid dynamics of $\mathbf{H}$, consisting of reset maps and local flows of $\mathbf{H}$, map $\pi^{-1}(x)$ to itself. For example, any Zeno state is a hybrid equilibrium despite Lemma 5.5; however, hybrid dynamics make no time progress at this kind of equilibrium. The following definition distinguishes those hybrid equilibria which are created from equilibria of vector fields in $\mathbf{H}$ in the standard sense.

Definition 7.2 *A point $x \in M_{\mathbf{H}}$ is called a standard equilibrium for $\Psi^{\mathbf{H}}$ if it is a hybrid equilibrium and for each $p \in \pi^{-1}(x)$, if $p \in D_i$, then p is an equilibrium for X_i (i.e. $X_i(p) = 0$). It is called a pure equilibrium if it is standard and belongs to $\pi(\mathrm{int}\, D)$.*

Note that the only dynamics involved in creating a pure equilibrium are those of a single vector field. We now define the notions of (Lyapunov) stability and asymptotic stability of hybrid equilibria in analogy with those from dynamical systems.

Definition 7.3 *An equilibrium x_* of $\Psi^{\mathbf{H}}$ is called (Lyapunov) stable if for every neighborhood U of x_* in $M_{\mathbf{H}}$ there exists a neighborhood V of x_* in U such that for every $x \in V$,*

$$\Psi_t^{\mathbf{H}}(x) \in U \quad \textit{for all } t \in [0, \tau_\infty(x)).$$

If V can be chosen so that in addition to the properties described above,

$$\lim_{t \to \tau_\infty(x)} \Psi_t^{\mathbf{H}}(x) = x_*,$$

then x_* is asymptotically stable.

The following examples serve as a warning.

Example 7.1 (stable + stable = unstable) Let $\mathbf{H}_{\mathbf{ssu}}$ be defined as follows (cf. [LLN]):

$$D_1 = \{1\} \times \{(x,y) : xy \geq 0\}, \quad D_2 = \{2\} \times \{(x,y) : xy \leq 0\},$$

$$G(1,2) = \{1\} \times \{(x,y) : x = 0\}, \quad G(2,1) = \{2\} \times \{(x,y) : y = 0\},$$

$$R_{(i,j)}(i,x,y) = (j,x,y) \qquad \text{where } (i,j) = (1,2) \text{ or } (2,1),$$

$$X_1(x,y) = (-x + 10y, -100x - y)^T, \quad X_2(x,y) = (-x + 100y, -10x - y)^T.$$

Then $\mathbf{0}$ is a spiral sink for both X_1 and X_2. However, it is not difficult to see that all executions spiral away from the origin. In fact, if p_0 is a point on, say, the x-axis, then the execution starting from p_0 first returns to the x-axis at a point p_1 such that $p_1 = \eta_{ssu} p_0$, where

$$\eta_{ssu} = \left(\frac{100}{\exp(\frac{\pi}{2\sqrt{1000}})\sqrt{1000}} \right)^4 > 1.$$

Example 7.2 (unstable + stable = stable) Define a hybrid system $\mathbf{H}_{\mathbf{uss}}$ by (domains are in polar coordinates):

$$D_1 = \{1\}\times\{(r,\theta) : r \geq 0,\ \alpha \leq \theta \leq \beta\},\quad D_2 = \{2\}\times\{(r,\theta) : r \geq 0,\ 0 \leq \theta \leq \alpha \text{ or } \beta \leq \theta \leq 2\pi\},$$

$$X_1(x,y) = (-x,y)^T,\quad X_2(x,y) = (-x-y, x-y)^T,$$

$$G(1,2) = \{1\} \times \{(x,bx) : x \geq 0,\ b = \tan\beta\},\quad G(2,1) = \{2\} \times \{(x,ax) : x \geq 0,\ a = \tan\alpha\},$$

$$R_{(1,2)}(1,x,y) = (2,x,y),\quad R_{(2,1)}(2,x,y) = (1,x,y).$$

We take $0 < \alpha < \beta < \pi/2$.

