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Effective Interference and Effective Bandwidth of Linear Multiuser Receivers in Asynchronous Systems *

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Abstract

The performance of *linear multiuser receivers* in terms of the *Signal-to-Interference Ratio* (SIR) achieved by the users has been analyzed in a *synchronous CDMA system* under random spreading sequences. In this paper, we extend these results to a symbol-asynchronous system and characterize the SIR for linear receivers — the *Matched Filter* receiver, the *MMSE* receiver and the *Decorrelator.* We first analyze receivers that demodulate a symbol with an observation window that extends over the duration of symbol of interest and then extend these results to the multiple symbol observation window. For each of the receivers, we characterize the limiting SIR achieved when the processing gain is large and also derive lower bounds on the SIR using the notion of *effective interference.* Applying the results to a power controlled system, we derive *effective bandwidths* of the users for these linear receivers and characterize the *user capacity region*: a set of users is supportable by a system if the sum of the effective bandwidths is less than the processing gain of the system. We show that while the effective bandwidth of the decorrelator and the MMSE receiver is higher in an asynchronous system than that in a synchronous system, it progressively decreases with the increase in the length of the observation window and is asymptotic to that of the synchronous system, when the observation window extends infinitely on both sides of the symbol of interest. Moreover, the performance gap between the MMSE receiver and the decorrelator is significantly wider in the asynchronous setting as compared to the synchronous case.

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1 Introduction

Multiuser receivers which utilize the structure of *multi-access interference* in order to improve performance among users have gained importance as they perform better than the *conventional matched filter receiver* (IS-95) in a *spread-spectrum system* [5, 6, 7, 8, 12, 13, 17, 21]. Much of the previous work has been focused on the ability of receivers to reject the worst case interference (the *near-far resistance* [5, 6]), rather than performance evaluation in a power controlled system.

Recently, [15, 16] characterized the performance of *linear multi-user receivers* in terms of the *Signal-to-Interference Ratio* (SIR) achieved by the users in power-controlled systems, under spreading sequences that are random but known perfectly to the receiver. Particular attention is paid to the linear minimum mean-square error (MMSE) receiver which maximizes the output SIR, although parallel results were derived for the conventional matched filter and the decorrelator. The main result showed that in a system with large processing gain and many users, the interference across users at the output of each of these linear receivers can be decoupled by ascribing an *effective interference* term to each interferer. Based on the notion of effective interference, the paper characterized the *user capacity* in a power controlled system by the notion of *effective bandwidths* of the users which is the fraction of resources consumed by a user for attaining the target SIR. Related results were independently derived in [20] for a system with equal received powers.

In this paper, we extend the results of [16] to linear multi-user receivers in an *asynchronous* CDMA system, where we assume that the receiver not only has acquired perfect knowledge of the signature sequences of the users but also their relative delays. We first analyze linear receivers that estimate the transmitted symbol by observing the received signal over that symbol interval only, that is, the observation window of the receiver is limited to one symbol duration. The output SIR is random, being a function of the random spreading sequences and the random relative delays between the asynchronous users. For each of the three receivers, we show that the random SIR converges to a deterministic limit and we characterize the limit by the solution of a fixed point equation that depends on the received power and relative delay distributions of the users. Focusing on the MMSE receiver, we obtain a lower bound on the limiting SIR using the notion of *effective interference*. As in the synchronous case, these results are applied to derive notions of effective bandwidths in an asynchronous power-controlled system.

In the synchronous case, the output SIR $\beta_{1,sync}$ of user 1 under the MMSE receiver has been shown to satisfy the fixed-point equation, asymptotically in a large system:

$$\beta_{1,sync} \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K} I(P_k, P_1, \beta_{1,sync})}$$

where N, K are the processing gain and number of users respectively, σ^2 is the background noise power per degree of freedom, P_k is the received power of user k, and

$$I(P_k, P_1, \beta) := \frac{P_k P_1}{P_1 + P_k \beta}.$$

This quantity can be interpreted as the *effective interference* of interferer k on user 1, and it depends only on the received power P_k of the user to be demodulated, received power P_1 of the interferer, and the attained SIR β. Thus, the above fixed-point equation says that in a large system with random spreading sequences, the effect of an interferer at the output of the MMSE receiver does not depend on the received powers of the other users, except through the attained SIR. This is surprising given the fact that the MMSE receiver structure depends on the powers of all users in the system. In the asynchronous case, we will show that an analogous concept arises: the effective interference of an interferer, with received power P_k and whose symbols are delayed by a fraction of τ_k with respect to the user to be demodulated, is given by:

$$I(\tau_k P_k, P_1, \beta) + I((1-\tau_k)P_k, P_1, \beta)$$

The above expression shows that the effect of the asynchrony is approximately equivalent to splitting each interferer into two virtual users, one for each symbol interfering with the symbol to be demodulated and with a power proportional to the amount of overlap. Unlike the synchronous case, however, this only yields a *lower bound* to the limiting SIR achieved, although numerical results will show that this bound is very tight.

In order to characterize the *user capacity* of the system we extend the concept of *effective bandwidth* of [16] to the asynchronous system. In an asynchronous system where the relative delays τ_k's are uniformly distributed, the effective bandwidth of a user is given by:

$$e_{mf}(\beta) = \beta; \quad e_{mmse}(\beta) = 2\left(1 - \frac{\ln(1+\beta)}{\beta}\right); \quad e_{dec}(\beta) = 2$$

where β is the SIR requirement of the user. The effective bandwidths can be interpreted as the fraction of the available degrees of freedom consumed by a user: A set of users can be admitted into the system if the sum of their effective bandwidths is less than the total number of degrees of freedom in the system. The effective bandwidths for the matched filter and the decorrelator give an asymptotically exact characterization of the user capacity region, while the effective bandwidth characterization for the MMSE receiver yields only an inner bound, due to the conservative nature of the effective interference bound.

We also extend the above results to linear receivers that estimate the transmitted symbol by observing the received signal over an observation window that spans more than one symbol interval and is symmetrical around the symbol to be demodulated. In such a system, we show that the effective interference of interferer k under the MMSE receiver is given by,

$$\frac{1}{T}\Big[I(\tau_k P_k, P_1, \beta) + (T-1)I(P_k, P_1, \beta) + I((1-\tau_k)P_k, P_1, \beta)\Big]$$

where T is the length of the observation window. The first and last term can be attributed to the effect of the two symbols partially overlapping with the observation window at the two ends, while the middle term corresponds to the effect of the $T-1$ completely overlapping symbols. Just as in the single-symbol observation window case, we use the notion of effective bandwidth to characterize the user capacity of the system, when the relative delay distribution is uniform. The effective bandwidths for the MMSE receiver and the decorrelator yield inner bounds to the user capacity region, and are given by,

$$e_{mf}(\beta, T) = \beta; \quad e_{mmse}(\beta, T) = \frac{1}{T}\left[(T-1)\left(\frac{\beta}{1+\beta}\right) + 2\left(1 - \frac{\ln(1+\beta)}{\beta}\right)\right]; \quad e_{dec}(\beta, T) = \frac{T+1}{T}$$

where β is the SIR requirement of the user. While the matched filter receiver has the same performance regardless of the size of the observation window, we see that the MMSE receiver and the decorrelator have an effective bandwidth that decreases as the observation window is enlarged. As the observation window extends infinitely on both sides, that is, as $T \to \infty$, we see that the effective bandwidth is asymptotic to that in the synchronous system:

$$e_{mf}^{sync}(\beta) = \beta; \quad e_{mmse}^{sync}(\beta) = \frac{\beta}{1+\beta}; \quad e_{dec}^{sync}(\beta) = 1.$$

The results in [16] were derived using the random matrix results in [9, 14]. But in the asynchronous system, due to the relative delays between the users, these results are no longer applicable. We use some stronger results on random matrices from [1] in order to derive the SIR achieved by users in an asynchronous system.

All the results stated above were derived under the assumption that the users were chip synchronous, though they are symbol asynchronous. This was necessary to simplify the analysis, which assumes a chip sampled discrete time model. We present some simulation results which compare the theoretical results derived here with the SIR achieved by the receivers in the completely asynchronous system. Not surprisingly, the SIR in the chip-synchronous case is lower than that in the completely asynchronous case. While the difference is small when the number of users per degree of freedom is small, the gap grows to about 2 dB when the number of users is large. Using the insights gained from the notion of the effective interference, we propose a heuristic lower bound for the SIR in the completely asynchronous system. This heuristic bound follows the simulated SIR pretty well in the sense that the maximum difference between the two is less than a dB. Another assumption that is made in deriving the results for multiple symbol observation window is that the signature sequences of users are not only independent across users but also independent from symbol to symbol for any particular user. Though this is a valid assumption for long spreading codes, which extend over multiple symbols (as in the IS-95 system), it is not valid when the signature sequences are repeated from symbol to symbol. So, in order to justify the utility of our results, we present some simulation results which show that the SIR achieved in the case of repeated signature sequences is identical to the theoretically predicted SIR (which required independence among the signature sequences).

The outline of this paper is as follows: In Section 2, we discuss the model of the *asynchronous multi-access spread-spectrum* system and the structure of *linear multi-user receivers.* In Section 3, the concept of *random spreading sequences* is explained, followed by a brief review of the results in the synchronous system. In Section 4, we present our main result for the MMSE receiver and the matched filter receiver, extending the notion of *effective interference* to asynchronous systems. In Section 5, we derive the corresponding results for the decorrelator. In Sections 6, we apply the results derived to study the performance under power control and define a notion of *effective bandwidths* for asynchronous systems. In Section 7, we generalize the above results to a system where the observation window extends over multiple symbols, to obtain the SIR achieved and to characterize the *user capacity* using the notion of *effective bandwidths.* In Section 8, we present some simulation results to compare the performance of the completely asynchronous system with the results derived for the chip synchronous (symbol asynchronous) system. We conclude the paper with some discussions in Section 9.

2 The Spread Spectrum Model

In a *spread-spectrum system*, each user's information is spread over a larger bandwidth by modulation onto its *signature* or *spreading sequence.* In this paper, we consider a system that has a *processing gain* of N. In a symbol synchronous situation, a chip-sampled discrete-time model of the received signal $\mathbf{r} \in \mathbb{R}^N$ in a multi-access spread spectrum system with K users is given by,

$$\mathbf{r} = \sum_{k=1}^{K} x_k \mathbf{s}_k + \mathbf{n} \tag{1}$$

where $x_k \in \mathbb{R}$ is the transmitted information symbol and $\mathbf{s}_k \in \mathbb{R}^N$ is the signature sequence of the k^{th} user. $\mathbf{n}$ is the background Gaussian noise $\mathcal{N}(0, \sigma^2 \mathbf{I}_N)$. The transmitted symbols, x_k are assumed to be independent and $\mathbb{E}[x_k] = 0$, $\mathbb{E}[x_k^2] = P_k$, where P_k is the received power of user k.

In this paper, we consider a *symbol asynchronous* multi-access spread spectrum system with a processing gain of N. Even though we allow the system to be symbol asynchronous, we will assume the system to be *chip synchronous*, to make the analysis tractable. In the first part of the paper, we also restrict ourselves to receivers that have an *observation window* of one symbol, N chips in length: receivers which estimate the user's information symbol by observing the received signal over N chips of the symbol. Without loss of generality, we focus on a symbol of user 1 and notice that a typical interferer will have two different symbols interfering with the symbol of user

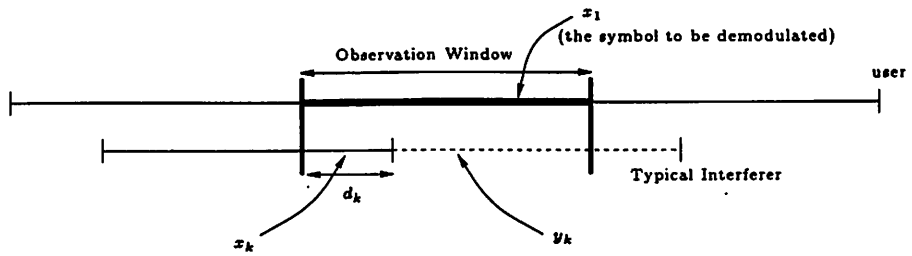

Figure 1: User 1 and a typical interferer within the observation window.

1 within the observation window, as shown in Figure 1. The observation window is marked in thick vertical lines and the two interfering symbols of a typical interferer are shown in solid and dotted line below the reference symbol of user 1. Since the system is corrupted by white noise and is assumed to be chip synchronous, the projections onto N waveforms which are matched to the N chip pulses (an orthogonal basis set) form a sufficient statistic for the received signal [18]. Thus, the sampled discrete-time model for the received signal $\mathbf{r} \in \mathbb{R}^N$ is given by,

$$\mathbf{r} = x_1\mathbf{s}_1 + \sum_{k=2}^{K} x_k\mathbf{u}_k + \sum_{k=2}^{K} y_k\mathbf{v}_k + \mathbf{n} \tag{2}$$

where $x_k, y_k \in \mathbb{R}$ are the two consecutive symbols of the k^{th} user which overlap with user 1 in the observation window, as shown in Figure 1. These have *effective signature sequences* $\mathbf{u}_k \in \mathbb{R}^N$ and $\mathbf{v}_k \in \mathbb{R}^N$, respectively. The effective signature sequences are completely determined by the original signature sequences $\mathbf{s}_k$ and the delays relative to user 1. If $d_k \in \mathbb{Z}^+$ denotes the relative delay in terms of number of chips of the k^{th} user with respect to user 1, then $\mathbf{u}_k$ has its first d_k elements to be the last d_k elements of $\mathbf{s}_k$ and the rest zeros. Similarly, $\mathbf{v}_k$ has the first d_k elements zero and the last $N - d_k$ elements to be the first $N - d_k$ elements of $\mathbf{s}_k$. That is,

$$(\mathbf{u}_k)_i = \begin{cases} (\mathbf{s}_k)_{(N-d_k+i)} & i \le d_k \\ 0 & N \le i < d_k \end{cases}$$

$$(\mathbf{v}_k)_i = \begin{cases} 0 & i \le d_k \\ (\mathbf{s}_k)_{(i-d_k)} & N \le i < d_k \end{cases}$$

The background noise $\mathbf{n}$ is still white gaussian, $\mathcal{N}(0, \sigma^2\mathbf{I}_N)$. The transmitted symbols x_k and y_k are assumed to be independent of each other and of the transmitted symbols of other users and $\mathbb{E}[x_k] = \mathbb{E}[y_k] = 0$ and $\mathbb{E}[x_k^2] = \mathbb{E}[y_k^2] = P_k$, where P_k is the received power of user k. Notice that the model for the synchronous system in eqn (1) can be arrived by setting the relative delays $d_k = 0$ in eqn (2).

In this paper, we restrict ourselves to the study of *linear demodulators* (*receivers*), that is, the estimate is a linear function of the received vector $\mathbf{r}$. Therefore, if $\hat{x}_1$ is the estimate of x_1, the symbol transmitted by user 1, then, a linear demodulator is captured by,

$$\hat{x}_1(\mathbf{r}) = \mathbf{c}_1^t \mathbf{r}$$

The information symbols transmitted by the user may be coded and in order to facilitate *soft decoding* by the channel decoder, we are interested in *demodulation* rather than *symbol detection*. From this point of view, the *Signal to Interference Ratio* (SIR) of the estimates is a relevant performance measure [16] and the SIR achieved by the linear receiver in the synchronous case [7, 18] is given by,

$$\mathrm{SIR}_{1,sync} = \frac{(\mathbf{c}_1^t \mathbf{s}_1)^2 P_1}{(\mathbf{c}_1^t \mathbf{c}_1)\sigma^2 + \sum_{k=2}^{K} (\mathbf{c}_1^t \mathbf{s}_k)^2 P_k} \tag{3}$$

The corresponding SIR in the asynchronous case would be,

$$\mathrm{SIR}_1 = \frac{(\mathbf{c}_1^t \mathbf{s}_1)^2 P_1}{(\mathbf{c}_1^t \mathbf{c}_1)\sigma^2 + \sum_{k=2}^{K} \left[(\mathbf{c}_1^t \mathbf{u}_k)^2 + (\mathbf{c}_1^t \mathbf{v}_k)^2\right] P_k} \tag{4}$$

The synchronous case was analyzed in [16, 20] and in this paper we consider the asynchronous situation. The conventional *matched filter receiver* simply projects the received vector onto the user's signature signature sequence, $\mathbf{s}_1$. This matched filter demodulator is optimal only if the total interference is white, which may not necessarily be the case in a multi-access system. In general, the *MMSE receiver* is the *optimal linear receiver* in the sense that it maximizes the SIR of user 1 by exploiting the structure of the interference [7, 12, 13]. The estimate $\hat{x}_{MMSE}$ of the MMSE demodulator [7] is given by,

$$\hat{x}_{MMSE}(\mathbf{r}) = \frac{P_1 \mathbf{s}_1^t (\mathbf{S}_1 D_1 \mathbf{S}_1^t + \sigma^2 I)^{-1} \mathbf{r}}{1 + P_1 \mathbf{s}_1^t (\mathbf{S}_1 D_1 \mathbf{S}_1^t + \sigma^2 I)^{-1} \mathbf{s}_1} \tag{5}$$

and the SIR achieved by user 1 is given by,

$$SIR_1 = P_1 \mathbf{s}_1^t (\mathbf{S}_1 D_1 \mathbf{S}_1^t + \sigma^2 I)^{-1} \mathbf{s}_1 \tag{6}$$

where $\mathbf{S}_1$ is a $N \times 2(K-1)$ matrix that has the effective signature sequences of the interferers, $\mathbf{u}_2, \ldots, \mathbf{u}_K, \mathbf{v}_2, \ldots, \mathbf{v}_K$ for its columns and $D_1 = \mathrm{diag}(P_2, \ldots, P_K, P_2, \ldots, P_K)$ which is the covariance matrix of $(x_2, \ldots, x_K, y_2, \ldots, y_K)^t$.

