A Categorical Theory of Hybrid Systems

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Abstract

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This dissertation uses the formalism of category theory to study hybrid phenomena. One begins with a collection of "non-hybrid" mathematical objects that have been well-studied, together with a notion of how these objects are related to one another; that is, one begins with a category C of the non-hybrid objects of interest. The objects being considered can be "hybridized" by considering *hybrid objects over* C consisting of pairs $(\mathscr{D}, \mathbf{A})$ where $\mathscr{D}$ is a small category of a specific form, termed a *D-category*, which encodes the discrete structure of the hybrid object and

$$\mathbf{A} : \mathscr{D} \to \mathsf{C}$$

is a functor encoding its continuous structure. The end result is the *category of hybrid objects* over C, denoted by $\mathsf{Hy}(\mathsf{C})$.

In Part I, the foundations for the theory of hybrid objects are established. After reviewing the basics of category theory, inasmuch as they will be needed in this dissertation, D-categories are formally introduced. Hybrid objects over a general category C are then defined along with the corresponding notion of a category of hybrid objects. Elementary properties of categories of this form are discussed. We then proceed to relate the formalism of hybrid objects to hybrid systems in their classical form, the end result of which is a categorical formulation of hybrid systems together with a constructive correspondence between classical hybrid systems and their categorical counterpart. Finally, executions or trajectories of both classical and categorical hybrid systems are introduced, and they are related to one another—again in a constructive fashion.

Part II applies the categorical theory of hybrid objects to obtain novel results related to the reduction and stability of hybrid systems. The geometric reduction of simple hybrid systems is first considered, e.g., conditions are given on when robotic systems undergoing impacts can be reduced. As an application of these results, it is shown that a three-dimensional bipedal robotic walker can be reduced to a two-dimensional bipedal walker; the result is walking gaits in three-dimensions based on corresponding walking gaits for a two dimensional biped—walking gaits that simultaneously stabilize the walker to

the upright position. Using hybrid objects, the reduction results for simple hybrid systems are generalized to general hybrid systems; to do so, many familiar geometric objects—manifolds, differential forms, *et cetera*—are first "hybrizied." The end result is a hybrid reduction theorem much in the spirt of the classical geometric reduction theorem. This part of the dissertation concludes with a partial characterization of Zeno behavior in hybrid systems. A new type of equilibria, *Zeno equilibria*, is introduced and sufficient conditions for the stability of these equilibria are given. Since the stability of these equilibria correspond to the existence of Zeno behavior, the end result is sufficient conditions for the existence of Zeno behavior.

The final portion of this dissertation, Part III, lays the groundwork for a categorical theory, not of hybrid systems, but of networked systems. It is shown that a network of tagged systems correspond to a network over the category of tagged systems and that taking the composition of such a network is equivalent to taking the limit; this allows us to derive necessary and sufficient conditions for the preservation of semantics, and thus illustrates the possible descriptive power of categories of hybrid and network objects.

Professor Shankar Sastry
Dissertation Committee Chair

To my mother,

Catherine A. Bartlett,

for always nurturing my talents, giving me the strength to pursue them and the wisdom to know how.

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Introduction

Category theory provides a framework for describing objects with like properties and for comparing objects with different properties. The concept of classifying objects based on the category in which they reside can be traced back to Aristotle and his work *Categories*, written in 350 BC. In modern mathematics, the concept of a category has been formalized into a common language. It is exactly for this reason that establishing a bridge between engineering and category theory can provide so many benefits.

Yet there remains skepticism about the true usefulness of category theory, especially in the areas of computer science and engineering where there is common reference to the nickname "abstract nonsense." In fact, to quote Mitchell [92],

> "A number of sophisticated people tend to disparage category theory as consistently as others disparage certain kinds of classical music. When obliged to speak of category theory they do so in an apologetic tone, similar to the way some say 'It was a gift—I've never even played it' when a record of Chopin Nocturnes is discovered in their possession."

The purpose of this dissertation is to dispel some of these concerns by demonstrating that hybrid systems, i.e., systems that display both discrete and continuous behavior, are naturally amendable to the formalisms of category theory.

Hybrid systems have the ability to model a wide range of phenomena, including: robotic systems undergoing impacts, biological systems, power systems, dynamical systems with non-smooth control laws, simplifying approximations of complex systems, networks of embedded and robotic systems, *et cetera*. Understanding hybrid systems on a deep level, therefore, has important and practical consequences. The yin to this yang is that a deep understanding of these systems is still lacking.

There is currently no unifying mathematical framework of hybrid systems—one that is analogous to the theory of continuous and discrete systems. This is due, in part, to the fact that hybrid systems represent a great increase in complexity over their discrete and continuous counterparts; this makes it difficult to analyze even the simplest hybrid models. In addition, this added complexity results in the existence of new behavior that is unique to hybrid systems, e.g., Zeno behavior, that can have unexpected and sometimes catastrophic consequences. This indicates that a new and more sophisticated theory is needed to describe hybrid phenomena.

This dissertation presents a categorical theory of hybrid systems—the theory of hybrid objects—which we claim provides a unifying mathematical framework for hybrid systems. The results and applications that will be presented support this thesis in that they demonstrate the following properties of hybrid objects:

Property I: Provide a common language for hybrid systems, i.e., marry the discrete and continuous components of a hybrid system in such a way that its underlying structure becomes apparent.

Property II: Relate hybrid systems to preexisting theory and constructions in mathematics.

Property III: Elucidate the relationship between hybrid systems.

Property IV: Provide novel and practical results that would not be possible without this mathematical framework.

This work, therefore, will be devoted to introducing the theoretical underpinnings of hybrid objects, with a special focus on their usefulness in understanding hybrid systems and other hybrid phenomena. Applications also will be presented with the express goal of establishing the practical usefulness of categories of hybrid objects—this should dispel concerns to the effect that these categories are nothing but "abstract nonsense."

Following is an overview of the general structure of this dissertation. The specific chapter dependencies can be seen in Table 0.1.

Part I: Foundations. The first portion of this dissertation is devoted to establishing the foundational principles underlying the rest of this work. These formulations support the claim that the categorical theory of hybrid objects display Properties I, II and III.

Chapter 1: Hybrid Objects. The first chapter is devoted to the formal introduction of the theory of hybrid objects, which is necessarily done on an abstract level. We begin by introducing the theory of categories, which is done in a self-contained, albeit brief, fashion. With these concepts in hand, a special class of small categories is introduced: D-categories, denoted by $\mathscr{D}$. Categories of this form describe the "discrete" component of hybrid objects, and are analogous to graphs. D-categories allow for the introduction of the notion of a hybrid object over a category C, $(\mathscr{D}, \mathbf{A})$, where

$$\mathbf{A} : \mathscr{D} \to \mathsf{C}$$

is a functor. The category of hybrid objects over C, $\mathsf{Hy}(\mathsf{C})$, can thus be formed. These are not the only hybrid objects of interest; cohybrid objects and network objects also will be introduced.

Chapter 2: Hybrid Systems. Having introduced the notion of a hybrid object over a category, this abstract concept is related to the standard formulation of a hybrid system. This relationship is established in a constructive manner, i.e., it is demonstrated how one can transform the components defining a hybrid system into the categorical framework for hybrid systems. These correspondences are bijective, indicating that no information is lost in the reformulation of hybrid systems to this setting; it simply serves the purpose of reframing hybrid systems so that they can be more easily reasoned about, i.e., it unifies,

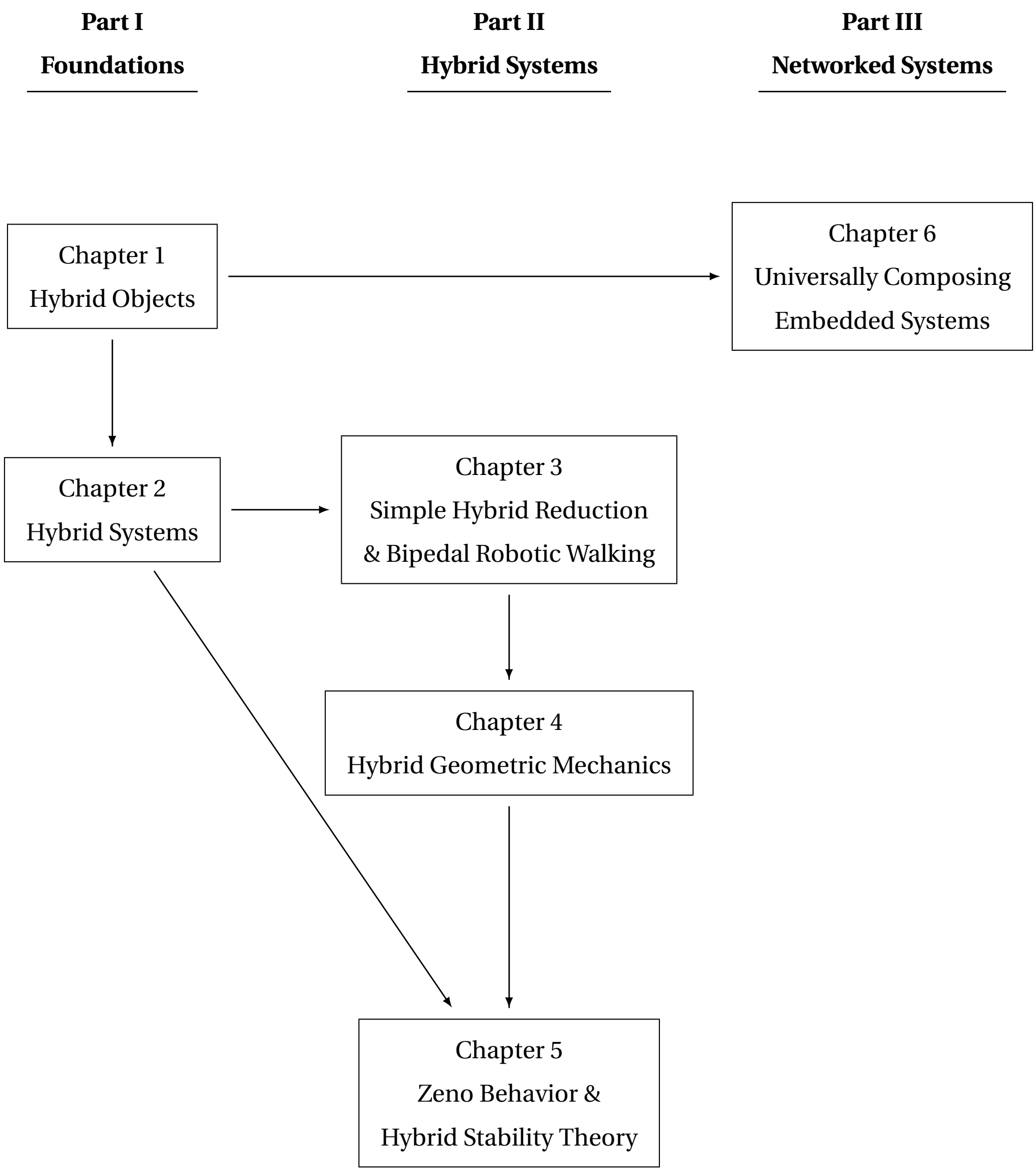

Table 0.1: Chapter dependency chart.

but clearly separates, the discrete and continuous components of a hybrid system. The latter half of this chapter is devoted to the categorical formulation of trajectories of hybrid systems; again, it is demonstrated that this is in agreement with the standard notion of an execution. Simple examples that clearly elucidate these concepts and reformulations are discussed throughout.

Part II: Hybrid Systems. The second part of this dissertation is devoted to applications of the theory of hybrid objects, thus supporting the claim that hybrid objects display Property IV.

Chapter 3: Simple Hybrid Reduction & Bipedal Robotic Walking. This chapter temporarily draws back from the categorical framework for hybrid systems with the goal of better understanding the relationship between mechanical systems undergoing impacts and hybrid systems. Simple hybrid systems are studied, with a special focus on Lagrangian hybrid systems and simple hybrid mechanical systems. We begin by investigating the generalization of Routhian reduction to a hybrid setting, giving explicit conditions on when this form of Lagrangian reduction can be carried out. The focus then shifts to Hamiltonian reduction, where conditions are given on when symplectic reduction can be carried out in the setting of simple hybrid systems. The chapter concludes with the crowning application of this dissertation: bipedal robotic walking. The results on the reduction of simple hybrid systems are utilized in order to reduce a three-dimensional bipedal robot to two-dimensions; we are able to provide walking gaits that allow the walker to converge to the upright position.

Chapter 4: Hybrid Geometric Mechanics. Drawing intuition from the study of simple hybrid systems, we use hybrid objects to extend the results presented in Chapter 3 to general hybrid systems. Due to the categorical and functorial nature of geometric objects, they can be extended to a hybrid setting through the framework of hybrid objects. Specific examples of this process are discussed, e.g., hybrid differential forms, hybrid Lie groups and hybrid Lie algebras. In a similar vein, the ingredients necessary to perform reduction are generalized to a hybrid setting, the end result of which is the hybrid analogue of the classical symplectic reduction theorem. The implications of this theorem to the geometric reduction of hybrid dynamics, i.e., hybrid Hamiltonian reduction, is established. This chapter, therefore, demonstrates the ability of hybrid objects to generalize geometry to a hybrid setting.

Chapter 5: Zeno Behavior & Hybrid Stability Theory. Zeno behavior is unique to hybrid systems, and thus provides a unique opportunity to better understand not only the similarities between hybrid and dynamical systems, but also their differences. In order to study Zeno behavior, a type of equilibria—again unique to hybrid systems—is first introduced: Zeno equilibria. The relationship between the stability of Zeno equilibria and Zeno behavior is first established for a simple class of hybrid systems: first quadrant hybrid systems. After revisiting the stability of dynamical systems—specifically, Lyapunov's second method—in a categorical light, conditions on the stability of Zeno equilibria for general hybrid systems are established, a corollary of which is sufficient conditions on the existence of Zeno behavior. The similarities between these conditions and the categorical formulation of Lyapunov's second method indicate that hybrid objects are fundamental in understanding the general stability properties of hybrid systems.

Part II: Networked Systems. The final portion of this dissertation investigates the possibility of using hybrid objects, and the related notion of network objects, to described networked systems. While this provides only the first tentative steps toward such a theoretical extension, it could lay the groundwork for a categorical theory of networked systems.

Chapter 6: Universally Composing Embedded Systems. The final chapter of this dissertation is devoted not to hybrid systems, but to networked systems. This indicates that hybrid objects, and the related notion of network objects, may be instrumental in the study of such systems. A heterogeneous network of embedded systems can be modeled mathematically by a network of tagged systems, which provides a denotational semantics for such systems. We establish, in a constructive fashion, how a network of tagged systems can be formulated as a network over the category of tagged systems. Taking the composition of this network corresponds to taking the limit of the corresponding functor. Therefore, composition is endowed with a universal property. With this important observation in hand, necessary and sufficient conditions on the preservation of semantics are derived—that is, when behavior is preserved by composition.

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Part I

Foundations

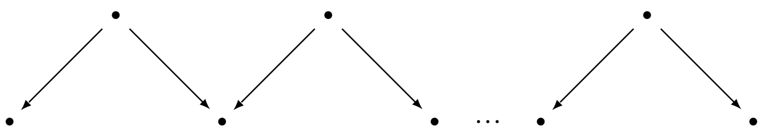
Chapter 1

Hybrid Objects

This chapter begins by introducing the basics of category theory in order to establish the necessary language in which to formulate the fundamental notion of a hybrid object over a category. After introducing category theory, and before introducing hybrid objects, it is necessary to introduce *D-categories*; these encode the discrete structure of a hybrid object. We then introduce hybrid objects over a category; this allows one to "hybridize" objects in a general category, and thus provides the foundation for our mathematical theory of hybrid systems. The chapter concludes by introducing other "hybrid" objects of interest: cohybrid objects over a category and networks over a category. Throughout the chapter, simple examples are introduced in order to highlight the concepts involved.

Before proceeding to our introduction of categories, we summarize in more detail the contents of this chapter; it is recommended that those not familiar with category theory first read Section 1.1. In addition, the motivation for the ideas introduced may seem opaque for those not familiar with hybrid systems; we refer the reader to Chapter 2 for this motivation. This dissertation, like most systems, is irrevocably nonlinear.

D-categories. Fundamental to our studies of hybrid objects is the notion of a D-category. These categories define the "discrete" structure of a hybrid object—the "D" stands for discrete—and dictate how the "continuous components" of a hybrid object interact. To be more specific, every D-category[1] $\mathscr{A}$ has the general form[2]

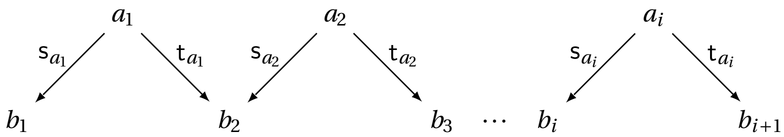


In no way is this structure accidental; the objects in the upper half of this diagram dictate the interaction between the objects in the lower half of the diagram.

[1] Categories of this form are denoted by calligraphic symbols.

[2] Where $\bullet$ denotes an arbitrary object in $\mathscr{A}$ together with its identity morphism and $\longrightarrow$ denotes an arbitrary (non-identity) morphism.

Directionality can be added to D-categories by picking a specific labeling of their morphisms; this defines an *oriented*[3] D-category. For example, the D-category above can be oriented as follows:



where s_{a_i} and t_{a_i} are morphisms indexed by a_i, with "s" standing for source in that b_i is the "source" of a_i, and "t" standing for target in that b_{i+1} is the "target" of a_i. Therefore, D-categories are in direct and formal analogy to graphs, e.g., the above D-category is obtained from or yields a graph of the form:

$$b_1 \xrightarrow{a_1} b_2 \xrightarrow{a_2} b_3 \quad \cdots \quad b_i \xrightarrow{a_i} b_{i+1}$$

and so the reader may prefer to think about D-categories as modified graphs. In fact, this is justified due to the isomorphism of categories: $\mathsf{Dcat} \cong \mathsf{Grph}$, where Dcat is the category of (oriented) D-categories and Grph is the category of (oriented) graphs. On the other hand, one should not make the mistake of assuming that the formalism of D-categories is unnecessary or extraneous; one *could not* work with graphs alone.

Hybrid objects. After introducing D-categories, we begin our exposition of hybrid objects and the categories thereof. Beginning with a category of "non-hybrid" objects of interest, C, the *hybrid objects over this category* are diagrams of a specific form, i.e., a hybrid object is a pair $(\mathscr{A}, \mathbf{A})$ where $\mathscr{A}$ is a D-category, and

$$\mathbf{A} : \mathscr{A} \to \mathsf{C}$$

is a functor. For example, a hybrid vector space is a functor $\mathbf{V} : \mathscr{V} \to \mathsf{Vect}_{\mathbb{R}}$, where $\mathscr{V}$ is a D-category and $\mathsf{Vect}_{\mathbb{R}}$ is the category of (real) vector spaces.