Then $\mathbf{0}$ is a saddle for X_1 and a spiral sink for X_2. If $p_0 \in G(2,1)$ is an arbitrary nonzero point let $p_1 \in G(2,1)$ be the first intersection of the forward execution from p_0 with $G(2,1)$. It is not difficult to check that $p_1 = \eta_{uss} p_0$, where

$$\eta_{uss} = e^{\beta-\alpha-2\pi}\sqrt{\frac{a(1+b^2)}{b(1+a^2)}}.$$

For a fixed b (and β), $\eta_{uss} \to 0$ as $a, \alpha \to 0$, so we can choose the parameters so that $\eta_{uss} < 1$. Then $\mathbf{0}$ is an asymptotically stable standard equilibrium for $\mathbf{H}_{\mathbf{uss}}$.

Example 7.3 (unstable + unstable = stable) Define a hybrid system $\mathbf{H}_{\mathbf{uus}}$ as follows: keep the domains, guards and resets the same as in the previous example, with a different choice of parameters which will be specified later. Let

$$X_1(x,y) = (-x,y)^T,\quad X_2(x,y) = (x-y, x+y)^T.$$

Let p_0 and p_1 have the same definition as in the previous example. Then $p_1 = \eta_{uus} p_0$, where

$$\eta_{uus} = e^{2\pi-\beta+\alpha}\sqrt{\frac{a(1+b^2)}{b(1+a^2)}}.$$

It is not difficult to see that $\eta_{uus} \to 0$ as $a \to 0$ (with b fixed), so $\eta_{uus} < 1$ for a sufficiently small. Then $\mathbf{0}$ is an unstable equilibrium for both X_1 and X_2 (a saddle for X_1, spiral source for X_2), but an asymptotically stable equilibrium for the hybrid dynamics.

Therefore, the situation is more complicated than in the case of a single dynamical systems. Our next goal is to formulate a stability result in terms of linearized data of the hybrid system at the equilibrium, which encompasses the above examples.

In the subsequent text, we use the following notation: if X is a vector field on a manifold M with local flow ϕ_t and $f : M \to \mathbf{R}$ a function, Xf will denote the derivative of f in the direction of X:

$$(Xf)(x) = \left.\frac{d}{dt}\right|_0 f(\phi_t x) = Tf(X(x)).$$

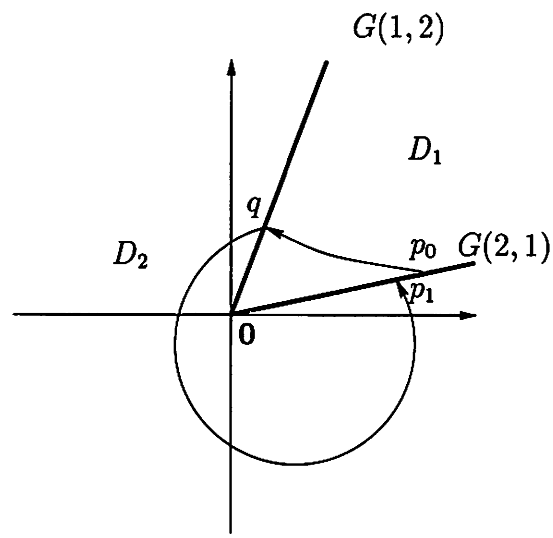

Figure 17: Unstable + unstable = stable.

For a map $h : (A, d_A) \to (B, d_B)$ between metric spaces, let

$$\mathrm{Lip}_p(f) = \sup_{q \in A - \{p\}} \frac{d_B(f(q), f(p))}{d_A(q, p)}.$$

This is the *Lipschitz constant of f at p*.

Lemma 7.4 (Poincaré maps) *Suppose $x_* \in M_{\mathbf{H}}$ is an isolated (not necessarily standard) equilibrium for $\Psi^{\mathbf{H}}$ and*

$$\pi^{-1}(x_*) = \{p_1, \dots, p_l\},$$

where $p_j \in D_{i_j}$. Suppose that for some neighborhood W of x_*, every execution starting in W has the same "itinerary", that is, if $x \in D_{i_j}$, then its hybrid trajectory visits

$$\pi(D_{i_j}), \pi(D_{i_{j+1}}), \dots, \pi(D_{i_l}), \pi(D_{i_1}), \dots$$

respectively, for all $1 \le j \le l$.