3 Performance and Random Spreading Sequences

The SIR described in Section 2 can be used to calculate the performance achieved with specific set of signature sequences assigned to the users. But it does not give any insight as to how the users interfere with each other to affect the performance because the output of the MMSE receiver has a complicated dependence on the powers of the interferers, as given by eqn (6), even though the interferers are additive at the input of the receiver. In practice, it is often more reasonable to assume that the spreading sequences are *randomly* and *independently* chosen [8, 16, 20]. These random sequences can be from a long *pseudo-random code* or sequences picked at random from a large look-up table. A similar situation arises when deterministic signature sequences are transmitted over a channel that has *multi-path fading* which randomizes the received signature sequences. Since the signature sequences are random, the SIR achieved β, being a function of the signature sequences is a random variable. In this paper, the signature sequences, though chosen randomly, are assumed to be known to the receiver.

In an asynchronous system, the various relative delays d_k are also assumed to be *random*. So, the performance measure as a random variable is a function of the random spreading sequences and random delays. Again, though the delays are random, it is assumed that the receiver has information of the delays of all the users. Practically, this means that the receiver has attained timing information of different users and this varies at a rate considerably lower than the symbol rate.

The random signature sequence of the k^{th} user can be modeled as $\mathbf{s}_k = \frac{1}{\sqrt{N}}(\nu_{k1}, \nu_{k2}, \ldots, \nu_{kN})^t$, where ν_{kl} are i.i.d, zero-mean unit variance random variables. The normalization of $\frac{1}{\sqrt{N}}$ ensures that the signature sequences are of unit norm on an average, that is, $\mathbb{E}[\|\mathbf{s}_k\|^2] = 1$. For technical reasons, we also require that the random variables have a finite fourth moment, $\mathbb{E}[\nu_{kl}^4] < \infty$. Practical situations like choice of ± 1 signature sequences can be obtained as particular cases of the final result. The results in this paper show that in the asymptotic regime, the SIR achieved is independent of the distribution of the random variables that constitute the signature sequences.

All the results described in this paper are *asymptotic* in nature, that is, we consider the limiting regime where the number of users is large, $K \to \infty$. This means that the processing gain also needs to be scaled up, else we will have the SIR to be zero with probability 1. Therefore, we have a system where $N, K \to \infty$, but with a fixed number of users per degree of freedom, α, that is, $K = \lfloor \alpha N \rfloor$. As we scale up the system, the empirical distribution of the powers of the users is assumed to converge to a fixed distribution (Cumulative Distribution Function), $F(P)$. The empirical distribution of the delays of the different users, relative to the observation window is also assumed to converge to a fixed distribution (CDF), $G(\tau)$, where the delay d relative to the reference user is given by $d = \lfloor \tau N \rfloor$. In a typical asynchronous system, we can assume that the arrivals are equally probable to be anywhere in $[0, N)$ and hence *uniform delay distribution* serves as a good model for the delays relative to a particular user. This paper analyzes the performance for a general delay distribution, which is later specialized to the uniform delay distribution.

The asymptotic performance in a synchronous CDMA system has been analyzed in [16]. Here, we review some of the important results of [16] which will help in comparison and also provide insight into the asynchronous problem.

Theorem 3.1 *In a synchronous system, as the processing gain $N \to \infty$, the SIR attained by user 1 in the MMSE receiver converges in probability to $\beta^*_{1,sync}$, which is the unique solution to the fixed point equation,*

$$\beta^*_{1,sync} = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \left\{ I(P, P_1, \beta^*_{1,sync}) \right\}} \tag{7}$$

where $\mathbb{E}_P$ denotes the expectation with respect to the power distribution $F(P)$ and

$$I(P, P_1, \beta^*_1) = \frac{PP_1}{P_1 + P\beta^*_1}$$

Heuristically, this means that in a large system, the SIR β_1 attained by user 1 is deterministic and approximately satisfies,

$$\beta_{1,sync} \approx \frac{P_1}{\sigma^2 + \frac{1}{N} \sum_{k=2}^{K} I(P_k, P_1, \beta_{1,sync})} \tag{8}$$

This result has an interesting interpretation: for a large system, the total interference at the output of the MMSE receiver can be decoupled into a sum of the background noise and an interference term from each of the other users, scaled down by the processing gain N. An important observation is that the interference term depends only on the received power of the interfering user, the received power of user 1 and the attained SIR and not on the powers of the other interfering users, except through the attained SIR, in spite of the fact that the MMSE receiver depends on the received amplitudes of all the users in the system. Therefore, as shown in [16], we can term $I(P_k, P_1, \beta_T)$ as the *effective interference* of k^{th} user at target SIR β_T.

The MMSE demodulator can be compared in performance with the commonly used matched filter receiver. In a matched filter, the asymptotic SIR achieved converges in probability to,

$$\beta^*_{1,MF} = \frac{P_1}{\sigma^2 + \alpha \int_0^\infty P dF(P)} \tag{9}$$

Hence, in a system of large processing gain N, the performance approximately satisfies,

$$\beta_{1,MF} \approx \frac{P_1}{\sigma^2 + \frac{1}{N} \sum_{k=2}^{K} P_k} \tag{10}$$

Thus, we have the interference term linear in the received powers of the users, which is not surprising as the interference in a finite system is additive across users. The following section extends the above results to the asynchronous system.

4 The Asynchronous System

In the synchronous system, if $\mathbf{S}$ denotes a matrix with the signature sequences of the interferers for its columns, then $\mathbf{S}$ has its elements identically distributed and the random matrix results of [9] are applied to prove Theorem 3.1. But in an asynchronous situation, some entries of $\mathbf{S}_1$ are zero (depending on the delays d_k) and we therefore need more powerful results than [9]. As the processing gain N of the system tends to infinity, if the empirical distribution of powers converges to a fixed distribution $F(P)$ and the empirical distribution of delays relative to user 1 also converges to a fixed distribution, $G(\tau)$, then the following theorem gives the asymptotic SIR achieved by user 1. The proof of this theorem requires some random matrix results from [1].

Theorem 4.1 *If $\beta_1^{(N)}$ is the SIR attained by user 1 for the MMSE receiver in an asynchronous system with a processing gain of N and an observation window of one symbol (N chips), then, as $N \to \infty$, $\beta_1^{(N)}$ converges in probability to β_1^*, where β_1^* is given by,*

$$\beta_1^* = \int_0^1 w(x)dx \tag{11}$$

and

$$w(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int_0^\tau w(z)dz\right) 1_{\{\tau \geq x\}} + I\left(P, P_1, \int_\tau^1 w(z)dz\right) 1_{\{\tau \leq x\}} \right\}} \tag{12}$$

where $\mathbb{E}_P$ and $\mathbb{E}_\tau$ denote the expectation with respect to the power distribution $F(P)$ and the delay distribution $G(\tau)$ respectively. $I(P, P_1, \beta) = \frac{PP_1}{P_1 + P\beta}$ is the effective interference term introduced in the synchronous case and $1_{\{\tau \geq x\}}$ is an indicator function that is 1 if $\tau \geq x$ and 0 otherwise. The solution to $w(x)$ exists and is unique in a class of function $w(x) \geq 0$.

The above theorem, which is proved in Appendix A gives a way to calculate the asymptotic SIR for a given distribution of delays and powers of the users. The function $w(x)$ can be solved numerically by the method of iterations, after suitably discretizing it over the interval $[0, 1]$. In a large system, if l is the chip of user 1 under consideration, heuristically,

$$w(l) \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K}\left\{ I\left(P_k, P_1, \frac{1}{N}\sum_{j=1}^{d_k} w(j)\right) 1_{\{d_k \geq l\}} + I\left(P_k, P_1, \frac{1}{N}\sum_{j=d_k+1}^{N} w(j)\right) 1_{\{d_k < l\}} \right\}} \tag{13}$$

The result can be interpreted as follows: the k^{th} interferer will have either the first or the second part overlapping with the chip l of user 1, depending on the value of the relative delay d_k. Therefore, only one of the two indicators is non-zero. Since the users are arranged in the increasing order of their delays, let K_1 be the first set of users who have the second symbol overlapping with chip l (that is, $d_k < l$ for $k \leq K_1$) and $K_2 = K - K_1$ be those users who have the first symbol overlapping with chip l. Therefore, we have,

$$w(l) \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K_1} I\left(P_k, P_1, \frac{1}{N}\sum_{j=d_k+1}^{N} w(j)\right) + \frac{1}{N}\sum_{k=K_1+1}^{K} I\left(P_k, P_1, \frac{1}{N}\sum_{j=1}^{d_k} w(j)\right)}$$

Therefore, following the argument in [16], we can have the interpretation: the *effective interference* of the k^{th} user at the l^{th} chip of user 1 is given by $I_k(P_k, P_1, l, w(\cdot))$, where,

$$I_k(P_k, P_1, l, w(\cdot)) = \begin{cases} I\left(P_k, P_1, \frac{1}{N}\sum_{j=1}^{d_k} w(j)\right) & d_k \geq l \\ I\left(P_k, P_1, \frac{1}{N}\sum_{j=d_k+1}^{N} w(j)\right) & d_k < l \end{cases} \tag{14}$$

and

$$w(l) \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K} I_k(P_k, P_1, l, w(\cdot))}$$

can be interpreted as the SIR achieved at the l^{th} chip and the total SIR is the average of the SIRs achieved in the N chips,

$$\beta_1 \approx \frac{1}{N}\sum_{l=1}^{N} w(l)$$

Hence, the above theorem shows that the interference by the k^{th} user at the l^{th} chip depends only on the received power of the k^{th} user, the received power of user 1 and the SIR achieved by user 1 at the various chips. Thus, unlike the synchronous situation where the effective interference term depends only on the overall SIR achieved, the asynchronous situation has the dependence on the SIR achieved at the chip level. The following theorem (derived in Appendix C) not only gives a lower bound on the SIR achieved but also provides simpler insights on how the interference from other users affects the performance by reducing the dependence to the overall SIR achieved. To derive the bound, we need to make a weak assumption on the delay distribution: in a common system, the arrival of a particular symbol has equal probability of being τN before or after the interval of the symbol to be demodulated. Therefore, for the purposes of the bound, we assume $G(\tau) = 1 - G(1-\tau)$, that is, the probability density function is symmetric about $\frac{1}{2}$.

Theorem 4.2 *In an asynchronous system, if the relative delay distribution $G(\tau)$ satisfies the condition $G(\tau) = 1 - G(1-\tau)$, then the asymptotic SIR β_1^* attained is lower bounded by γ_1^*, which is the unique solution of the fixed point equation,*

$$\gamma_1^* = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \{ I(\tau P, P_1, \gamma_1^*) + I((1-\tau)P, P_1, \gamma_1^*) \}}. \tag{15}$$

where

$$I(P, P_1, \gamma_1^*) = \frac{P P_1}{P_1 + P\gamma_1^*},$$

is the effective interference of an user of power P at SIR γ_1^*, as developed in the synchronous case.

Heuristically, in a large system,

$$\gamma_1^* \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K} I(\tau_k P_k, P_1, \gamma_1^*) + I((1-\tau_k)P_k, P_1, \gamma_1^*)}$$

An interesting interpretation can be given to the above expression by making the following observation: the first symbol of the k^{th} interferer has $\tau_k N$ part of its signature sequence overlapping within the observation window. This means that it has an effective power of $\tau_k P_k$ interfering in the observation window. Similarly, the second part of the k^{th} interferer has an effective power of $(1-\tau_k)P_k$. But since the two parts do not spread over the entire range of N, eqn (7) of the synchronous case cannot be applied. But the above theorem shows that the effective interference, assuming that the two parts spread over the entire range, but with a proportionally reduced power serves as a lower bound to the SIR achieved. Also the numerical calculations and simulation (as will be explained in Figure 2 later) in a specific case of equal powers and uniform delay distribution show that the bound is very close to the actual SIR achieved. The following proposition helps to give the interpretation of effective interference.

Proposition 4.3 *The equation*

$$x = \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K} [I(\tau_k P_k, P_1, x) + I((1-\tau_k)P_k, P_1, x)]} \tag{16}$$

has a unique fixed point x^* and the monotonicity property: $x^* \geq x$ if and only if,

$$\frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K} [I(\tau_k P_k, P_1, x) + I((1-\tau_k)P_k, P_1, x)]} \geq x$$

Proof: See Appendix B □

Therefore, if user 1 has an SIR requirement β_T, then it suffices to check,

$$\frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K}\left[I(\tau_k P_k, P_1, \beta_T) + I((1-\tau_k)P_k, P_1, \beta_T)\right]} \geq \beta_T$$

Based on this, we can interpret $I((1-\tau_k)P_k, P_1, \gamma_1^*)$ and $I(\tau_k P_k, P_1, \gamma_1^*)$ as the *effective interference* offered by the first $(\mathbf{u}_k)$ and second $(\mathbf{v}_k)$ part of the signature sequence respectively of the symbols of the k^{th} user interfering within the observation window. Therefore, the effective interference of the k^{th} interferer on user 1 at SIR requirement β_1 is given by,

$$\boxed{I(\tau_k P_k, P_1, \beta_1) + I((1-\tau_k)P_k, P_1, \beta_1).} \tag{17}$$

In order to compare the performance of the MMSE receiver, we also evaluate the asymptotic SIR achieved by the conventional matched filter receiver by the following proposition.

Proposition 4.4 *If $\beta_{1,MF}^{(N)}$ denotes the SIR attained in the asynchronous situation by user 1 in the matched filter receiver when the spreading sequence is of length N, then, as $N \to \infty$, $\beta_{1,MF}^{(N)}$ converges in probability to $\beta_{1,MF}^*$ given by,*

$$\beta_{1,MF}^* = \frac{P_1}{\sigma^2 + \alpha\int_0^\infty P\,dF(P)} \tag{18}$$

In a large system, the above theorem can be interpreted as,

$$\beta_{1,MF}^* \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K} P_k}$$

The above proposition shows that the matched filter receiver in the asynchronous case has the same asymptotic performance as in the synchronous situation. From the definition of $I(P, P_1, \gamma_1^*)$, we observe that $I(\tau_k P_k, P_1, \gamma_1^*) + I((1-\tau_k)P_k, P_1, \gamma_1^*) \leq P_k$ and hence the MMSE receiver performs better than the matched filter receiver (which is consistent with the fact that the MMSE receiver maximizes SIR among all the linear receivers). Another observation to be made is: while the interference in the matched filter receiver, being linear in powers of the interferer, grows unbounded as the received power increases, the MMSE receiver has the total effective interference from any user bounded by $\frac{2P_1}{\beta_1}$. This is the well known *near-far resistance* of the MMSE receiver [7], which the matched filter receiver lacks.

Typically, the relative delays of the symbols with respect to the reference are equally probable to be anywhere in $[0, N)$ and hence uniform distribution is a good example of the empirical delay distribution to consider. A lower bound for the asymptotic SIR achieved by the MMSE receiver under uniform delay distribution can be derived as a special case of Theorem 4.2. In addition, if we assume that power control is done and all the users have equal received powers, then, the SIR achieved by the MMSE receiver β_1^* is lower bounded by γ_1^*, which is the solution to the following fixed point equation,

$$\gamma_1^* = \frac{P}{\sigma^2 + \frac{2\alpha P}{\gamma_1^*}\left(1 - \frac{\ln(1+\gamma_1^*)}{\gamma_1^*}\right)}. \tag{19}$$

The first plot in the top left corner of Figure 2 compares the lower bound γ^* to the actual limiting SIR β^* obtained by numerically solving $w(x)$ given by eqn (12) by method of iterations, when the signal-to-noise ratio $\frac{P}{\sigma^2} = 20$dB. The maximum difference between the proposed bound and the actual SIR is less that half a dB over a wide range of α, the number of users per degree of freedom. The limiting SIR for the synchronous case is also plotted for comparison. In the remaining three plots, we would like to give a sense of how fast the convergence is to the asymptotic limit β^*. The plots compares the actual SIRs attained for several realizations with the asymptotic SIR achieved. The simulation assumes that the elements of the signature sequences are equiprobable ± 1 random variables of variance $\frac{1}{\sqrt{N}}$ and the relative delays are uniformly distributed in $[0, N)$. For different spreading lengths and for each value of α, 1000 samples of the SIR attained are computed using eqn (6).We plot the average of the sample points of the simulated SIRs and also the 1 standard deviation spread around the mean (that is, simulated $\beta \pm \sigma_\beta$). From the plots, we notice that the average of the simulated SIRs is overlapping with the predicted SIR given by eqn (12). Moreover, by observing the variance spread in $N = 32, 64, 128$, we notice that the variance progressively reduces as N increases (roughly halved on doubling N). For $N = 128$, the spread is less than a dB around the mean. So, we have that the average of the simulated SIRs coincides with the predicted asymptotic limit and the variance decreases with increasing N, thus reinforcing Theorem 4.1.

