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Reachability analysis for discrete time stochastic hybrid systems

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Abstract. A model for discrete time stochastic hybrid systems is proposed in this paper. With reference to the introduced class of systems, a methodology for probabilistic reachability analysis is studied, which can be useful for safety verification. This methodology is based on the interpretation of the safety verification problem as an optimal control problem for a certain controlled Markov process. In particular, this allows to characterize through some optimal cost function the set of initial conditions for the system such that its state can be maintained within a given "safe" set with sufficiently high probability. The proposed methodology is applied to the problem of regulating the average temperature in a room by a thermostat controlling a heater.

1 Introduction

Engineering systems like air traffic control systems or infrastructure networks, and natural systems like biological networks exhibit complex behaviors which can often be naturally described by hybrid dynamical models– systems with interacting discrete and continuous dynamics. In many situations the system dynamics are uncertain, and the evolution of discrete and continuous dynamics as well as the interactions between them are of stochastic nature.

An important problem in hybrid systems theory is that of reachability analysis. In general, a reachability analysis problem consists of evaluating whether the system will eventually reach a pre-specified set within a certain time horizon, starting from a given set of initial conditions. This problem often arises in connection to the safety verification scenarios, where the system is declared to be "safe" if its state does not enter some unsafe set. In a stochastic setting, the safety verification problem can be formulated as that of estimating the probability that the state of the system remains within the safe set for a given time horizon. If the evolution of the state can be influenced by some control input, the problem becomes verifying if it is possible to keep the state of the system within the safe set with sufficiently high probability by selecting a suitable control input.

Reachability analysis for stochastic hybrid systems has been a recent focus of research, e.g., in [1–4]. Most of the approaches address the problem of reachability analysis for continuous time stochastic hybrid systems (CTSHS), wherein the effect of control actions is not directly taken into account. The theory of CTSHS, developed for instance in [5–8], is used in [1, 2] to address the theoretical issues regarding measurability and computability of the reach events. On the computational side, a stochastic approximation method is used in [3] to compute the probability of entering into the unsafe set. More recently, in [4], certain functions of the state of the system known as barrier certificates are used to compute an upper bound on the probability that the state enters in the unsafe set. In the discrete time framework, [9] computes the reach probability using randomized algorithms.

This study adopts the discrete time setting in order to gain a deeper understanding of the theoretical and computational issues associated with the reachability analysis of stochastic hybrid systems. The present work extends the above mentioned approaches by developing a methodology to compute the maximum probability of remaining in a safe set for a certain time horizon for a discrete time stochastic hybrid system (DTSHS) whose dynamics is affected by a control input. The approach is based on formulating the reachability analysis problem as an optimal control problem and computing the optimal cost function and the corresponding optimal policy using dynamic programming. The maximum probability of remaining in a safe set for a certain time horizon can then be obtained. In addition, the optimal value function directly enables us to compute the maximal safe set for a specified threshold probability, which is the set of all initial conditions such that the probability of exiting from the safe set during a certain time horizon is greater than or equal to the threshold probability.

The paper is organized as follows: Section 2 introduces a model for a DTSHS. This model is inspired by the stochastic hybrid systems models previously introduced in [6, 7, 5, 10, 8, 11] in continuous time. An equivalent representation of the DTSHS in the form of a controlled Markov process is derived. In Section 3, the notion of stochastic reachability for an execution of a DTSHS is introduced. The problem of determining probabilistic maximal safe sets for a DTSHS is formulated as a stochastic reachability analysis problem, which can be solved by dynamic programming. The representation of the DTSHS as a controlled Markov process is useful in this respect. In Section 4 we apply the proposed methodology to the problem of regulating the temperature of a room by a thermostat that controls a heater. Concluding remarks are drawn in Section 5.

2 Discrete time stochastic hybrid system

In this section, we introduce a definition of discrete time stochastic hybrid system (DTSHS). This definition is inspired by the continuous time stochastic hybrid system (SHS) model described in [12].

The hybrid state of the DTSHS is characterized by a discrete and a continuous component. The discrete state component takes values in a finite set $\mathcal{Q}$. In

each mode $q \in \mathcal{Q}$, the continuous state component takes values in the Euclidean space $\Re^{n(q)}$, whose dimension is determined by the map $n : \mathcal{Q} \to \mathbb{N}$. Thus the hybrid state space is $\mathcal{S} := \cup_{q \in \mathcal{Q}} \{q\} \times \Re^{n(q)}$. Let $\mathcal{B}(\mathcal{S})$ be the σ-field generated by the subsets of $\mathcal{S}$ of the form $\cup_q \{q\} \times A_q$, where A_q is a Borel set in $\Re^{n(q)}$. It can be shown (see [13, page 58]) that $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ is a Borel space.
The continuous state evolves according to a probabilistic law that depends on the discrete state. A jump in the discrete state may occur during the continuous state evolution, according to some probabilistic law. This will then cause a modification of the probabilistic law governing the continuous state evolution. A control input can affect both the continuous and discrete probabilistic evolutions. After a jump in the discrete state has occurred, the continuous state is subject to a probabilistic reset that is also influenced by some control input. Following the reference continuous time SHS model in [12], we distinguish this latter input from the former one. We call them reset input and transition input, respectively.

Definition 1 (DTSHS). *A discrete time stochastic hybrid systems (DTSHS) is a collection* $\mathcal{H} = (\mathcal{Q}, n, \mathcal{U}, \Sigma, T_x, T_q, R)$, *where*