Morphisms between hybrid objects can be defined; these are functors of a very specific form, $\vec{F} : \mathscr{A} \to \mathscr{B}$, between D-categories together with a natural transformation:

$$\vec{f} : \mathbf{A} \xrightarrow{\cdot} \mathbf{B} \circ \vec{F}.$$

The result of combining this data is the category of hybrid objects over C, $\mathsf{Hy}(\mathsf{C})$. This will be our main object of study. In this light, we devote some energy to establishing some fundamental constructions relating to categories of this form. For example, given a functor $F : \mathsf{C} \to \mathsf{D}$, there is an induced functor:

$$\mathsf{Hy}(F) : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Hy}(\mathsf{D})$$

between categories of hybrid objects over C and D, respectively. Equally important will be the notion of an *element* of a hybrid object, e.g., an element of a hybrid vector space is a *hybrid vector*.

Cohybrid and network objects. Our studies do not end with $\mathsf{Hy}(\mathsf{C})$. There are many other interesting "hybrid" categories that naturally arise. One of these is the category of cohybrid objects over C, $\mathsf{CoHy}(\mathsf{C})$.

[3] These are the only type of D-categories that will be considered, so the prefix "oriented" will often be dropped.

The objects of this category are *contravariant* functors $\mathbf{A} : \mathscr{A} \to \mathsf{C}$. These categories frequently appear when dealing with contravariant functors between categories; if $F : \mathsf{C} \to \mathsf{D}$ is contravariant, then there is an induced contravariant functor:

$$\mathsf{Hy}(F) : \mathsf{Hy}(\mathsf{C}) \to \mathsf{CoHy}(\mathsf{D}).$$

There is also the notion of an element of a cohybrid objects. Concretely, the dual to a hybrid vector space is a cohybrid object over the category of vector spaces $\mathbf{V}^{\star} : \mathscr{V} \to \mathsf{Vect}_{\mathbb{R}}$, and an element of such a cohybrid object is a *hybrid covector*.

The final category of "hybrid" objects of interest appears not in hybrid systems, but in networked systems. That is, a *network over a category* C is a functor $\mathbf{N} : \mathfrak{N} \to \mathsf{C}$, where $\mathfrak{N}$ is the opposite to a D-category, or a D^{op}-category. The end result is the category of networks over C, $\mathsf{Net}(\mathsf{C})$. These categories are important in the study of networked systems—as the name suggests—and so will be instrumental in Chapter 6.

1.1 Categories

The goal of this section is to introduce the basics of category theory in order to provide the necessary framework in which to introduce our categorical framework for hybrid systems and the more general notion of a hybrid object over a category. While this review is self-contained, it is clearly not possible to briefly introduce all of the elementary category theory in a concise fashion. We refer the reader to [74] for any missing details, although there are many other good references on category theory; see [5], [21] and [92].

Definition 1.1. A **category** C consists of the following data:

- A class of *objects* $A, B, C, \ldots$, denoted by $\mathsf{Ob}(\mathsf{C})$,
- For all $A, B \in \mathsf{Ob}(\mathsf{C})$, a set of *morphisms* $\mathsf{Hom}_{\mathsf{C}}(A, B)$; a morphism $f \in \mathsf{Hom}_{\mathsf{C}}(A, B)$ is often written as $f : A \to B$ and in such a case the *domain* of f, $\mathsf{dom}(f)$, is A and the *codomain* of f, $\mathsf{cod}(f)$, is B,
- For all $A, B, C \in \mathsf{Ob}(\mathsf{C})$ with morphisms $f \in \mathsf{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathsf{Hom}_{\mathsf{C}}(B, C)$, there exists a morphism $g \circ f \in \mathsf{Hom}_{\mathsf{C}}(A, C)$ given by *composition*,

satisfying the axioms:

Associativity: For morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

Existence of identity: For all $A \in \mathsf{Ob}(\mathsf{C})$, there exists an identity morphism $\mathrm{id}_A : A \to A$ which satisfies, for every $f : A \to B$,

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A.$$

Category	**Objects**	**Morphisms**
Grp	Groups	Group homomorphisms
Ab	Abelian groups	Group homomorphisms
$\mathsf{Vect}_{\mathbb{R}}$	Real vector spaces	Linear maps
Top	Topological spaces	Continuous functions
Met	Metric spaces	Nonexpansive functions
Man	Smooth manifolds	Smooth functions

Table 1.1: Important categories.

A category is called **small** if its class of objects, $\mathsf{Ob}(\mathsf{C})$, is a set.

Remark 1.1. There are variants on the definition of a category. The most important of these is that it is not always required that the set of morphisms between two objects in a category form a set, but rather a class. Categories of this form are termed *quasi-categories*, the most important example of which is CAT, the category of all categories.

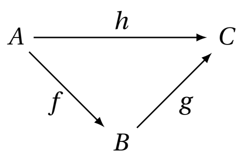
Example 1.1. One of the most fundamental examples of a category is the category of sets, Set, defined with

Objects: Sets,
Morphisms: Functions between sets.

The composition operation in this category is the usual composition of functions.

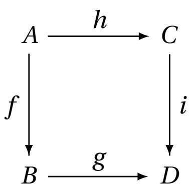
The category Set is fundamental because it allows one to endow many familiar collections of objects with the structure of a category; these are termed *concrete categories* [5]. Some examples can be found in Table 1.1; in all of the above examples, composition is given by the standard composition of functions in Set.

1.1.1 Commuting diagrams. Collections of objects and morphisms in a category are commonly displayed in the form of a diagram. That is, for $A, B, C \in \mathsf{Ob}(\mathsf{C})$ and morphisms $f : A \to B$, $g : B \to C$ and $h : A \to C$, it is often useful to display this data in the form:



(1.1)

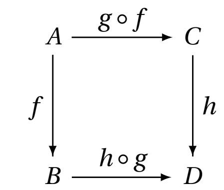
A diagram of this form is said to *commute* if $h = g \circ f$. Another canonical example of a commuting diagram is a commuting square:



Requiring this diagram to commute is equivalent to requiring that $g \circ f = i \circ h$.

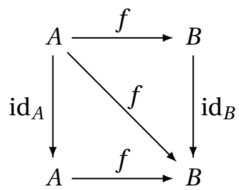
To provide an explicit example of the useful visual nature of diagrams, and especially commuting diagrams, the two axioms of a category can be restated as follows:

Associativity: For morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, the following diagram



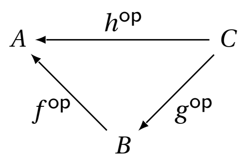
commutes.

Existence of identity: For all $A \in \mathsf{Ob}(\mathsf{C})$, there exists an identity morphism $\mathrm{id}_A : A \to A$ such that, for every $f : A \to B$, the following diagram



commutes.

1.1.2 Opposite categories. To provide an example of a category obtained from another category, let C be a category. We can then define the *opposite category to* C, denoted by C^{op}. The objects are the same as C, but the morphisms are reversed. That is, if $f : A \to B$ in C, then there is by definition a corresponding morphism in C^{op} given by $f^{\mathsf{op}} : B \to A$. Composition in C^{op} is defined by $f^{\mathsf{op}} \circ g^{\mathsf{op}} := (g \circ f)^{\mathsf{op}}$. Commuting diagrams allow us to visualize the difference between C and C^{op}. Specifically, a commuting diagram of the form (1.1) in C becomes a commuting diagram of the form:



in C^{op}.

These categories will play an important role when considering categories of cohybrid objects (to be introduced in Definition 1.12).

1.1.3 Distinguished morphisms. In the category of sets, Set, there is a well-understood notion of injective, surjective and bijective functions. These concepts can be extended to arbitrary categories through morphisms termed *monomorphisms*, *epimorphisms*, and *isomorphisms*. For a category C, there are the following classes of morphisms.

Monomorphisms: A morphism $m: A \to B$ is a monomorphism if for every object D and every pair of morphisms $f_1, f_2: D \to A$, i.e., for every diagram:

$$D \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \xrightarrow{m} B,$$

the following condition holds:

$$m \circ f_1 = m \circ f_2 \qquad \Rightarrow \qquad f_1 = f_2.$$

Epimorphisms: A morphism $e: A \to B$ is an epimorphism if for every object C and every pair of morphisms $g_1, g_2: B \to C$, i.e., for every diagram:

$$A \xrightarrow{e} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C,$$

the following condition holds:

$$g_1 \circ e = g_2 \circ e \qquad \Rightarrow \qquad g_1 = g_2.$$

Isomorphisms: A morphism $f: A \to B$ is an isomorphism if there exists a morphism $f^{-1}: B \to A$ such that:

$$f \circ f^{-1} = \mathrm{id}_B, \qquad f^{-1} \circ f = \mathrm{id}_A.$$

The morphism f^{-1} is unique.

Two objects A and B of C are isomorphic, denoted by $A \cong B$, if there exists an isomorphism $f: A \to B$.

Example 1.2. In the category of sets, Set, the monomorphisms are injective functions, the epimorphisms are surjective functions and the isomorphisms are bijective functions.

1.1.4 Distinguished objects. The above definitions dealt with properties of morphisms in a category. There are also some important properties that objects of a category C can display. Of special interest are the following distinguished classes of objects:

Terminal Objects: An object $*$ of C is a terminal object if for every object A of C there exists a unique morphism $A \to *$.

Initial Objects: An object $\emptyset$ of C is an initial object if for every object B of C there exists a unique morphism $\emptyset \to B$.

Zero Objects: An object 0 of C is a zero object if it is both an initial and terminal object.

Example 1.3. In the category of sets, Set, the empty set is the (unique in this case) initial object and every set consisting of a single point is a terminal object. There are no zero objects.

1.1.5 Functors. It is often important to investigate the relationship between multiple categories; this relationship is established by functors.

Definition 1.2. A **covariant functor** F between two categories C and D is given by

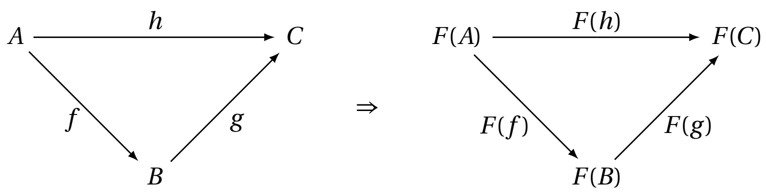
- An *object function* (also denoted by) F which associates to each object A of C an object $F(A)$ in D,

- ◇ A *morphism function* (also denoted by) F which associates to each morphism $f : A \to B$ in C a morphism $F(f) : F(A) \to F(B)$ in D,

satisfying the following two axioms:

- ◇ $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for every $A \in \mathsf{Ob}(\mathsf{C})$,
- ◇ $F(g \circ f) = F(g) \circ F(f)$ for morphisms $f : A \to B$ and $g : B \to C$ in C.

The last axiom in the definition of a functor requires that functors "preserve commuting diagrams." For example:



where the implication is on the commutativity of the diagram.

Example 1.4. Taking the power set of a set yields a functor $\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$ given on objects of Set, i.e., sets, by associating to a set X its power set $\mathscr{P}(X)$. To a morphism, i.e., a function, between sets $f : X \to Y$, we obtain a function $\mathscr{P}(f)$ where $\mathscr{P}(f)(U) = f(U)$ for $U \in \mathscr{P}(X)$.

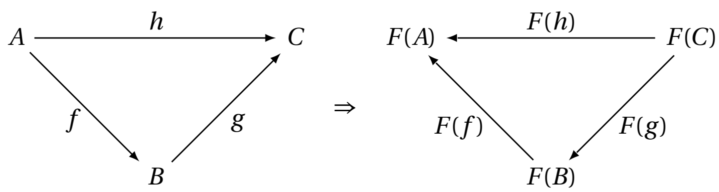
1.1.6 Contravariant functors. A contravariant functor can be thought of as a functor that "reverses" arrow. It again consists of an object function and a morphism function, except the condition on the morphism function given in Definition 1.2 becomes:

- ◇ A *morphism function* (as denoted by) F which associates to each morphism $f : A \to B$ in C a morphism $F(f) : F(B) \to F(A)$ in D.

We also require that the first axiom in Definition 1.2 holds, while the second axiom becomes:

- ◇ $F(g \circ f) = F(f) \circ F(g)$ for morphisms $f : A \to B$ and $g : B \to C$ in C.

The last of these two conditions can be visualized best by commuting diagrams:



where, again, the implication is on the commutativity of the diagram.

Notation 1.1. All functors are assumed to be covariant unless otherwise stated.

Example 1.5. The process of associating to a vector space its dual and to a linear map its dual results in a contravariant functor

$$(\,-\,)^\star : \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{R}}$$

where V maps to $V^\star$ and $f : V \to W$ maps to $f^\star : W^\star \to V^\star$.

1.1.7 Distinguished functors. Just as there are distinguished morphisms, e.g., monomorphisms and epimorphisms, there are also distinguished functors. Specifically, a functor:

$$F : \mathsf{C} \to \mathsf{D},$$

is

Full: if for every pair of objects A and B of C and morphism $f : F(A) \to F(B)$ in D there exists a morphism $g : A \to B$ in C such that $f = F(g)$. More compactly:

$$f : F(A) \to F(B) \qquad \Rightarrow \qquad \exists \quad g : A \to B \quad \text{s.t.} \quad f = F(g).$$

If the functor F is full, for any two objects A and B of C, the morphism function:

$$\begin{array}{rcl} F : \mathsf{Hom}_{\mathsf{C}}(A,B) & \to & \mathsf{Hom}_{\mathsf{D}}(F(A),F(B)) \\ g : A \to B & \mapsto & F(g) : F(A) \to F(B) \end{array}$$

is surjective.

Faithful: if for every pair of objects A and B and morphisms $f_1, f_2 : A \to B$,

$$F(f_1) = F(f_2) \qquad \Rightarrow \qquad f_1 = f_2.$$

If the functor F is faithful, for any two objects A and B of C, the morphism function:

$$F : \mathsf{Hom}_{\mathsf{C}}(A,B) \to \mathsf{Hom}_{\mathsf{D}}(F(A),F(B))$$

is injective.

Fully Faithful: if it is full and faithful.

Surjective on Objects: if for all objects X of D, there exists an object A of C such that $F(A) = X$.

Surjective: if it is surjective on objects and full, i.e., surjective on objects and morphisms.

Essentially Surjective: if for any object X of D there exists an object A of C such that $F(A) \cong X$.

Injective on Objects: if for any two objects A, B of C:

$$F(A) = F(B) \qquad \Rightarrow \qquad A = B.$$

Injective: if it is injective on objects and faithful, i.e., injective on objects and morphisms.

Bijective: if it is bijective on objects and fully faithful, i.e., bijective on objects and morphisms.

1.1.8 Forgetful functors. As indicated in Example 1.1, it is often the case that objects of a category C are sets together with some additional structure. More specifically, suppose that every object A of C is a set together with some additional structure, i.e., satisfying some additional axioms, and every morphism of C is a function together with some additional structure, i.e., satisfying some additional axioms. In this case, there is a *forgetful functor*:

$$U : \mathsf{C} \to \mathsf{Set},$$

given by viewing $U(A)$ as only a set, i.e., forgetting about any additional structure it may have, and viewing $U(f)$ as a function (again forgetting about any additional structure is may have). In this case, we often write $a \in A$ if $a \in U(A)$. Categories of this form are related to *concrete categories* [5] (if U is faithful, then C is a concrete category).

Example 1.6. For the category of vector spaces $\mathsf{Vect}_{\mathbb{R}}$, there is a forgetful functor:

$$U : \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Set},$$

given by forgetting about the vector space structure of a vector space and the linearity of a morphism between vector spaces.

1.1.9 Subcategories. Let D be a category. A subcategory of this category is a category C such that $\mathsf{Ob}(\mathsf{C}) \subseteq \mathsf{Ob}(\mathsf{D})$ and $\mathsf{Hom}_{\mathsf{C}}(A,B) \subseteq \mathsf{Hom}_{\mathsf{D}}(A,B)$ for all $A,B \in \mathsf{Ob}(\mathsf{C})$. It follows that there is an inclusion functor $I : \mathsf{C} \to \mathsf{D}$ which is the identity on objects and morphisms, i.e., the object function is the identity and the morphism function is the identity. A special class of subcategories that is of interest are *full subcategories*; these are subcategories in which the inclusion functor is a full functor. In particular, this implies that for any two objects A and B in C:

$$\mathsf{Hom}_{\mathsf{C}}(A,B) = \mathsf{Hom}_{\mathsf{D}}(A,B).$$

So, when defining a full subcategory of a category D, one need only specify the objects of this category.

Example 1.7. The category of abelian groups, Ab, is a full subcategory of the category of groups, Grp.

1.1.10 The category of categories. Functors can be thought of as "morphisms between categories." In fact, we can define the quasi-category of all categories, CAT, with

> **Objects:** All categories,
> **Morphisms:** Functors between categories.

This is technically not a category as defined in Paragraph 1.1 since the collection of functors $\mathsf{Hom}_{\mathsf{CAT}}(\mathsf{C},\mathsf{D})$ does not form a set. Regardless, the category of all categories can still be (at least conceptually) useful. For example, we can give a notion of when two categories are isomorphic.

Definition 1.3. Two categories C and D are **isomorphic**, denoted by $\mathsf{C} \cong \mathsf{D}$, if there exists two functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $F \circ G = \mathrm{Id}_{\mathsf{D}}$ and $G \circ F = \mathrm{Id}_{\mathsf{C}}$ where Id is the identity functor.

There is a useful characterization of when two categories are isomorphic based upon the properties of a functor between these categories: two categories C and D are isomorphic iff there exists a bijective functor $F : \mathsf{C} \to \mathsf{D}$.

1.1.11 The category of small categories. We can restrict the categories in CAT being considered in order to get a category in the classical sense. Let Cat be the category of all small categories, with

> **Objects:** All small categories,
> **Morphisms:** Functors between small categories.

In this case, the collection $\mathsf{Hom}_{\mathsf{Cat}}(\mathsf{C},\mathsf{D})$ forms a set. This category is very important in the study of hybrid objects over a category since it can be thought of as the "category of all indexing categories."

1.1.12 Natural transformations. Natural transformations can be viewed as "morphisms between functors." As such, they play a vital role in all of category theory, and especially categories of hybrid objects.

Definition 1.4. Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A **natural transformation** $\tau : F \overset{\cdot}{\to} G$ from F to G consists of a collection of morphisms $\tau_A : F(A) \to G(A)$ in D such that for every $f : A \to B$ in C, the following diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\tau_B} & G(B) \end{array}$$

commutes.