5 The Decorrelator

The previous sections compared the performance of the MMSE receiver with the conventional matched filter receiver. But among all the linear multi-user detectors, the first receiver to be proposed was the *decorrelator* [5, 6]. This receiver was shown to be *optimal* in the *worst case scenario*, when the powers of the interferers tend to infinity, in the sense that it attains the optimal near-far resistance, both in the synchronous and asynchronous systems [5, 6]. In the

Actual SIR attained in the Asynchronous MMSE Demodulator.

Average of the simulated SIRs in the Asynchronous MMSE Demodulator.

Bound on the SIR achieved in the MMSE receiver attained using the notion of Effective Interference.

Actual SIR achieved in the Synchronous MMSE demodulator.

Mean + 1 STD of the simulated SIRs ($\beta + \sigma_\beta$, simulated)

Mean - 1 STD of the simulated SIRs ($\beta - \sigma_\beta$, simulated)

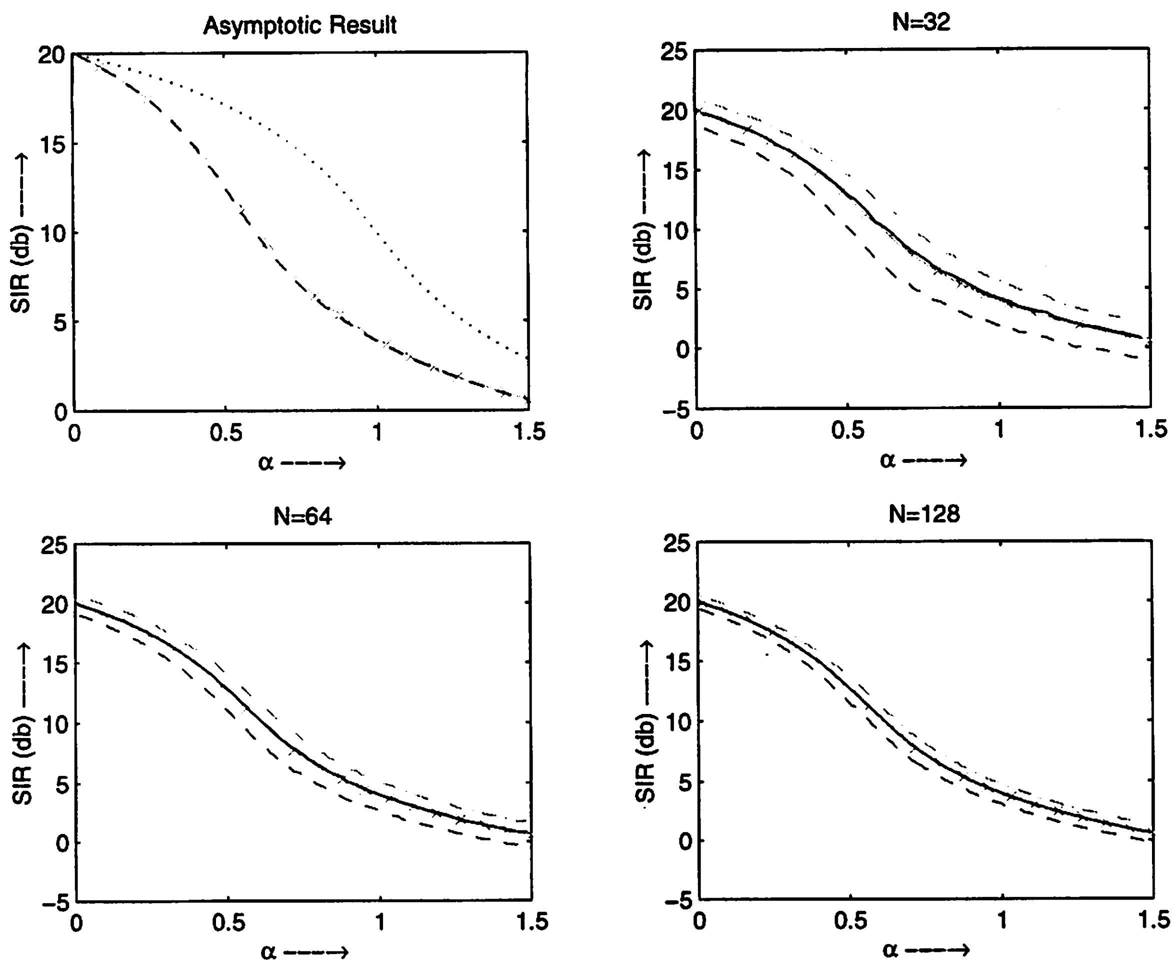

Figure 2: Comparison of the randomly generated SIRs (for processing gains $N = 32, 64, 128$) with actual SIR achieved and the bound proposed.

synchronous case, the SIR achieved by user 1 in the decorrelator, as the processing gain $N \to \infty$, converges in probability to $\beta^*_{1,dec,sync}$ given by,

$$\beta^*_{1,dec,sync} = \begin{cases} \frac{P_1(1-\alpha)}{\sigma^2} & \alpha < 1 \\ 0 & \alpha \geq 1 \end{cases} \tag{20}$$

This shows that the decorrelator in the synchronous system has an SIR which decreases linearly (with a slope of 1) with the increase in the number of users per degree of freedom. In an asynchronous system, the received signal with the reference fixed to the demodulated symbol of user 1 is given by eqn (2),

$$\mathbf{r} = x_1\mathbf{s}_1 + \sum_{k=2}^{K} x_k\mathbf{u}_k + \sum_{k=2}^{K} y_k\mathbf{v}_k + \mathbf{n}$$

If we define a $N \times 2K-1$ matrix $\mathbf{S}$ which has the signature sequences $\mathbf{s}_1, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_K, \mathbf{v}_2, \mathbf{v}_3, \ldots,$ $\mathbf{v}_K$ for its columns and $\mathbf{z} = [x_1, x_2, \ldots, x_K, y_2, \ldots, y_K]^t$ to be the vector of the corresponding information symbols, then, we have,

$$\mathbf{r} = \mathbf{S}\mathbf{z} + \mathbf{n} \tag{21}$$

The decorrelating filter [5, 6] can be described by an overall filter $(\mathbf{S}\mathbf{S}^t)^{-1}\mathbf{S}^t$ (if the inverse does not exist, then the pseudo-inverse is used in its place). In the absence of the external noise $\mathbf{n}$, this would give perfect estimates of the information symbols and hence is a *zero-forcing linear filter* [5, 18]. If $\mathbf{w} = (\mathbf{S}\mathbf{S}^t)^{-1}\mathbf{S}^t\mathbf{n}$ denotes the colored noise at the output of the filter, then it has a covariance matrix $\Sigma = (\mathbf{S}\mathbf{S}^t)^{-1}\sigma^2$. Therefore, we can describe the effect of the overall system as corrupting the information symbol z_k by noise w_k, a zero mean gaussian random variable of variance Σ_{kk}. The SIR attained by user 1 is therefore given by,

$$SIR_1 = \frac{P_1}{\Sigma_{11}} \tag{22}$$

Theorem 5.1 *In an asynchronous system, if $\beta^{(N)}_{1,dec}$ is the SIR of a decorrelator for user 1, then, as the processing gain $N \to \infty$, $\beta^{(N)}_{1,dec}$ converges in probability to $\beta^*_{1,dec}$ given by,*

$$\beta^*_{1,dec} = \begin{cases} \frac{P_1(1-2\alpha)}{\sigma^2} & \alpha < \frac{1}{2} \\ 0 & \alpha \geq \frac{1}{2} \end{cases} \tag{23}$$

Proof: See Appendix D □

Thus, we have that the SIR of the decorrelator for an asynchronous system also decreases linearly with the increase of number of users per degree of freedom, but with a slope of 2. Unlike

——— SIR achieved by the MMSE receiver in the Asynchronous system.

.... SIR achieved by the decorrelator in the Asynchronous system.

— - — Lower bound on the SIR achieved by the MMSE receiver in the Asynchronous system, derived from the notion of effective interference.

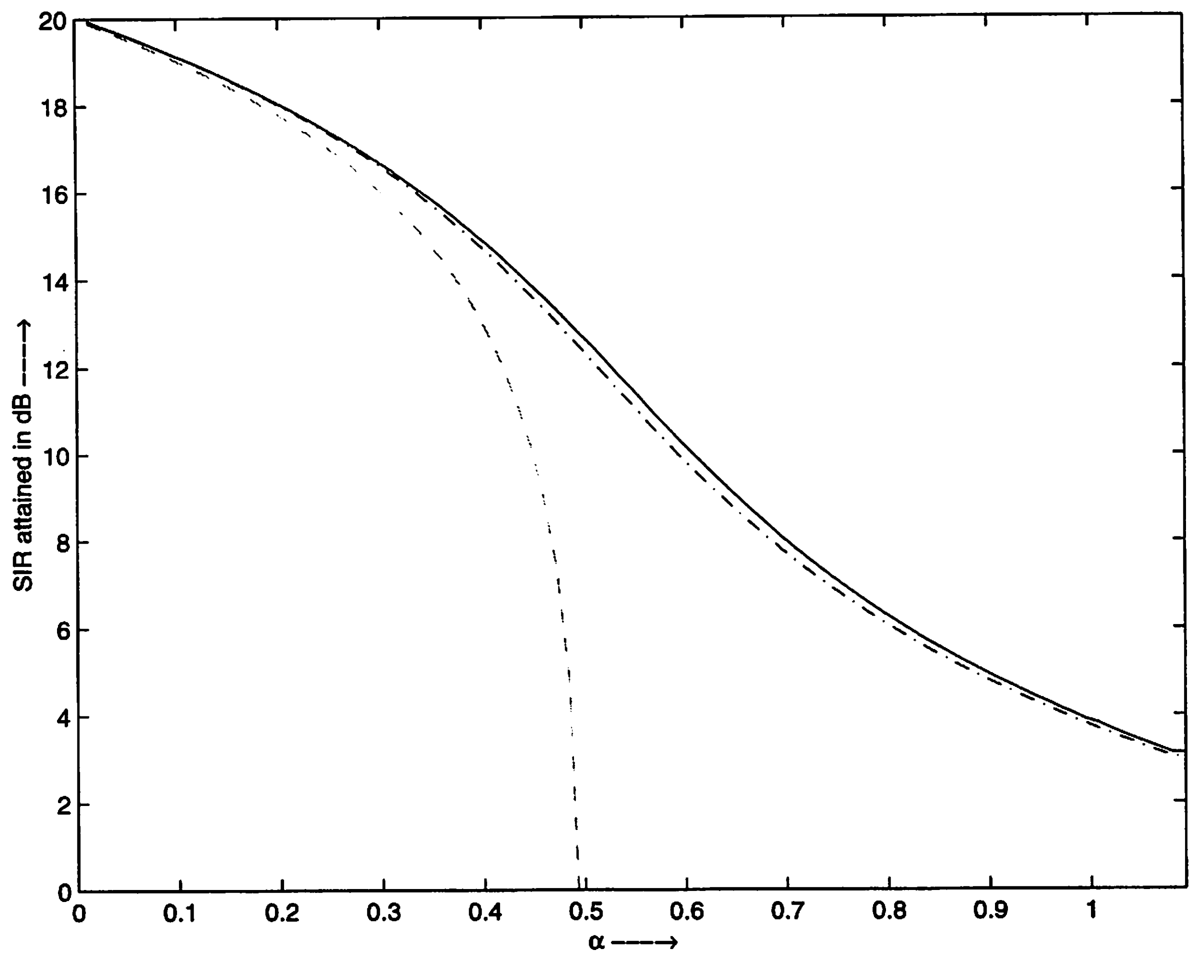

Figure 3: Comparison of the SIR achieved in the MMSE receiver and the decorrelator in an asynchronous system with equal received powers and $\frac{P}{\sigma^2}$ =20dB.

Theorem 4.2 where the SIR of the MMSE receiver derived was a lower bound, the SIR derived for the decorrelator in Theorem 5.1 is the actual SIR achieved by user 1.

In order to compare the performance of the MMSE receiver and the decorrelator, we plot the SIR achieved in the two receivers for a special case of ± 1 signature sequences and uniform delay distribution in Figure 3. The received powers of all the users are equal and the users have an SNR $\frac{P}{\sigma^2} = 20$dB. The SIR achieved by the MMSE receiver is plotted by numerically solving for $w(x)$ in eqn (12) in Theorem 4.1 and the SIR for the decorrelator is given by Theorem 5.1. The bound as given by eqn (15) is also plotted in the figure. We notice that the performance of the MMSE and the decorrelator are very close when the number of users per degree of freedom is small. But as the number of users per degree of freedom increases, the decorrelator degrades very rapidly and $\beta_{dec} \to 0$ as $\alpha \to \frac{1}{2}$, whereas, the MMSE receiver performs significantly better than the decorrelator in that region and beyond $\alpha = \frac{1}{2}$. The MMSE receiver has a more graceful degradation than the decorrelator and is therefore preferred in a system that needs to operate when the number of users form a significant fraction (roughly around 0.4 or higher) of the total processing gain of the system.

6 User Capacity and Performance under Power Control

In the previous sections, we analyzed the SIR achieved when the interferers had arbitrary received powers. We now apply the results to analyze the performance in a power-controlled system. For given SIR requirements of the users and given received power constraints, we characterize the number of users that can be admitted into the system under appropriate power control. We define the *user capacity* of the system as the number of users that can be admitted into the system without any power constraints. The capacity regions of the various demodulators will be analyzed in the asymptotic regime, when the processing gain $N \to \infty$. We first focus on the case when all users have a common SIR requirement β, and then extend the results to the case when users can have different requirements.

The matched filter receiver has exactly the same SIR in the synchronous and asynchronous cases and hence the derivations for the capacity region are identical to those found in [16]. It can be shown that the asymptotic minimal power solution is to assign equal received powers to all users. For a given power constraint $\bar{P}$, the maximum number of users with requirement β supportable can then be obtained by solving eqn (18) for α:

$$\boxed{\alpha_{max,MF}(\bar{P},\beta) = \frac{1}{\beta} - \frac{\sigma^2}{\bar{P}} \text{ users per degree of freedom.}}$$

Therefore, the user capacity of the matched filter receiver as $\bar{P} \to \infty$ is,

$$\boxed{C_{MF}(\beta) = \frac{1}{\beta} \text{ users per degree of freedom.}} \tag{24}$$

For the MMSE receiver, we find that the actual limiting SIR achieved, given by Theorem 4.1, does not allow easy calculations of the capacity region. Therefore, we use the lower bound on the SIR proposed in Theorem 4.2 and calculate the above mentioned quantities. So, the actual capacity region would be larger than the one derived from the bound, that is, more users can be admitted per degree of freedom than suggested by the bound. But in common situations, the difference between the bound and the actual SIR attained is shown by numerical calculations (Figure 2) to be small and hence, we can expect the user capacity to be only slightly greater than the one derived below.

In an asynchronous system, the relative delay with the reference window fixed to different users may be different, resulting in different expressions for the SIRs of different users. But in what follows, we consider the situation when the relative delay distributions are same for all the users and the *uniform delay distribution* is the delay distribution that satisfies this condition. Moreover, in a large system, the removal of any one user does not affect the empirical power distribution, and hence we have similar expressions for the SIRs of each user, with the ratio of the SIRs being equal to the ratio of the received powers. Thus, for the scenario when the users have common SIR requirement, it suffices to assign equal received powers to all users. Since the SIR achieved in the MMSE receiver is a decreasing function of α, we can expect it to saturate just like the matched filter receiver, but at a possibly higher value of α. Indeed, for a given power constraint $\bar{P}$, a lower bound on the number of users with requirement β supportable can be obtained by setting $P = \bar{P}$ and solving eqn (19) for α:

$$\boxed{\alpha_{max}(\bar{P}, \beta) = \frac{1}{2\left(1 - \frac{\ln(1+\beta)}{\beta}\right)} \left(1 - \frac{\sigma^2 \beta}{\bar{P}}\right) \text{ users per degree of freedom.}}$$

Hence, the user capacity of the MMSE receiver without power constraints is lower bounded by

$$\boxed{C_{MMSE}(\beta) = \frac{1}{2\left(1 - \frac{\ln(1+\beta)}{\beta}\right)} \text{ users per degree of freedom.}} \tag{25}$$

Contrasting eqns (24) and (25), we note that if α is feasible for both the matched filter and the MMSE receivers, the MMSE receiver achieves the target SIR with a lower power consumption than the matched filter receiver and also yields a higher capacity. Also, if $\alpha < 2$, then arbitrarily high SIRs β can be achieved without saturating the MMSE receiver, whereas the matched filter receiver saturates as the required SIR $\beta \to \frac{1}{\alpha}$.