For each j, let

$$A_j = \overline{\mathrm{im}\, R_{(i_{j-1}, i_j)}} \cap \pi^{-1}(W), \quad B_j = \overline{G(i_j, i_{j+1})} \cap \pi^{-1}(W),$$

and define a map

$$h_j : A_j \to B_j$$

as follows: $h_j(p)$ is the first intersection of the positive X_{i_j}-orbit of $p \in A_j$ with B_j. Let $\tau_j(p)$ be the time it takes for this orbit of p to reach B_j, and suppose that τ_j is bounded above on some neighborhood of p_j in A_j (for all $1 \le j \le l$). Let

$$\mu_j = \mathrm{Lip}_{p_j} h_j,$$

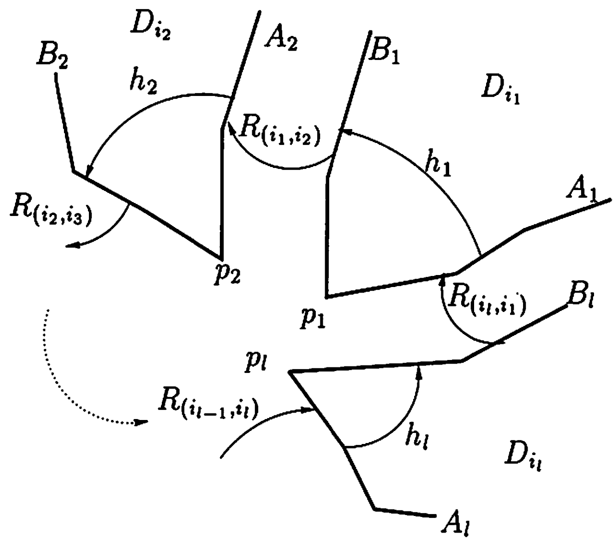

Figure 18: Theorem 7.4.

$$\nu_j = \|T_{p_j} R_{(i_j,i_{j+1})}\|$$

and

$$\eta = \prod_{j=1}^{l} \mu_j \, \nu_j.$$

If $\eta < 1$*, then* x_* *is asymptotically stable. If* $\dim \mathbf{H} = 2$ *and* $\eta > 1$*, then* x_* *is unstable.*

PROOF: For each j define a first-return map $P_j : A_j \to A_j$ by

$$P_j = h_{j+1} \circ R_{j+1} \circ \cdots \circ h_l \circ R_l \circ h_1 \circ R_1 \circ \cdots \circ h_j \circ R_j,$$

where $R_j = R_{(i_j,i_{j+1})}$. Then $P_j(p_j) = p_j$, and

$$\mathrm{Lip}_{p_j} P_j \leq \mu_1 \cdots \mu_l \cdot \nu_1 \cdots \nu_l < 1.$$

Therefore, there exists a ball V_j around p_j in A_j such that for all $p \in V_j$:

$$\|P_j(p) - P_j(p_j)\| \leq c_j \|p - p_j\|,$$

where $c_j < 1$. Thus P_j maps V_j into itself and for all $p \in V_j$,

$$\|P_j^m(p) - p_j\| = \|P_j^m(p) - P_j^m(p_j)\| \leq c_j^m \|p - p_j\| \to 0,$$

as $m \to \infty$. Since this is true for all $j = 1, \ldots, l$, and the times τ_j are bounded in V_j (if we take V_j sufficiently small), it follows that in $M_{\mathbf{H}}$, every execution in $\pi(\bigcup V_j)$ converges to x_*.

Observe that if $\dim \mathbf{H} = 2$, then for all j, $\mathrm{Lip}_{p_j} P_j$ is *equal* to η (because the norms is the absolute value and $|ab| = |a||b|$), so $\eta > 1$ clearly implies instability. ∎

To see how things (such as boundedness of τ and smoothness of h) can go wrong even in a very simple situation, consider the following example.