- $\mathcal{Q} := \{q_1, q_2, \dots, q_k\}$, *for some* $k \in \mathbb{N}$, *represents the discrete state space;*
- $n : \mathcal{Q} \to \mathbb{N}$ *assigns to each discrete state value* $q \in \mathcal{Q}$ *the dimension of the continuous state space* $\Re^{n(q)}$. *The hybrid state space is then given by* $\mathcal{S} := \cup_{q \in \mathcal{Q}} \{q\} \times \Re^{n(q)}$;
- $\mathcal{U}$ *is a Borel space representing the transition control space;*
- Σ *is a Borel space representing the reset control space;*
- $T_x : \mathcal{B}(\Re^{n(\cdot)}) \times \mathcal{S} \times \mathcal{U} \to [0,1]$ *is a Borel-measurable stochastic kernel on* $\Re^{n(\cdot)}$ *given* $\mathcal{S} \times \mathcal{U}$, *which assigns to each* $s = (q, x) \in \mathcal{S}$ *and* $u \in \mathcal{U}$ *a probability measure on the Borel space* $(\Re^{n(q)}, \mathcal{B}(\Re^{n(q)}))$: $T_x(dx|(q,x),u)$;
- $T_q : \mathcal{Q} \times \mathcal{S} \times \mathcal{U} \to [0,1]$ *is a discrete stochastic kernel on* $\mathcal{Q}$ *given* $\mathcal{S} \times \mathcal{U}$, *which assigns to each* $s \in \mathcal{S}$ *and* $u \in \mathcal{U}$, *a probability distribution over* $\mathcal{Q}$: $T_q(q|s,u)$;
- $R : \mathcal{B}(\Re^{n(\cdot)}) \times \mathcal{S} \times \Sigma \times \mathcal{Q} \to [0,1]$ *is a Borel-measurable stochastic kernel on* $\Re^{n(\cdot)}$ *given* $\mathcal{S} \times \Sigma \times \mathcal{Q}$, *that assigns to each* $s = (q,x) \in \mathcal{S}$, $\sigma \in \Sigma$, *and* $q' \in \mathcal{Q}$, *a probability measure on the Borel space* $(\Re^{n(q')}, \mathcal{B}(\Re^{n(q')}))$: $R(dx|(q,x),\sigma,q')$. □

In order to define the semantic of a DTSHS, we need first to specify how the system is initialized and how the reset and transition inputs are selected.

The system initialization can be specified through some probability measure $\pi : \mathcal{B}(\mathcal{S}) \to [0,1]$ on the Borel space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. When the initial state of the system is $s \in \mathcal{S}$, then, the probability measure π is concentrated at $\{s\}$.

As for the choice of the reset and transition inputs, we need to specify which is the rule to determine their values at every time step during the DTSHS evolution (control policy). Here, we consider DTSHS evolving over a finite horizon $[0, N]$ $(N < \infty)$.
If the values for the control inputs are determined based on the values taken by inputs and state up to the current time step, then the policy is said to be a feedback policy.

Definition 2 (Feedback policy). *Let $\mathcal{H} = (\mathcal{Q}, n, \mathcal{U}, \Sigma, T_x, T_q, R)$ be a DTSHS. A feedback policy μ for $\mathcal{H}$ is a sequence $\mu = (\mu_0, \mu_1, \dots, \mu_{N-1})$ of universally measurable maps $\mu_k : \mathcal{S} \times (\mathcal{S} \times \mathcal{U} \times \Sigma)^k \to \mathcal{U} \times \Sigma$, $k = 0, 1, \dots, N-1$, where $\mathcal{S} = \cup_{q \in \mathcal{Q}} \{q\} \times \Re^{n(q)}$. We denote the set of feedback policies as $\mathcal{M}$.* □

Definition 3 (Execution). *Consider a DTSHS $\mathcal{H} = (\mathcal{Q}, n, \mathcal{U}, \Sigma, T_x, T_q, R)$. A stochastic process $\{\mathbf{s}(k) = (\mathbf{q}(k), \mathbf{x}(k)),\ k \in [0, N]\}$ with values in $\mathcal{S} = \cup_{q \in \mathcal{Q}} \{q\} \times \Re^{n(q)}$ is an execution of $\mathcal{H}$ associated with a policy $\mu \in \mathcal{M}$ and an initial distribution π if its sample paths are obtained according to the following algorithm, where all the random extractions involved are independent:*

DTSHS algorithm:

extract from $\mathcal{S}$ a value $s_0 = (q_0, x_0)$ for the random variable $\mathbf{s}(0) = (\mathbf{q}(0), \mathbf{x}(0))$ according to π;

set k=0

while $k < N$ do

> *set $(u_k, \sigma_k) = \mu_k(s_k, s_{k-1}, u_{k-1}, \sigma_{k-1}, \dots)$;*
>
> *extract from Q a value q_{k+1} for the random variable $\mathbf{q}(k+1)$ according to $T_q(\cdot\,|(q_k, x_k), u_k)$;*
>
> *if $q_{k+1} = q_k$, then*
>
> > *extract from $\Re^{n(q_{k+1})}$ a value x_{k+1} for $\mathbf{x}(k+1)$ according to $T_x(\cdot\,|(q_k, x_k), u_k)$*
>
> *else*
>
> > *extract from $\Re^{n(q_{k+1})}$ a value x_{k+1} for $\mathbf{x}(k+1)$ according to $R(\cdot\,|(q_k, x_k), \sigma_k, q_{k+1})$*
>
> *set $s_{k+1} = (q_{k+1}, x_{k+1})$*
>
> $k \to k+1$

end □

If the values for the control inputs are determined only based on the value taken by the state at the current time step, i.e., $(u_k, \sigma_k) = \mu_k(s_k)$, then the policy is said to be a Markov policy.

Definition 4 (Markov Policy). *Consider a DTSHS $\mathcal{H} = (\mathcal{Q}, n, \mathcal{U}, \Sigma, T_x, T_q, R)$. A Markov policy μ for $\mathcal{H}$ is a sequence $\mu = (\mu_0, \mu_1, \dots, \mu_{N-1})$ of universally measurable maps $\mu_k : \mathcal{S} \to \mathcal{U} \times \Sigma$, $k = 0, 1, \dots, N-1$, where $\mathcal{S} = \cup_{q \in \mathcal{Q}} \{q\} \times \Re^{n(q)}$. We denote the set of Markov policies as $\mathcal{M}_m$.*

Note that Markov policies are a subset of the feedback policies: $\mathcal{M}_m \subseteq \mathcal{M}$.