In case of the decorrelator, if all the users require an SIR of β and power control is employed, then we can deduce from Theorem 5.1 that the maximum number of users supportable at requirement β and power constraint $\bar{P}$ is

$$\boxed{\alpha_{max,dec}(\bar{P},\beta) = \frac{1}{2} - \frac{\beta\sigma^2}{2\bar{P}} \text{ users per degree of freedom}}$$

Thus, the user capacity of the decorrelator is,

$$\boxed{C_{dec} = \frac{1}{2} \text{ users per degree of freedom.}} \tag{26}$$

It is important to note that the above expressions for the decorrelator are exact unlike those for the MMSE receiver. Contrasting with the user capacity of the decorrelator in the synchronous system [16], we note that the effect of asynchrony in the decorrelator is to reduce the user capacity of the system by a factor of 2.

The above results for the single class of users can be generalized to a system where there are J classes of users, with the j^{th} class requiring an SIR of β_j. Let the number of users in the j^{th} class be denoted by $\lfloor \alpha_j N \rfloor$ and we consider the regime where $N \to \infty$. The synchronous situation was analyzed and the effective bandwidths of the matched filter, the MMSE receiver and the decorrelator were derived in [16].

In an asynchronous system, the matched filter receiver has the same performance as the synchronous system and hence the same *user capacity region* in terms of the users per degree of freedom, that is, The capacity constraint on the feasible values of $(\alpha_1, \dots, \alpha_J)$ in the matched filter receiver is linear.

If there is no power constraint, then the user capacity region is given by

$$\boxed{\sum_{j=1}^{J} \alpha_j \beta_j < 1} \tag{27}$$

If class j user has a power constraint of $\bar{P}_j$, then,

$$\boxed{\sum_{j=1}^{J} \alpha_j \beta_j \leq \min_{1 \leq j \leq J} \left[1 - \frac{\beta_j \sigma^2}{\bar{P}_j}\right]} \tag{28}$$

Hence, as explained in [16], the above equation tells that a j^{th} class user consumes β_j part of the resources, that is, the *effective bandwidth* of the matched filter receiver is,

$$\boxed{e_{mf}(\beta_j) = \beta_j \text{ degrees of freedom per user.}} \tag{29}$$

Using the effective interference lower bound on the SIR under the MMSE receiver, we can obtain an inner bound on the user capacity region. Assuming a uniform delay distribution, it can be deduced from Theorem 4.2 and the monotonicity property of the fixed-point equation that an SIR of *at least* β_j can be achieved if we assign powers P_j to users in class j such that:

$$\frac{P_k}{\sigma^2 + \sum_{j=1}^{J} \frac{2\alpha_j P_k}{\beta_k}\left[1 - \frac{P_k}{\beta_k P_j}\ln\left(1 + \frac{\beta_k P_j}{P_k}\right)\right]} = \beta_k, \quad \text{for } k = 1, \ldots, J \tag{30}$$

The above equations implies $\frac{\beta_j}{P_j}$ is a constant across the classes and further simplification yields,

$$P_k = \frac{\beta_k \sigma^2}{1 - \sum_{j=1}^{J} 2\alpha_j \left(1 - \frac{\ln(1+\beta_j)}{\beta_j}\right)} \quad \text{for } k = 1, \ldots, J. \tag{31}$$

Hence, if the linear constraint:

$$\boxed{\sum_{j=1}^{J} 2\left(1 - \frac{\ln(1+\beta_j)}{\beta_j}\right)\alpha_j < 1} \tag{32}$$

is satisfied, then the SIR requirements can be attained with some received powers. Thus, this constraint specifies an inner bound on the interference-limited user capacity region, i.e. when there are no power limitations. If class j user has a power constraint of $\bar{P}_j$ such that $P_j \leq \bar{P}_j$ for all users in class j, then the sufficient condition for the SIR requirements to be satisfied becomes

$$\boxed{\sum_{j=1}^{J} 2\left(1 - \frac{\ln(1+\beta_j)}{\beta_j}\right)\alpha_j \leq \min_{1\leq j\leq J}\left[1 - \frac{\beta_j \sigma^2}{\bar{P}_j}\right]} \tag{33}$$

Note that the constraints are linear in $(\alpha_1, \ldots, \alpha_J)$ and therefore, the MMSE receiver (under uniform delay distribution) has an *effective bandwidth*,

$$\boxed{e_{mmse}(\beta) = 2\left(1 - \frac{\ln(1+\beta)}{\beta}\right) \text{ degrees of freedom per user.}} \tag{34}$$

The actual capacity region for a requirement of β_k will at least be the region given above.

The decorrelator always consumes 2 degrees of freedom per user and hence we have,

$$\boxed{e_{dec}(\beta) = 2 \text{ degrees of freedom per user.}} \tag{35}$$

and the *capacity region* is given by,

$$\boxed{\sum_{j=1}^{J} 2\alpha_j \leq 1} \tag{36}$$

in the absence of power constraint and

$$\boxed{\sum_{j=1}^{J} 2\alpha_j \leq \min_{1\leq j\leq J} \left[1 - \frac{\beta_j \sigma^2}{\bar{P}_j}\right]} \tag{37}$$

when the j^{th} class user has a power constraint of $\bar{P}_j$.

The matched filter receiver has an effective bandwidth linear in the SIR required. So, the matched filter receiver cannot provide arbitrary high SIRs when the number of degrees of freedom per user are restricted. But the effective bandwidth in the MMSE receiver is upper bounded by 2 degrees of freedom per user and hence arbitrary high SIRs can be achieved if the number of users are less than the half the total degrees of freedom available. Moreover, the MMSE receiver coincides with the decorrelator when arbitrary high SIRs are required. These facts are effectively captured in Figure 4, where we plot the effective bandwidths of the three linear receivers analyzed in the synchronous and asynchronous systems. From the plot, we notice that when the SIR requirement is small, the matched filter receiver has a lower effective bandwidth than the decorrelator, but when the SIR requirement is large, the decorrelator outperforms the matched filter. This cross-over occurs at a higher value of required SIR in the asynchronous system than in the synchronous system. The MMSE receiver follows the matched filter receiver for small values of SIR and is asymptotic to that of the decorrelator when the SIR requirement is high. Observe also that the improvement gain of the MMSE receiver over the decorrelator is more significant in the asynchronous case than in the synchronous case. This is because while the decorrelator loses an entire degree of freedom in the asynchronous system due to an additional interfering symbol per user, the MMSE receiver fares better as it takes advantage of the fact that the overlapping symbols are only partial and hence their energy is only a fraction of that of the interfering symbol in the synchronous case. This can also be seen directly from the expression for the effective interference eqn (17).

7 Multiple Symbol Observation Window

In the previous sections, we considered the observation window limited to the symbol to be demodulated and analyzed the performance of different receivers. In this section, we extend the results to a situation when the observation window spans more than one symbol, that is, we estimate the symbol of interest by observing over T symbols intervals.

The observation window is assumed to be symmetric about the symbol to be demodulated, that is, if the observation window extends over T symbols, then T is an odd integer and the symbol to be demodulated has $\frac{T-1}{2}$ symbols on either side of it. This kind of demodulation (compared

——— (solid)	effective bandwidth of the MMSE receiver in the Asynchronous system.
— — (dashed)	effective bandwidth of the decorrelator in the Asynchronous system.
— · — (dash-dot)	effective bandwidth of the MMSE receiver in the Synchronous system.
· · · · · · (dotted)	effective bandwidth of the decorrelator in the Synchronous system.
—×—	effective bandwidth of the Matched Filter receiver in the Synchronous and the Asynchronous systems.

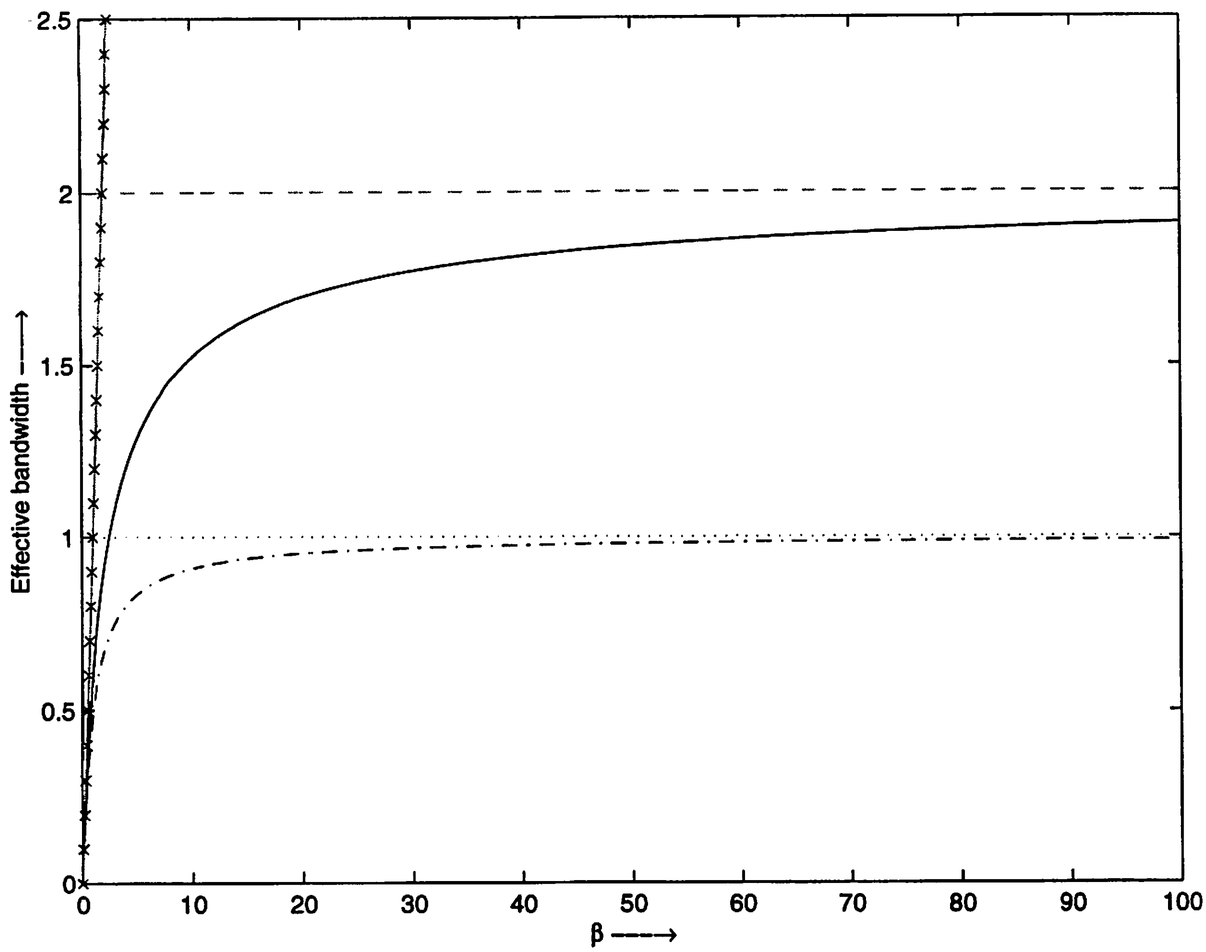

Figure 4: Comparison of the effective bandwidths of the linear multi-user receivers in synchronous and asynchronous systems.

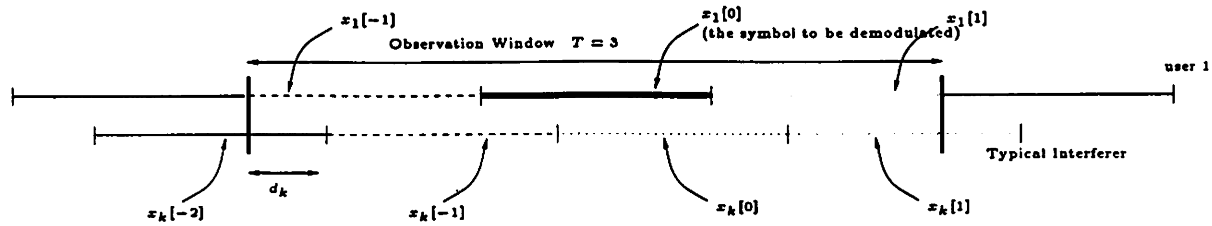

Figure 5: User 1 and a typical interferer within the observation window, $T = 3$.

to the single-symbol observation window) leads to a delay of $\frac{T-1}{2}$ symbols, increased complexity and an increase in the order of the filter which results in a slower rate of convergence when an adaptive system is used. In spite of the above drawbacks, enlarging of the observation window might be a good compromise if the performance can be significantly improved in an asynchronous system. Figure 5 shows $x_1[0]$, the symbol to be demodulated and an observation window (marked in the figure by solid vertical lines) that spans 1 symbol on either side (i.e. $T = 3$). A typical interferer has $T+1$ symbols, $x_k[i]$, $i = -\frac{T+1}{2}, \ldots, \frac{T-1}{2}$ interfering within the observation window, with effective signature sequences that are non-zero only over a particular span (shown in the figure by the corresponding solid/broken line) which is determined by the relative delay d_k of the k^{th} user and the symbol $x_k[i]$ interfering. The $T-1$ symbols of user 1 (that is, $x_1[-1]$ and $x_1[1]$ in Figure 5) interfere with effective signature sequences that are non-zero only over a span of N chips.

The symbols transmitted by the users are assumed to be independent, with the power of the k^{th} user being constant over the observation interval, that is, $\mathbb{E}[x_k[i]] = 0$, $\mathbb{E}[x_k[i]x_l[j]] = P_k\delta_{kl}\delta_{ij}$. As explained in Section 3, the spreading sequences are assumed to be randomly chosen. In what follows, we assume that the spreading sequence of any user is independent from symbol to symbol, in addition to being independent of the signature sequences of the other users. This is a valid assumption in the case of long spreading codes which extend over many symbols, as in the IS-95 standard. But in some situations, it is more reasonable to choose a signature sequence and repeat it from symbol to symbol. We will analyze the system where the signature sequences are independent from symbol to symbol and then compare the results by means of simulation to that of a system with repeated signature sequences. The signature sequences have unit norm on an average, that is, $\mathbb{E}[\|\mathbf{s}_k\|^2] = 1$ and the relative delays are also assumed to be random. As usual, the receiver is assumed to have knowledge of the signature sequences and relative delays.

Theorem 7.1 *If $\beta_1^{(N)}$ denotes the SIR attained by user 1 for the MMSE receiver in the asynchronous system of processing gain N and an observation window of $T \geq 1$ symbols (T being an odd integer), symmetric about the symbol to be demodulated, then as $N \to \infty$, $\beta_1^{(N)}$ converges in* *probability to β_1^*, where β_1^* is given by,*

$$\beta_1^* = \int_{\frac{T-1}{2}}^{\frac{T+1}{2}} w(x)dx \tag{38}$$

where $w(x) \geq 0$ in $[0, T]$ is given by

$$w(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau I\left(P, P_1, \int_{C(x,\tau)} w(z)dz\right)} \tag{39}$$

the region of integration $C(x, \tau)$ is given by,

$$C(x,\tau) = \begin{cases} [0,\tau] & x \in [0,\tau] \\ [\tau+i-1, \tau+i] & x \in [\tau+i-1,\tau+i] \text{ for } i = 1, \ldots, (T-1) \\ [\tau+T-1, T] & x \in [\tau+T-1, T] \end{cases}$$

and $\mathbb{E}_P$, $\mathbb{E}_\tau$ denote the expectations with respect to the power distribution $F(P)$ and the delay distribution $G(\tau)$ respectively. $I(P, P_1, \beta) = \frac{PP_1}{P_1 + P\beta}$ is the effective interference introduced in the synchronous case. The solution to $w(x)$ exists and is unique in a class of functions $w(x) \geq 0$.

The above theorem is an extension of the results in Section 4. The proof of Theorem 7.1 is analogous to the proof of Theorem 4.1 and follows on the same lines from the results of [1], lemma A.1, Theorem A.2, and lemma A.3, with the regions of integration $C(x, \tau)$ depending on the regions where the signature sequence of interferer's symbols have non-zero interfering power, which in turn depends on the relative delays. Following the derivation in the last part of Appendix C, we have a lower bound for the SIR achieved, which is given by the following theorem.

Theorem 7.2 *If the delay distribution satisfies $G(\tau) = G(1-\tau)$ and the observation window is an odd integer $T \geq 1$, symmetric about the symbol to be demodulated, then the asymptotic SIR β_1^* achieved by the MMSE receiver can be lower bounded by γ_1^*, which is the unique solution of the fixed point equation,*

$$\gamma_1^* = \frac{P_1}{\sigma^2 + \frac{\alpha}{T}\mathbb{E}_P \mathbb{E}_\tau \left[I(\tau P, P_1, \gamma_1^*) + (T-1) I(P, P_1, \gamma_1^*) + I((1-\tau)P, P_1, \gamma_1^*)\right]} \tag{40}$$

where $I(P, P_1, \gamma_1^*) = \frac{PP_1}{P_1 + P\gamma_1^*}$ is the effective interference of an user of power P, at SIR γ_1^*.