Example 7.4 Let $n = 2$, $p_* = \mathbf{0}$, $X(x, y) = (-x, y)$ and thus $\phi_t(x, y) = (e^{-t}x, e^t y)$. Assume A is the graph of $x \mapsto x^2$ $(x \geq 0)$, and $B : y = x$ $(x \geq 0)$. Then it is not difficult to check that for $p = (x, x^2) \in A - \{\mathbf{0}\}$, $\tau(p) = \frac{1}{2}\log\frac{1}{x}$, which is unbounded as $x \to 0+$, and

$$h(p) = \sqrt{x}\,(1, 1),$$

which is neither smooth not Lipschitz at $\mathbf{0}$.

In the subsequent text we will use the following notation. For a piecewise smooth $(n-1)$-dimensional submanifold A of $\mathbf{R}^n$ with piecewise smooth boundary ∂A, and a point $p \in \partial A$ at which the boundary of A is smooth, denote by T_p^+A the set of all vectors $v \in T_pA$ which point "inside" A. More precisely, $v \in T_p^+A$ if there exists $\epsilon > 0$ and a smooth curve $c : [0, \epsilon] \to A$ such that $c(0) = p$, $\dot{c}(0) = v$ and $c(t) \in A - \partial A$ for all $0 < t \leq \epsilon$.

Lemma 7.5 *Let X be a smooth vector field in $\mathbf{R}^n$ with flow $\{\phi_t\}$, and assume p_* is an isolated standard equilibrium for X. Let*

$$f : U - \{p_*\} \to \mathbf{R}$$

be a smooth submersion, where U is a bounded neighborhood of p_*. Suppose A and B are two closed sets which are also $(n-1)$-dimensional submanifolds of U with boundary, and assume the following holds:

(a) $p_* \in A \cap B$;

(b) $0 < m_- \leq Xf \leq m_+$ *on* $U - \{p_*\}$;

(c) $a_- \leq f \leq a_+$ *on A and* $B = \overline{f^{-1}(b)}$, *where* $a_+ < b$.

(d) *There exists $\tau_* > 0$ such that*

$$e^{\tau_* L}(T_{p_*}^+A) \subset T_{p_*}^+B,$$

where $L = T_{p_*}X$. Recall that $T_{p_*}\phi_t = e^{tL}$.

Then for every $p \in A$, the forward X-orbit of p, $\mathcal{O}_+(p)$, reaches B, defining a map

$$h : A \to B$$

by

$$h(p) = \mathcal{O}_+(p) \cap B = \phi_{\tau(p)}(p),$$

where $\tau(p)$ is the smallest $t \geq 0$ such that $\phi_t(p) \in B$. Moreover, τ is bounded, h is Lipschitz and its Lipschitz constant at p_* is

$$\begin{aligned} \mathrm{Lip}_{p_*} h &= \|e^{\tau_* L}|_{T_{p_*}A}\| & (3) \\ &= \sqrt{\lambda_{\max}[(e^{\tau_* L}S)^T e^{\tau_* L}S]}, & (4) \end{aligned}$$

where $\lambda_{\max}$ denotes the largest (in absolute value) eigenvalue, and S is an $n \times (n-1)$ matrix whose columns form an orthonormal basis of $T_{p_*}A$ and belong to $T_{p_*}^+A$.

PROOF: Let us first show that for all $p \in A - \{p_*\}$, the forward orbit of p reaches B. Since $f(p) \in [a_-, a_+]$, we have, for $t > 0$,

$$\begin{aligned} f(\phi_t p) &= f(p) + \int_0^t (Xf)(\phi_s p)\, ds \\ &\in [a_- + m_- t, a_+ + m_+ t], \end{aligned}$$

so since f is continuous on $U - \{p_*\}$ and $a_+ < b$, there exists a unique $\tau(p) \in [\frac{b-f(p)}{m_+}, \frac{b-f(p)}{m_-}]$ such that $f(\phi_{\tau(p)}p) = b$ and $f(\phi_t p) < b$, for all $t \in [0, \tau(p))$. This shows that $\mathcal{O}_+(p)$ reaches B as well as that τ is a bounded function on $A - \{p_*\}$; namely,