Remark 1. It is worth noticing that the map T_q can model both the spontaneous jumps that might occur during the continuous state evolution, and the forced jumps that must occur when the continuous state exits some prescribed set.
As for spontaneous jumps, if at some hybrid state $(q, x) \in \mathcal{S}$ a jump to the

discrete state q' is allowed by the control input $u \in \mathcal{U}$, then this is modeled by $T_q(q'|(q,x),u) > 0$. T_q also encodes a possible delay in the actual occurrence of a jump: if $T_q(q'|(q,x),u) = 1$, then the jump must occur, the smaller is $T_q(q'|(q,x),u)$, the more likely is that the jump will be postponed to a later time.
The invariant set $Dom(q)$ of a discrete state $q \in \mathcal{Q}$, namely the set of all the admissible values for the continuous state within mode q, can be expressed in terms of T_q by forcing $T_q(q'|(q,x),u)$ to be zero irrespectively of the value of the control input u in $\mathcal{U}$, for all the continuous state values $x \in \Re^{n(q)}$ outside $Dom(q)$. Thus $Dom(q) := \Re^{n(q)} \setminus \{x \in \Re^{n(q)} : T_q(q'|(q,x),u) = 0, \forall u \in \mathcal{U}\}$. □

Define the stochastic kernel $\tau_x : \mathcal{B}(\Re^{n(\cdot)}) \times \mathcal{S} \times \mathcal{U} \times \Sigma \times \mathcal{Q} \to [0,1]$ on $\Re^{n(\cdot)}$ given $\mathcal{S} \times \mathcal{U} \times \Sigma \times \mathcal{Q}$, which assigns to each $s = (q,x) \in \mathcal{S}$, $u \in \mathcal{U}$, $\sigma \in \Sigma$ and $q' \in \mathcal{Q}$ a probability measure on the Borel space $(\Re^{n(q')}, \mathcal{B}(\Re^{n(q')}))$ as follows:

$$\tau_x(dx'\,|(q,x),u,\sigma,q') = \begin{cases} T_x(dx'|(q,x),u), & \text{if } q' = q \\ R(dx'|(q,x),\sigma,q'), & \text{if } q' \neq q. \end{cases}$$

In the DTSHS algorithm, τ_x is used to extract a value for the continuous state at time $k+1$ given the values taken by the hybrid state and the control inputs at time k, and the value extracted for the discrete state at time $k+1$.

Based on τ_x we can define the Borel-measurable stochastic kernel $T_s : \mathcal{B}(\mathcal{S}) \times \mathcal{S} \times \mathcal{U} \times \Sigma \to [0,1]$ on $\mathcal{S}$ given $\mathcal{S} \times \mathcal{U} \times \Sigma$, which assigns to each $s = (q,x) \in \mathcal{S}$, $(u,\sigma) \in \mathcal{U} \times \Sigma$ a probability measure on the Borel space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ as follows:

$$T_s(ds'\,|s,(u,\sigma)) = \tau_x(dx'\,|s,u,\sigma,q')T_q(q'|s,u), \tag{1}$$

$s, s' = (q',x') \in \mathcal{S}$, $(u,\sigma) \in \mathcal{U} \times \Sigma$. Then, the DTSHS algorithm can be rewritten in a more compact form as:

extract from $\mathcal{S}$ *a value* s_0 *for the random variable* $\mathbf{s}(0)$ *according to* π*;*

set $k=0$

while $k < N$ *do*

   set $(u_k, \sigma_k) = \mu_k(s_k, s_{k-1}, u_{k-1}, \sigma_{k-1}, \dots)$*;*

   extract from $\mathcal{S}$ *a value* s_{k+1} *for* $\mathbf{s}(k+1)$ *according to* $T_s(\cdot\,|s_k,(u_k,\sigma_k))$*;*

   $k \to k+1$

end □

This shows that a DTSHS $\mathcal{H} = (\mathcal{Q}, n, \mathcal{U}, \Sigma, T_x, T_q, R)$ can be described as a controlled Markov process with state space $\mathcal{S} = \cup_{q \in \mathcal{Q}}\{q\} \times \Re^{n(q)}$, control space $\mathcal{A} := \mathcal{U} \times \Sigma$, and controlled transition probability function $T_s : \mathcal{B}(\mathcal{S}) \times \mathcal{S} \times \mathcal{A} \to [0,1]$ defined in (1). This will be referred to in the following as "embedded controlled Markov process" (see, e.g., [14] for an extensive treatment on controlled Markov processes).

As a consequence of this representation of $\mathcal{H}$, the execution $\{\mathbf{s}(k) = (\mathbf{q}(k), \mathbf{x}(k)), k \in [0,N]\}$ associated with $\mu \in \mathcal{M}$ and π is a stochastic process defined on the

canonical sample space $\Omega = \mathcal{S}^N$, endowed with its product topology $\mathcal{B}(\Omega)$, with probability measure P_π^μ uniquely defined by the transition kernel T_s, the policy $\mu \in \mathcal{M}$, and the initial pobability measure π (see [15, Proposition 7.45]). When π is concentrated at $\{s\}$, $s \in \mathcal{S}$, we shall write simply P_s^μ.
From the embedded Markov process representation of a DTSHS it also follows that the execution of a DTSHS associated with a Markov policy μ and an initial condition π is a Markov process.

Example 1 (The thermostat). Consider the problem of regulating the temperature of a room by a thermostat that can switch a heater on and off.
The state of the controlled system is naturally described as a hybrid state. The discrete state component is represented by the heater being in either in the "on" or in the "off" condition. The continuous state component is represented by the average temperature of the room.
We next show how the controlled system can be described through a DTSHS model $\mathcal{H} = (\mathcal{Q}, n, \mathcal{U}, \Sigma, T_x, T_q, R)$. We then formulate the temperature regulation problem with reference to this model.

As for the state space of the DTSHS, the discrete component of the hybrid state space is $\mathcal{Q} = \{\texttt{ON}, \texttt{OFF}\}$, whereas $n : \mathcal{Q} \rightarrow \mathbb{N}$ defining the continuous component of the hybrid state space is the constant map $n(q) = 1, \forall q \in \mathcal{Q}$

We assume that the heater can be turned on or off, and that this is the only available control on the system. We then define $\Sigma = \emptyset$ and $\mathcal{U} = \{0, 1\}$ with the understanding that 1 means that a switching command is issued, 0 that no switching command is issued.