Heuristically, in a large system,

$$\gamma_1^* \approx \frac{P_1}{\sigma^2 + \frac{1}{NT}\sum_{k=2}^{T} \left[I(\tau_k P_k, P_1, \gamma_1^*) + (T-1) I(P_k, P_1, \gamma_1^*) + I((1-\tau_k)P_k, P_1, \gamma_1^*)\right]} \tag{41}$$

The first part of the interference in the denominator is from noise. The remaining three can be explained by referring back to the Figure 5 : The first term is due to $d_k = \lfloor \tau_k N \rfloor$ part of the symbol interfering within the observation window, the second term is due to the $(T-1)$ complete symbols interfering and the last term is due to the $N - d_k$ chips of the interfering symbol in the observation window. Therefore, just as in the single symbol observation case, we have that the bound has virtual interferers which have power proportional to the fraction of the interfering symbol within the observation window. Since,

$$I(P_k, P_1, \gamma_1^*) \leq I(\tau_k P_k, P_1, \gamma_1^*) + I((1-\tau_k)P_k, P_1, \gamma_1^*),$$

we have that a better SIR is achieved by a longer observation window. Since eqn (41) above has the monotonicity property as required by Proposition 4.3, we have the interpretation that

$$\boxed{\frac{1}{T}\left[I(\tau_k P_k, P_1, \beta_1) + (T-1)I(P_k, P_1, \beta_1) + I((1-\tau_k)P_k, P_1, \beta_1)\right]} \tag{42}$$

is the *effective interference* of the k^{th} user at SIR requirement β_1. This gives a picture on how the interfering users affect the performance. As observed in the other cases, the effective interference depends only on the received power of the interferer, the received power of the user and the SIR achieved.

Since the decorrelator can be obtained from the MMSE receiver when the SNR $\frac{P_k}{\sigma^2} \to \infty$ for each user, we can use Theorem 7.1 to obtain a lower bound for the performance achieved when the observation window spans over T symbols. It should be noted that while the bound is tight for $T = 1$, it is not for $T > 1$.

Proposition 7.3 *In an asynchronous system of processing gain N, if the decorrelator estimating the symbol of user 1 by observing over T symbols attains an SIR $\beta_{1,dec}^{(N)}$, then, as the processing gain $N \to \infty$, $\beta_{1,dec}^{(N)}$ converges in probability to $\beta_{1,dec}^*$ which is lower bounded by $\gamma_{1,dec}^*$ given by,*

$$\gamma_{1,dec}^* = \begin{cases} \frac{P_1}{\sigma^2}\left[1 - \left(\frac{T+1}{T}\right)\alpha\right] & \alpha < \frac{T}{T+1} \\ 0 & \alpha \geq \frac{T}{T+1} \end{cases} \tag{43}$$

From eqns (40) and (43), we notice that as $T \to \infty$, that is, as the observation window extends infinitely on both sides of the symbol of interest, the SIR achieved is asymptotic to the corresponding SIR achieved in the synchronous system.

In order to compare the above results, we obtain the SIR achieved by the matched filter receiver. Since the symbols which do not overlap with the symbol to be demodulated are orthogonal to the effective signature sequence of the symbol to be demodulated and the matched filter receiver simply projects the received vector onto the direction of the signature sequence of the symbol to be demodulated, we have that the matched filter does not gain in performance by the observation window being increased to more than one symbol. Therefore, the asymptotic SIR achieved is exactly the same as the asynchronous and the synchronous systems, given by eqn (9).

In a common system, since the empirical delay distribution converges to a uniform delay distribution, we can specialize the above results to this delay distribution. In addition, if we consider all users to have the same power P, the MMSE receiver attains an SIR that is lower bounded by γ^*, which satisfies the fixed point equation,

$$\gamma^* = \frac{P}{\sigma^2 + \alpha\left(\frac{T-1}{T}\right)\left(\frac{P}{1+\gamma^*}\right) + \frac{2\alpha P}{T\gamma^*}\left(1 - \frac{\ln(1+\gamma^*)}{\gamma^*}\right)}. \tag{44}$$

In order get a feel for the advantage of increasing the observation window to more than one symbol, we plot some numerical results in Figure 6. In these, we consider a system that has α users per degree of freedom, all with equal received powers and the relative delay distribution is uniform. The actual SIR achieved for a particular T is computed by numerically solving for $w(x)$ in eqn (39) for a specific case of uniform delay distribution. The bound was evaluated by solving for γ^* in eqn (44). In each of the four plots, apart from the actual SIR and the bound on the SIR for a particular T, we also plot the SIR achieved in the synchronous system, in order to help in comparison. From the plots, we note that when the observation window is increased from $T = 1$ to $T = 3$, there is a very large gain in the performance achieved by the user and the incremental gain achieved reduces as T is increased further. Therefore, we note that a delay of 2-3 symbols in demodulation and increased complexity may be a good compromise as we can achieve an SIR that is very close to the one achieved in a synchronous system.

We also note that the effective interference bound in $T = 3, 5$ is not as close as in the single symbol observation window. The is because of the fact that the bound gives equal weight to the interference from the symbols that directly overlap with the symbol of interest and to those that are farther away from the reference symbol. Since the bound does not take into account the fact that effect of the symbols that are overlapping with the symbol to be demodulated have a higher effect than the rest of the symbols, it is not very tight. But again as T increases, the relative contribution from these tail symbols goes down and the bound gets tighter, and finally converges to the SIR achieved in the synchronous system.

In order to compare the the various kinds of receivers analyzed so far, we derive the effective bandwidths for the three linear receivers for an observation window that extends over T symbols. The results here follow from similar arguments as in Section 6. We consider a system consisting of J classes of users, with the j^{th} class requiring an SIR of β_j and the number of users in the j^{th} class equal to $\lfloor \alpha_j N \rfloor$. The effective bandwidth characterization of the capacity region without power.

——— SIR achieved by the MMSE receiver over T-symbol observation window in an Asynchronous system.

— — The bound on the SIR attained using Effective Interference for the MMSE receiver over T-symbol observation window in an Asynchronous system.

— - — SIR achieved by the MMSE receiver in the Synchronous system.

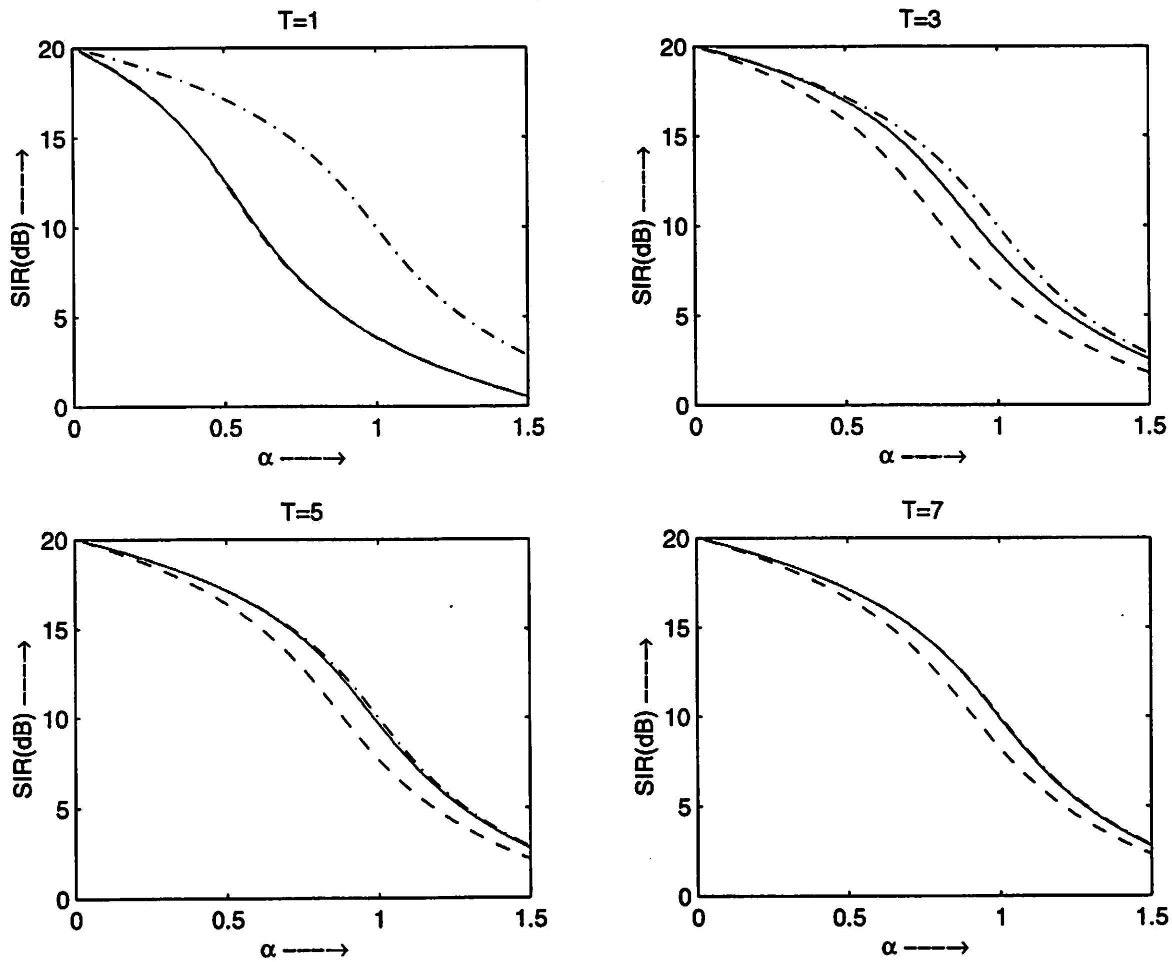

Figure 6: Comparison of the SIR achieved by users in a system of equal received powers and $\frac{P}{\sigma^2}$ =20dB for observation windows spanning over 1, 3, 5 and 7 symbols.

constraint is of the form:

$$\sum_{j=1}^{J} e(\beta_j, T)\alpha_j \leq 1$$

and with received power constraints $\bar{P}_j$ is given by

$$\sum_{j=1}^{J} e(\beta_j, T)\alpha_j \leq \min_{1\leq j\leq J} \left[1 - \frac{\beta_j \sigma^2}{\bar{P}_j}\right]$$

where the effective bandwidth function $e(\beta, T)$ depends on the receiver:

$$\boxed{e_{mf}(\beta, T) = \beta \text{ degrees of freedom per user.}} \tag{45}$$

$$\boxed{e_{mmse}(\beta, T) = \frac{1}{T}\left[(T-1)\left(\frac{\beta}{1+\beta}\right) + 2\left(1 - \frac{\ln(1+\beta)}{\beta}\right)\right] \text{ degrees of freedom per user.}} \tag{46}$$

$$\boxed{e_{dec}(\beta, T) = \frac{T+1}{T} \text{ degrees of freedom per user.}} \tag{47}$$

It is worth emphasizing again that while the characterization for the matched filter receiver is asymptotically exact, those for the MMSE receiver and the decorrelator are inner bounds on the user capacity region and are therefore conservative.

If we look at the expressions for the effective bandwidths of the MMSE receiver and the decorrelator given by eqns (46), (47), we notice that the effective bandwidths decrease with the increase in the length of the observation window T and as $T \to \infty$, the MMSE and the decorrelator have the effective bandwidth asymptotically approaching the respective effective bandwidths in a synchronous system:

$$e_{mf}^{sync}(\beta) = \beta; \quad e_{mmse}^{sync}(\beta) = \frac{\beta}{1+\beta}; \quad e_{dec}^{sync}(\beta) = 1.$$

All the analysis in this section was made with the assumption that the signature sequences are independent across users and across symbols of any particular user. Here, we compare the theoretical results obtained above with simulation results for the sequences that are random and independent across users, but repeated for each user. In Figure 7, we consider an asynchronous system in which the users have equal received powers and a SNR $\frac{P}{\sigma^2} = 20$dB, with an observation window of $T = 3$. The actual SIR, on the assumption that the signature sequences are independent across symbols is calculated by numerically solving for $w(x)$ in eqn (39). In the plot, we compare this with the average of the simulated SIRs (500 sample points) for the case when the signature sequences were randomly chosen once and then repeated for the other symbols transmitted by the

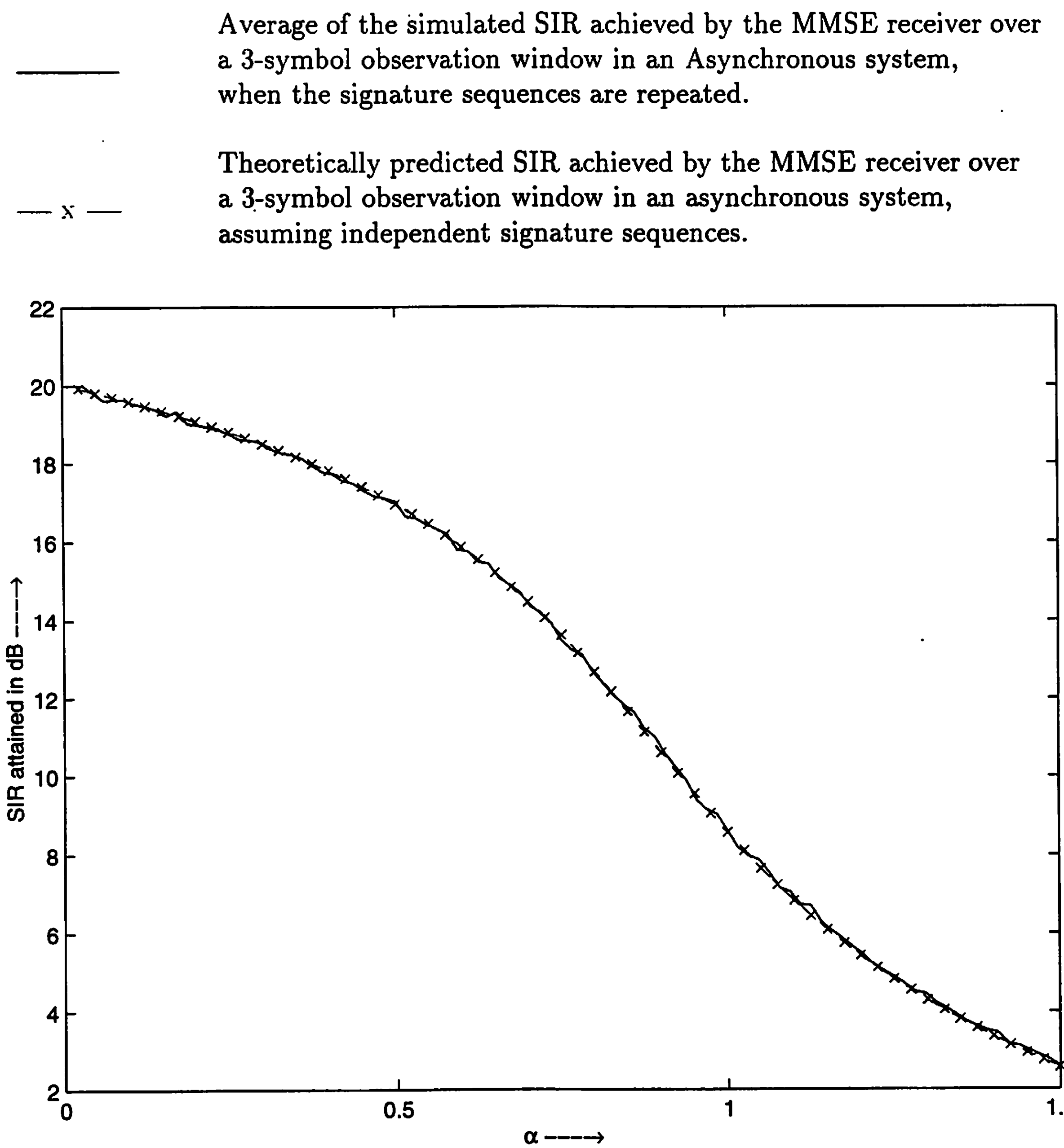

Figure 7: A comparison of the SIR achieved for an observation window of $T = 3$ with the system when the signature sequences are repeated.

user within the observation window. The figure shows that the results derived above are almost identical to the performance in the case of repeated signature sequences. The variance of the simulated SIRs for the case of repetition codes is found to reduce with increasing N, similar to those shown in Figure2.

8 Chip Synchronous versus Completely Asynchronous

In all the sections so far, we analyzed the asynchronous CDMA system with the assumption that the users were chip synchronous. In this section, we compare the results derived for the chip synchronous system with some simulations of the chip asynchronous system. In addition to this, based on some insights gained in the previous sections, we also propose a heuristic lower bound for the SIR achieved in the chip asynchronous system.

In Figure 8, we compare the SIR achieved by the MMSE receiver under different assumptions, when the observation window is one symbol duration and the received powers of all the users are equal. The curve plotted in the chip synchronous system is the result derived in this paper for a single symbol observation window, when the relative delay between users in terms of number of chips is an integer which is uniformly picked in $[0, N)$. The SIR achieved in the synchronous system as derived in [16] is also plotted for comparison. The solid curve for the completely asynchronous system is the average of 500 sample values of the SIR achieved by user 1 in a simulation when the processing gain is $N = 64$ and the relative delay is a real number uniform in $[0, L]$. In all these cases, the SNR for each user is assumed to be $\frac{P}{\sigma^2} = 20$dB. From the figure, we notice that the results in the chip synchronous system, is tight for small α and in general form a conservative estimate for the SIR achieved in the completely asynchronous system.