$$\frac{b - a_+}{m_+} \leq \tau(p) \leq \frac{b - a_-}{m_-}.$$

Next, let us show that $\tau(p) \to \tau_*$ as $p \to p_*$. Observe that $\phi_{\tau_*}A$ and B are tangent to each other at p_*. Therefore,

$$\frac{\ell(\phi_{\tau_*}p, h(p))}{d(h(p), p_*)} \to 0,$$

as $p \to p_*$, where $\ell(\phi_r p, \phi_s p)$ denotes the arclength of the indicated segment of the X-orbit of p. So in particular $\ell(\phi_{t_*}p, h(p)) = \ell(\phi_{\tau_*}p, \phi_{\tau(p)}p) \to 0$ as $p \to p_*$, and thus, for $p \in A - \{p_*\}$ we obtain:

$$\begin{aligned} |f(\phi_{\tau_*}p) - f(h(p))| &\leq m_+\, \ell(\phi_{\tau_*}p, h(p)) \\ &\to 0, \end{aligned}$$

as $p \to p_*$. Hence:

$$m_-\, |\tau(p) - \tau_*| \leq \left| \int_{\tau(p)}^{\tau_*} (Xf)(\phi_t p)\, dt \right| = |f(\phi_{\tau_*}p) - f(h(p))| \to 0,$$

as $p \to p_*$. This shows that $\tau(p) \to \tau_*$.

By the implicit function theorem, τ and h are smooth functions on $A - \{p_*\}$, and

$$T_p h(v) = d\tau(v) X(h(p)) + T_p \phi_{\tau(p)}(v), \tag{5}$$

for all $p \in A - \{p_*\}$ and $v \in T_p A$. Since τ is bounded, it follows that

$$p \mapsto \|T_p \phi_{\tau(p)}\|$$

is bounded, so to prove that h is Lipschitz it remains to show that $\|d\tau(v) X(h(p))\|$ is bounded (or better: converges to 0) at p_*, with $|v| = 1$.

Let $g : V \to \mathbf{R}$ be a submersion defined on some neighborhood V of p_* such that g is constant on B and $\delta := \inf_{p \in V} \inf_{|v|=1} |d_p g(v)| > 0$. This means that the "co-norm" or "minimum norm" of $d_p g$ is bounded below by δ on V. Then $dg(T_p h(v)) = 0$ and by (5)

$$\begin{aligned} d\tau(v)(Xg)(h(p)) &= -dg(T_p \phi_{\tau(p)}(v)) \\ &\to -dg(e^{\tau_* L} v) \\ &= 0, \end{aligned} \tag{6}$$

as $p \to p_*$. Therefore, for $p \in A - \{p_*\}$, $v \in T_pA$, $|v| = 1$:

$$\begin{aligned}
\|d\tau(v)X(h(p))\| &= |d\tau(v)(Xg)(h(p))| \frac{\|X(h(p))\|}{(Xg)(h(p))} \\
&= \frac{1}{\delta}|d\tau(v)(Xg)(h(p))| \\
&\to 0,
\end{aligned}$$

as $p \to p_*$. This proves that h is Lipschitz as well as formula (3). The expression for $\|e^{\tau_* L}|_{T_{p_*}A}\|$ in (4) follows from the following lemma. ∎

Lemma 7.6 *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a linear isomorphism and let E be a k-dimensional subspace of $\mathbf{R}^n$, where $0 < k < n$. Choose an $n \times k$ matrix S whose columns form an orthonormal (relative to the standard Euclidean inner product on $\mathbf{R}^n$) basis for E. Define the norm of the restriction of F to E by:*

$$\|F|_E\| = \sup\{|F(v)| : v \in E,\ |v| = 1\}.$$

Then:

$$\|F|_E\| = \sqrt{\lambda_{\max}[(FS)^T FS]}.$$

PROOF: Note first that $S^T S$ is the $k \times k$ identity matrix. If $v \in E$, then $v = Su$ for a unique $u \in \mathbf{R}^k$, so if $|v| = 1$, then $1 = v^T v = u^T S^T S u = u^T u = |u|^2$, hence $|u| = 1$ also. Therefore we have:

$$\begin{aligned}
\|F|_E\| &= \sup\{|F(v)| : v \in E,\ |v| = 1\} \\
&= \sup\{|FS(u)| : u \in \mathbf{R}^k,\ |u| = 1\} \\
&= \sup\{\sqrt{u^T S^T F^T F S u} : u \in \mathbf{R}^k,\ |u| = 1\} \\
&= \sup\{\sqrt{(S^T F^T F S u) \cdot u} : u \in \mathbf{R}^k,\ |u| = 1\} \\
&= \sqrt{\lambda_{\max}[(FS)^T FS]},
\end{aligned}$$

as claimed. ∎

The following main theorem is an analog of the linearization theorem for stability of equilibria of a single dynamical system. In the hybrid case, the linearized data include, besides the derivatives of the vector fields at the equilibrium, the tangent spaces at the equilibrium of guards and images of resets involved in the hybrid dynamics near the equilibrium.

Theorem 7.7 (Stability via Linearization) *Let $x_* \in M_{\mathbf{H}}$ be an isolated standard equilibrium for $\Psi^{\mathbf{H}}$ and*

$$\pi^{-1}(x_*) = \{p_1, \ldots, p_l\},$$

where $p_j \in D_{i_j}$ and $1 \le j \le l$. Suppose that there exists a bounded neighborhood W of x_* and for each $1 \le j \le l$ a smooth function

$$f_j : U_j - \{p_j\} \to \mathbf{R},$$

where U_j is a neighborhood of $D_{i_j} \cap \pi^{-1}(W)$ in $\{i_j\} \times \mathbf{R}^n$, such that:

(a) *$p_j \in A_j \cap B_j$, where*

$$A_j = \overline{\mathrm{im} R_{(i_{j-1},i_j)}} \cap U_j, \quad B_j = \overline{G(i_j, i_{j+1})} \cap U_j,$$

for all $1 \le j \le l$. Assume further that A_j and B_j are differentiable at p_j.

(b) *$a_j^- \le f_j \le a_j^+$ on A_j, and $B_j = \overline{f_j^{-1}(b_j)}$, for all j, for some numbers $a_j^- \le a_j^+ < b_j$.*

(c) *$0 < m_j^- \le X_{i_j} f_j \le m_j^+$ on $U_j - \{p_j\}$ $(1 \le j \le l)$.*

(d) *For each j there exists $\tau_j > 0$ such that*

$$e^{\tau_j L_j}(T_{p_j}^+ A_j) \subset T_{p_j}^+ B_j,$$

where $L_j = T_{p_j} X_{i_j}$.

For $1 \le j \le l$, let S_j be an $n \times (n-1)$-matrix whose columns form an orthonormal basis for $T_{p_j} A_j$ and belong to $T_{p_j}^+ A_j$. Let

$$\mu_j = \sqrt{\lambda_{\max}[(e^{\tau_j L_j} S_j)^T e^{\tau_j L_j} S_j]},$$

and

$$\nu_j = \| T_{p_j} R_{(i_j, i_{j+1})} \|.$$

Define

$$\eta_{\mathbf{H}}(x_*) = \prod_{j=1}^{l} \mu_j \nu_j.$$

If $\eta_{\mathbf{H}}(x_*) < 1$, then x_* is an asymptotically stable hybrid equilibrium. If $\dim \mathbf{H} = 2$ *and $\eta_{\mathbf{H}}(x_*) > 1$, then x_* is unstable.*

Remarks.

(i) Condition (b) says that B_j is the closure of a level set of f_j while A_j is "almost" a level set of f_j. The function f_j measures the progress trajectories of X_{i_j} make towards B_j, starting from A_j.

(ii) Condition (c) says that the time-τ_j map of the linearization of the flow of X_{i_j} at p_j (i.e. $T\phi_t^{i_j}$) maps $T_{p_j}^+ A_j$ to $T_{p_j}^+ B_j$. This means that at least on the level of linearizations, B_j is reachable from A_j in a bounded amount of time.

(iii) Note that we do not require $l \geq 2$, i.e. there may be only one state involved in creating the hybrid dynamics near x_*. However, the theorem doesn't work for pure equilibria (that is, we do not allow $x_* \in \operatorname{int} M_{\mathbf{H}}$).