1.1.13 Composing natural transformations. Let $F, G, H : \mathsf{C} \to \mathsf{D}$ be functors. Natural transformations $\tau : F \overset{\cdot}{\to} G$ and $\upsilon : G \overset{\cdot}{\to} H$, can be composed "objectwise." That is, composing τ and υ results in a natural transformation:

$$\upsilon \bullet \tau : F \overset{\cdot}{\to} H,$$

defined objectwise by: $(\upsilon \bullet \tau)_A := \upsilon_A \circ \tau_A$ for all $A \in \mathsf{Ob}(\mathsf{C})$.

A natural transformation $\tau : F \overset{\cdot}{\to} G$ is a *natural isomorphism* if it is objectwise an isomorphism, i.e., $\tau_A : F(A) \to G(A)$ is an isomorphism for every object A of C. Equivalently, a natural transformation τ is a natural isomorphism if there exists a natural transformation $\tau^{-1} : G \overset{\cdot}{\to} F$ such that:

$$\tau \bullet \tau^{-1} = \mathrm{id}_G, \qquad \tau^{-1} \bullet \tau = \mathrm{id}_F,$$

where id_G and id_F are natural transformations that are objectwise the identity.

Two functors F and G are *isomorphic* , $F \cong G$, if there exists a natural isomorphism $\tau : F \overset{\cdot}{\to} G$.

Using the notion of natural isomorphisms, an equivalence of categories can be defined; this turns out frequently to be a better notion of equivalence between categories than requiring the categories to be isomorphic.

Definition 1.5. A functor $F : \mathsf{C} \to \mathsf{D}$ is an **equivalence of categories** if there exists a functor $G : \mathsf{D} \to \mathsf{C}$ and natural isomorphisms:

$$F \circ G \cong I_{\mathsf{D}}, \qquad G \circ F \cong I_{\mathsf{C}}.$$

Two categories C and D are *equivalent*, written $\mathsf{C} \approx \mathsf{D}$, if there exists an equivalence of categories $F : \mathsf{C} \to \mathsf{D}$ (or $G : \mathsf{D} \to \mathsf{C}$); F is an equivalence of categories iff F is fully faithful and essentially surjective.

1.1.14 Natural transformations between contravariant functors. If $F, G : \mathsf{C} \to \mathsf{D}$ are contravariant functors, then a natural transformation $\tau : F \overset{\cdot}{\to} D$ between these functors is again a collection of mor-

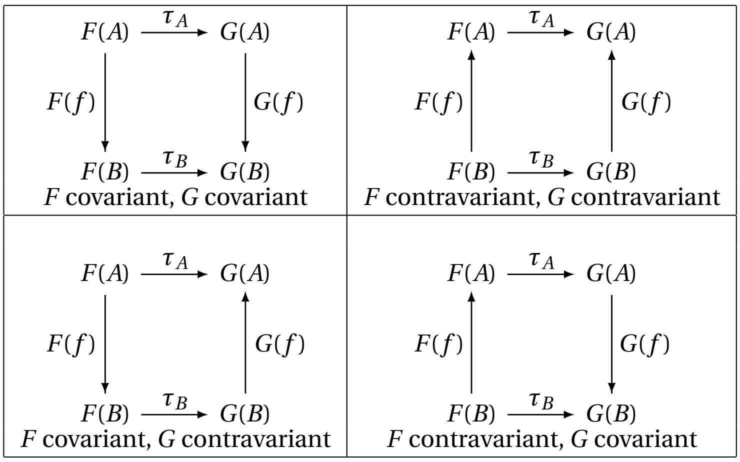
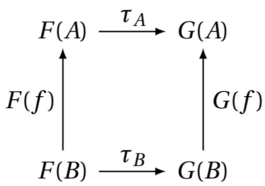


Table 1.2: Different variations of natural transformations.

phisms $\tau_A : F(A) \to G(A)$, except we now require that for every $f : A \to B$ in C the following diagram:



commutes. Natural transformations also can be defined when considering mixed covariant/contravariant functors as illustrated in Table 1.2.

1.1.15 Diagrams. A diagram (or J-diagram) in a category C is a functor $F : \mathsf{J} \to \mathsf{C}$ for some small category J (an indexing category). We can form the category of all J-diagrams in the category C, denoted by C^{J}, with

Objects: Functors $F : \mathsf{J} \to \mathsf{C}$,
Morphisms: Natural transformations.

Categories of this form are commonly referred to as *functor categories.*

1.1.16 The constant functor. A very important, yet simple, functor is the *constant functor*, Δ_{J}. This is a functor:

$$\Delta_{\mathsf{J}} : \mathsf{C} \to \mathsf{C}^{\mathsf{J}},$$

given on objects $A \in \mathsf{Ob}(\mathsf{C})$ by

$$\Delta_{\mathsf{J}}(A)(a) = A \xrightarrow{\Delta_{\mathsf{J}}(A)(\alpha) = \mathrm{id}_A} \Delta_{\mathsf{J}}(A)(b) = A$$

for $\alpha : a \to b$ in J. On morphisms $f : A \to B$ in C, $\Delta_{\mathsf{J}}(f)_a := f$ for every object a of J.

1.1.17 Basic diagrams. Diagrams play a central role in the theory of hybrid objects, except we will restrict our attention to a specific class of small categories termed D-categories. In preparation, we now enumerate some of the basic diagrams of interest in category theory.

$(\bullet)$: A category consisting of a single object and an identity morphism. A functor $F : (\bullet) \to \mathsf{C}$ can be identified with an object of C, i.e., it is just the object $F(\bullet) \in \mathsf{Ob}(\mathsf{C})$. Therefore, the category $\mathsf{C}^{(\bullet)} = \mathsf{Ob}(\mathsf{C})$.

$(\bullet \to \bullet)$: A category consisting of two objects, the identity morphisms for these objects and a non-identity morphism. A functor

$$F : (\bullet \to \bullet) \to \mathsf{C}$$

is just a diagram:

$$F(\bullet \to \bullet) = A \xrightarrow{f} B$$

in C. Therefore, the category $\mathsf{C}^{(\bullet\to\bullet)}$ can be identified with the morphisms in C.

$(\bullet \rightrightarrows \bullet)$: A category with two objects and two non-identity morphisms. A functor

$$F : (\bullet \rightrightarrows \bullet) \to \mathsf{C}$$

is just a diagram:

$$F(\bullet \rightrightarrows \bullet) = A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B$$

in C. Diagrams of this form are important when considering *equalizers* and *coequalizers*.

$(\bullet \leftarrow \bullet \to \bullet)$: A category with three objects and two non-identity morphisms. A functor

$$F : (\bullet \leftarrow \bullet \to \bullet) \to \mathsf{C}$$

is just a diagram:

$$F(\bullet \leftarrow \bullet \to \bullet) = A \xleftarrow{f} B \xrightarrow{g} C$$

in C. Diagrams of this form are important when considering *pushouts*.

$(\bullet \to \bullet \leftarrow \bullet)$: A category with three objects and two non-identity morphisms. A functor

$$F : (\bullet \to \bullet \leftarrow \bullet) \to \mathsf{C}$$

is just a diagram:

$$F(\bullet \to \bullet \leftarrow \bullet) = A \xrightarrow{f} B \xleftarrow{g} C$$

in C. Diagrams of this form are important when considering *pullbacks*.

1.2 D-categories

In this section, we introduce an important class of small categories: *D-categories*. These categories are very simple small categories that essentially can be thought of as graphs. In fact, we will demonstrate that the category of (oriented) D-categories is isomorphic to the category of (oriented) graphs:

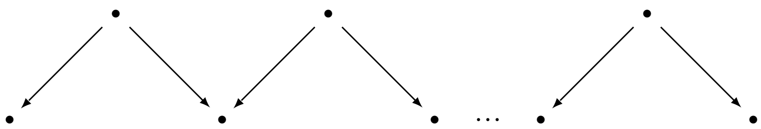
$$\mathsf{Dcat} \cong \mathsf{Grph}\,.$$

The proof of this fact is constructive in nature, i.e., it is shown how to obtain a graph from a D-category and a D-category from a graph.

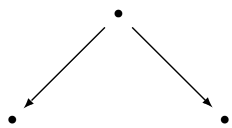
The motivation for considering D-categories is that they play a fundamental role in defining hybrid objects over a category. The motivation for the name D-categories is that they define the "discrete" structure of a hybrid object over a category.

1.2.a Axioms and Orientations

We must define a specific type of small category, termed a *D-category*, in order to introduce hybrid objects. This is a small category in which every diagram has the form:



That is, a D-category has as its basic atomic unit a diagram of the form:



and any other diagram in this category must be obtainable by gluing such atomic units along the codomain of a morphism (and not the domain). More formally, consider the following:

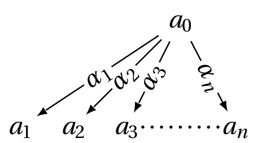
Definition 1.6. A **D-category** is a small category $\mathscr{D}$ satisfying the following two axioms:

AD1 Every object in $\mathscr{D}$ is either the domain of a non-identity morphism in $\mathscr{D}$ or the codomain of a non-identity morphism but never both, i.e., for every diagram

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$$

in $\mathscr{D}$, all but one morphism must be the identity (the longest chain of composable non-identity morphisms is of length one).

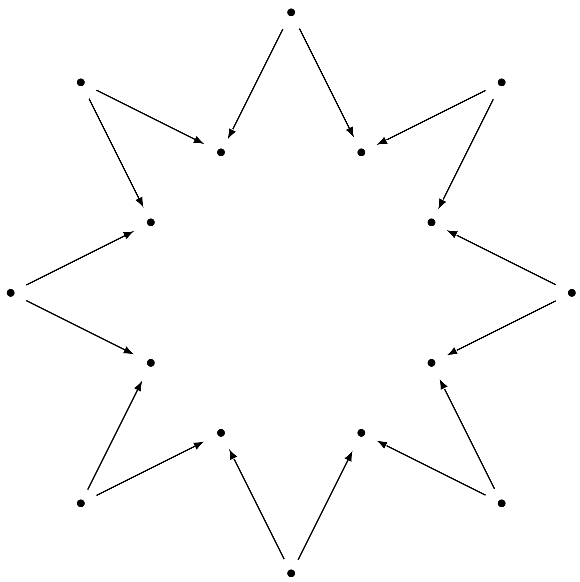
AD2 If an object in $\mathscr{D}$ is the domain of a non-identity morphism, then it is the domain of exactly two non-identity morphisms, i.e., for every diagram in $\mathscr{D}$ of the form



consisting of all morphisms with domain a_0, either all of the morphisms are the identity or two and only two morphisms are not the identity.

Remark 1.2. We could form the category of D-categories with objects D-categories and morphisms all functors. This being said, we actually do not consider this category as it does not yet have enough structure, i.e., we will consider D-categories that are *oriented* and functors between D-categories that preserve these orientations.

Example 1.8. An example of a D-category is given in the following diagram:



This D-category can be justifiably thought of as a "cycle" D-category.

1.2.1 Important objects in D-categories. Let $\mathscr{D}$ be a D-category. We use $\mathsf{Mor}(\mathscr{D})$ to denote the morphisms of $\mathscr{D}$, i.e.,

$$\mathsf{Mor}(\mathscr{D}) = \bigcup_{(a,b)\in\mathsf{Ob}(\mathscr{D})\times\mathsf{Ob}(\mathscr{D})} \mathsf{Hom}_{\mathscr{D}}(a,b),$$

and $\mathsf{Mor}_{\not{\mathrm{id}}}(\mathscr{D})$ to denote the set of non-identity morphisms of $\mathscr{D}$, i.e.,

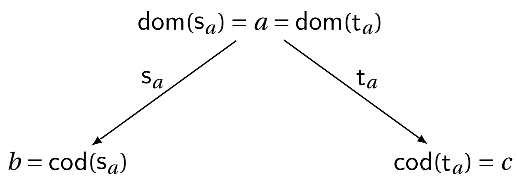
$$\mathsf{Mor}_{\not{\mathrm{id}}}(\mathscr{D}) = \{\alpha \in \mathsf{Mor}(\mathscr{D}) : \alpha \neq \mathrm{id}\}.$$

For a morphism $\alpha : a \to b$ in $\mathscr{D}$, recall from Definition 1.1 that its domain is denoted by $\mathsf{dom}(\alpha) = a$ and its codomain is denoted by $\mathsf{cod}(\alpha) = b$.

For D-categories, there are two sets of objects that are of particular interest; these are subsets of $\mathsf{Ob}(\mathscr{D})$. The first of these is termed the *edge set of* $\mathscr{D}$, denoted by $\mathsf{E}(\mathscr{D})$, and defined to be:

$$\mathsf{E}(\mathscr{D}) = \{a \in \mathsf{Ob}(\mathscr{D}) :\ a = \mathsf{dom}(\alpha),\ a = \mathsf{dom}(\beta),\ \alpha,\beta \in \mathsf{Mor}_{\not{\mathrm{id}}}(\mathscr{D}),\ \alpha \neq \beta\}.$$

That is, for all $a \in \mathsf{E}(\mathscr{D})$ there are two and only two morphisms (which are not the identity) $\alpha, \beta \in \mathsf{Mor}(\mathscr{D})$ such that $a = \mathsf{dom}(\alpha)$ and $a = \mathsf{dom}(\beta)$, so we denote these morphisms by s_a and t_a (the specific choice will define an *orientation*). Conversely, given a morphism $\gamma \in \mathsf{Mor}_{\not{\mathrm{id}}}(\mathscr{D})$, there exists a unique $a \in \mathsf{E}(\mathscr{D})$ such that $\gamma = \mathsf{s}_a$ or $\gamma = \mathsf{t}_a$. Therefore, every object $a \in \mathsf{E}(\mathscr{D})$ sits in a diagram of the form:



(1.2)

Note that giving all diagrams of this form (for which there is one for each $a \in \mathsf{E}(\mathscr{D})$) gives all the objects in $\mathscr{D}$, i.e., every object of $\mathscr{D}$ is the domain or codomain of s_a or t_a for some $a \in \mathsf{E}(\mathscr{D})$.

Define the *vertex set of* $\mathscr{D}$ by:

$$\mathsf{V}(\mathscr{D}) = (\mathsf{E}(\mathscr{D}))^c,$$

where here $(\mathsf{E}(\mathscr{D}))^c$ is the complement of $\mathsf{E}(\mathscr{D})$ in the set $\mathsf{Ob}(\mathscr{D})$. It follows by definition that

$$\begin{aligned} \mathsf{E}(\mathscr{D}) \cap \mathsf{V}(\mathscr{D}) &= \emptyset, \\ \mathsf{E}(\mathscr{D}) \cup \mathsf{V}(\mathscr{D}) &= \mathsf{Ob}(\mathscr{D}). \end{aligned}$$

The above choice of morphisms s_a and t_a can be used to define an orientation on a D-category.

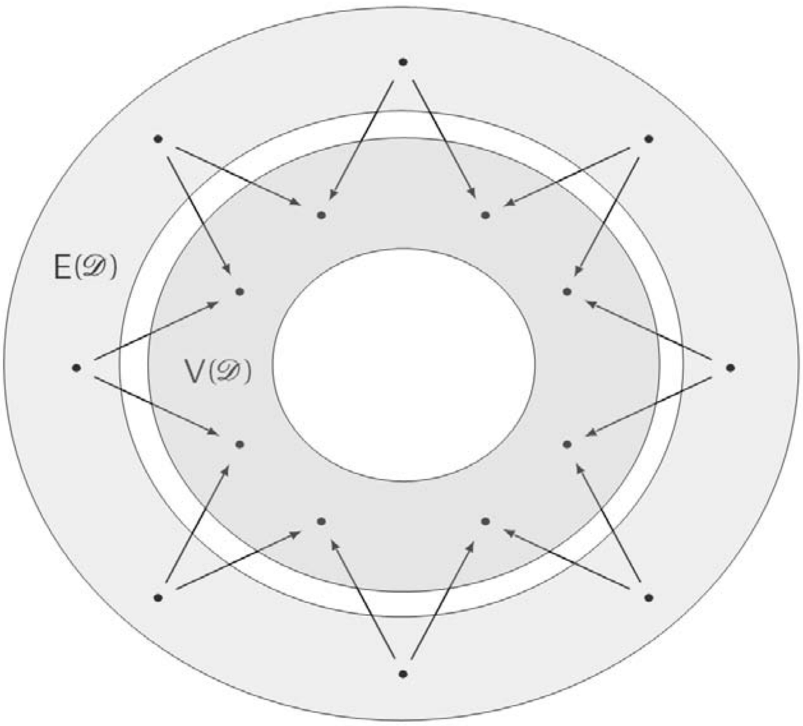


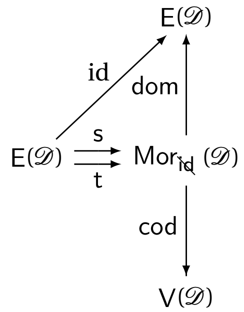
Figure 1.1: The edge and vertex sets for a D-category.

Example 1.9. For the D-category introduced in Example 1.8, the edge and vertex sets can be seen in Figure 1.1; in this figure "•" is now just an object, not an object together with its identity morphism.

Definition 1.7. An **orientation** of a D-category $\mathscr{D}$ is a pair of functions (s, t) between sets:

$$\mathsf{E}(\mathscr{D}) \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathsf{Mor}_{\not{\mathsf{id}}}(\mathscr{D}),$$

that fit into a diagram



(1.3)

in which the top triangle commutes.

Notation 1.2. We will always assume that a given D-category has an orientation. Therefore, we will not explicitly say "oriented D-category" since all D-categories considered will be oriented.

The notion of a D-category, together with an orientation thereon, can be summarized succinctly as follows:

Definition 1.8. A **D-category** is a small category $\mathscr{D}$ such that:

- There exist two subsets of $\mathsf{Ob}(\mathscr{D})$, $\mathsf{E}(\mathscr{D})$ and $\mathsf{V}(\mathscr{D})$, termed the *edge set* and *vertex set*, satisfying:

$$\begin{aligned} \mathsf{E}(\mathscr{D}) \cap \mathsf{V}(\mathscr{D}) &= \emptyset, \\ \mathsf{E}(\mathscr{D}) \cup \mathsf{V}(\mathscr{D}) &= \mathsf{Ob}(\mathscr{D}), \end{aligned}$$

- There exists a pair of functions:

$$\mathsf{E}(\mathscr{D}) \overset{\mathsf{s}}{\underset{\mathsf{t}}{\rightrightarrows}} \mathsf{Mor}_{\not{\mathsf{id}}}(\mathscr{D}),$$

  such that:

$$\begin{aligned} \mathsf{s}(\mathsf{E}(\mathscr{D})) \cap \mathsf{t}(\mathsf{E}(\mathscr{D})) &= \emptyset, \\ \mathsf{s}(\mathsf{E}(\mathscr{D})) \cup \mathsf{t}(\mathsf{E}(\mathscr{D})) &= \mathsf{Mor}_{\not{\mathsf{id}}}(\mathscr{D}). \end{aligned}$$

  and the diagram in (1.3) is well-defined and commutes; the pair (s, t) is termed an *orientation* of $\mathscr{D}$.