In order to show the dependence of the SIR achieved by the receivers for large values of α, in Figure 9, we plot the performance achieved by the matched filter and the MMSE receivers in the chip synchronous and the chip asynchronous systems when all the users have equal received powers and the background noise is such that $\frac{P}{\sigma^2} = 20$dB. The average of the simulated (500 sample points each) SIRs for the chip synchronous and the completely asynchronous systems with $N = 32$ under the assumptions stated in the previous paragraph is also plotted. From the figure, we notice that for small values of α, the SIR achieved by the MMSE receiver in the completely asynchronous system is close to the SIR achieved in the chip synchronous system and for large values of α, the performance is similar to that of the matched filter receiver.

In Figure 9, we notice that at large α, the completely asynchronous system has an SIR about 1.76dB greater than that in the chip synchronous system. This can be explained as follows: when α is large and the attained SIR is small, the MMSE receiver has a performance close to that of the matched filter receiver. In [11], it was shown that the matched filter has a performance in which the

——— SIR achieved by the MMSE receiver in an Asynchronous system, when the users are allowed to be chip asynchronous.

— — SIR achieved by the MMSE receiver in an Asynchronous system, when the users are chip synchronous.

— · — SIR achieved by the MMSE receiver in the Synchronous system.

· · · · · · Heuristic bound on the SIR achieved in the completely asynchronous system, based on the notion of effective interference.

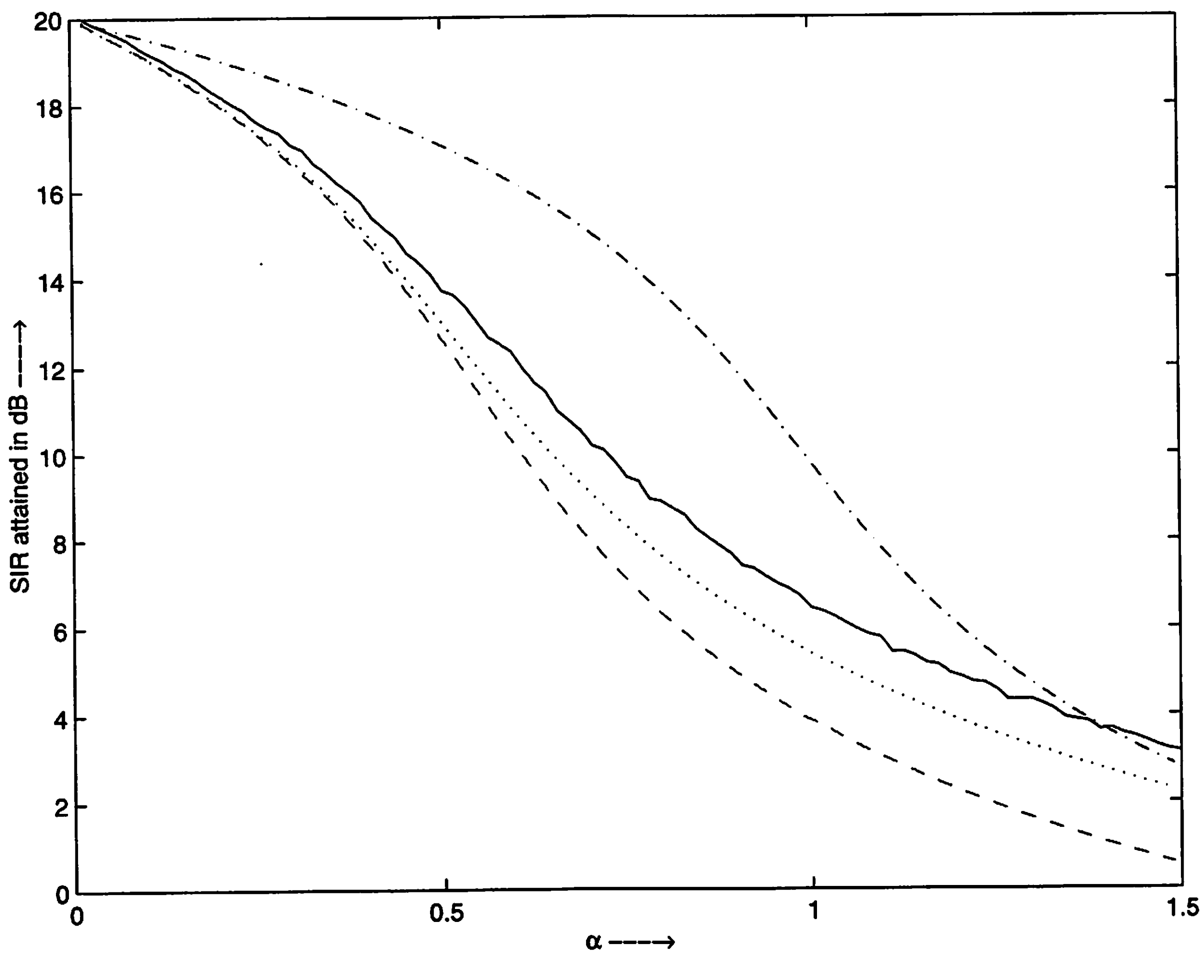

Figure 8: Comparison of the SIR achieved by users in a system of equal received powers and $\frac{P}{\sigma^2}$ =20dB.

SIR achieved by the MMSE receiver in an Asynchronous system, when the users are allowed to be chip asynchronous.

SIR achieved by the matched filter receiver in an Asynchronous system, when the users are allowed to be chip asynchronous.

SIR achieved by the MMSE receiver in an Asynchronous system, when the users are chip synchronous.

SIR achieved by the matched filter receiver in a Synchronous system.

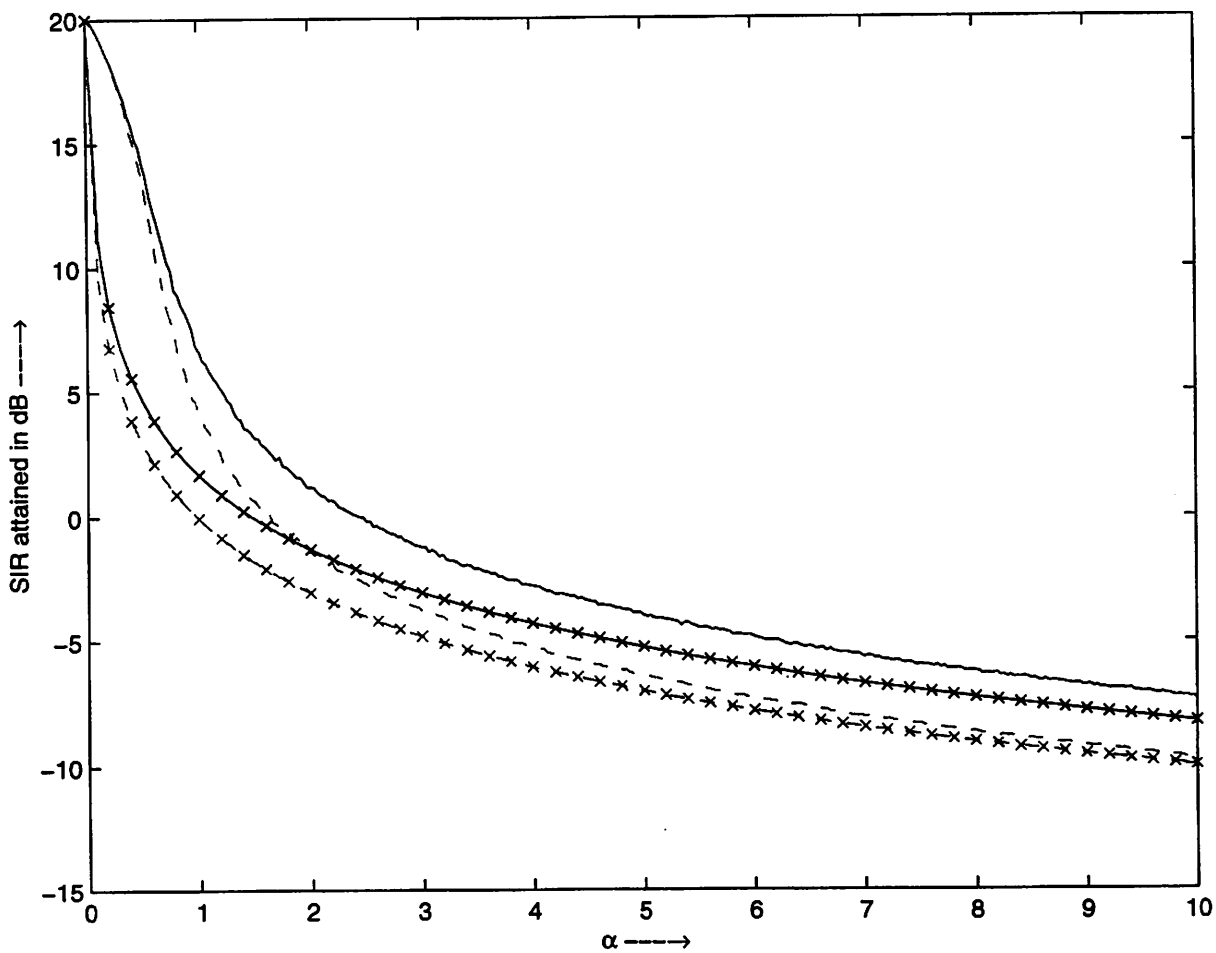

Figure 9: Comparison of the SIR achieved by users in a system of equal received powers and $\frac{P}{\sigma^2}$ =20dB.

interferers in a completely asynchronous system have an effective power of $\frac{2P}{3}$. Therefore, for large values of α, the matched filter (and hence the MMSE receiver) in the completely asynchronous system has an SIR that is about 1.76dB ($10\log_{10}(3/2)$) greater than that in the chip synchronous system. Based on this and the notion of effective interference, we propose a heuristic lower bound for the SIR achieved by the MMSE receiver in the completely asynchronous case. Since a user of power P interferers with an effective power of $\frac{2P}{3}$ in a chip asynchronous system for the matched filter, we can heuristically predict the effective interference, at the output of the MMSE receiver for user 1 due to an interferer of power P and relative delay τ, at a target SIR of β_1 to be,

$$\boxed{I\left(\tau\frac{2P}{3}, P_1, \beta_1\right) + I\left((1-\tau)\frac{2P}{3}, P_1, \beta_1\right)}$$

This is plotted as the dotted line in Figure 8. As seen in the figure, it is a decent lower bound to the simulation results, with the maximum difference being less than a dB. The heuristic bound as expected approaches that of the matched filter receiver as $\alpha \to \infty$.

In the case of a decorrelator, as shown in Figure 10, we plot the SIR achieved when the system is chip synchronous and chip asynchronous. The system is assumed to have users with equal received powers and an input SNR of $\frac{P}{\sigma^2} = 20dB$. The plots show that the results derived in the previous sections under the assumption that the users are chip synchronous is coincident with the mean (of 500 sample points simulated at $N = 64$) of the SIR achieved when the users are completely asynchronous. Therefore, with the decorrelator as the receiver, in an asynchronous system of processing gain N, we can admit approximately $\frac{N}{2}$ users.

9 Summary of results and Conclusion

In an asynchronous CDMA system, we have characterized the *Signal-to-Interference* Ratio of the users by the notion of *effective interference* and the *user capacity* of the system by the notion of *effective bandwidths.* To conclude the paper, we compare the three receivers, the matched filter receiver, the MMSE receiver and the decorrelator in terms of their effective interference and bandwidths, in the single symbol and multi-symbol observation windows. When the observation window spans over the symbol to be demodulated, the effective interference of the matched filter receiver is equal to the power of the interfering user in the asynchronous system (exactly as in the synchronous system). In the MMSE receiver, the effective interference term is non-linear and depends on the received power as well as the SIR attained. In the synchronous case, it was given by,

$$I(P, P_1, \beta) = \frac{PP_1}{P_1 + P\beta}$$

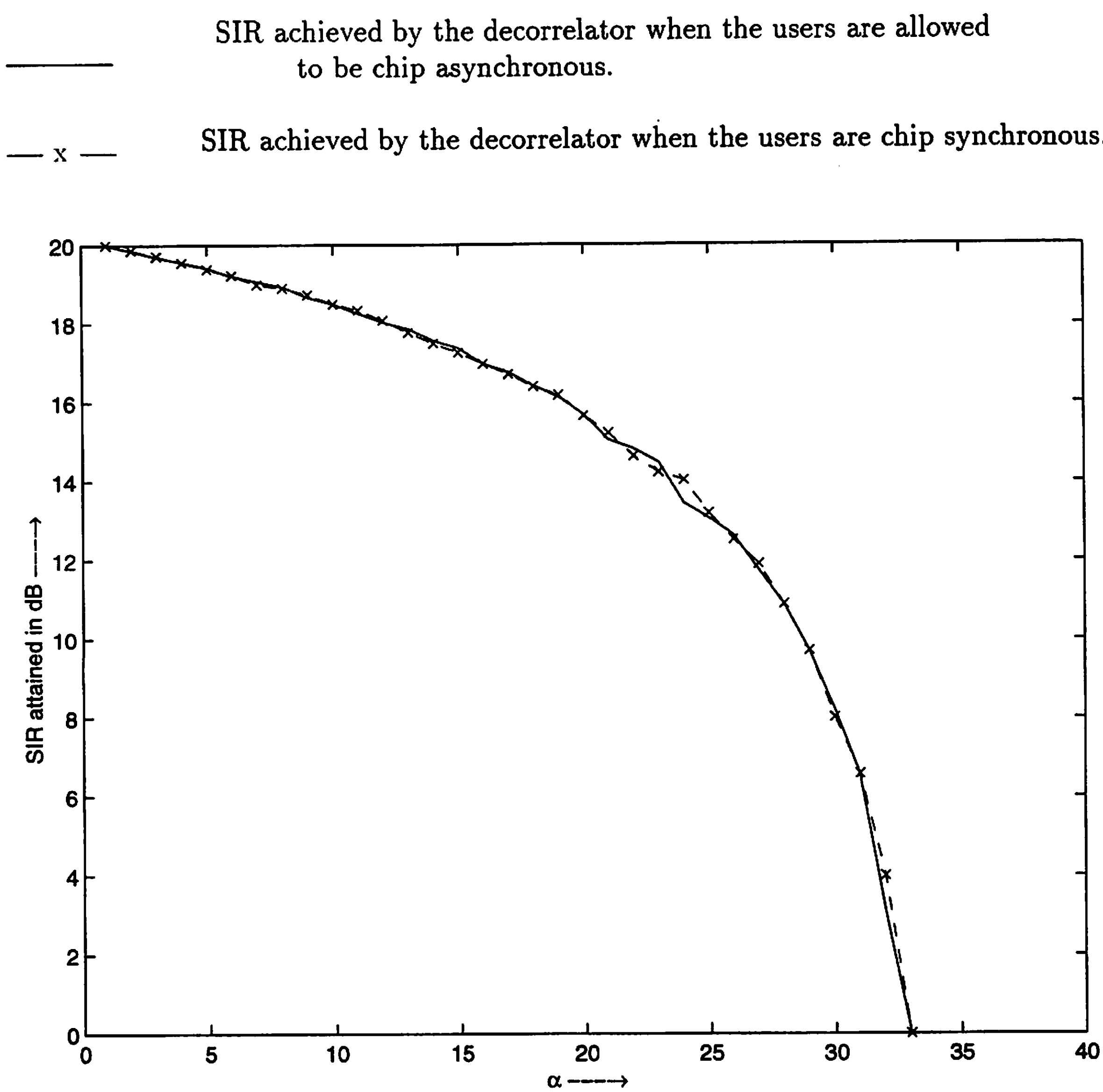

Figure 10: Comparison of the SIR achieved by users in a decorrelator, when the users have equal received powers and $\frac{P}{\sigma^2}$ =20dB.

In the asynchronous system, when the observation window spans over T symbols, we have that the effective interference of the MMSE receiver is given by,

$$I_\tau(P, P_1, \beta) = \frac{1}{T}\left[I(\tau P, P_1, \beta) + (T-1)I(P, P_1, \beta) + I((1-\tau)P, P_1, \beta)\right]$$

Therefore, as the observation window is enlarged, the MMSE receiver and the decorrelator have an effective interference term that is decreasing and is finally asymptotic to the effective interference in the synchronous system when $T \to \infty$. The dependence of the effective interference on the length of the observation window in the MMSE receiver is captured in Figure 11, where we plot the effective interference for various lengths of the observation window, when the received powers of all the users are equal to P. For comparison, we also plot the corresponding effective interference term in the synchronous system. The effective interference of the MMSE receiver is asymptotic to that of the decorrelator as $\beta \to \infty$.