As for the continuous state evolution, in the stochastic model proposed in [16], the average temperature of the room evolves in the two different modes according to the following stochastic differential equations (SDEs)

$$d\mathbf{x}(t) = \begin{cases} -\frac{a}{C}(\mathbf{x}(t) - x_a)dt + \frac{1}{C}d\mathbf{w}(t), & \text{if the heater is off} \\ -\frac{a}{C}(\mathbf{x}(t) - x_a)dt + \frac{r}{C}dt + \frac{1}{C}d\mathbf{w}(t), & \text{if the heater is on} \end{cases} \tag{2}$$

where a is the average heat loss rate; C is the average thermal capacity of the room; x_a is the ambient temperature; r is the rate of heat gain supplied by the heater; $\mathbf{w}(t)$ is a standard Wiener process modeling the noise affecting the temperature evolution. By applying the constant-step Euler-Maruyama discretization scheme [17] to the SDEs in (2), with time step Δt, we obtain the stochastic difference equation

$$\mathbf{x}(k+1) = \begin{cases} \mathbf{x}(k) - \frac{a}{C}(\mathbf{x}(k) - x_a)\Delta t + \mathbf{n}(k), & \text{if the heater is off} \\ \mathbf{x}(k) - \frac{a}{C}(\mathbf{x}(k) - x_a)\Delta t + \frac{r}{C}\Delta t + \mathbf{n}(k) & \text{if the heater is on,} \end{cases} \tag{3}$$

where, $\{\mathbf{n}(k), k \geq 0\}$ is a sequence of i.i.d. Gaussian random variables with zero mean and variance $\nu^2 := \frac{1}{C^2}\Delta t$.

Let $\mathcal{N}(\cdot; m, \sigma^2)$ denote the probability measure over $\Re$ associated with a Gaussian density function with mean m and variance σ^2. Then, the continuous transition kernel T_x implicitly defined in (3) can be expressed as follows:

$$T_x(\cdot\,|(q,x),u) = \begin{cases} \mathcal{N}(\cdot; x - \frac{a}{C}(x - x_a)\Delta t, \nu^2), & q = \texttt{OFF} \\ \mathcal{N}(\cdot; x - \frac{a}{C}(x - x_a)\Delta t + \frac{r}{C}\Delta t, \nu^2), & q = \texttt{ON} \end{cases} \tag{4}$$

Note that the evolution of temperature within each mode is uncontrolled and so the continuous transition kernel T_x does not depend on the value u of the transition control input.

We assume that it takes some (random) time for the heater to actually switch between its two operating conditions, after a switching command has been issued. This is modeled by defining the discrete transition kernel T_q as follows

$$T_q(q'|(q,x),0) = \begin{cases} 1, & q' = q \\ 0, & q' \neq q \end{cases}$$

$$T_q(q'|(q,x),1) = \begin{cases} \alpha, & q' = \texttt{OFF},\ q = \texttt{ON} \\ 1-\alpha, & q' = q = \texttt{ON} \\ \beta, & q' = \texttt{ON},\ q = \texttt{OFF} \\ 1-\beta, & q' = q = \texttt{OFF} \end{cases} \tag{5}$$

$\forall x \in \Re$, where $\alpha \in [0,1]$ represents the probability of switching from the ON to the OFF mode in one time-step. Similarly for $\beta \in [0,1]$.

We assume that the actual switching between the two operating conditions of the heater takes a time step. During this time step the temperature keeps evolving according to the dynamics characterizing the starting condition. This is modeled by defining the reset kernel as follows

$$R(\cdot\,|(q,x),q') = \begin{cases} \mathcal{N}(\cdot; x - \frac{a}{C}(x - x_a)\Delta t, \nu^2), & q = \texttt{OFF} \\ \mathcal{N}(\cdot; x - \frac{a}{C}(x - x_a)\Delta t + \frac{r}{C}\Delta t, \nu^2), & q = \texttt{ON}. \end{cases} \tag{6}$$

Let $\bar{x}^-$, $\bar{x}^+ \in \Re$, with $\bar{x}^- < \bar{x}^+$.
Consider the (stationary) Markov policy $\mu_k : \mathcal{S} \to \mathcal{U}$ defined by

$$\mu_k((q,x)) = \begin{cases} 1, & q = \texttt{ON},\ x \geq \bar{x}^+ \text{ or } q = \texttt{OFF},\ x \leq \bar{x}^- \\ 0, & q = \texttt{ON},\ x < \bar{x}^+ \text{ or } q = \texttt{OFF},\ x > \bar{x}^- \end{cases}$$

that switches the heater on when the temperature drops below $\bar{x}^-$ and off when the temperature goes beyond $\bar{x}^+$.

Suppose that initially the heater is off and the temperature is uniformly distributed in the interval between $\bar{x}^-$ and $\bar{x}^+$, independently of the noise process affecting its evolution. In Figure 1, we report some sample paths of the execution of the DTSHS associated with this policy and initial condition. We plot only the continuous state realizations.

The temperature is measured in Fahrenheit degrees ($°F$) and the time in minutes (min). The time horizon N is taken to be 600 min. The discretization time step Δt is chosen to be 1 min. The parameters in equations (4) and (6) are assigned the following values: $x_a = 10.5°F$, $a/C = 0.1\ min^{-1}$, $r/C = 10°F/min$, and $\nu = 1°F$. The switching probabilities α and β in equation (5) are each chosen to be equal to 0.8. Finally, $\bar{x}^-$ and $\bar{x}^+$ are set equal to $70°F$ and $80°F$, respectively.

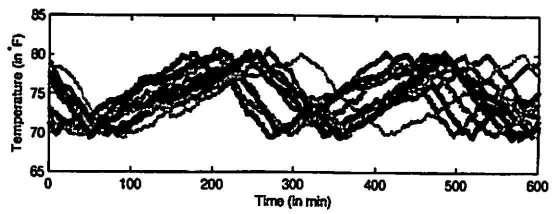

Fig. 1. Sample paths of the temperature for the execution corresponding to a Markov policy starting with the heater off and temperature uniformly distributed on $[70, 80]°F$.

Note that some of the sample paths exit the set $[70, 80]°F$. This is due partly to the delay in turning the heater on/off and partly to the noise entering the system. If the objective is keeping the temperature within the set [70, 80]°F, more effective control policies can be found.

In the following section we consider the problem of determining those initial conditions for the system such that it is possible to keep the temperature of the room within prescribed limits over a certain time horizon $[0, N]$, by appropriately acting on the only available control input. Due to the stochastic nature of the controlled system, we relax our requirement to that of keeping the temperature within prescribed limits over $[0, N]$ with sufficiently high probability. We shall see how this problem can be formulated as a stochastic reachability analysis problem. □

3 Stochastic reachability

In general terms, a reachability analysis problem consists in evaluating if a given system will eventually reach some set during some time horizon, starting from some initial condition or set of initial conditions. This problem arises, for instance, in connection with those safety verification problems where the unsafe conditions for the system can be characterized in terms of its state entering some unsafe set: if the state of the system cannot enter the unsafe set, then the system is declared to be "safe".