Remark 1.3. By requiring that the diagram in (1.3) is well-defined we are imposing the condition that $\mathsf{dom}(\mathsf{Mor}_{\not{\mathsf{id}}}(\mathscr{D})) = \mathsf{E}(\mathscr{D})$ and $\mathsf{cod}(\mathsf{Mor}_{\not{\mathsf{id}}}(\mathscr{D})) = \mathsf{V}(\mathscr{D})$. In addition, for every $a \in \mathsf{E}(\mathscr{D})$, there is a corresponding diagram (1.2) in which $b, c \in \mathsf{V}(\mathscr{D})$.

To verify that the (oriented) D-categories, as defined in 1.8, satisfy the axioms of a D-category as given in Definition 1.6, we demonstrate the following:.

Lemma 1.1. *A D-category, as defined in 1.8, satisfies* **AD1** *and* **AD2**.

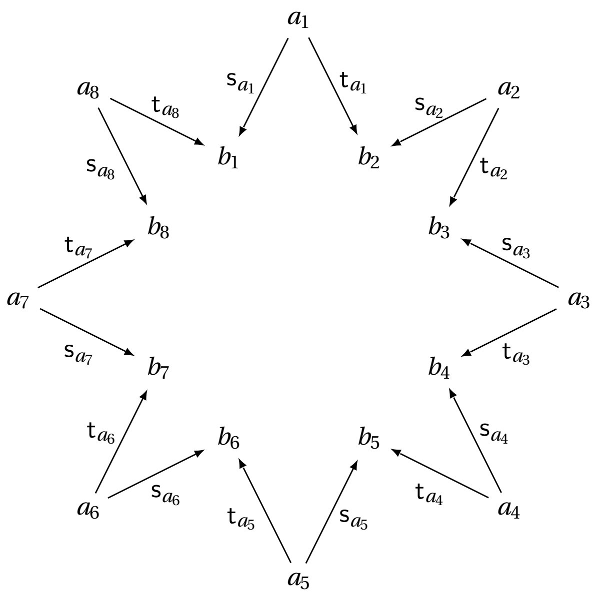
Proof. Beginning with **AD1**, we argue by way of contradiction. Suppose that there are two morphisms

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c$$

with $\alpha \neq \mathrm{id}$ and $\beta \neq \mathrm{id}$. Then, since $\mathsf{s}(\mathsf{E}(\mathscr{D})) \cup \mathsf{t}(\mathsf{E}(\mathscr{D})) = \mathsf{Mor}_{\not\mathrm{id}}(\mathscr{D})$, $\alpha = \mathsf{s}_a$ or t_a and $\beta = \mathsf{s}_b$ or t_b. Since $b = \mathsf{cod}(\alpha)$, and because (1.3) is well-defined, it follows that $b \in \mathsf{V}(\mathscr{D})$. But $b = \mathsf{dom}(\beta)$ and so, again because of the fact that (1.3) is well-defined, it follows that $b \in \mathsf{E}(\mathscr{D})$. Since $\mathsf{E}(\mathscr{D}) \cap \mathsf{V}(\mathscr{D}) = \emptyset$ we have established the desired contradiction.

To show that **AD2** holds, let $a = \mathsf{dom}(\alpha)$ with $\alpha \neq \mathrm{id}$. Then $a \in \mathsf{E}(\mathscr{D})$ by the fact that (1.3) is well-defined; moreover $a = \mathsf{dom}(\mathsf{s}_a)$ and $a = \mathsf{dom}(\mathsf{t}_a)$ by the commutativity of this diagram. Therefore, a is the domain of two non-identity morphisms. Now, for any other non-identity morphism β such that $a = \mathsf{dom}(\beta)$, since $\mathsf{s}(\mathsf{E}(\mathscr{D})) \cup \mathsf{t}(\mathsf{E}(\mathscr{D})) = \mathsf{Mor}_{\not\mathrm{id}}(\mathscr{D})$, it follows that $\beta = \mathsf{s}_a$ or $\beta = \mathsf{t}_a$. Therefore, a is the domain of exactly two non-identity morphisms. □

Example 1.10. We can pick an orientation for the D-category given in Example 1.8. This orientation is displayed in the following diagram:

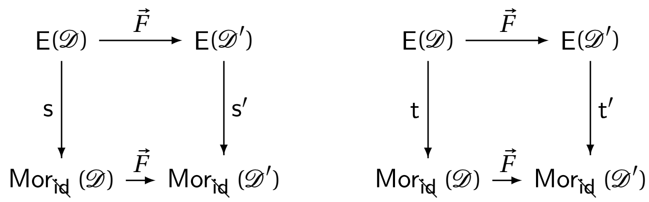


This is by no means the only orientation that we could impose; it was chosen because it makes this D-category into a "directed cycle" D-category or a D-cycle. D-categories of this form will be fundamental in the study of Zeno behavior in hybrid systems (cf. Chapter 5).

1.2.2 The category of D-categories. Define the category of (oriented) D-categories, Dcat, to have objects D-categories. A morphism between two D-categories, $\mathscr{D}$ and $\mathscr{D}'$ (with orientations (s,t) and $(\mathsf{s}',\mathsf{t}')$, respectively) is a functor $\vec{F}: \mathscr{D} \to \mathscr{D}'$ such that

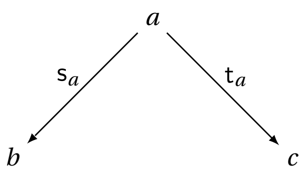
$$\vec{F}(\mathsf{E}(\mathscr{D})) \subseteq \mathsf{E}(\mathscr{D}'), \qquad \vec{F}(\mathsf{V}(\mathscr{D})) \subseteq \mathsf{V}(\mathscr{D}'), \tag{1.4}$$

and the following diagrams

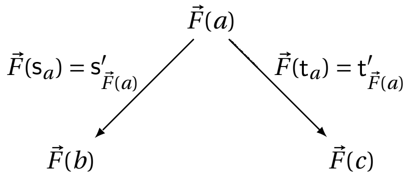


(1.5)

commute. By requiring these diagrams to commute, it implies that for all diagrams of the form:

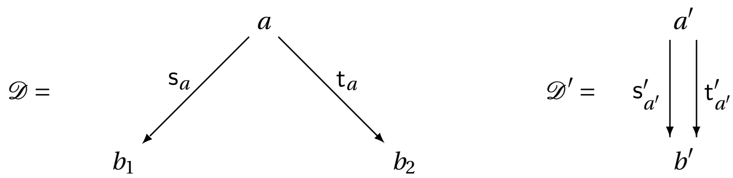


in $\mathscr{D}$, i.e., $a \in \mathsf{E}(\mathscr{D})$ and $b, c \in \mathsf{V}(\mathscr{D})$, there are corresponding diagrams:



in $\mathscr{D}'$, where $\vec{F}(a) \in \mathsf{E}(\mathscr{D}')$ and $\vec{F}(b), \vec{F}(c) \in \mathsf{V}(\mathscr{D}')$.

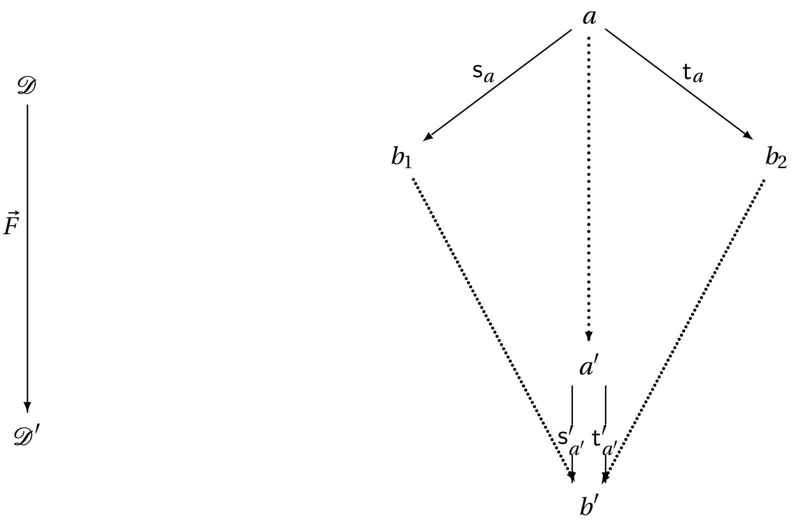
Example 1.11. Let $\mathscr{D}$ and $\mathscr{D}'$ be the D-categories given by the following diagrams:



There is a morphism $\vec{F} : \mathscr{D} \to \mathscr{D}'$ of D-categories given by:

$$\vec{F}(a) = a', \quad \vec{F}(b_1) = \vec{F}(b_2) = b', \quad \vec{F}(\mathsf{s}_a) = \mathsf{s}'_{a'}, \quad \vec{F}(\mathsf{t}_a) = \mathsf{t}'_{a'}.$$

This morphism can be visualized by the following diagram:



Example 1.12. Let $\mathscr{D}''$ be the D-category given by the following diagram:

$$\mathscr{D}'' = \begin{array}{ccccccc} & & a_1 & & & a_2 & \\ & \swarrow^{\mathsf{s}_{a_1}} & & \searrow^{\mathsf{t}_{a_1}} & \swarrow^{\mathsf{s}_{a_2}} & & \searrow^{\mathsf{t}_{a_2}} \\ b_1 & & & b_2 & & & b_3 \end{array}$$

In this case, there is *not* a morphism from $\mathscr{D}''$ to $\mathscr{D}$ as any such morphism would not preserve the orientations of these D-categories.

1.2.3 Elementary properties. At this point, we verify some elementary properties of D-categories.

Lemma 1.2. *For any two objects a, b in $\mathscr{D}$, if $a \cong b$ then $a = b$.*

Proof. We argue by contradiction. If $a \cong b$ and $a \neq b$, then there exist two non-identity morphisms:

$$a \xrightarrow{\alpha} b \xrightarrow{\alpha^{-1}} a.$$

This violates **AD1**. □

Using this result, we characterize equivalences between D-categories.

Lemma 1.3. *A morphism $\vec{F} : \mathscr{D} \to \mathscr{D}'$ is an equivalence of categories iff it is an isomorphism of categories.*

1.2.b D-categories and Graphs

We now turn our attention to relating D-categories to graphs.

1.2.4 Oriented graphs. A (directed or oriented) graph is a pair $\Gamma = (Q, E)$, where Q is a set of vertices and E is a set of edges (assumed to be disjoint), together with a pair of functions:

$$E \underset{\mathsf{tar}}{\overset{\mathsf{sor}}{\rightrightarrows}} Q$$

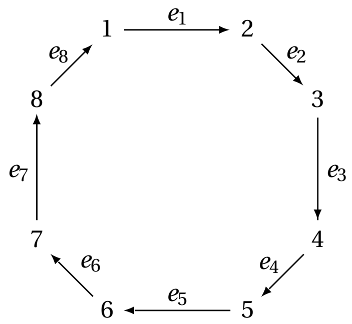
called the source and target functions; for $e \in E$, $\mathsf{sor}(e)$ is the source of e and $\mathsf{tar}(e)$ is the target of e.

A morphism of graphs is a pair $F = (F_Q, F_E) : \Gamma = (Q, E) \to \Gamma' = (Q', E')$, where $F_Q : Q \to Q'$ and $F_E : E \to E'$, such that the following diagrams commute:

$$\begin{CD} E @>{F_E}>> E' \\ @V{\mathsf{sor}}VV @VV{\mathsf{sor}'}V \\ Q @>{F_Q}>> Q' \end{CD} \qquad \begin{CD} E @>{F_E}>> E' \\ @V{\mathsf{tar}}VV @VV{\mathsf{tar}'}V \\ Q @>{F_Q}>> Q' \end{CD} \tag{1.6}$$

Thus we have defined the category of graphs, Grph.

Example 1.13. An example of a graph is given by the following directed cycle graph:

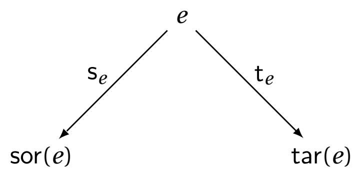


A graph of this form is often denoted by C_8.

1.2.5 D-categories from graphs. Given a graph, $\Gamma = (Q, E)$, we can associate to this graph a D-category $\mathscr{D}_\Gamma$. Define the objects of $\mathscr{D}_\Gamma$ by defining

$$\mathsf{E}(\mathscr{D}_\Gamma) := E, \qquad \mathsf{V}(\mathscr{D}_\Gamma) := Q, \qquad \mathsf{Ob}(\mathscr{D}_\Gamma) = \mathsf{E}(\mathscr{D}_\Gamma) \cup \mathsf{V}(\mathscr{D}_\Gamma).$$

To define the morphisms of $\mathscr{D}_\Gamma$ we define, for every $e \in E$, morphisms:



We complete the description of $\mathscr{D}_\Gamma$ by defining an identity morphism on each object of $\mathscr{D}_\Gamma$. Note that in the definition of $\mathscr{D}_\Gamma$, we gave it a canonical orientation; namely, (s, t) where s_e and t_e are defined as above for every $e \in E$.

Given a morphism $F = (F_Q, F_E) : \Gamma \to \Gamma'$, we can define a functor $\vec{F} : \mathscr{D}_\Gamma \to \mathscr{D}_{\Gamma'}$ by defining it on objects and morphisms as follows:

$$\vec{F}(a) := \begin{cases} F_E(a) & \text{if} \quad a \in \mathsf{E}(\mathscr{D}_\Gamma) \\ F_Q(a) & \text{if} \quad a \in \mathsf{V}(\mathscr{D}_\Gamma) \end{cases} \qquad \vec{F}(\gamma) := \begin{cases} \mathsf{s}'_{\vec{F}(e)} & \text{if} \quad \gamma = \mathsf{s}_e \\ \mathsf{t}'_{\vec{F}(e)} & \text{if} \quad \gamma = \mathsf{t}_e \end{cases}$$

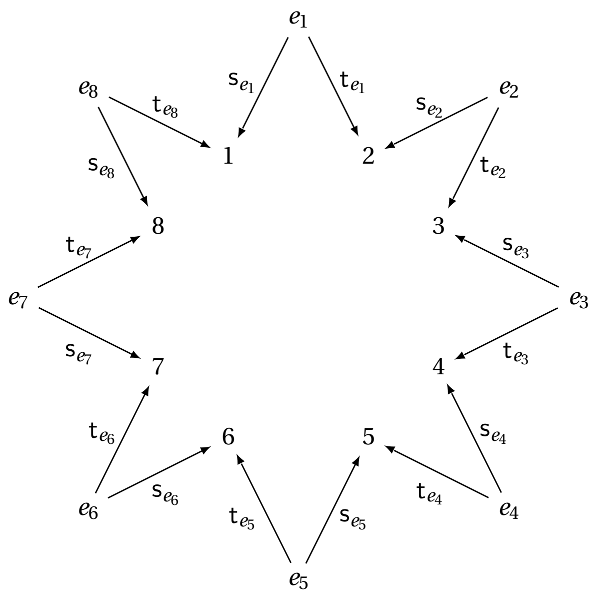
Of course, $\vec{F}$ is defined on identity morphisms in the obvious fashion: $\vec{F}(\mathrm{id}_a) := \mathrm{id}_{\vec{F}(a)}$. It follows by the commutativity of (1.6) that $\vec{F}$ is a valid morphism of D-categories.

The method of associating a D-category to a graph defines a functor:

$$\mathsf{dcat} : \mathsf{Grph} \to \mathsf{Dcat}$$

We will introduce the inverse of this construction, but first consider the following:

Example 1.14. The D-category obtained from the graph C_8 is just the D-category given in Example 1.10. To make explicit the fact that this D-category is obtained from the graph C_8, we could denote it by $\mathscr{D}_{C_8}$, and label its objects and morphisms as follows:



This is in accordance with the construction given in the previous paragraph.

1.2.6 Graphs from D-categories. Given a D-category $\mathscr{D}$, we can obtain a graph from this D-category,

$$\Gamma_{\mathscr{D}} = (Q_{\mathscr{D}}, E_{\mathscr{D}}) := (\mathsf{V}(\mathscr{D}), \mathsf{E}(\mathscr{D})),$$

with source and target functions:

$$E_{\mathscr{D}} \underset{\mathsf{tar}=\mathsf{cod}(\mathsf{t}_{(-)})}{\overset{\mathsf{sor}=\mathsf{cod}(\mathsf{s}_{(-)})}{\rightrightarrows}} Q_{\mathscr{D}}$$

For a morphism between D-categories, $\vec{F} : \mathscr{D} \to \mathscr{D}'$, we obtain a morphism between the graphs $\Gamma_{\mathscr{D}}$ and $\Gamma_{\mathscr{D}'}$:

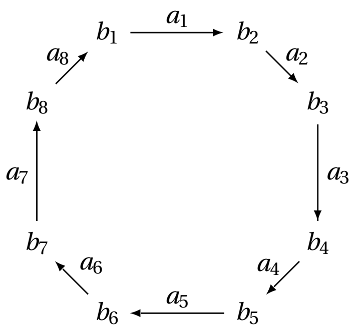
$$F := (\vec{F}|_{Q_{\mathscr{D}}}, \vec{F}|_{E_{\mathscr{D}}}) = (\vec{F}|_{\mathsf{V}(\mathscr{D})}, \vec{F}|_{\mathsf{E}(\mathscr{D})}).$$

It follows that F is a valid morphism of graphs; (1.6) commutes because (1.5) commutes.

The result of these constructions is a functor:

$$\mathsf{grph} : \mathsf{Dcat} \quad \to \quad \mathsf{Grph}.$$

Example 1.15. The graph obtained from the D-category given in Example 1.10 is just the graph C_8. To make explicit that this graph was obtained from this D-category, we could label the vertices and edges of this graph as follows:



This is in accordance with the construction given in the previous paragraph.

We now introduce a very important, although fairly obvious, result. Its importance lies in the fact that many of the properties that the category of graphs displays—which is a fair number—the category of D-categories will inherit. This will be made explicit, for example, in Appendix A

Theorem 1.1. *There is an isomorphism of categories:*

$$\mathsf{Dcat} \cong \mathsf{Grph},$$

where this isomorphism is given by the functor $\mathrm{grph} : \mathsf{Dcat} \to \mathsf{Grph}$ *with inverse* $\mathrm{dcat} : \mathsf{Grph} \to \mathsf{Dcat}$.