Assuming perfect power control, we have been able to define effective bandwidths which characterize the amount of resource consumed by a user to achieve the target SIR. When the observation window extends over T symbols, we have,

$$e_{mf}(\beta) = \beta; \quad e_{mmse}(\beta) = \frac{1}{T}\left[(T-1)\left(\frac{\beta}{1+\beta}\right) + 2\left(1 - \frac{\ln(1+\beta)}{\beta}\right)\right]; \quad e_{dec}(\beta) = \frac{T+1}{T}$$

In Figure 12, we compare the effective bandwidths of the MMSE receiver in the synchronous system and the asynchronous system for different lengths of the observation window. We note that the effective bandwidth of the MMSE receiver decreases with the increasing length of the observation window and is asymptotic to that of the synchronous system as $T \to \infty$. On the other hand, the matched filter receiver has the same effective interference in the synchronous and asynchronous systems. As the SIR requirement β increases, the effective bandwidth under the MMSE receiver approaches that under the decorrelator. However, the performance gap between these two receivers is wider when T is small.

This paper analyzed the asynchronous CDMA system with the assumption that the users were chip synchronous and characterized the limiting SIR for linear multiuser receivers. We provided a heuristic lower bound for the completely asynchronous system and compared the results derived by means of simulation (Figure 8). The bounds derived in the multiple symbol observation window ($T \geq 3$, Figure 6) are not as close as in the single symbol observation window. The notion of effective interference might prove of great utility in deriving tighter bounds on the asymptotic SIR, which will give better characterization of user capacity regions.

Synchronous	————
Asynchronous $T = 1$	— —
Asynchronous $T = 3$	— · —
Asynchronous $T = 5$	· · · · · · ·

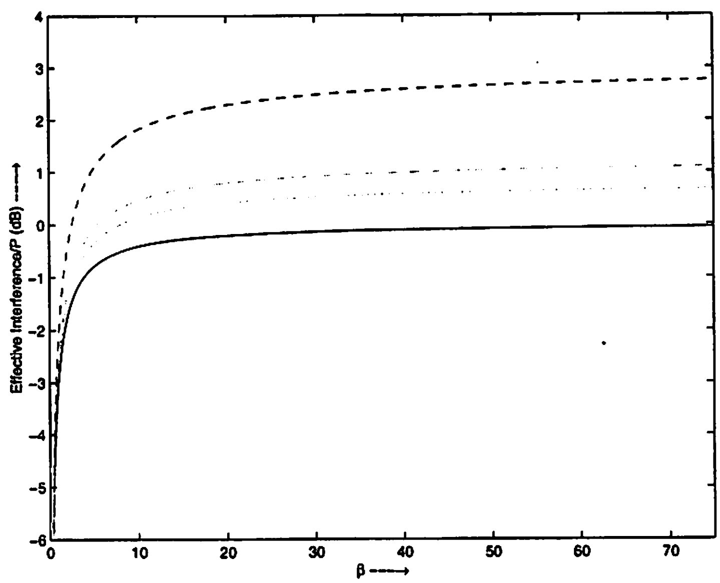

Figure 11: Effective Interference for the MMSE receiver in Synchronous and Asynchronous systems.

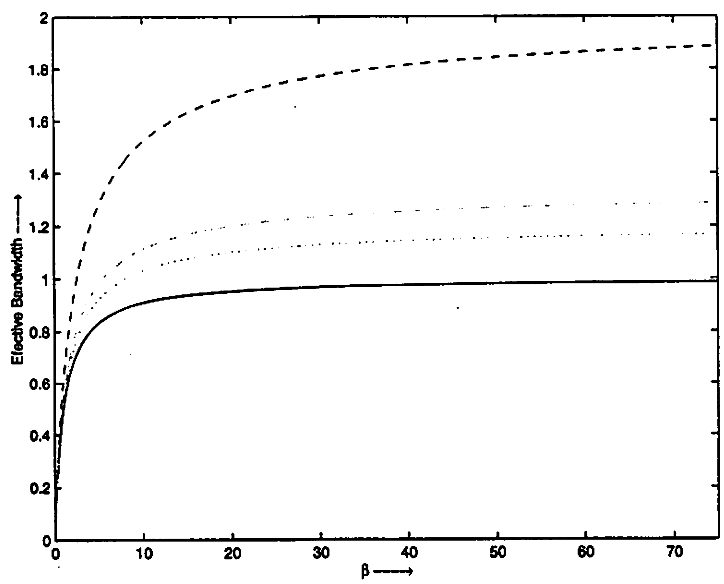

Figure 12: Effective Bandwidths for the MMSE receiver in Synchronous and Asynchronous systems.

Acknowledgements

We would like to thank Professor Dilip Sarwate for pointing out reference [11] regarding the performance difference of the matched filter receiver in the chip-asynchronous and completely asynchronous cases.

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A Proof of the Theorem 4.1

Theorem 3.1 in [16] was proved using the results of convergence of the empirical distribution of eigenvalues for large random matrices as derived in [9, 14]. But these results require the random matrices to have i.i.d entries, which is not satisfied by the matrices that arise in an asynchronous system because of the relative delays between users. Hence, in this section, to prove Theorem 4.1, we use some stronger results from [1], where large random matrices with non-i.i.d random variables are treated.

The SIR achieved by user 1 in the MMSE receiver for an asynchronous system is given by eqn (6). If $\mathbf{K}_z = \mathbf{S}_1 D_1 \mathbf{S}_1^t + \sigma^2 I$ denotes the covariance matrix of the interference, then we have that the SIR of user 1 is given by,

$$\beta_1 = P_1 \mathbf{s}_1^t \mathbf{K}_z^{-1} \mathbf{s}_1 \tag{48}$$

When the signature sequences are randomly chosen, the above performance measure depends in a simple way only on the eigenvalues of the covariance matrix which captured by the following lemma which is proved in [9].

Lemma A.1 *If* $\mathbf{s}_1 = \frac{1}{\sqrt{N}}(\nu_{11}, \dots, \nu_{iN})^t$, *where* ν_{1j} *are i.i.d zero mean, unit variance random variables with finite fourth moment, and independent of* $\mathbf{A}$, *a* $N \times N$ *symmetric matrix, then,*

$$\mathbb{E}\left[\mathbf{s}^t \mathbf{A} \mathbf{s}\right] = \frac{1}{N} tr\ \mathbf{A}$$

and

$$Var[\mathbf{s}^t \mathbf{A} \mathbf{s}] \leq \frac{C_1}{N} \lambda_{max}^2(\mathbf{A})$$

for some constant C_1 *which depends only on the fourth moment of* ν_{11}.

The above lemma is true for any $N \times N$ symmetric matrix $\mathbf{A}$ which is independent of $\mathbf{s}_1$. Since $\mathbf{s}_1$ is independent of $\mathbf{K}_z$, applying the above lemma to the SIR of user 1 in eqn (48), we have,

$$\mathbb{E}\left[\mathbf{s}_1^t \mathbf{K}_z^{-1} \mathbf{s}_1\right] = \frac{1}{N} \mathbb{E}\left[\text{tr } \mathbf{K}_z^{-1}\right]$$

Since $\lambda_{max}(\mathbf{K}_z^{-1}) \leq \frac{1}{\sigma^2}$, we have that $\text{Var}[\mathbf{s}^t \mathbf{A} \mathbf{s}] \to 0$ as $N \to \infty$ and from Chebychev's inequality,

$$\mathbf{s}_1^t \mathbf{K}_z^{-1} \mathbf{s}_1 - \frac{1}{N} \text{tr } \mathbf{K}_z^{-1} \xrightarrow{\mathcal{P}} 0 \tag{49}$$

Therefore, if we could get the limiting value of $\frac{1}{N}\text{tr } \mathbf{K}_z^{-1}$, then, we have an expression to which the limiting SIR converges in probability. Since the elements of $\mathbf{S}_1$ are not i.i.d, the results of [9, 14] cannot be applied. The following theorem captures the essence of Corrolary 10.1.2 of [1], which is the key tool in proving Theorem 4.1.

Theorem A.2 *Let $\mathbf{A}_n$ be a $n \times m_n$ matrix with independent random entries which have zero-mean and satisfy the condition*

$$v_n(x,y) = n\,Var(\mathbf{A}_n)_{ij} < B$$

for some uniform bound $B < \infty$ and $\frac{i}{n} \le x \le \frac{(i+1)}{n}$ and $\frac{j}{n} \le y \le \frac{(j+1)}{n}$. As $n \to \infty$, if $\frac{m_n}{n} \to c$ and

$$v(x,y) = \lim_{n\to\infty} v_n(x,y) \quad \text{for } \frac{i}{n} \le x \le \frac{(i+1)}{n} \text{ and } \frac{j}{n} \le y \le \frac{(j+1)}{n}$$

is a bounded function for $x \in [0,1], y \in [0,c]$, then, the limiting eigenvalue distribution $H(x)$ of $\mathbf{A}_n\mathbf{A}_n^t$ is given by,

$$\int_0^\infty (1+tx)^{-1} dH(x) = \int_0^1 u(x,t)dx \tag{50}$$

and $u(x,t)$ satisfies the equation,

$$u(x,t) = \frac{1}{1 + t\displaystyle\int_0^c \frac{v(x,y)\;dy}{1 + t\int_0^1 u(z,t)v(x,y)dz}} \tag{51}$$

The solution of eqn (51) exists and is unique in the class of functions $u(x,t) \ge 0$, analytical on t and continuous on $x \in [0,1]$.

In an asynchronous system, if the received powers of the users belong to a set with finite number of discrete power levels, then, we can use the above theorem to find the limiting SIR achieved by user 1. Assume that as $N \to \infty$, the limiting power distribution of the users has M discrete power levels, $P_1, \ldots, P_M$ occuring with probability masses $q_1, \ldots, q_M$ respectively. We also have that the empirical relative delay distribution converges to $G(\tau)$. Now, if we rearrange the $\mathbf{S}_1 D^{\frac{1}{2}}$ matrix by grouping users of different power levels into different blocks and within each block, arranging the users in the increasing order of their delays, then from the Strong Law of Large Numbers, we have,

$$v(x,y) = \begin{cases} P_m & 0 \le x \le G(\tau) \\ 0 & \text{Otherwise} \end{cases} \quad \text{if } 2\alpha\sum_{i=1}^m q_i - 2\alpha q_m \le y \le 2\alpha\sum_{i=1}^m q_i - \alpha q_m$$

$$v(x,y) = \begin{cases} P_m & G(\tau) < x \le 1 \\ 0 & \text{Otherwise} \end{cases} \quad \text{if } 2\alpha\sum_{i=1}^m q_i - \alpha q_m < y \le 2\alpha\sum_{i=1}^m q_i$$

for $m = 1, \ldots, M$. Therefore, for a system that has M discrete power levels, from Theorem A.2,

$$\begin{aligned} u_M(x,t) &= \frac{1}{1 + t\int\limits_0^c \frac{v(x,y)\,dy}{1 + t\int\limits_0^1 u_M(z,t)v(x,y)dz}} \\ &= \frac{1}{1 + \alpha t \sum\limits_{m=1}^{M} \mathbb{E}_\tau \left[\frac{P_m}{1 + tP_m \int\limits_0^\tau u_M(z,t)dz} 1_{\tau \geq x} + \frac{P_m}{1 + tP_m \int\limits_\tau^1 u_M(z,t)dz} 1_{\tau < x} \right] q_m} \\ &= \frac{1}{1 + \alpha t \mathbb{E}_P \mathbb{E}_\tau \left[\frac{P}{1 + tP \int\limits_0^\tau u_M(z,t)dz} 1_{\tau \geq x} + \frac{P}{1 + tP \int\limits_\tau^1 u_M(z,t)dz} 1_{\tau < x} \right]} \end{aligned}$$

where $\mathbb{E}_P$ is the expectation with respect to the probability masses q_i. Therefore, from Theorem A.2, we have that the limiting eigenvalue distribution of $\mathbf{S}_1 D_1 \mathbf{S}_1^t$ converges for the system which has finite number of power levels. From lemma A.1, we derived eqn (49) which showed that the limiting SIR achieved by user 1 converges in probability to the trace of $\mathbf{K}_z^{-1}$. Since the trace of $\mathbf{K}_z^{-1}$ depends on the eigenvalue distribution of $\mathbf{S}_1 D_1 \mathbf{S}_1^t$, we have the following lemma which is derived in [16].

Lemma A.3 *As $N, K \to \infty$, with $\frac{K}{N} = \alpha$, the SIR $\beta_1^{(N)}$ converges in probability to β_1^*,*

$$\beta_1^* = \int\limits_0^\infty \frac{P_1}{\lambda + \sigma^2} dH^*(\lambda)$$

where H^* is the limiting eigenvalue distribution of the random matrix $\mathbf{S}_1 D_1 \mathbf{S}_1^t$

Now, using eqn (50), in a system with M power levels, we have that the limiting SIR converges in probability to β_M^* given by,

$$\beta_M^* = \frac{P_1}{\sigma^2} \int\limits_0^1 u_M(x, \frac{1}{\sigma^2}) dx$$

Now, if we define $w_M(x) = \frac{P_1}{\sigma^2} u(x, 1/\sigma^2)$, then,

$$w_M(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int\limits_0^\tau w_M(z)dz\right) 1_{\{\tau \geq x\}} + I\left(P, P_1, \int\limits_\tau^1 w_M(z)dz\right) 1_{\{\tau \leq x\}} \right\}} \tag{52}$$

and the SIR of user 1 converges in probability to

$$\beta_M^* = \int_0^1 w_M(x)dz \tag{53}$$

In order to extend the above result to a more general case where the limiting power distribution is $F(P)$, we start by approximating $F(P)$ by staircase functions, thus reducing it to the case where there are finite number of power levels. For a given M, we can define upper and lower staircase approximations for $F(P)$, of step height $\frac{1}{M}$ by

$$\begin{aligned}\overline{P}_m &= \sup\{P : F(P) \leq \frac{m-1}{M}\} \text{ occurring with probability } q_m = \frac{1}{M} \text{ for } m = 1, \ldots, M \\ \underline{P}_m &= \inf\{P : F(P) \geq \frac{m}{M}\} \text{ occurring with probability } q_m = \frac{1}{M} \text{ for } m = 1, \ldots, M\end{aligned}$$

From the above construction of the staircase functions, we have that $\overline{P}_m - \underline{P}_m \leq \frac{1}{M}$. These two approximations will have corresponding $\overline{v}^M(x,y)$, $\overline{H}_M^*(\lambda)$, $\overline{w}_M(x)$ and $\underline{v}^M(x,y)$, $\underline{H}_M^*(\lambda)$, $\underline{w}_M(x)$ defined and from the derivations above, as $N \to \infty$,

$$\overline{\beta}_M^{(N)} \xrightarrow{\mathcal{P}} \overline{\beta}_M^* = \int_0^1 \overline{w}_M(x)dx$$

where $\overline{w}_M(x)$ satisfies the fixed point equation,

$$\overline{w}_M(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_{\overline{P}} \mathbb{E}_\tau \left\{ I\left(\overline{P}, P_1, \int_0^\tau \overline{w}_M(z)dz\right) 1_{\{\tau \geq x\}} + I\left(\overline{P}, P_1, \int_\tau^1 \overline{w}_M(z)dz\right) 1_{\{\tau \leq x\}} \right\}}$$

and

$$\underline{\beta}_M^{(N)} \xrightarrow{\mathcal{P}} \underline{\beta}_M^* = \int_0^1 \underline{w}_M(x)dx$$

where

$$\underline{w}_M(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_{\underline{P}} \mathbb{E}_\tau \left\{ I\left(\underline{P}, P_1, \int_0^\tau \underline{w}_M(z)dz\right) 1_{\{\tau \geq x\}} + I\left(\underline{P}, P_1, \int_\tau^1 \underline{w}_M(z)dz\right) 1_{\{\tau \leq x\}} \right\}}$$

and $\mathbb{E}_{\overline{P}}, \mathbb{E}_{\underline{P}}$ are expectations with respect to the upper and lower staircase distributions. From the definition of $I(P, P_1, \beta)$, and the fact that $\overline{P}_m - \underline{P}_m \leq \frac{1}{M}$, we have,

$$\left| I\left(\overline{P}, P_1, \int_0^\tau \overline{w}_M(z)dz\right) - I\left(\underline{P}, P_1, \int_0^\tau \underline{w}_M(z)dz\right) \right| \leq \frac{1}{M}$$

and

$$\left| I\left(\overline{P}, P_1, \int_{\tau}^{1} \overline{w}_M(z)dz\right) - I\left(\underline{P}, P_1, \int_{\tau}^{1} \underline{w}_M(z)dz\right) \right| \leq \frac{1}{M}$$

Therefore, we have

$$\lim_{M\to\infty} |\overline{\beta}^* - \underline{\beta}^*| = 0 \tag{54}$$

The SIR achieved by user 1 at any spreading length N, as given by eqn (6 is monotonically non-increasing with the increasing powers of the interferers. Therefore, since $\overline{P}_m \geq \underline{P}_m$, we have that $\overline{\beta}_M^{(N)} \leq \underline{\beta}_M^{(N)}$. Moreover, if $\beta^{(N)}$ denotes the SIR achieved by user 1 in the system which has a power distribution $F(P)$, we have,

$$\overline{\beta}_M^{(N)} \leq \beta^{(N)} \leq \underline{\beta}_M^{(N)}$$

Since $\limsup \beta^{(N)} \leq \underline{\beta}_M^*$ and $\liminf \beta^{(N)} \geq \overline{\beta}_M^*$, from eqn (54), we have, that as $N, M \to \infty$,

$$\beta^{(N)} \xrightarrow{\mathcal{P}} \beta^* = \int_0^1 w(x)dx$$

where

$$w(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int_0^{\tau} w(z)dz\right) 1_{\{\tau \geq x\}} + I\left(P, P_1, \int_{\tau}^{1} w(z)dz\right) 1_{\{\tau \leq x\}} \right\}}$$

and $\mathbb{E}_P$ is expectation with respect to $F(P)$ to which the upper and lower staircases converge weakly to as $M \to \infty$. □

B Proof of Proposition 4.3

Define a function $f(x)$,

$$f(x) = \frac{1}{P_1}\left(\sigma^2 x + \frac{1}{N}\sum_{k=2}^{K} x\left[I(\tau_k P_k, P_1, x) + I((1-\tau_k)P_k, P_1, x)\right]\right)$$

The function $f(x)$ is continuous and strictly increasing in x. To prove that it has a fixed point solution x^*, we note that $f(0) = 0$ and $f(\infty) = \infty$ and hence, there is at least one point such that $f(x) = 1$. Since the function is strictly increasing, the fixed point solution is unique. From the monotonicity of $f(x)$, we have,

$$\begin{aligned} x^* \geq x &\Leftrightarrow f(x) \leq 1 \\ &\Leftrightarrow \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{k=2}^{K}\left[I(\tau_k P_k, P_1, x) + I((1-\tau_k)P_k, P_1, x)\right]} \geq x \end{aligned}$$

□

C Proof of Theorems 4.2 and 7.2

We prove the bound proposed for the SIR achieved in Theorem 4.2 by noting some of the properties of $w(x)$ when $G(\tau) = 1 - G(1-\tau)$, which are given by the following lemma.