If the evolution of the state of the system can be affected by some control input, then the problem becomes evaluating if it is possible to maintain the state of the system outside the unsafe set by selecting an appropriate control policy. The answer to this question will obviously depend on the system initialization:

for some initial conditions (in particular if the system is initialized in the unsafe set...), there is no chance of guaranteeing safety.

In a stochastic setting, the problem translates into that of verifying if it is possible to maintain the state of the system outside the unsafe set with sufficiently high probability, by choosing an appropriate control policy. Different initial conditions are characterized by a different probability of entering the unsafe set: if the system starts from an initial condition that corresponds to a probability ϵ of entering the unsafe set, then the system is "safe with probability $1-\epsilon$".

One can then define sets of initial conditions corresponding to different safety levels, that is sets of states such that the value for the probability of entering the unsafe set starting from them is smaller or equal to a given value ϵ. These sets are called *maximal probabilistic safe sets.* The attribute "maximal" underlines the fact that they are obtained by computing for each state the minimal probability of entering the unsafe set starting from that state: each single policy would generate probabilistic safe sets that are subsets of the maximal probabilistic safe sets.

With reference to the introduced stochastic hybrid model $\mathcal{H}$, a reachability analysis problem consists in determining the probability that the execution associated with some given policy $\mu \in \mathcal{M}$ and initial distribution π will enter a Borel set $A \in \mathcal{B}(\mathcal{S})$ during the time horizon $[0, N]$:

$$p_\pi^\mu(A) := P_\pi^\mu(\mathbf{s}(k) \in A \text{ for some } k \in [0, N]). \tag{7}$$

Let $I_C : \mathcal{S} \to \{0, 1\}$ denote the indicator function of a set $C \subseteq \mathcal{S}$: $I_C(s) = 1$, if $s \in C$, 0, if $s \notin C$.

$p_\pi^\mu(A)$ can be expressed as $p_\pi^\mu(A) = P_\pi^\mu\big(\max_{k\in[0,N]} \mathbf{1}_A(\mathbf{s}(k)) = 1\big)$. If the probability π is concentrated at $\{s\}$, $s \in \mathcal{S}$, then this is the probability of entering A starting from s, which we denote by $p_s^\mu(A)$.

Suppose that A represents an unsafe set for $\mathcal{H}$. Fix $\epsilon \in (0, 1)$.
The probabilistic safe set that guarantees a safety level $1-\epsilon$, when the control policy is $\mu \in \mathcal{M}$, is defined as

$$S^\mu(\epsilon) = \{s \in \mathcal{S} : p_s^\mu(A) \leq \epsilon\}, \tag{8}$$

whereas the maximal probabilistic safe set with safety level $1-\epsilon$ is

$$S^\star(\epsilon) = \{s \in \mathcal{S} : \inf_{\mu\in\mathcal{M}} p_s^\mu(A) \leq \epsilon\}, \tag{9}$$

from which it appears evident that $S^\mu(\epsilon) \subseteq S^\star(\epsilon)$, for each $\epsilon \in (0, 1)$.

In the rest of the section, we show that (i) the problem of computing $p_s^\mu(A)$ for a Markov policy $\mu \in \mathcal{M}_m$ can be solved by using a backward iterative procedure; and (ii) the problem of computing $S^\star(\epsilon)$ can be reduced to an optimal control problem. This, in turn, can be solved by dynamic programming, and admits as solution an optimal control policy that is Markov. These results are obtained based on the representation of $p_\pi^\mu(A)$ as a multiplicative cost function.

The probability $p_\pi^\mu(A)$ defined in (7) can be expressed as $p_\pi^\mu(A) = 1 - p_\pi^\mu(\bar{A})$, where $\bar{A}$ denotes the complement of A in $\mathcal{S}$ and $p_\pi^\mu(\bar{A}) := P_\pi^\mu(\mathbf{s}(k) \in \bar{A} \text{ for all } k \in [0, N])$. Observe now that

$$\prod_{k=0}^{N} \mathbf{1}_{\bar{A}}(s_k) = \begin{cases} 1, & \text{if } s_k \in \bar{A} \text{ for all } k \in [0, N] \\ 0, & \text{otherwise,} \end{cases}$$

$s_k \in \mathcal{S}$, $k \in [0, N]$.
Then, $p_\pi^\mu(\bar{A}) = P_\pi^\mu(\prod_{k=0}^{N} \mathbf{1}_{\bar{A}}(\mathbf{s}(k)) = 1) = E_\pi^\mu[\prod_{k=0}^{N} \mathbf{1}_{\bar{A}}(\mathbf{s}(k))]$. From this expression it follows that

$$p_\pi^\mu(\bar{A}) = \int_{\mathcal{S}} E_\pi^\mu\Big[\prod_{k=0}^{N} \mathbf{1}_{\bar{A}}(\mathbf{s}(k)) \,|\, \mathbf{s}(0) = s\Big] \pi(ds), \tag{10}$$

where the conditional mean $E_\pi^\mu[\prod_{k=0}^{N} \mathbf{1}_{\bar{A}}(\mathbf{s}(k)) \,|\, \mathbf{s}(0) = s]$ is well defined over the support of the probability measure π representing the distribution of $\mathbf{s}(0)$.

3.1 Backward reachability computations

We next show how it is possible to compute $p_\pi^\mu(\bar{A})$ through a backward iterative procedure for a Markov policy $\pi \in \mathcal{M}_m$. The motivation is that, in view of the embedded Markov process representation of a DTSHS, the only policies we need to consider for the maximal safe set computation are in fact Markov policies.

Consider a Markov policy $\mu = (\mu_0, \mu_1, \dots, \mu_{N-1})$, with $\mu_k : \mathcal{S} \to \mathcal{U} \times \Sigma$, $k = 0, 1, \dots, N-1$.