Proof. We first verify that $\mathrm{dcat} \circ \mathrm{grph} = \mathrm{Id}_{\mathsf{Dcat}}$. On objects, this holds since

$$\begin{aligned} \mathsf{Mor}_{\not{\mathsf{Id}}}\,(\mathrm{dcat} \circ \mathrm{grph}(\mathscr{D})) &= \{\mathsf{s}_e\}_{e \in E_{\mathscr{D}}} \cup \{\mathsf{t}_e\}_{e \in E_{\mathscr{D}}} = \{\mathsf{s}_e\}_{e \in \mathsf{E}(\mathscr{D})} \cup \{\mathsf{t}_e\}_{e \in \mathsf{E}(\mathscr{D})} = \mathsf{Mor}_{\not{\mathsf{Id}}}\,(\mathscr{D}), \\ \mathsf{Ob}(\mathrm{dcat} \circ \mathrm{grph}(\mathscr{D})) &= E_{\mathscr{D}} \cup Q_{\mathscr{D}} = \mathsf{V}(\mathscr{D}) \cup \mathsf{E}(\mathscr{D}) = \mathsf{Ob}(\mathscr{D}), \end{aligned}$$

and the identity morphisms of $\mathscr{D}$ and $\mathrm{dcat} \circ \mathrm{grph}(\mathscr{D})$ are the same by definition. Consider a morphism $\vec{F} : \mathscr{D} \to \mathscr{D}'$ of D-categories. For all $a \in \mathsf{Ob}(\mathscr{D})$,

$$\begin{aligned} \mathrm{dcat} \circ \mathrm{grph}(\vec{F})(a) &= \begin{cases} \vec{F}(a) & \text{if} \quad a \in E_{\mathscr{D}} = \mathsf{E}(\mathscr{D}_{\Gamma_{\mathscr{D}}}) = \mathsf{E}(\mathscr{D}) \\ \vec{F}(a) & \text{if} \quad a \in Q_{\mathscr{D}} = \mathsf{V}(\mathscr{D}_{\Gamma_{\mathscr{D}}}) = \mathsf{V}(\mathscr{D}) \end{cases} \\ &= \vec{F}(a), \end{aligned}$$

and for all $\gamma \in \mathsf{Mor}(\mathscr{D})$,

$$\begin{aligned} \mathrm{dcat} \circ \mathrm{grph}(\vec{F})(\gamma) &= \begin{cases} \mathsf{s}'_{\vec{F}(a)} & \text{if} \quad \gamma = \mathsf{s}_a \\ \mathsf{t}'_{\vec{F}(a)} & \text{if} \quad \gamma = \mathsf{t}_a \end{cases} \\ &= \begin{cases} \vec{F}(\gamma) & \text{if} \quad \gamma = \mathsf{s}_a \\ \vec{F}(\gamma) & \text{if} \quad \gamma = \mathsf{t}_a \end{cases} \\ &= \vec{F}(\gamma). \end{aligned}$$

Therefore $\mathrm{dcat} \circ \mathrm{grph} = \mathrm{Id}_{\mathsf{Dcat}}$.

Next we verify that $\text{grph} \circ \text{dcat} = \text{Id}_{\mathsf{Grph}}$. For a graph $\Gamma = (Q, E)$, we have

$$\text{grph} \circ \text{dcat}(\Gamma) = \Gamma_{\mathscr{D}_\Gamma} = (Q_{\mathscr{D}_\Gamma}, E_{\mathscr{D}_\Gamma}) = (\mathsf{V}(\mathscr{D}_\Gamma), \mathsf{E}(\mathscr{D}_\Gamma)) = (Q, E) = \Gamma.$$

For a morphism of graphs $F = (F_Q, F_E) : \Gamma \to \Gamma'$,

$$\text{grph} \circ \text{dcat}(F)_Q = \vec{F}|_Q = F_Q, \qquad \text{grph} \circ \text{dcat}(F)_E = \vec{F}|_E = F_E.$$

□

1.3 Hybrid Objects

The starting point for theory of hybrid objects is the observation that systems that display both continuous and discrete behavior, i.e., hybrid systems, can be represented by a D-category together with a functor. This relates hybrid systems to the two most fundamental objects in category theory: a functor and a natural transformation.

In this section, and from this point on, we will denote D-categories by the calligraphic symbols: $\mathscr{A}, \mathscr{B}, \mathscr{C}$, *et cetera.*

Using the notion of a D-category, we have the following definition of a hybrid object over a category.

Definition 1.9. Let C be a category. A **hybrid object over** C is a pair $(\mathscr{A}, \mathbf{A})$, where $\mathscr{A}$ is a D-category and

$$\mathbf{A} : \mathscr{A} \to \mathsf{C}$$

is a (covariant) functor.

For a hybrid object $(\mathscr{A}, \mathbf{A})$ over C, the category C is called the *target category*, the functor $\mathbf{A}$ is called the *continuous component* of the hybrid object, and the category $\mathscr{A}$ is called its *discrete component.*

Notation 1.3. We denote the value of a functor $\mathbf{A} : \mathscr{A} \to \mathsf{C}$ on objects and morphisms of $\mathscr{A}$ by $\mathbf{A}_a$ and $\mathbf{A}_\alpha$, i.e., $\mathbf{A}_a = \mathbf{A}(a)$ and $\mathbf{A}_\alpha = \mathbf{A}(\alpha)$. This is done to notationally differentiate the "continuous" portion of a hybrid object from other functors.

Example 1.16. A (real) *hybrid vector space* is a hybrid object $(\mathscr{V}, \mathbf{V})$ over $\mathsf{Vect}_{\mathbb{R}}$, i.e.,

$$\mathbf{V} : \mathscr{V} \to \mathsf{Vect}_{\mathbb{R}}.$$

In particular, $\mathbf{V}_a$ is a vector space for every object a of $\mathscr{V}$ and $\mathbf{V}_\alpha : \mathbf{V}_a \to \mathbf{V}_b$ is a linear map for every $\alpha : a \to b$ in $\mathscr{V}$.

Having defined hybrid objects, there is a natural definition of morphisms between hybrid objects.

Definition 1.10. Let $(\mathscr{A},\mathbf{A})$ and $(\mathscr{B},\mathbf{B})$ be two hybrid objects over the category C. A **morphism of hybrid objects**, or just a **hybrid morphism**, is a pair

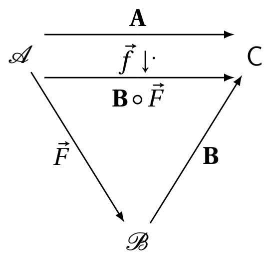
$$(\vec{F},\vec{f}) : (\mathscr{A},\mathbf{A}) \to (\mathscr{B},\mathbf{B}), \tag{1.7}$$

where $\vec{F} : \mathscr{A} \to \mathscr{B}$ is a morphism in Dcat and $\vec{f}$ is a natural transformation

$$\vec{f} : \mathbf{A} \stackrel{.}{\to} \mathbf{B} \circ \vec{F} \tag{1.8}$$

in $\mathsf{C}^{\mathscr{A}}$.

A morphism $(\vec{F},\vec{f}) : (\mathscr{A},\mathbf{A}) \to (\mathscr{B},\mathbf{B})$ of hybrid objects can be visualized in the following diagram:



and has, like a hybrid object, both a discrete and a continuous component, which justifies the term "hybrid morphism." The *discrete* component is given by the functor $\vec{F} : \mathscr{A} \to \mathscr{B}$, and the *continuous* component is given by the natural transformation $\vec{f}$.

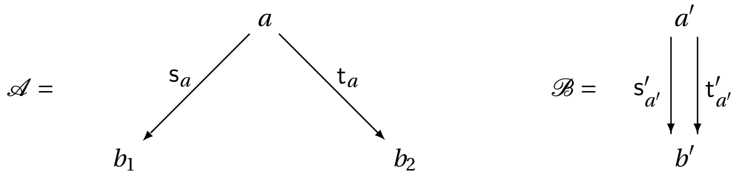
As morphisms of hybrid objects play a central role, we devote some energy to discussing their meaning. First, we introduce some notation and examples.

Notation 1.4. Often, hybrid objects are simply denoted by

$$\mathbf{A} : \mathscr{A} \to \mathsf{C}.$$

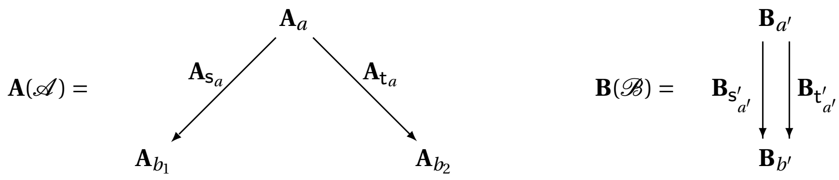
It is clear that the corresponding hybrid object is the pair $(\mathscr{A},\mathbf{A})$. We will often only be interested in a single hybrid object and its relation to hybrid objects with the same discrete structure, i.e., the same D-category. In this case, we will denote such a hybrid object by $\mathbf{A}$ and a morphism between it and another hybrid object, $\mathbf{B}$, by $\vec{f} : \mathbf{A} \stackrel{.}{\to} \mathbf{B}$; that is, $\mathbf{A}$ represents the hybrid object $(\mathscr{A},\mathbf{A})$, $\mathbf{B}$ represents the hybrid object $(\mathscr{A},\mathbf{B})$ and $\vec{f}$ represents the hybrid morphism $(\vec{\mathrm{Id}}_{\mathscr{A}},\vec{f})$, where $\vec{\mathrm{Id}}_{\mathscr{A}}$ is the identity functor (or the identity morphism of $\mathscr{A}$ in Dcat).

Example 1.17. Consider the D-categories $\mathscr{A}$ and $\mathscr{B}$ given by the following diagrams:

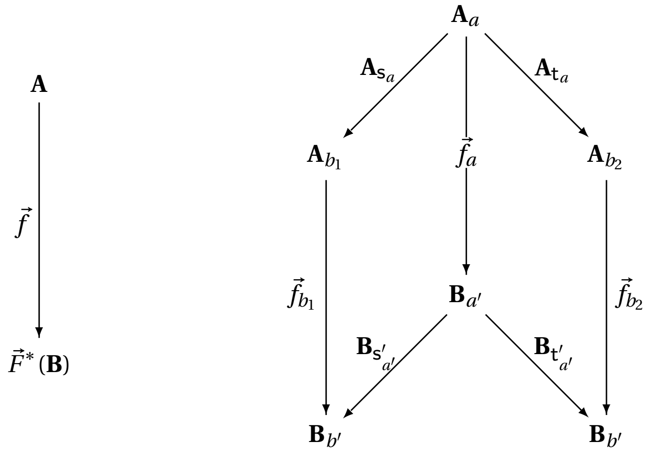


Let $\vec{F} : \mathscr{A} \to \mathscr{B}$ be the morphism of D-categories given in Example 1.11.

For $\mathbf{A} : \mathscr{A} \to \mathsf{C}$ and $\mathbf{B} : \mathscr{B} \to \mathsf{C}$, which can be visualized in the following diagrams:



A morphism $\vec{f} : \mathbf{A} \dot{\to} \vec{F}^*(\mathbf{B})$ in $\mathsf{C}^{\mathscr{A}}$ consists of three morphisms $\vec{f}_a$, $\vec{f}_{b_1}$ and $\vec{f}_{b_2}$ in C such that the following diagram

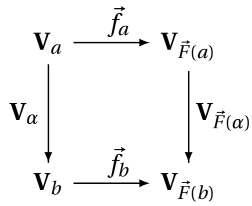


commutes. The end result is a morphism of hybrid objects: $(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$.

Example 1.18. For two hybrid vector spaces $(\mathscr{V}, \mathbf{V})$ and $(\mathscr{V}', \mathbf{V}')$, a hybrid morphism between these hybrid objects consists of a functor $\vec{F} : \mathscr{V} \to \mathscr{V}'$ between their discrete components and a *hybrid linear map*, i.e., natural transformation:

$$\vec{f} : \mathbf{V} \dot{\to} \mathbf{V}' \circ \vec{F}.$$

That is, for every $\alpha : a \to b$ in $\mathscr{V}$, there is a commuting diagram:



where $\vec{f}_a$ and $\vec{f}_b$ are linear maps.

Morphisms of hybrid objects can be defined in an equivalent and possible more enlightening way through the use of pullbacks of functors.

1.3.1 Pullbacks.

The *pullback* of a functor $\vec{F} : \mathscr{A} \to \mathscr{B}$ is a functor:

$$\vec{F}^* : \mathsf{C}^{\mathscr{B}} \to \mathsf{C}^{\mathscr{A}}$$

given on objects, i.e., functors $\mathbf{B} : \mathscr{B} \to \mathsf{C}$, and morphisms, i.e., natural transformations $\vec{g} : \mathbf{B} \stackrel{\cdot}{\to} \mathbf{B}'$, of $\mathsf{C}^{\mathscr{B}}$ by:

$$\vec{F}^*(\mathbf{B}) = \mathbf{B} \circ \vec{F}, \qquad \vec{F}^*(\vec{g}) = \vec{g} \circ \vec{F},$$

where $\vec{F}^*(\vec{g})$ is the natural transformation given on objects a of $\mathscr{A}$ by

$$(\vec{F}^*(\vec{g}))_a = \vec{g}_{\vec{F}(a)} : \mathbf{B}_{\vec{F}(a)} \to \mathbf{B}'_{\vec{F}(a)}.$$

This implies that for a morphism of hybrid objects (1.7),

$$\vec{f} : \mathbf{A} \stackrel{\cdot}{\to} \vec{F}^*(\mathbf{B}),$$

which is simply a reformulation of (1.8). This is the notation we will most frequently use.

1.3.2 Composing hybrid morphisms. Given two hybrid morphisms $(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ and $(\vec{G}, \vec{g}) : (\mathscr{B}, \mathbf{B}) \to (\mathscr{C}, \mathbf{C})$, the composite morphism is given by:

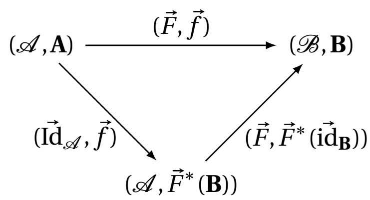
$$(\vec{G}, \vec{g}) \circ (\vec{F}, \vec{f}) := (\vec{G} \circ \vec{F}, \vec{F}^*(\vec{g}) \bullet \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{C}, \mathbf{C}).$$

Specifically, the composite morphism is just the standard composition of functors and objectwise composition of natural transformations, i.e.,

$$\vec{F}^*(\vec{g}) \bullet \vec{f} : \mathbf{A} \stackrel{\cdot}{\to} (\vec{G} \circ \vec{F})^*(\mathbf{C}) = \vec{F}^*(\vec{G}^*(\mathbf{C})),$$

in $\mathsf{C}^{\mathscr{A}}$ is defined objectwise by $(\vec{F}^*(\vec{g}) \bullet \vec{f})_a = \vec{F}^*(\vec{g})_a \circ \vec{f}_a = \vec{g}_{\vec{F}(a)} \circ \vec{f}_a$.

1.3.3 Decomposing hybrid morphisms. Every morphism $(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ has a canonical factorization:



into its continuous and discrete component.

1.3.4 Categories of hybrid objects. Utilizing hybrid objects and hybrid morphisms, we have the following:

Definition 1.11. Let C be a category. The **category of hybrid objects over the category** C, denoted by $\mathsf{Hy}(\mathsf{C})$, has as

Objects: Hybrid objects over C, i.e., pairs $(\mathscr{A}, \mathbf{A})$, where $\mathbf{A} : \mathscr{A} \to \mathsf{C}$.
Morphisms: Morphisms of hybrid objects, i.e., pairs

$$(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B}),$$

where $\vec{F} : \mathscr{A} \to \mathscr{B}$ is a morphism in Dcat and $\vec{f} : \mathbf{A} \stackrel{\cdot}{\to} \vec{F}^*(\mathbf{B})$ is a morphism in $\mathsf{C}^{\mathscr{A}}$.

It is useful to know that the collection of morphisms between any two objects in $\mathsf{Hy}(\mathsf{C})$ form a set. This makes $\mathsf{Hy}(\mathsf{C})$ a category in the classical sense.

Lemma 1.4. *For any two objects* $(\mathscr{A},\mathbf{A})$ *and* $(\mathscr{B},\mathbf{B})$ *of* $\mathsf{Hy}(\mathsf{C})$*,* $\mathsf{Hom}_{\mathsf{Hy}(\mathsf{C})}((\mathscr{A},\mathbf{A}),(\mathscr{B},\mathbf{B}))$ *forms a set.*

Proof. Since $\mathsf{Hom}_{\mathsf{Cat}}(\mathscr{A},\mathscr{B})$ is a set and

$$\mathsf{Hom}_{\mathsf{Dcat}}(\mathscr{A},\mathscr{B}) \subseteq \mathsf{Hom}_{\mathsf{Cat}}(\mathscr{A},\mathscr{B}),$$

it follows that $\mathsf{Hom}_{\mathsf{Dcat}}(\mathscr{A},\mathscr{B})$ is a set. Moreover, $\mathsf{Hom}_{\mathsf{C}^{\mathscr{A}}}(\mathbf{A},\vec{F}^*(\mathbf{B}))$ form sets for any $\vec{F} \in \mathsf{Hom}_{\mathsf{Dcat}}(\mathscr{A},\mathscr{B})$. So

$$\mathsf{Hom}_{\mathsf{Hy}(\mathsf{C})}((\mathscr{A},\mathbf{A}),(\mathscr{B},\mathbf{B})) \subseteq \mathsf{Hom}_{\mathsf{Dcat}}(\mathscr{A},\mathscr{B}) \times \left(\bigcup_{\vec{F}\in \mathsf{Hom}_{\mathsf{Dcat}}(\mathscr{A},\mathscr{B})} \mathsf{Hom}_{\mathsf{C}^{\mathscr{A}}}(\mathbf{A},\vec{F}^*(\mathbf{B})) \right)$$

is a set. □

Example 1.19. We already have introduced the notion of a hybrid vector space. The collection of all hybrid vector spaces forms the category of hybrid vector spaces: $\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})$.

1.3.5 Left comma categories.

The notion of a hybrid object over a category has not yet appeared in the literature as it was originally formulated by the author. There is a notion that is "close" to this one, that of a *left comma category* (as introduced by Saunders Mac Lane [74], where is was referred to as a super comma category[4]), and it in fact supports the terminology "hybrid object over a category." We briefly discuss left comma categories, comparing and contrasting them to categories of hybrid objects.

Let C be a category. The left comma category $(\mathsf{Cat} \cdot\!\downarrow\!\cdot\, \mathsf{C})$ is a category with

Objects: Pairs (A, A), where A is a small category and $A : \mathsf{A} \to \mathsf{C}$ is a functor,
Morphisms: Pairs $(F, f) : (\mathsf{A}, A) \to (\mathsf{B}, B)$, where $F : \mathsf{A} \to \mathsf{B}$ is a functor and $f : B \circ F \stackrel{\bullet}{\to} A$ is a natural transformation.