Lemma C.1 *The function $w(x)$ is symmetric about $\frac{1}{2}$ for all delay distributions which satisfy $G(\tau) = 1 - G(1-\tau)$ (that is, have probability density function symmetric about $\frac{1}{2}$).*

Proof:

To prove that $w(x)$ is symmetric about half, we need to prove $w(x) = w(1-x)$. From eqn (12), we have,

$$\begin{aligned} w(1-x) &= \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int_0^\tau w(z)dz\right) 1_{\{\tau \geq 1-x\}} + I\left(P, P_1, \int_\tau^1 w(z)dz\right) 1_{\{\tau \leq 1-x\}} \right\}} \\ &= \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int_0^{1-\tau} w(z)dz\right) 1_{\{\tau \leq x\}} + I\left(P, P_1, \int_{1-\tau}^1 w(z)dz\right) 1_{\{\tau \geq x\}} \right\}} \\ &= \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int_\tau^1 w(1-z)dz\right) 1_{\{\tau \leq x\}} + I\left(P, P_1, \int_0^\tau w(1-z)dz\right) 1_{\{\tau \geq x\}} \right\}} \end{aligned}$$

The second equality follows from the fact that $G(\tau) = 1 - G(1-\tau)$. Since $w(1-x)$ also satisfies the above fixed point equation which has a unique solution $w(x)$, we have, $w(1-x) = w(x)$, that is, $w(x)$ is symmetric about $\frac{1}{2}$. □

Lemma C.2 *The function $w(x)$ is a non decreasing function in $x \in [0, \frac{1}{2}]$.*

From lemma C.1, this implies that the function $w(x)$ is non-increasing in $[\frac{1}{2}, 1]$.

Proof:

Since Theorem 4.1 gives an expression for $w(x)$, to prove the above theorem, it suffices to show,

$$x_1 \leq x_2 \leq \frac{1}{2} \Longrightarrow w(x_1) \leq w(x_2)$$

Notice that the dependence on the argument x in eqn (12) appears only within the two expectations in the interference terms. If we denote $\mathcal{G}_x(P, P_1, \tau)$ to be,

$$\mathcal{G}_x(P, P_1, \tau) = I\left(P, P_1, \int_0^\tau w(z)dz\right) 1_{\{\tau \geq x\}} + I\left(P, P_1, \int_\tau^1 w(z)dz\right) 1_{\{\tau \leq x\}},$$

if $x_1 \leq x_2$, then,

$$\mathcal{G}_{x_1}(P, P_1, \tau) - \mathcal{G}_{x_2}(P, P_1, \tau) = \left[I\left(P, P_1, \int_0^\tau w(z)dz\right) - I\left(P, P_1, \int_\tau^1 w(z)dz\right)\right] 1_{\{x_1 \leq \tau \leq x_2\}}$$

The function $w(x)$ is symmetric about $\frac{1}{2}$ and we have $2\int_0^{\frac{1}{2}} w(z)dz = \int_0^1 w(z)dz$. From the fact that $w(x) \geq 0$ for $x \in [0, 1]$, we have $\int_0^1 w(z)dz \geq 2\int_0^\tau w(z)dz$ if $\tau \leq \frac{1}{2}$, that is, $\int_0^\tau w(z)dz \leq \int_\tau^1 w(z)dz$ if $\tau \leq \frac{1}{2}$. Now from the decreasing property of $I(P, P_1, y)$ with respect to y, we have,

$$I\left(P, P_1, \int_0^\tau w(z)dz\right) - I\left(P, P_1, \int_\tau^1 w(z)dz\right) \geq 0 \text{ for all } \tau \leq \frac{1}{2}$$

and thus $\mathcal{G}_{x_1}(P, P_1, \tau) - \mathcal{G}_{x_2}(P, P_1, \tau) \geq 0$ for all $\tau \leq \frac{1}{2}$ and $x_1 \leq x_2 \leq \frac{1}{2}$

Therefore, if $x_1 \leq x_2 \leq \frac{1}{2}$, we have $w(x_1) \leq w(x_2) \leq w(\frac{1}{2})$: $w(x)$ is increasing in $x \in [0, \frac{1}{2}]$. □

Lemma C.3 *Let $G(x)$ denote the distribution of the random variable X and for some odd integer T, let $g(t)$ denote a non-negative function for $t \in [0, T]$ which is non-decreasing in $[0, \frac{T}{2}]$, symmetric about $\frac{T}{2}$ and has an area $\int_0^T g(t)dt = \mu$. Among all such functions, $\bar{g}(t) = \frac{\mu}{T}$ for all $t \in [0, T]$ minimizes the function,*

$$\mathcal{F}(g) = \mathbb{E}_X\left[\frac{1}{1 + \int_0^X g(t)dt} + \sum_{i=1}^{T-1}\left(\frac{1}{1 + \int_{X+i-1}^{X+i} g(t)dt}\right) + \frac{1}{1 + \int_{X+T-1}^{T} g(t)dt}\right]$$

Proof:

The above lemma is proved using the concept of majorization [10] : $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ is said to majorize $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$,

$$\mathbf{u} \preceq \mathbf{v} \text{ if } \begin{cases} \sum_{i=1}^{k} u_{[i]} \leq \sum_{i=1}^{k} v_{[i]}, & k = 1, \dots, n-1 \\ \sum_{i=1}^{n} u_{[i]} = \sum_{i=1}^{n} v_{[i]} \end{cases}$$

where $(u_{[1]}, \dots, u_{[n]})$ denotes the elements of $\mathbf{u}$ arranged in decreasing order, $u_{[1]} \geq \dots \geq u_{[n]}$. Now, define $\mathbf{y}(g)$ such that,

$$y_k(g) = \begin{cases} \int_0^x g(t)dt & i = 1 \\ \int_{x+i-1}^{x+i} g(t)dt & i = 2, \dots, T-1 \\ \int_{x+T-1}^{T} g(t)dt & i = T \end{cases} \quad \text{and} \quad \mathcal{F}(g) = \sum_{k=1}^{T} \frac{1}{1 + y_k(g)}$$

The function $\sum_{k=1}^{T} \frac{1}{1+y_k}$ is schur convex and therefore from the properties of majorization,

$$\mathcal{F}(g_1) \leq \mathcal{F}(g_2) \text{ if } \mathbf{y}(g_1) \preceq \mathbf{y}(g_2).$$

that is, $\mathcal{F}(g_1) \leq \mathcal{F}(g_2)$ if $\mathbf{y}(g_1)$ majorizes $\mathbf{y}(g_2)$.

Among all non-negative functions $g(t)$ that are symmetric about $\frac{T}{2}$, non-decreasing in $[0, \frac{T}{2}]$ and a constant area μ, it can be seen that $\bar{g}(t) = \frac{\mu}{T}$ has the property,

$$\mathbf{y}(\bar{g}) \preceq \mathbf{y}(g) \text{ for all } g \tag{55}$$

Therefore, $\bar{g}(t)$ for all $t \in [0, T]$ minimizes the function $\mathcal{F}(g)$. □

Lemma C.3 plays a important role in attaining the bound on the SIR. If we set $T = 1$, then, the above lemma proves that between two functions $g_1(t)$ and $g_2(t)$ with same area in $[0, 1]$, $\mathcal{F}(g)$ is lesser for the one which has more area concentrated in the regions near zero and one. That is,

$$\frac{\mu}{2} \geq \int_0^x g_1(t)dt \geq \int_0^x g_2(t)dt \;\; \forall \;\; x \leq \frac{1}{2} \quad \Rightarrow \mathcal{F}(g_1) \leq \mathcal{F}(g_2)$$

Proof of Theorem 4.2

The three lemma stated above provide enough footing to derive the bound from the actual SIR attained. To prove Theorem 4.2, we begin from eqn (12),

$$w(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int_0^\tau w(z)dz\right) 1_{\{\tau \geq x\}} + I\left(P, P_1, \int_\tau^1 w(z)dz\right) 1_{\{\tau \leq x\}} \right\}}$$

Cross multiplying and integrating both sides with respect to x from 0 to 1,

$$\begin{aligned}
&\sigma^2 \int_0^1 w(x)dx \\
&= P_1 - \alpha \int_0^1 \mathbb{E}_P \mathbb{E}_\tau \left\{ I\left(P, P_1, \int_0^\tau w(z)dz\right) 1_{\{\tau \geq x\}} + I\left(P, P_1, \int_\tau^1 w(z)dz\right) 1_{\{\tau \leq x\}} \right\} w(x)dx \\
&= P_1 - \alpha \mathbb{E}_P \left[\int_0^1 \int_x^1 \frac{PP_1 \, w(x)}{P_1 + P\int_0^\tau w(z)dz} dG(\tau)\, dx + \int_0^1 \int_0^x \frac{PP_1 \, w(x)}{P_1 + P\int_\tau^1 w(z)dz} dG(\tau)\, dx \right] \\
&= P_1 - \alpha \mathbb{E}_P \left[\int_0^1 \frac{PP_1 \int_0^\tau w(x)dx}{P_1 + P\int_0^\tau w(z)dz} dG(\tau) + \int_0^1 \frac{PP_1 \int_\tau^1 w(x)dx}{P_1 + P\int_\tau^1 w(z)dz} dG(\tau) \right]
\end{aligned}$$

where the last step follows from the fact that the first integral is over the triangular area $\tau = [0, x]$ and $x = [0, 1]$, which is equivalent to the integral over $x = [\tau, 1]$ and $\tau = [0, 1]$. The second integral follows from a similar argument over the other half of the square in $\tau - x$ plane. Therefore,

$$\sigma^2 \int_0^1 w(x)dx = (1 - 2\alpha)P_1 + \alpha P_1^2 \mathbb{E}_P \mathbb{E}_\tau \left[\frac{1}{P_1 + P\int_0^\tau w(z)dz} + \frac{1}{P_1 + P\int_\tau^1 w(z)dz} \right] \tag{56}$$

Setting $T = 1$ in lemma C.3, we have that the right hand side is minimized when $w(x) = \int_0^1 w(z)dz$ for all $x \in [0, 1]$. Using this, and since $\beta_1^* = \int_0^1 w(z)dz$, we have,

$$\sigma^2 \beta_1^* \geq P_1 - \alpha \beta_1^* \mathbb{E}_P \mathbb{E}_\tau \left\{ I(\tau P, P_1, \beta_1^*) + I((1 - \tau)P, P_1, \beta_1^*) \right\}$$

Therefore, using the monotonicity property of proposition 4.3, β_1^* is lower bounded by the unique solution γ_1^* of the fixed point equation,

$$\gamma_1^* = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left[I(\tau P, P_1, \gamma_1^*) + I((1 - \tau)P, P_1, \gamma_1^*) \right]}$$

where $I(P, P_1, \gamma_1^*)$ is the effective interference from an interferer of power P at SIR γ_1^* defined as

$$I(P, P_1, \gamma_1^*) = \frac{PP_1}{P_1 + P\gamma_1^*} \qquad \square$$

Proof of Theorem 7.2

In the case of multiple symbol observation window, we observe that the function $w(x)$ defined in eqn (39) is symmetric about $\frac{T}{2}$ and is non-decreasing in $[0, \frac{T}{2}]$. If we cross-multiply the two sides of eqn (39) and integrate between 0 and T, we have,

$$\begin{aligned}
\sigma^2 \int_0^T w(x)dx &= P_1 T - \alpha \int_0^T \mathbb{E}_{P,\tau} I\left(P, P_1, \int_{C(x,\tau)} w(z)dz\right) w(x)dx \\
&= TP_1 - (T+1)P_1\alpha + \alpha P_1^2 \mathbb{E}_{P,\tau}\left[\frac{1}{P_1 + P\int_0^\tau w(z)dz} + \right. \\
&\quad \left. + \sum_{i=1}^{T-1} \frac{1}{P_1 + P\int_{\tau+i-1}^{\tau+i} w(z)dz} + \frac{1}{P_1 + P\int_{x+T-1}^{T} w(z)dz}\right]
\end{aligned}$$

From lemma C.3, we have that the right hand side is minimized when $w(x) = \frac{1}{T}\int_0^T w(z)dz$ for all $x \in [0, T]$ and hence,

$$\int_{\frac{T-1}{2}}^{\frac{T+1}{2}} w(x)dx = \frac{1}{T}\int_0^T w(x)dx$$

Thus, we arrive at the bound as stated in Theorem 7.2

□

D Proof of Theorem 5.1

To prove this theorem, we shall rely on a geometric interpretation of the decorrelator.The decorrelator can be defined as a linear function of the received vector, $\mathbf{r}$, which maximizes the SIR, subject to the constraint that the estimate is independent of the other interfering symbols. So, if $\hat{x}_{dec}(\mathbf{r})$ denotes the estimate of user 1's information symbol, then,

$$\begin{aligned}
\hat{x}_{dec}(\mathbf{r}) &= \mathbf{c}^t\mathbf{r} \\
&= (\mathbf{c}^t\mathbf{s}_1)x_1 + \sum_{k=2}^{K}\left(x_k\mathbf{c}^t\mathbf{u}_k + y_k\mathbf{c}^t\mathbf{v}_k\right) + \mathbf{c}^t\mathbf{n}
\end{aligned}$$

Since the estimate is independent of all the interfering symbols, $x_2, \ldots, x_K, y_2, \ldots, y_K$, we have $\mathbf{c}^t\mathbf{u}_k = \mathbf{c}^t\mathbf{v}_k = 0$ for all $i = 2, \ldots, K$. Therefore, if we define $\mathcal{C} = \text{span}\{\mathbf{u}_2, \ldots, \mathbf{u}_K, \mathbf{v}_2, \ldots, \mathbf{v}_K\}^{\perp}$, then, the decorrelator can be obtained by constraining the vector $\mathbf{c} \in \mathcal{C}$ and maximizing the SIR achieved. From the fact that $\|\mathbf{s}_1 - \mathbf{c}\|^2 = \|\mathbf{s}_1\|^2 + \|\mathbf{c}\|^2 - 2\mathbf{c}^t\mathbf{s}_1$, we have that the maximum is achieved when $\mathbf{c}$ is the projection of $\mathbf{s}_1$ on to the subspace $\mathcal{C}$ and has a norm $\|\mathbf{c}\|$. If we denote this optimal $\mathbf{c}$ by $\mathbf{h}$, then the SIR of the decorrelator is given by,

$$\beta_1 = \frac{P_1}{\sigma^2}\frac{(\mathbf{h}^t\mathbf{s}_1)^2}{\mathbf{h}^t\mathbf{h}} = \frac{P_1}{\sigma^2}\mathbf{h}^t\mathbf{h} \tag{57}$$

When the spreading sequences are independently and randomly chosen, $\mathcal{C}$ (comprising of $2(K-1)$ signature sequences) will have a dimension of $\max\{N - 2(K-1), 0\}$, with high probability, as $N \to \infty$. Moreover from the independence of the signature sequences, we have that $\mathbf{s}_1$ is independent of the subspace $\mathcal{C}$. Hence, from lemma A.1, we have that $\mathbf{h}^t\mathbf{h} \to 1 - 2\alpha$ in probability as $N, K \to \infty, \frac{K}{N} = \alpha$ and $\alpha < 1$. If $\alpha \geq 1$, then, $\mathbf{h}^t\mathbf{h} \to 0$ in probability, thus proving the theorem. □