For each $k \in [0, N]$, define the map $V_k^\mu : \mathcal{S} \to [0, 1]$ as follows

$$V_k^\mu(s) := \mathbf{1}_{\bar{A}}(s) \int_{\mathcal{S}^{N-k}} \prod_{l=k+1}^{N} \mathbf{1}_{\bar{A}}(s_l) \prod_{h=k+1}^{N-1} T_s(ds_{h+1} | s_h, \mu_h(s_h)) T_s(ds_{k+1} | s, \mu_k(s)), \tag{11}$$

$\forall s \in \mathcal{S}$, where T_s is the controlled transition function of the embedded Markov process, and $\int_{\mathcal{S}^0}(\dots) = 1$.
If s belongs to the support of π, then, $E_\pi^\mu\big[\prod_{l=k}^{N} \mathbf{1}_{\bar{A}}(\mathbf{s}(l)) \,|\, \mathbf{s}(k) = s\big]$ is well-defined and equal to the right-hand-side of (11), so that

$$V_k^\mu(s) = E_\pi^\mu\Big[\prod_{l=k}^{N} \mathbf{1}_{\bar{A}}(\mathbf{s}(l)) \,|\, \mathbf{s}(k) = s\Big]. \tag{12}$$

Hence, $V_k^\mu(s)$ denotes the probability of keeping outside A during the (residual) time horizon $[k, N]$ starting from s at time k, under policy μ applied from π.

By (10) and (12), $p_\pi^\mu(\bar{A})$ can be expressed as

$$p_\pi^\mu(\bar{A}) = \int_{\mathcal{S}} V_0^\mu(s) \pi(ds).$$

If π is concentrated at $\{s\}$, $p_s^\mu(\bar{A}) = V_0^\mu(s)$. Since $p_s^\mu(A) = 1 - p_s^\mu(\bar{A})$, then the probabilistic safe sets with safety level $1-\epsilon$, $\epsilon \in (0,1)$, defined in (8) can be computed as

$$S^\mu(\epsilon) = \{s \in \mathcal{S} : V_0^\mu(s) \geq 1-\epsilon\}.$$

Following [18], we prove the following lemma.

Lemma 1. *If μ is a Markov policy, then the maps $V_k^\mu : \mathcal{S} \to [0,1]$, $k = 0, 1 \ldots, N$, can be computed by the backward recursion:*

$$\begin{aligned} V_k^\pi(s) =& \mathit{1}_{\bar{A}}(s)\big[T_q(q|s, u_k^\pi(s)) \int_{\Re^{n(q)}} V_{k+1}^\pi((q,x'))T_x(dx'|s, u_k^\pi(s)) \\ &+ \sum_{q' \neq q} T_q(q'|s, u_k^\pi(s)) \int_{\Re^{n(q')}} V_{k+1}^\pi((q',x'))R(dx'|s, \sigma_k^\pi(s), q')\big], \end{aligned}$$

for all $s = (q,x), s' = (q',x') \in \mathcal{S}$, where $\mu_k(s) = (u_k^\mu(s), \sigma_k^\mu(s)) \in \mathcal{U} \times \Sigma$, with the initialization $V_N^\mu(s) = \mathit{1}_{\bar{A}}(s)$, $s \in \mathcal{S}$.

Proof. From the definition of V_k^μ in equation (11), we immediately get that $V_N^\mu(s) = \mathbf{1}_{\bar{A}}(s)$, $s \in \mathcal{S}$. Next, for $k < N$

$$\begin{aligned} V_k^\mu(s) = \mathbf{1}_{\bar{A}}(s) &\int_{\mathcal{S}^{N-k}} \prod_{l=k+1}^{N} \mathbf{1}_{\bar{A}}(s_l) \prod_{h=k+1}^{N-1} T_s(ds_{h+1}|s_h, \mu_h(s_h)) T_s(ds_{k+1}|s, \mu_k(s)) \\ = \mathbf{1}_{\bar{A}}(s) &\int_{\mathcal{S}} \mathbf{1}_{\bar{A}}(s_{k+1}) \Big(\int_{\mathcal{S}^{N-k-1}} \prod_{l=k+2}^{N} \mathbf{1}_{\bar{A}}(s_l) \prod_{h=k+2}^{N-1} T_s(ds_{h+1}|s_h, \mu_h(s_h)) \\ &T_s(ds_{k+2}|s_{k+1}, \mu_{k+1}(s_{k+1}))\Big) T_s(ds_{k+1}|s, \mu_k(s)) \\ = \mathbf{1}_{\bar{A}}(s) &\int_{\mathcal{S}} V_{k+1}^\mu(s_{k+1}) T_s(ds_{k+1}|s, \mu_k(s)). \end{aligned}$$

Recalling the definition of T_s the thesis immediately follows. □

3.2 Maximal probabilistic safe set computation

The calculation of the maximal probabilistic safe set $S^*(\epsilon)$ defined in (9) amounts to finding the feedback policy $\mu^* \in \mathcal{M}$ that minimizes the probability that the execution associated with it starting at s enters the unsafe set A, for all $s \in \bar{A}$, and grouping those states s which give a sufficiently high safety guarantee.

Definition 5 (Optimal policy). *Consider a DTSHS $\mathcal{H} = (\mathcal{Q}, n, \mathcal{U}, \Sigma, T_x, T_q, R)$. Let $A \in \mathcal{B}(\mathcal{S})$ be an unsafe set. A policy $\mu^* \in \mathcal{M}$ is optimal if $p_s^{\mu^*}(A) = \inf_{\mu \in \mathcal{M}} p_s^\mu(A)$, $\forall s \in \mathcal{S} \setminus A$.*

Remark 2. Note that for every $s \in A$, the probability of entering the unsafe set A starting from s is 1, irrespectively of the policy chosen: $\inf_{\mu \in \mathcal{M}} p_s^\mu(A) = 1$, $\forall s \in A$. Moreover, $p_s^{\mu^*}(A) = \inf_{\mu \in \mathcal{M}} p_s^\mu(A) = 1 - \sup_{\mu \in \mathcal{M}} p_s^\mu(\bar{A})$.

Let us define the cost-to-go at time k for any arbitrary policy $\mu \in \mathcal{M}$ as

$$J_k^\mu = E\left[\prod_{l=k}^{N} \mathbf{1}_{\bar{A}}(s_l)|s_0, \ldots, s_k\right].$$

Theorem 1. *The solution $\mu^* \in \mathcal{M}_m$ of the following recursion*

$$V_N^*(s) = \mathbf{1}_{\bar{A}}(s)$$
$$V_k^*(s) = \sup_{a \in \mathcal{A}} \mathbf{1}_{\bar{A}}(s) \int_S V_{k+1}^\mu(s_{k+1}) T_s(ds_{k+1}|s,a), \text{ for } k \in [0, N-1]$$

has the form $\mu_k^*(s) = \arg\sup_{a \in \mathcal{A}} \mathbf{1}_{\bar{A}}(s) \int_S V_{k+1}^*(s_{k+1}) T_s(ds_{k+1}|s,a)$, and is such that for any $\mu \in \mathcal{M}$ and $k \in [0,N]$, $V_k^*(s_k) \geq J_k^\mu$ a.s.