If we consider the left comma category $(\mathsf{Dcat} \cdot\!\downarrow\!\cdot\, \mathsf{C})$, then this category has as

Objects: Pairs $(\mathscr{A},\mathbf{A})$ where $\mathscr{A}$ is a small category and $\mathbf{A} : \mathscr{A} \to \mathsf{C}$ is a functor,
Morphisms: Pairs $(\vec{F}, f) : (\mathscr{A},\mathbf{A}) \to (\mathscr{B},\mathbf{B})$ where $\vec{F} : \mathscr{A} \to \mathscr{B}$ is a functor and $f : \mathbf{B} \circ \vec{F} \stackrel{\bullet}{\to} \mathbf{A}$ is a natural transformation.

This is "almost" the category of hybrid objects over C, except the direction of the natural transformations are reversed, i.e.,

$$f : \mathbf{B} \circ \vec{F} \stackrel{\bullet}{\to} \mathbf{A} \text{ for } (\mathsf{Dcat} \cdot\!\downarrow\!\cdot\, \mathsf{C}) \qquad f : \mathbf{A} \stackrel{\bullet}{\to} \mathbf{B} \circ \vec{F} \text{ for } \mathsf{Hy}(\mathsf{C}).$$

This is a not-so-subtle difference that has important ramifications. We only note that the motivation for considering categories of hybrid objects rather than left comma categories can be seen when the notion of a trajectory is introduced in Section 2.3.

[4]We avoid the name "super comma category" since there is an entire area of "super" mathematics and we want to prevent the possibility of confusion.

1.3.6 Pushforwards. Consider a functor $G : \mathsf{C} \to \mathsf{D}$. For a D-category $\mathscr{A}$, this induces a functor:

$$G_* : \mathsf{C}^{\mathscr{A}} \to \mathsf{D}^{\mathscr{A}}$$

given on objects, i.e., functors $\mathbf{A} : \mathscr{A} \to \mathsf{C}$ by

$$G_*(\mathbf{A}) = G \circ \mathbf{A}.$$

This functor can be visualized as follows:

$$\begin{array}{ccc} \mathscr{A} & & \\ {\scriptstyle \mathbf{A}}\downarrow & \searrow^{G_*(\mathbf{A})} & \\ \mathsf{C} & \xrightarrow{G} & \mathsf{D} \end{array}$$

From a natural transformation $\vec{f} : \mathbf{A} \dot{\to} \mathbf{A}'$ in $\mathsf{C}^{\mathscr{A}}$, we obtain a natural transformation:

$$G_*(\vec{f}) : G_*(\mathbf{A}) \dot{\to} G_*(\mathbf{A}')$$

defined objectwise by: $G_*(\vec{f})_a := G(\vec{f_a}) : G(\mathbf{A}_a) \to G(\mathbf{A}'_a)$. The relationship between pushforwards and pullbacks is given as follows:

Lemma 1.5. *For $G : \mathsf{C} \to \mathsf{D}$ and $\vec{F} : \mathscr{A} \to \mathscr{B}$, the following diagram*

$$\begin{array}{ccc} \mathsf{C}^{\mathscr{B}} & \xrightarrow{G_*} & \mathsf{D}^{\mathscr{B}} \\ {\scriptstyle \vec{F}^*}\downarrow & & \downarrow{\scriptstyle \vec{F}^*} \\ \mathsf{C}^{\mathscr{A}} & \xrightarrow{G_*} & \mathsf{D}^{\mathscr{A}} \end{array} \tag{1.9}$$

commutes.

Proof. For $\mathbf{B} : \mathscr{B} \to \mathsf{C}$,

$$\vec{F}^*(G_*(\mathbf{B})) = \vec{F}^*(G \circ \mathbf{B}) = G \circ \mathbf{B} \circ \vec{F} = G \circ \vec{F}^*(\mathbf{B}) = G_*(\vec{F}^*(\mathbf{B})).$$

For $\vec{g} : \mathbf{B} \dot{\to} \mathbf{B}'$ in $\mathsf{C}^{\mathscr{B}}$, it is enough to check that the commutativity condition holds objectwise:

$$\vec{F}^*(G_*(\vec{g}))_a = G_*(\vec{g})_{\vec{F}(a)} = G(\vec{g}_{\vec{F}(a)}) = G(\vec{F}^*(\vec{g})_a) = G_*(\vec{F}^*(\vec{g}))_a.$$

□

1.3.7 Functors between categories of hybrid objects. Consider a functor $G : \mathsf{C} \to \mathsf{D}$ between two categories. Using the pushforward of the functor G, this induces a functor:

$$\mathsf{Hy}(G) : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Hy}(\mathsf{D})$$

between categories of hybrid objects. On objects $(\mathscr{A},\mathbf{A})$ and morphisms $(\vec{F},\vec{f}):(\mathscr{A},\mathbf{A})\to(\mathscr{B},\mathbf{B})$ of $\mathsf{Hy}(\mathsf{C})$, the functor $\mathsf{Hy}(G)$ is given by:

$$\begin{aligned}\mathsf{Hy}(G)(\mathscr{A},\mathbf{A}) &= (\mathscr{A},G_*(\mathbf{A})),\\ \mathsf{Hy}(G)(\vec{F},\vec{f}) &= (\vec{F},G_*(\vec{f})):(\mathscr{A},G_*(\mathbf{A}))\to(\mathscr{B},G_*(\mathbf{B})),\end{aligned}$$

where $G_*(\vec{f})$ is well-defined by (1.9).

Example 1.20. Recall from Example 1.6 that there is a forgetful functor $U:\mathsf{Vect}_{\mathbb{R}}\to\mathsf{Set}$. This induces a "forgetful" functor:

$$\mathsf{Hy}(U):\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})\to\mathsf{Hy}(\mathsf{Set})$$

between hybrid categories.

1.3.8 Elements of hybrid objects. Consider a set S. We can regard the elements of S as morphisms from the set with one point, $*$, to S:

$$S\cong\mathsf{Hom}_{\mathsf{Set}}(*,S),$$

where the isomorphism is given by sending an element $s\in S$ to $e_s:*\to S$ with $e_s(*):=s$, and a morphism $e:*\to S$ to $e(*)\in S$. This inspires the definition of elements of a hybrid object.

Suppose that $\mathbf{A}:\mathscr{A}\to\mathsf{C}$ is a hybrid object and C is a category such that there is a forgetful functor $U:\mathsf{C}\to\mathsf{Set}$. Define the elements of $(\mathscr{A},\mathbf{A})$ by[5]

$$\mathrm{Elem}_{\mathsf{Hy}(\mathsf{C})}(\mathscr{A},\mathbf{A}):=\mathsf{Hom}_{\mathsf{Set}^{\mathscr{A}}}(\Delta_{\mathscr{A}}(*),U_*(\mathbf{A})). \tag{1.10}$$

So the elements of a hybrid object are natural transformations $\vec{e}:\Delta_{\mathscr{A}}(*)\stackrel{\cdot}{\to}U_*(\mathbf{A})$, i.e., morphisms:

$$(\vec{\mathrm{Id}}_{\mathscr{A}},\vec{e}):(\mathscr{A},\Delta_{\mathscr{A}}(*))\to(\mathscr{A},U_*(\mathbf{A}))=\mathsf{Hy}(U)(\mathscr{A},\mathbf{A}),$$

in $\mathsf{Hy}(\mathsf{Set})$. In particular, for $\vec{e}\in\mathrm{Elem}_{\mathsf{Hy}(\mathsf{C})}(\mathscr{A},\mathbf{A})$, the following diagram must commute:

$$\begin{array}{ccc} * & \xrightarrow{\vec{e}_a} & U(\mathbf{A}_a)\\ {\scriptstyle \mathrm{id}_*}\downarrow & & \downarrow{\scriptstyle U(\mathbf{A}_\alpha)}\\ * & \xrightarrow{\vec{e}_b} & U(\mathbf{A}_b)\end{array}$$

for all $\alpha:a\to b$ in $\mathscr{A}$.

By slight abuse of notation, we will identify $\vec{e}_a$ with $\vec{e}_a(*)\in\mathbf{A}_a$; with this notation, we write $\vec{e}\in(\mathscr{A},\mathbf{A})$. Note that $\vec{e}\in(\mathscr{A},\mathbf{A})$ must satisfy the following properties:

- $\vec{e}_a\in\mathbf{A}_a$ for all objects a of $\mathscr{A}$,
- $\vec{e}_b=\mathbf{A}_\alpha(\vec{e}_a)$ for all $\alpha:a\to b$ in $\mathscr{A}$.

[5] It follows that the elements of a hybrid object form a set, i.e., a hybrid object has a set of elements.

Note that $\vec{e}$ inherits the structure of the objects of C objectwise. Therefore, the elements of a hybrid object can be thought of as "vectors" in an abstract sense of the word; this is one of the motivations for the vector notation.

Lemma 1.6. *If* $\vec{f}:\mathbf{A}\dot{\to}\mathbf{A}'$ *and* $\vec{e}\in(\mathscr{A},\mathbf{A})$ *then* $\vec{f}(\vec{e})\in(\mathscr{A},\mathbf{A}')$ *where* $\vec{f}(\vec{e})_a := \vec{f}_a(\vec{e}_a)$.

Proof. It trivially follows that $\vec{f}(\vec{e})_a \in \mathbf{A}'_a$. The second condition follows from the fact that:

$$\mathbf{A}'_\alpha(\vec{f}(\vec{e})_a) = \mathbf{A}'_\alpha(\vec{f}_a(\vec{e}_a)) = \vec{f}_b(\mathbf{A}_\alpha(\vec{e}_a)) = \vec{f}_b(\vec{e}_b) = \vec{f}(\vec{e})_b.$$

□

Example 1.21. Let $(\mathscr{V},\mathbf{V})$ be a hybrid vector space. A *hybrid vector* is an element of this hybrid object:

$$\vec{v}\in(\mathscr{V},\mathbf{V}).$$

In particular, a hybrid vector must satisfy the following properties:

- $\vec{v}_a\in\mathbf{V}_a$ for all objects a of $\mathscr{V}$, i.e., $\vec{v}_a$ is a vector,
- $\vec{v}_b = \mathbf{V}_\alpha(\vec{v}_a)$ for all $\alpha : a\to b$ in $\mathscr{V}$.

In fact, we can justify the use of the term *hybrid vector* since we have two operations: *hybrid vector addition* and *hybrid scalar multiplication*. These are operations:

$$\begin{array}{rcl} \mathrm{Elem}_{\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})}(\mathscr{V},\mathbf{V})\times\mathrm{Elem}_{\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})}(\mathscr{V},\mathbf{V}) & \to & \mathrm{Elem}_{\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})}(\mathscr{V},\mathbf{V}) \\ (\vec{v},\vec{w}) & \mapsto & \vec{v}+\vec{w} \\ \mathbb{R}\times\mathrm{Elem}_{\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})}(\mathscr{V},\mathbf{V}) & \to & \mathrm{Elem}_{\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})}(\mathscr{V},\mathbf{V}) \\ (r,\vec{v}) & \mapsto & r\vec{v} \end{array}$$

defined objectwise by $(\vec{v}+\vec{w})_a := (\vec{v}_a+\vec{w}_a)$ and $(r\vec{v})_a := r\vec{v}_a$. It follows from the additivity and homogeneity of linear maps that $\vec{v}+\vec{w}$ and $r\vec{v}$ are again hybrid vectors.

More generally, the set of elements of a hybrid vector space, $\mathrm{Elem}_{\mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})}(\mathscr{V},\mathbf{V})$, is again a vector space. The vector addition identity element is given by $\vec{0}$ which is defined objectwise $\vec{0}_a = 0_{\mathbf{V}_a}$ where $0_{\mathbf{V}_a}$ is the vector addition identity element for $\mathbf{V}_a$. Similarly, there is a scalar multiplication identity element $\vec{1}$. The axioms of a vector space are easy to verify since they hold objectwise.

1.4 Cohybrid Objects

Thus far, we have only considered covariant functors. Contravariant functors will also arise naturally in a hybrid setting as they naturally arise in category theory. In order to deal with these functors in a systematic fashion, we introduce categories of cohybrid objects. First, we discuss:

1.4.1 Covariant functor categories. For a category C, we use ${}^{\mathscr{A}}\mathsf{C}$ to denote the category of *contravariant* functors $\mathbf{A} : \mathscr{A} \to \mathsf{C}$, with morphisms natural transformations between contravariant functors (see Paragraph 1.1.14). Given a morphism $\vec{F} : \mathscr{A} \to \mathscr{B}$ of D-categories, it induces a morphism of contravariant functor categories:

$$\vec{F}^* : {}^{\mathscr{B}}\mathsf{C} \to {}^{\mathscr{A}}\mathsf{C}.$$

Given a covariant functor $G : \mathsf{C} \to \mathsf{D}$, there is a corresponding covariant functor:

$$G_* : {}^{\mathscr{A}}\mathsf{C} \to {}^{\mathscr{A}}\mathsf{D}$$

for every D-category $\mathscr{A}$. Similarly, if G is contravariant, it induces functors:

$$G_* : {}^{\mathscr{A}}\mathsf{C} \to \mathsf{D}^{\mathscr{A}}, \qquad G_* : \mathsf{C}^{\mathscr{A}} \to {}^{\mathscr{A}}\mathsf{D}.$$

In all cases $\vec{F}^* \circ G_* = G_* \circ \vec{F}^*$.

Definition 1.12. Let C be a category. The **category of cohybrid objects over the category** C, denoted by $\mathsf{CoHy}(\mathsf{C})$, is given by:

Objects: *Cohybrid objects over* C, which are pairs $(\mathscr{A}, \mathbf{A})$ where $\mathscr{A}$ is a D-category and $\mathbf{A}$ is a *contravariant* functor $\mathbf{A} : \mathscr{A} \to \mathsf{C}$.
Morphisms: Pairs $(\vec{F}^{\mathsf{op}}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ where

- $\vec{F}^{\mathsf{op}} : \mathscr{A} \to \mathscr{B}$ is the morphism in $\mathsf{Dcat}^{\mathsf{op}}$ corresponding to the morphism $\vec{F} : \mathscr{B} \to \mathscr{A}$ in Dcat,
- $\vec{f} : \vec{F}^*(\mathbf{A}) \stackrel{\bullet}{\to} \mathbf{B}$ in ${}^{\mathscr{B}}\mathsf{C}$.

1.4.2 Relating categories of hybrid and cohybrid objects. The numerous morphisms in the different categories used to define categories of hybrid objects leave a lot of freedom for dualization. To better understand this, we relate categories of hybrid and cohybrid objects.

Specifying a contravariant functor $\mathbf{A} : \mathscr{A} \to \mathsf{C}$ is equivalent to specifying a covariant functor $\mathbf{A}^{\mathsf{op}} : \mathscr{A} \to \mathsf{C}^{\mathsf{op}}$ given by:

$$\mathbf{A}^{\mathsf{op}}\left(a \xrightarrow{\ \alpha\ } a'\right) = \mathbf{A}^{\mathsf{op}}_a = \mathbf{A}_a \xrightarrow{\mathbf{A}^{\mathsf{op}}_\alpha = (\mathbf{A}_\alpha)^{\mathsf{op}}} \mathbf{A}^{\mathsf{op}}_{a'} = \mathbf{A}_{a'}.$$

Therefore, the objects of $\mathsf{CoHy}(\mathsf{C})$ are in bijective correspondence with the objects of $\mathsf{Hy}(\mathsf{C}^{\mathsf{op}})$ and hence the objects of $\mathsf{Hy}(\mathsf{C}^{\mathsf{op}})^{\mathsf{op}}$.

For a morphism $(\vec{F}^{\mathsf{op}}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ of cohybrid objects, $\vec{f} : \vec{F}^*(\mathbf{A}) \stackrel{\bullet}{\to} \mathbf{B}$ in ${}^{\mathscr{B}}\mathsf{C}$. That is, for every $\beta : b \to b'$ in $\mathscr{B}$, there is a corresponding commuting diagram:

$$\begin{array}{ccc}
\mathbf{A}_{\vec{F}(b)} & \xrightarrow{\ \vec{f}_b\ } & \mathbf{B}_b \\
\Big\uparrow{\scriptstyle \mathbf{A}_{\vec{F}(\beta)}} & & \Big\uparrow{\scriptstyle \mathbf{B}_\beta} \\
\mathbf{A}_{\vec{F}(b')} & \xrightarrow{\ \vec{f}_{b'}\ } & \mathbf{B}_{b'}
\end{array}$$

in C. Taking opposites yields a commuting diagram:

$$\begin{array}{ccc} \mathbf{A}^{\mathsf{op}}_{\vec{F}(b)} & \xleftarrow{\vec{f}^{\mathsf{op}}_b} & \mathbf{B}^{\mathsf{op}}_b \\ {\scriptstyle \mathbf{A}^{\mathsf{op}}_{\vec{F}(\beta)}}\downarrow & & \downarrow{\scriptstyle \mathbf{B}^{\mathsf{op}}_{\beta}} \\ \mathbf{A}^{\mathsf{op}}_{\vec{F}(b')} & \xleftarrow{\vec{f}^{\mathsf{op}}_{b'}} & \mathbf{B}^{\mathsf{op}}_{b'} \end{array}$$

in C^{op}. Therefore, $\vec{f}^{\mathsf{op}} : \mathbf{B}^{\mathsf{op}} \dot{\to} \vec{F}^*(\mathbf{A}^{\mathsf{op}})$ in $\mathsf{C}^{\mathscr{B}}$ and associated to $(\vec{F}^{\mathsf{op}}, \vec{f})$ in $\mathsf{CoHy}(\mathsf{C})$ is a morphism

$$(\vec{F}, \vec{f}^{\mathsf{op}}) : (\mathscr{B}, \mathbf{B}^{\mathsf{op}}) \to (\mathscr{A}, \mathbf{A}^{\mathsf{op}})$$

in $\mathsf{Hy}(\mathsf{C}^{\mathsf{op}})$. Since the direction of $(\vec{F}^{\mathsf{op}}, \vec{f})$ and $(\vec{F}, \vec{f}^{\mathsf{op}})$ are opposite to one another, we conclude that:

Proposition 1.1. *For every category* C*,*

$$\mathsf{CoHy}(\mathsf{C}) \cong \mathsf{Hy}(\mathsf{C}^{\mathsf{op}})^{\mathsf{op}}.$$

The motivation for considering categories of cohybrid objects is that they arise naturally in the context of contravariant functors. That is, the functors induced from contravariant functors are functors between categories of hybrid objects and categories of cohybrid objects.