PROOF: Follows immediately from lemmas 7.4 and 7.5. ∎

Let us now test the above theorem on our warning examples given at the beginning of the section.

Example 7.5 It is not difficult to see that $\mathbf{H}_{\mathbf{ssu}}$, $\mathbf{H}_{\mathbf{uss}}$ and $\mathbf{H}_{\mathbf{uus}}$ satisfy the conditions of Theorem 7.7, with the function $\arctan(y/x)$ as f_j (for all j). A simple computation shows:

$$\eta_{\mathbf{H_{ssu}}}(\mathbf{0}) = \eta_{ssu} > 1, \quad \eta_{\mathbf{H_{uss}}}(\mathbf{0}) = \eta_{uss} < 1, \quad \text{and} \quad \eta_{\mathbf{H_{uus}}}(\mathbf{0}) = \eta_{uus} < 1,$$

affirming the statements made in those examples.

Example 7.6 Define a 3-dimensional hybrid system $\mathbf{H}$ by: $D_1 = \{1\} \times S$, $D_2 = \{2\} \times \overline{\mathbf{R}^3 - S}$, where

$$S = \{(x,y,z) : x \geq 0,\ y \geq x^2,\ z \in \mathbf{R}\} \cup \{(x,y,x) : x \leq 0,\ y \geq -x(x-c),\ z \in \mathbf{R}\},$$

and

$$G(1,2) = \{(x,y,z \in D_1 : y = x^2\}, \quad G(2,1) = \{(x,y,z) \in D_2 : y = -x(x-c)\},$$

for some constant c. Let $X_1(x,y,z) = (-x-y, x-y, -\lambda_1 z)$ and $X_2(x,y,z) = (x-y, x+y, \lambda_2 z)$, where $0 < \lambda_2 \leq 1 \leq \lambda_1$. Then it is not difficult to check that

$$\eta_{\mathbf{H}}(\mathbf{0}) = e^{-2\gamma},$$

where $\gamma = \arctan c$, so if $c > 0$, then $\mathbf{0}$ is asymptotically stable.

Example 7.7 The following example illustrates some limitations of the stability theorem.

Let $\mathbf{H}$ be a 3-dimensional hybrid system with

$$D_1 = \{1\} \times K \times \mathbf{R} \quad \text{and} \quad D_2 = \{2\} \times \overline{\mathbf{R}^2 - K} \times \mathbf{R},$$

where $K = [0,\infty) \times [0,\infty)$. Let

$$G(1,2) = \{(x,y,z) \in D_1 : x = 0\}, \quad G(2,1) = \{(x,y,z) \in D_2 : y = 0\},$$

and

$$X_1(x,y,z) = (x-y, x+y, -\lambda_1 z), \quad X_2(x,y,z) = (-x-y, x-y, \lambda_2 z),$$

where $\lambda_1, \lambda_2 > 0$. The resets are identity maps.

Then the full trajectories of X_1 are spirals around the z-axis which increase in radius and converge to the xy-plane. The full trajectories of X_2 are also spirals around the z-axis, but they decrease in radius and diverge from the xy-plane. It is not difficult to check that, with notation from Theorem 7.7, $\mu_1 = e^{\pi/2}$, $\mu_2 = e^{3\pi\lambda_2/2}$, so $\eta_{\mathbf{H}}(\mathbf{0}) > 1$ and the theorem is inconclusive.

However, the flows can be decoupled into their xy- and z-parts the analysis of which shows that if $\lambda_1 > 3\lambda_2$, then $\mathbf{0}$ is an asymptotically stable hybrid equilibrium of $\mathbf{H}$. The reason Theorem 7.7 does not provide the same answer, intuitively speaking, is because it is not able to measure the small amount of contraction around $\mathbf{0}$ in the flows of both X_1 and X_2, which turns out to be sufficient for asymptotic stability. Namely, on $G(2,1)$ the flow of X_1 contracts in only one direction (and expands in the other) and similarly for the flow of X_2 on $G(1,2)$.

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