Proof. As previously assumed, we deal with a Borel space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ and with Borel measurable stochastic kernels. The one-stage cost function $\mathbf{1}_{\bar{A}}(s)$ is Borel measurable, non negative and bounded for all $s \in \mathcal{S}$. In particular, $V_N^\mu(s) = \mathbf{1}_{\bar{A}}(s)$ is Borel measurable. For all $k, 0 \leq k \leq N-1$, it can be directly checked that the mapping

$$V_k^\mu(s) = \mathbf{1}_{\bar{A}}(s) \int_S V_{k+1}^\mu(s_{k+1}) T_s(ds_{k+1}|s, \mu_k(s)), \text{ for } k \in [0, N-1]$$

verifies the *monotonicity assumption* on its function V_{k+1}^μ(cf. [15], Sec. 6.1). The cost function $V_k^\mu(s)$ belongs to the set of extended real-valued, universally measurable functions on $\mathcal{S}$(cf. [15], 7.46.1 and 7.30). Moreover, consider the mapping

$$V_k^*(s) = \sup_{a \in \mathcal{A}} \mathbf{1}_{\bar{A}}(s) \int_S V_{k+1}^*(s_{k+1}) T_s(ds_{k+1}|s,a).$$

The functions $V_k^*(s)$ are universally measurable and lower semianalytic. This holds because the product of a lower semianalytic function by a positive, Borel measurable function is lower semianalytic; furthermore, the integration of a lower semianalytic function with respect to a stochastic kernel and its supremization with respect to one of its arguments (in this specific instance, the control at time k) is lower semianalytic (cf. [15], 7.30, 7.47 and 7.48). The preceding measurability assumptions provide a solid ground for the *exact selection assumption* to hold ([15], Sec. 6.2): for all $s \in \mathcal{S}$, there exists an optimal markovian policy $\mu^* \in \mathcal{M}_m$ such that, for all $0 \leq k \leq N-1$

$$V_k^{\mu^*}(s) = V_k^*(s) = \sup_{a \in \mathcal{A}} V_k^\mu(s).$$

In particular (cf. Prop. 6.1, [15]): $V_0^*(s) = \sup_{a \in \mathcal{A}} V_0^\mu(s)$. □

Then the maximal probabilistic safe set $S^*(\epsilon)$ with safety level $1-\epsilon$ defined in (9) can be determined as $S^*(\epsilon) = \{s \in \mathcal{S} : V_0^*(s) \geq 1-\epsilon\}$.

4 The thermostat example

In this section we show the applicability of our methodology to the problem of regulating the temperature of a room by a thermostat controlling a heater. The DTSHS description of the system was given in Example 1 of Section 2. The system parameters and time horizon are set equal to the values reported at the end of Example 1. Three safe sets are considered: $\bar{A}_1 = (70, 80)°F$, $\bar{A}_2 = (72, 78)°F$, and $\bar{A}_3 = (74, 76)°F$. The dynamic programming recursion described in Section 3.2 is used to compute the optimal policies and the maximal probabilistic safe sets. The implementation is done in MATLAB. The temperature is discretized into 100 equally spaced values within the safe set.

Figures 2 show the plots of 100 temperature sample paths resulting from sampling the initial temperature from the uniform distribution over the safe sets, and using the corresponding optimal policy. The initial operating mode is chosen at random between the equiprobable ON and OFF values.
It can be observed from each of the plots that the optimal policy computed by the dynamic programming recursion leads to an optimal behavior in the following sense: regardless of the initial state, most of the temperature sample paths tend toward the middle of the corresponding safe set. As for the $\bar{A}_1$ and $\bar{A}_2$ safe sets, the temperature actually remain confined within the safe set in almost all the sample paths, whereas this is not the case for $\bar{A}_3$. This is expected because the set $\bar{A}_3$ is too small to enable the optimal policy to specify an action that is effective enough to offset the drifts and the randomness in the execution in order to maintain the temperature within the safe set.

The maximal probability of remaining in the safe set $p_{\pi}^{\mu^*}(\bar{A}_i)$ for π uniform over $\mathcal{Q} \times \bar{A}_i$, $i = 1, 2, 3$, is computed. The value is 0.991 for $\bar{A}_1$, 0.978 for $\bar{A}_2$ and 0.802 for $\bar{A}_3$.

The maximal probabilistic safe sets $S^{\mu^*}(\epsilon)$ corresponding to different values for ϵ are also calculated. A maximal safe set corresponding to ϵ is a set of all initial conditions for which the probability of remaining in the safe set for a pre-specified time horizon is greater than or equal to $1 - \epsilon$. Figure 3 gives the maximal probabilistic safe sets for different $1 - \epsilon$ values for each of the three considered cases when the heater is initially either on or off. As expected, the maximal probabilistic safe sets get smaller as the $1 - \epsilon$ value for the safety guarantee grows. In the third case, when the safe set is $\bar{A}_3$, there is no policy that can guarantee a safety probability greater than equal to about 0.86.

$\mu_k^* : \mathcal{S} \to \mathcal{U}$ at times $k \in \{1, 250, 500, 575, 580, 585, 590, 595, 599\}$ for the three cases are shown in the figure 4, as a function of the continuous state and discrete state (the red crossed line refers to the OFF state, whereas the blue circled line refers to the ON state).