1.4.3 Contravariant functors and categories of hybrid objects. Let $G : \mathsf{C} \to \mathsf{D}$ be a contravariant functor between two categories. This induces a contravariant functor:

$$\mathsf{Hy}(G) : \mathsf{Hy}(\mathsf{C}) \to \mathsf{CoHy}(\mathsf{D}).$$

This functor is given on objects $(\mathscr{A}, \mathbf{A})$ and morphisms $(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ of $\mathsf{Hy}(\mathsf{C})$ by:

$$\begin{array}{rcl} \mathsf{Hy}(G)(\mathscr{A}, \mathbf{A}) & := & (\mathscr{A}, G_*(\mathbf{A})), \\ \mathsf{Hy}(G)(\vec{F}, \vec{f}) & := & (\vec{F}^{\mathsf{op}}, G_*(\vec{f})) : (\mathscr{B}, G_*(\mathbf{B})) \to (\mathscr{A}, G_*(\mathbf{A})), \end{array}$$

where $\vec{F}^{\mathsf{op}} : \mathscr{B} \to \mathscr{A}$ is the morphism in $\mathsf{Dcat}^{\mathsf{op}}$ corresponding to the morphism $\vec{F} : \mathscr{A} \to \mathscr{B}$ in Dcat.

Similarly, there is a contravariant functor:

$$\mathsf{CoHy}(G) : \mathsf{CoHy}(\mathsf{C}) \to \mathsf{Hy}(\mathsf{D}),$$

defined in an analogous manner. Finally, if G is covariant then there is a covariant functor:

$$\mathsf{CoHy}(G) : \mathsf{CoHy}(\mathsf{C}) \to \mathsf{CoHy}(\mathsf{D})$$

defined in a manner analogous to the induced functor given in Paragraph 1.3.7.

Example 1.22. The functor that associates to a vector space its dual:

$$(\ -\ )^{\star} : \mathsf{Vect}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{R}}$$

induces a contravariant functor:

$$\mathsf{Hy}((\,-\,)^{\star}) : \mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}}) \to \mathsf{CoHy}(\mathsf{Vect}_{\mathbb{R}}).$$

In particular, for a hybrid vector space $(\mathscr{V}, \mathbf{V})$, the corresponding cohybrid object

$$\mathsf{Hy}((\,-\,)^{\star})(\mathscr{V}, \mathbf{V}) := (\mathscr{V}, \mathbf{V}^{\star})$$

is the *dual hybrid vector space* to the hybrid vector space $(\mathscr{V}, \mathbf{V})$ and so $\mathbf{V}^{\star}$ is a contravariant functor:

$$\mathbf{V}^{\star} : \mathscr{V} \to \mathsf{Vect}_{\mathbb{R}}.$$

This motivates the terminology "cohybrid object."

1.4.4 Elements of cohybrid objects. Suppose that $\mathbf{A} : \mathscr{A} \to \mathsf{C}$ is a cohybrid object, i.e., $\mathbf{A}$ is a contravariant functor, and C is a category such that there is a forgetful functor $U : \mathsf{C} \to \mathsf{Set}$. This yields a covariant functor $\mathsf{CoHy}(U) : \mathsf{CoHy}(\mathsf{C}) \to \mathsf{CoHy}(\mathsf{Set})$ where $\mathsf{CoHy}(U)(\mathscr{A}, \mathbf{A}) := (\mathscr{A}, U_*(\mathbf{A}))$.

As with hybrid objects, define the elements of $(\mathscr{A}, \mathbf{A})$ by

$$\mathrm{Elem}_{\mathsf{CoHy}(\mathsf{C})}(\mathscr{A}, \mathbf{A}) := \mathsf{Hom}_{{}^{\mathscr{A}}\mathsf{Set}}(\Delta^{\mathsf{op}}_{\mathscr{A}}(*), U_*(\mathbf{A})), \tag{1.11}$$

where $\Delta^{\mathsf{op}}_{\mathscr{A}}$ is the contravariant constant functor defined in the obvious manner. Therefore, elements of a cohybrid object are natural transformations $\vec{\omega} : \Delta^{\mathsf{op}}_{\mathscr{A}}(*) \xrightarrow{\cdot} U_*(\mathbf{A})$ in ${}^{\mathscr{A}}\mathsf{Set}$, i.e., we have a commuting diagram:

$$\begin{array}{ccc} * & \xrightarrow{\vec{\omega}_a} & U(\mathbf{A}_a) \\ {\scriptstyle \mathrm{id}_*}\uparrow & & \uparrow {\scriptstyle U(\mathbf{A}_\alpha)} \\ * & \xrightarrow{\vec{\omega}_b} & U(\mathbf{A}_b) \end{array}$$

for every $\alpha : a \to b$ in $\mathscr{A}$. Therefore, again identifying $\vec{\omega}_a$ with $\vec{\omega}_a(*) \in \mathbf{A}_a$ and writing $\vec{\omega} \in (\mathscr{A}, \mathbf{A})$ in this case, an element $\vec{\omega} \in (\mathscr{A}, \mathbf{A})$ must satisfy the following properties:

- $\vec{\omega}_a \in \mathbf{A}_a$ for all objects a of $\mathscr{A}$,
- $\mathbf{A}_\alpha(\vec{\omega}_b) = \vec{\omega}_a$ for all $\alpha : a \to b$ in $\mathscr{A}$.

Again, $\vec{\omega}$ inherits the structure of the objects of C objectwise.

Example 1.23. For the dual hybrid vector space $(\mathscr{V}, \mathbf{V}^{\star})$ to the hybrid vector space $(\mathscr{V}, \mathbf{V})$, an element of this cohybrid object:

$$\vec{\omega} \in (\mathscr{V}, \mathbf{V}^{\star}),$$

is a *hybrid covector*.

In particular, it must satisfy the conditions:

- $\vec{\omega}_a \in \mathbf{V}_a^\star$ for all objects a of $\mathscr{V}$, i.e., $\vec{\omega}_a : \mathbf{V}_a \to \mathbb{R}$ is a covector,
- $\mathbf{V}_\alpha^\star(\vec{\omega}_b) = \vec{\omega}_a$, i.e., $\vec{\omega}_b \circ \mathbf{V}_\alpha = \vec{\omega}_a$, for all $\alpha : a \to b$ in $\mathscr{V}$.

This implies that, for the covariant functor $\Delta_{\mathscr{V}}(\mathbb{R}) : \mathscr{V} \to \mathsf{Vect}_{\mathbb{R}}$, a hybrid covector corresponds to a natural transformation:

$$\vec{\omega} : \mathbf{V} \xrightarrow{\cdot} \Delta_{\mathscr{V}}(\mathbb{R}),$$

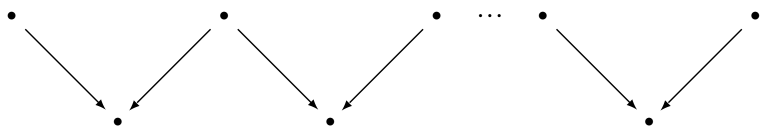
which is the hybrid analogue of the condition that, for a vector space V, a covector is a linear map

$$w : V \to \mathbb{R}.$$

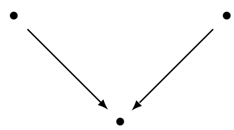
1.5 Network Objects

There is one more important "hybrid" object that will be important when considering networked systems, or networks of systems. These can be thought of as the dual to hybrid objects; their building block is not D-categories but D^{op}-categories. While these are just the opposite to D-categories, we devote some time to their development as they, along with the notion of a category of network objects, will be fundamental to Chapter 6.

1.5.1 D^{op}-categories. The opposite of a D-category, or a D^{op}*-category*, is given by reversing all of the arrows in a D-category. Therefore, a D^{op}-category has the general form:



with its basic atomic unit a diagram of the form:



and any other diagram in this category must be obtainable by gluing such atomic units along the *domain* of a morphism (and not the codomain, as was the case for D-categories).

Notation 1.5. In order to differentiate between D-categories and D^{op}-categories, we denote D^{op}-categories by $\mathfrak{N}$, i.e., $\mathfrak{N} = \mathscr{N}^{\mathrm{op}}$ for some D-category $\mathscr{N}$.

1.5.2 Oriented D^{op}-categories. Just as with D-categories, D^{op}-categories can be oriented. All of the formalisms are the same except that the arrows are reversed. In particular, let $\mathfrak{N}$ be a D^{op}-category such that $\mathfrak{N} = \mathscr{N}^{\mathrm{op}}$. Then

$$\mathsf{E}(\mathfrak{N}) = \mathsf{E}(\mathscr{N}), \qquad \mathsf{V}(\mathfrak{N}) = \mathsf{V}(\mathscr{N}),$$

since the objects in a category and its opposite category are the same. An *orientation* of a D^{op}-category is obtained from the orientation (s,t) of $\mathscr{N}$, i.e., it is given by $(\mathsf{s}^{\mathsf{op}},\mathsf{t}^{\mathsf{op}})$. That is, an orientation of a D^{op}-category is a pair of maps $(\mathsf{s}^{\mathsf{op}},\mathsf{t}^{\mathsf{op}})$ between sets:

$$\mathsf{E}(\mathfrak{N}) \underset{\mathsf{t}^{\mathsf{op}}}{\overset{\mathsf{s}^{\mathsf{op}}}{\rightrightarrows}} \mathsf{Mor}_{\not{\mathsf{id}}}(\mathfrak{N})$$

that fit into a diagram

$$\begin{array}{ccc} & & \mathsf{E}(\mathfrak{N}) \\ & \overset{\mathsf{id}}{\nearrow} & \uparrow \mathsf{cod} \\ \mathsf{E}(\mathfrak{N}) & \underset{\mathsf{t}^{\mathsf{op}}}{\overset{\mathsf{s}^{\mathsf{op}}}{\rightrightarrows}} & \mathsf{Mor}_{\not{\mathsf{id}}}(\mathfrak{N}) \\ & & \downarrow \mathsf{dom} \\ & & \mathsf{V}(\mathfrak{N}) \end{array}$$

in which the top triangle commutes. Therefore, for every $a \in \mathsf{E}(\mathfrak{N})$, there is a diagram in $\mathfrak{N}$:

$$\begin{array}{ccc} b = \mathsf{dom}(\mathsf{s}^{\mathsf{op}}_a) & & \mathsf{dom}(\mathsf{t}^{\mathsf{op}}_a) = c \\ & \overset{\mathsf{s}^{\mathsf{op}}_a}{\searrow} \quad \overset{\mathsf{t}^{\mathsf{op}}_a}{\swarrow} & \\ & \mathsf{cod}(\mathsf{s}^{\mathsf{op}}_a) = a = \mathsf{cod}(\mathsf{t}^{\mathsf{op}}_a) & \end{array}$$

where $b, c \in \mathsf{V}(\mathfrak{N})$.

Morphisms between D^{op}-categories are defined in a way analogous to morphisms between D-categories (see 1.2.2). In particular, for $\vec{F} : \mathfrak{N} \to \mathfrak{M}$, for every diagram of the form:

$$\begin{array}{ccc} b & & c \\ & \overset{\mathsf{s}^{\mathsf{op}}_a}{\searrow} \quad \overset{\mathsf{t}^{\mathsf{op}}_a}{\swarrow} & \\ & a & \end{array}$$

in $\mathfrak{N}$, i.e., $a \in \mathsf{E}(\mathfrak{N})$ and $b, c \in \mathsf{V}(\mathfrak{N})$, there are corresponding diagrams:

$$\begin{array}{ccc} \vec{F}(b) & & \vec{F}(c) \\ & \overset{\vec{F}(\mathsf{s}^{\mathsf{op}}_a)}{\searrow} \quad \overset{\vec{F}(\mathsf{t}^{\mathsf{op}}_a)}{\swarrow} & \\ & \vec{F}(a) & \end{array}$$

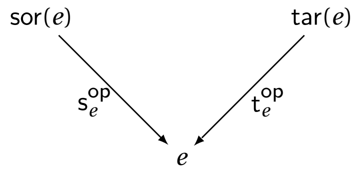
in $\mathfrak{M}$, where $\vec{F}(a) \in \mathsf{E}(\mathfrak{M})$ and $\vec{F}(b), \vec{F}(c) \in \mathsf{V}(\mathfrak{M})$.

We denote the category of D^{op}-categories by $\mathsf{D}^{\mathsf{op}}\mathsf{cat}$. Note that $\mathsf{D}^{\mathsf{op}}\mathsf{cat} \cong \mathsf{Dcat}$, where the isomorphism is given by sending $\mathfrak{N}$ to $\mathscr{N}$ with its inverse given by sending $\mathscr{N}$ to $\mathfrak{N}$.

1.5.3 Graphs and D^{op}-categories. Since $\mathsf{Dcat} \cong \mathsf{Grph}$ and $\mathsf{D}^{\mathsf{op}}\mathsf{cat} \cong \mathsf{Dcat}$, it follows that:

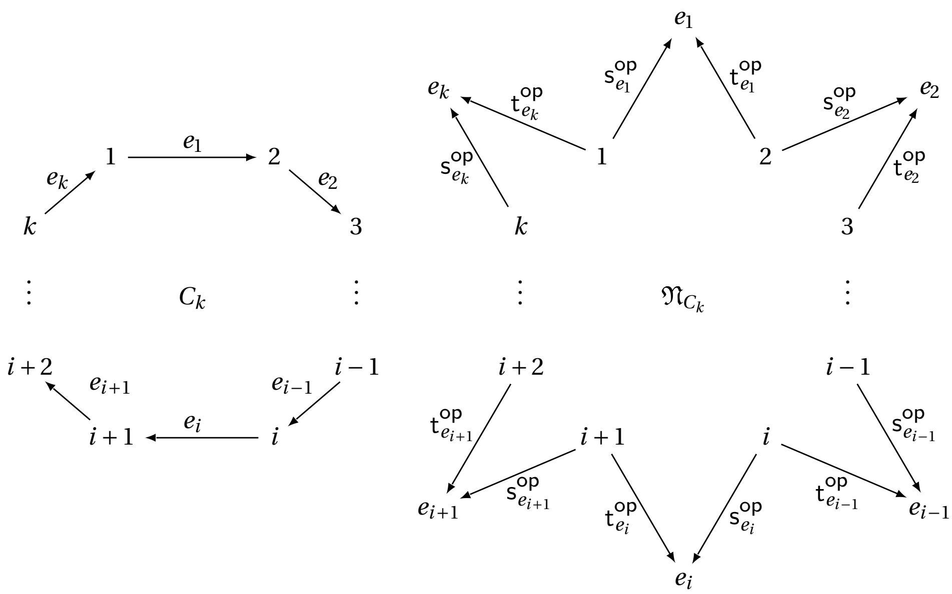
$$\mathsf{D}^{\mathsf{op}}\mathsf{cat} \cong \mathsf{Grph}.$$

For example, given a graph Γ, the corresponding D^{op}-category, $\mathfrak{N}_\Gamma$, is given by associating to every $e \in E$ a diagram of the form



in $\mathfrak{N}_\Gamma$. The identity morphisms must be added to each object in $\mathfrak{N}_\Gamma$ in order to complete the definition.

Example 1.24. The following diagram shows a directed cycle graph, $\Gamma = C_k$, and the associated D^{op}-category $\mathfrak{N}_{C_k}$:



1.5.4 Networks over a category. We now define the notion of a network over a category.

Definition 1.13. Let C be a category. The **category of networks over the category** C, denoted by $\mathsf{Net}(\mathsf{C})$, has as

Objects: *Networks over* C, which are pairs $(\mathfrak{N}, \mathbf{N})$ where $\mathfrak{N}$ is a D^{op}-category and $\mathbf{N}$ is a *covariant* functor $\mathbf{N} : \mathfrak{N} \to \mathsf{C}$.

Morphisms: Pairs $(\vec{F}, \vec{f}) : (\mathfrak{N}, \mathbf{N}) \to (\mathfrak{M}, \mathbf{M})$ where

- $\diamond$ $\vec{F} : \mathfrak{N} \to \mathfrak{M}$ is the morphism in $\mathsf{D}^{\mathsf{op}}\mathsf{cat}$,
- $\diamond$ $\vec{f} : \mathbf{N} \stackrel{\bullet}{\to} \vec{F}^*(\mathbf{M})$ in $\mathsf{C}^{\mathfrak{N}}$.

We refer the reader to Chapter 6 for examples of network objects. Constructions similar to the cases of categories of hybrid and cohybrid objects also can be introduced, although doing so would be repetitious.

Chapter 2

Hybrid Systems

Hybrid systems effectively describe systems with discrete and continuous behavior; as such, they are able to model a wide range of phenomena. With this expressiveness comes an increase in complexity; current models are difficult to manipulate as they consist of many different mathematical objects—a graph, domains, guards, reset maps and vector fields. This presents obvious difficulties in understanding and analyzing hybrid systems, and so indicates the need for a more coherent mathematical description of hybrid systems. We claim that the theory of hybrid objects can provide such a description.

We begin by introducing the "standard" model of a hybrid system. The first half of this chapter is devoted to transforming this standard model, in a constructive manner, into the framework of hybrid objects, i.e., it is demonstrated how hybrid systems can be viewed categorically. The end result is that a hybrid system can be represented by a triple:

$$(\mathscr{M}, \mathbf{M}, \mathbf{X}),$$

were $(\mathscr{M}, \mathbf{M})$ is a hybrid manifold, i.e., $\mathbf{M} : \mathscr{M} \to \mathsf{Man}$, and $\mathbf{X}$ is a collection of vector fields (indexed by $\mathsf{V}(\mathscr{M})$) on this hybrid manifold. Therefore, the categorical formulation of hybrid systems is in direct analogy with dynamical systems, i.e., pairs (M, X) where M is a manifold and X is a vector field on that manifold. This analogy is further extended by defining the category of hybrid systems utilizing the category of dynamical systems.

It is important to consider not only hybrid systems, but how hybrid systems *execute* or *evolve*. The latter half of this chapter is devoted to this topic. We again begin by introducing the "standard" notion of an execution for a hybrid system, and demonstrate in a constructive fashion how to obtain from this its categorical analogue. That is, we show that specifying an execution of a hybrid system is equivalent to specifying a morphism of categorical hybrid systems

$$(\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}, \mathbf{X}),$$

where $(\mathscr{I}, \mathbf{I}, \mathbf{d/dt})$ is a hybrid system which, roughly speaking, consists of a collection of "intervals" together with "unit clocks" on these intervals. This again nicely parallels the notion of a trajectory of a dynamical system, which is just a morphism of dynamical systems

$$(I, \mathrm{d}/\mathrm{dt}) \to (M, X),$$

with I an interval and d/dt a "unit clock."

The categorical formulation of hybrid systems and trajectories thereof is fundamental in understanding these systems in a broad mathematical context. For example, it is through this formulation of hybrid systems that we are able to generalize reduction to a hybrid setting and give sufficient conditions for the existence of Zeno behavior. Without these constructions, these results would have been difficult to derive. Aside from these results, the categorical formulation of hybrid systems is useful simply because of the manner in which it simplifies the representation of a hybrid system. We hope that the subsequent chapters of this dissertation will provide support for these claims.

Related work. There is a wealth of literature on hybrid systems. Instead of providing a complete review of the subject, we only note that the formulation of the "standard tuple" defining a hybrid systems introduced here is drawn from the series of papers [68, 69, 82, 83, 118, 119]; our definition differs slightly, but we think that it accurately describes a wide range of hybrid phenomena. Hybrid systems have also been studied in a categorical context, most notably in [56]. The need for a unifying mathematical framework in which to study hybrid systems was remarked upon in [103] and [104]; although the constructions in that work are different than ours, there are many philosophical similarities. Finally, concepts from geometry are used freely throughout this chapter; we refer the reader to [79] for more on the subject.