The obtained result is quite intuitive. For example, at time $k = 599$ and in mode OFF, the optimal policy prescribes to stay in same mode for most of the states except near the lower boundary of the safe set, in which case it prescribes to change the mode to ON since there is possibility of entering into the unsafe set. However, at earlier times (for instance, time $k = 1$) and in mode OFF, the optimal policy prescribes to change the mode even for states that are distant

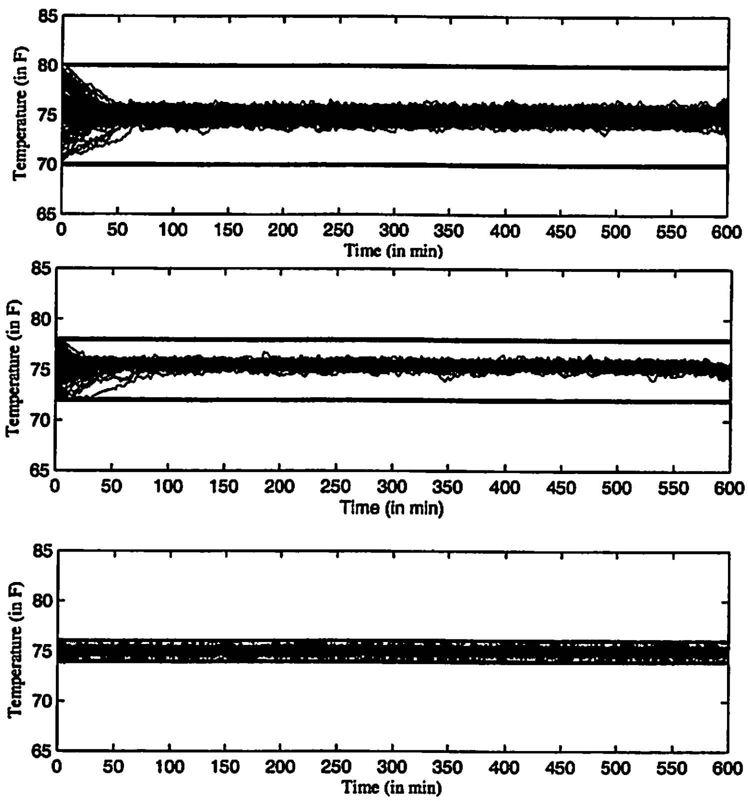

Fig. 2. Sample paths of the temperature for the execution corresponding to the optimal policy, when the safe set is: $\bar{A}_1$ (top), $\bar{A}_2$ (middle), and $\bar{A}_3$ (bottom).

from the safe set boundary. Similar comments apply to mode ON. From the left column of Figure 4 it can observed that the optimal policy is not stationary. More precisely, it has a steady-state behavior for most of the times, except for a transient behavior towards the end of the time horizon. By comparing the columns of Figure 4, this transient period gets progressively smaller for $\bar{A}_2$ and $\bar{A}_3$. It is interesting to note the behavior of the optimal policy corresponding to the safe set $\bar{A}_1$ at $k = 580$ and $k = 575$. For example, for $k = 580$, the optimal policy for OFF mode fluctuates between actions 0 and 1 when the temperature value is around $75°F$. This is because the values taken by the optimal cost function on the residual time horizon for the two control actions are almost equal.

It should be noted that the results obtained refer to the case of switching probabilities $\alpha = \beta = 0.8$. Different choices of switching probabilities may yield qualitatively different optimal control policies.

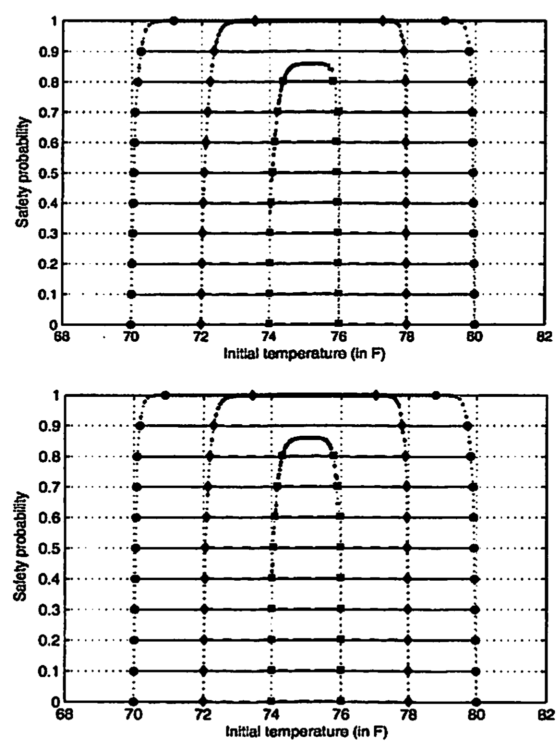

Fig. 3. Maximal probabilistic safe sets: heater initially off (top) and on (bottom).

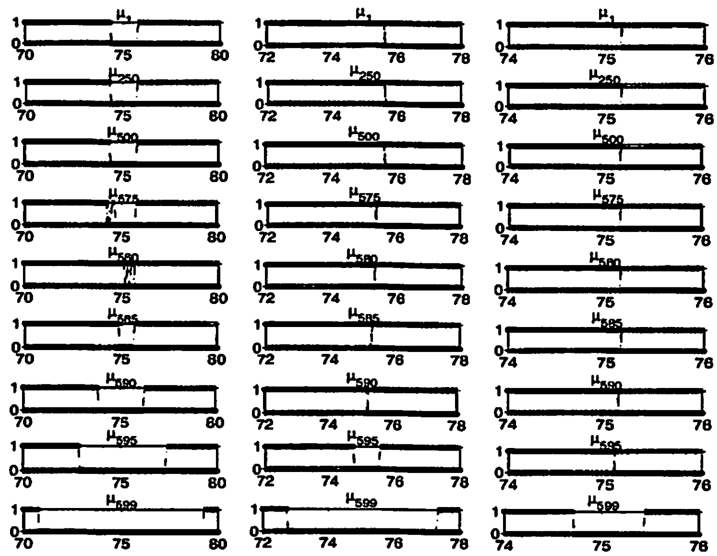

Fig. 4. Optimal control policy as a function of the temperature at different times during the control time horizon. The darker (blue) circled line corresponds to the OFF mode and the lighter (red) crossed lines correspond to the ON mode. The left, middle, and right columns refer to the safe sets $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$ respectively.

5 Final remarks

In this paper we proposed a model for controlled discrete time stochastic hybrid systems. With reference to such a model, we described the notion of stochastic reachability, and discussed how the problem of safety verification can be reinterpreted in terms of the introduced stochastic reachability notion. By an appropriate reformulation of the safety verification problem for the stochastic hybrid system as that of determining a feedback policy that optimizes some multiplicative cost function for a certain controlled Markov process, we were able to suggest a solution based on dynamic programming. Temperature regulation of a room by a heater that can be repeatedly switched on and off was presented as a simple example to illustrate the model capabilities and the reachability analysis methodology.

Further work is needed to extend the current approach to the infinite horizon and partial information cases. The more challenging problem of stochastic reachability analysis for continuous time stochastic hybrid systems is an interesting subject of future research.

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