The categorical formulation of hybrid systems introduced here first appeared in [14], and has since appeared in [13, 18].

2.1 Hybrid Systems

This section formally introduces hybrid systems and the corresponding notion of a hybrid space. We characterize the relationship between hybrid spaces and hybrid manifolds. Utilizing hybrid manifolds, we define "categorical" hybrid systems and characterize the relationship between these systems and "classical" hybrid systems. We conclude by introducing the category of hybrid systems. All of these concepts are illustrated through a series of examples.

Definition 2.1. A **hybrid system** is a tuple:

$$\mathfrak{H} = (\Gamma, D, G, R, X),$$

where

- $\Gamma = (Q, E)$ is an oriented graph (possibly infinite).
- $D = \{D_i\}_{i \in Q}$ is a set of *domains* where D_i is a smooth manifold.

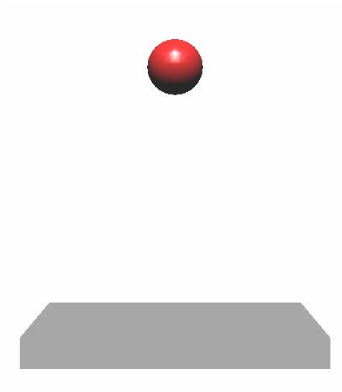


Figure 2.1: The bouncing ball.

- ⋄ $G = \{G_e\}_{e\in E}$ is a set of *guards*, where $G_e \subseteq D_{\mathsf{sor}(e)}$ is an embedded submanifold of $D_{\mathsf{sor}(e)}$.
- ⋄ $R = \{R_e\}_{e\in E}$ is a set of *reset maps*; these are smooth maps $R_e : G_e \to D_{\mathsf{tar}(e)}$.
- ⋄ $X = \{X_i\}_{i\in Q}$ is a collection of vector fields, i.e., X_i is a vector field on the manifold D_i.

2.1.1 Hybrid spaces. As with dynamical systems, it is sometimes desirable to consider the underlying "space" of a hybrid system. This amounts to "forgetting" about the vector field on each domain. More specifically, we can define a *(smooth) hybrid space* to be a tuple:

$$\mathbb{H} = (\Gamma, D, G, R).$$

It will be seen that hybrid spaces are just hybrid objects over the category of manifolds: hybrid manifolds.

Example 2.1. The quintessential example of a hybrid system is given by the one-dimensional bouncing ball; see Figure 2.1. While this system has, arguably, been over-studied, we will utilize it in order to illustrate non-trivial ideas in a trivial setting.

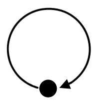
A ball bouncing in one-dimension is naturally modeled as a hybrid system:

$$\mathfrak{H}^{\mathrm{ball}} = (\Gamma^{\mathrm{ball}}, D^{\mathrm{ball}}, G^{\mathrm{ball}}, R^{\mathrm{ball}}, X^{\mathrm{ball}}).$$

That is, we consider a ball dropped from some positive height, say x_1, above a surface defined by $x_1 = 0$. Since the velocity of the ball will reset when it impacts the floor, the graph for this hybrid system is given by:

$$\Gamma^{\mathrm{ball}} = (Q^{\mathrm{ball}}, E^{\mathrm{ball}}), \qquad Q^{\mathrm{ball}} = \{1\}, \qquad E^{\mathrm{ball}} = \{e = (1,1)\}.$$

That is, by a graph of the form:



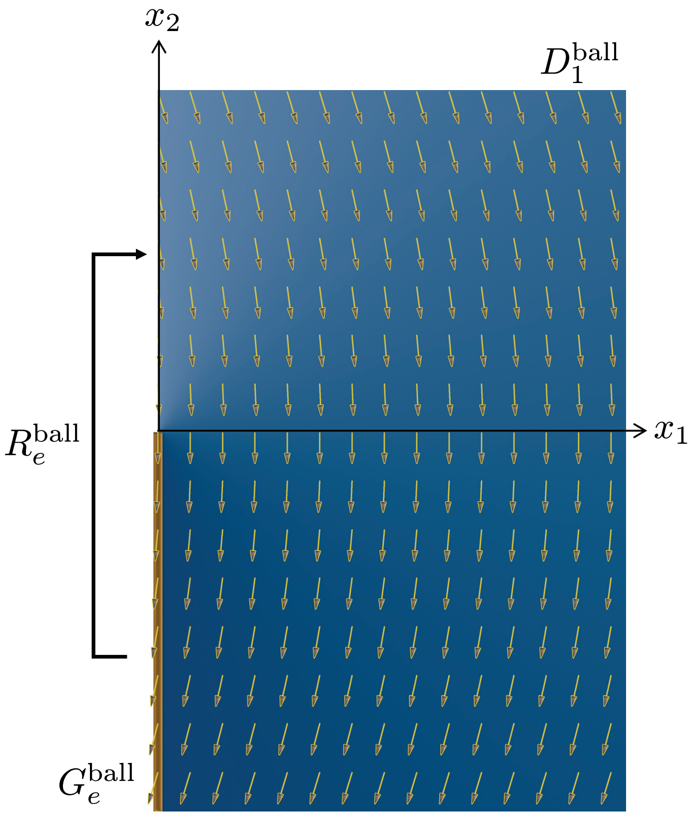


Figure 2.2: The hybrid model of a bouncing ball.

Since the phase space of the bouncing ball will consist of two variables, the position x_1 and velocity x_2, the domain for the hybrid system is given by:

$$D_1^{\text{ball}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \geq 0 \right\},$$

and $D^{\text{ball}} = \{D_1^{\text{ball}}\}$. The guard condition encodes the fact that a transition in the velocities of the system should occur when the position is zero and the velocity is "downward pointing." Therefore,

$$G_e^{\text{ball}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \leq 0 \right\},$$

and $G^{\text{ball}} = \{G_e^{\text{ball}}\}$. The reset map for the system is given by:

$$R_e^{\text{ball}}(x_1, x_2) = \begin{pmatrix} x_1 \\ -r x_2 \end{pmatrix},$$

where $0 \leq r \leq 1$ is the coefficient of restitution for the ball; this map encodes the fact that when the ball impacts the ground, its velocity is reversed and scaled down by the amount of energy lost through impact.

Finally, the vector field for this system is given by:

$$X_1^{\text{ball}}(x_1, x_2) = \begin{pmatrix} x_2 \\ -g \end{pmatrix},$$

where g is the acceleration due to gravity. A graphical representation of this system can be seen in Figure 2.2.

Example 2.2. The classical example of a system that models a physical system being controlled is the *two water tanks hybrid system*. This system models two tanks that are draining at rates v_1 and v_2 together with a spout that is inputting water into one of the two tanks at a rate of w. The control objective is to ensure that there is water in each tank at all times.

The hybrid system modeling the two water tanks is given by:

$$\mathfrak{H}^{\text{tank}} = (\Gamma^{\text{tank}}, D^{\text{tank}}, G^{\text{tank}}, R^{\text{tank}}, X^{\text{tank}}).$$

The graph for the hybrid system, Γ^{tank}, is given by:

$$1 \underset{e_2}{\overset{e_1}{\rightleftarrows}} 2.$$

To define the remaining portion of the hybrid system, we consider two state variables, x_1 and x_2, where x_1 is the level of water in tank 1 and x_2 is the level of water in tank 2. Since the goal is to keep the water in both tanks at all time, we are interested in domains for the hybrid system that capture the fact that the level of water in each tank must be greater than or equal to zero. Specifically, define $D^{\text{tank}} = \{D_1^{\text{tank}}, D_2^{\text{tank}}\}$ where

$$D_1^{\text{tank}} = D_2^{\text{tank}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 \geq 0 \right\}.$$

The guard sets for this system should capture the fact that the water spout will switch from one tank to the other tank if it detects that the other tank is empty. That is, if water is inflowing into tank 1 and the level of water in tank 2 decreases to 0, i.e., $x_2 = 0$, then the spout should start inflowing water into tank 2. This yields the first guard expression:

$$G_{e_1}^{\text{tank}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 = 0 \right\}.$$

Conversely, if water is inflowing into tank 2 and the system detects that $x_1 = 0$, the spout should switch from tank 2 to tank 1. This yields the second guard expression:

$$G_{e_2}^{\text{tank}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \geq 0 \right\},$$

and $G^{\text{tank}} = \{G_1^{\text{tank}}, G_2^{\text{tank}}\}$.

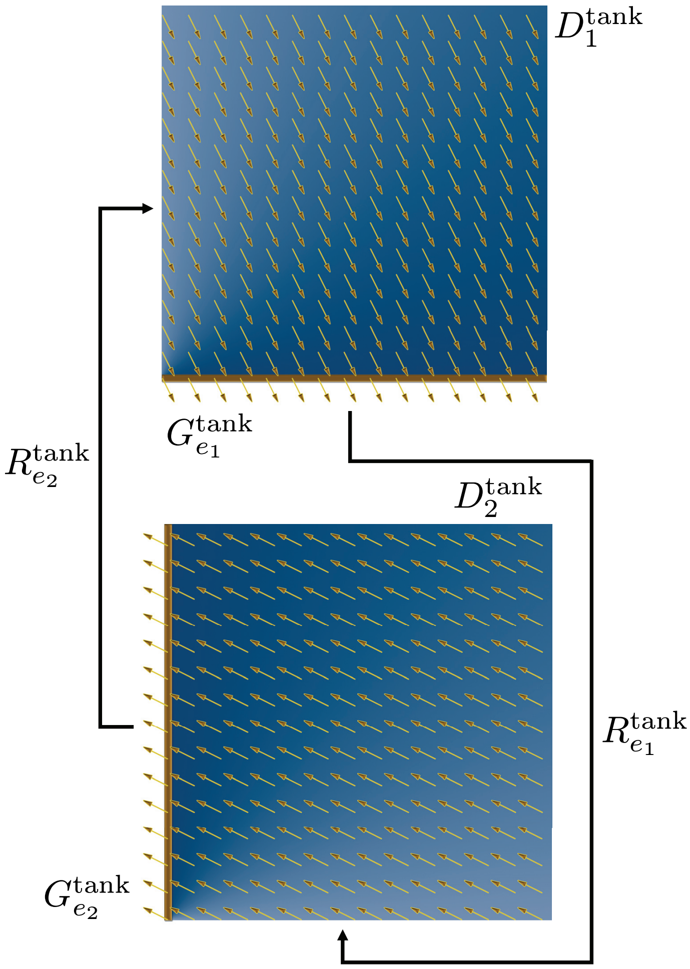


Figure 2.3: The hybrid model of a two water tank system.

It is assumed that the transition from one tank to the other is made in zero time, so the reset maps for the system are the identity:

$$R_{e_1}^{\text{tank}}(x_1, x_2) = R_{e_2}^{\text{tank}}(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Finally, if water is inflowing into the first tank at a rate w, and since water is flowing out of each tank at a rate of v_1 and v_2, the ODE governing the system is given by:

$$X_1^{\text{tank}}(x_1, x_2) = \begin{pmatrix} w - v_1 \\ -v_2 \end{pmatrix}$$

Similarly, if water is inflowing into the second tank, the resulting dynamics are given by:

$$X_2^{\text{tank}}(x_1, x_2) = \begin{pmatrix} -v_1 \\ w - v_2 \end{pmatrix}$$

Setting $X^{\text{tank}} = \{X_1^{\text{tank}}, X_2^{\text{tank}}\}$ completes the description of the hybrid system $\mathfrak{H}^{\text{tank}}$. This hybrid model can be seen in Figure 2.3

To make the behavior of the model interesting, we will assume that

$$\max\{v_1, v_2\} < w < v_1 + v_2.$$

That is, we assume that the inflow is greater than the outflow of each tank, and that the total outflow of the system is greater than the inflow to the system. This assumption results in *Zeno behavior*—this type of behavior will be discussed in detail in Chapter 5, where we will prove that the two water tanks hybrid system is Zeno (see Example 5.4).

2.1.2 Hybrid manifolds. A (smooth) *hybrid manifold* is given by a pair $(\mathscr{M}, \mathbf{M})$, where $\mathscr{M}$ is a D-category and $\mathbf{M}$ is a functor to the category of smooth manifolds, Man,

$$\mathbf{M} : \mathscr{M} \to \mathsf{Man}. \tag{2.1}$$

That is, a hybrid manifold is a hybrid object over the category of (smooth) manifolds.

In physical systems, it often is the case that for every $a \in \mathsf{E}(\mathscr{M})$, and hence every diagram

$$\mathsf{cod}(\mathsf{s}_a) \xleftarrow{\ \mathsf{s}_a\ } a \xrightarrow{\ \mathsf{t}_a\ } \mathsf{cod}(\mathsf{t}_a)$$

in $\mathscr{M}$, the corresponding diagram in Man is given by:

$$\mathbf{M}_{\mathsf{cod}(\mathsf{s}_a)} \xhookleftarrow{\ \mathbf{M}_{\mathsf{s}_a} = \iota\ } \mathbf{M}_a \xrightarrow{\ \mathbf{M}_{\mathsf{t}_a}\ } \mathbf{M}_{\mathsf{cod}(\mathsf{t}_a)} \tag{2.2}$$

where $\mathbf{M}_a \subseteq \mathbf{M}_{\mathsf{cod}(\mathsf{s}_a)}$ is an embedded submanifold and $\mathbf{M}_{\mathsf{s}_a} = \iota$ is the natural inclusion. We denote hybrid manifolds of this form by $\mathbf{M}^{\boldsymbol{\iota}}$.

Although we do not explicitly assume that $\mathbf{M}_{\mathsf{s}_a}$ is an inclusion, this often is the case, as the following proposition indicates.

Proposition 2.1. *There is a bijective correspondence:*

$$\{\text{Hybrid Spaces, } \mathbb{H} = (\Gamma, D, G, R)\} \leftrightarrow \{\text{Hybrid Manifolds, } (\mathscr{M}, \mathbf{M}^{\boldsymbol{\iota}})\}.$$

Proof. Given a hybrid space $\mathbb{H} = (\Gamma, D, G, R)$, we define the corresponding hybrid manifold by $\mathbf{M}^{(D,G,R)} : \mathscr{M}_\Gamma \to \mathsf{Man}$, where $\mathscr{M}_\Gamma = \text{dcat}(\Gamma)$ is the D-category obtained from the graph Γ and $\mathbf{M}^{(D,G,R)}$ is defined for every $e \in \mathsf{E}(\mathscr{M}_\Gamma) = E$ by

$$\mathbf{M}^{(D,G,R)}\left(\begin{array}{ccc} & e & \\ \swarrow^{\mathsf{s}_e} & & \searrow^{\mathsf{t}_e} \\ \mathsf{sor}(e) & & \mathsf{tar}(e) \end{array}\right) :=$$

$$\begin{array}{ccc} & \mathbf{M}^{(D,G,R)}_e := G_e & \\ \mathbf{M}^{(D,G,R)}_{\mathsf{s}_e} := \iota \;\swarrow & & \searrow\; \mathbf{M}^{(D,G,R)}_{\mathsf{t}_e} := R_e \\ \mathbf{M}^{(D,G,R)}_{\mathsf{sor}(e)} := D_{\mathsf{sor}(e)} & & \mathbf{M}^{(D,G,R)}_{\mathsf{tar}(e)} := D_{\mathsf{tar}(e)} \end{array}$$

It is clear that $\mathbf{M}^{(D,G,R)}$ is a hybrid manifold.

Conversely, consider a hybrid manifold $\mathbf{M}^{\boldsymbol{\iota}} : \mathscr{M} \to \mathsf{Man}$. Let $\Gamma_{\mathscr{M}} = \mathrm{grph}(\mathscr{M}) = (Q_{\mathscr{M}}, E_{\mathscr{M}})$ be the graph obtained from the D-category $\mathscr{M}$. We define

$$\mathbb{H}_{(\mathscr{M},\mathbf{M}^{\boldsymbol{\iota}})} = (\Gamma_{\mathscr{M}}, D_{\mathbf{M}^{\boldsymbol{\iota}}}, G_{\mathbf{M}^{\boldsymbol{\iota}}}, R_{\mathbf{M}^{\boldsymbol{\iota}}}),$$

where

$$\begin{aligned} D_{\mathbf{M}^{\boldsymbol{\iota}}} &:= \{\mathbf{M}^{\boldsymbol{\iota}}_b\}_{b \in \mathsf{V}(\mathscr{M}) = Q_{\mathscr{M}}}, \\ G_{\mathbf{M}^{\boldsymbol{\iota}}} &:= \{\mathbf{M}^{\boldsymbol{\iota}}_a\}_{a \in \mathsf{E}(\mathscr{M}) = E_{\mathscr{M}}}, \\ R_{\mathbf{M}^{\boldsymbol{\iota}}} &:= \{\mathbf{M}^{\boldsymbol{\iota}}_{\mathsf{t}_a}\}_{a \in \mathsf{E}(\mathscr{M}) = E_{\mathscr{M}}}. \end{aligned}$$

□

Example 2.3. The hybrid space for the bouncing ball is given by:

$$\mathbb{H}^{\mathrm{ball}} = (\Gamma^{\mathrm{ball}}, D^{\mathrm{ball}}, G^{\mathrm{ball}}, R^{\mathrm{ball}}).$$

We will construct the associated hybrid object $(\mathscr{M}^{\mathrm{ball}}, \mathbf{M}^{\mathrm{ball}})$. The D-category associated with the graph Γ^{ball} is given by:

$$\mathscr{M}^{\mathrm{ball}} = \quad \begin{array}{c} a \\ \mathsf{s}_a \downarrow\;\downarrow \mathsf{t}_a \\ b \end{array}$$

together with the identity morphisms $\mathrm{id}_a : a \to a$ and $\mathrm{id}_b : b \to b$. The functor

$$\mathbf{M}^{\mathrm{ball}} : \mathscr{M}^{\mathrm{ball}} \to \mathsf{Man}$$

takes the following values:

$$\mathbf{M}^{\mathrm{ball}}\left(\begin{array}{c} a \\ \mathsf{s}_a \downarrow\;\downarrow \mathsf{t}_a \\ b \end{array}\right) = \quad \begin{array}{c} \mathbf{M}^{\mathrm{ball}}_a = G^{\mathrm{ball}}_e \\ \mathbf{M}^{\mathrm{ball}}_{\mathsf{s}_a} = \iota \downarrow\;\downarrow \mathbf{M}^{\mathrm{ball}}_{\mathsf{t}_a} = R^{\mathrm{ball}}_e \\ \mathbf{M}^{\mathrm{ball}}_b = D^{\mathrm{ball}}_1 \end{array}$$

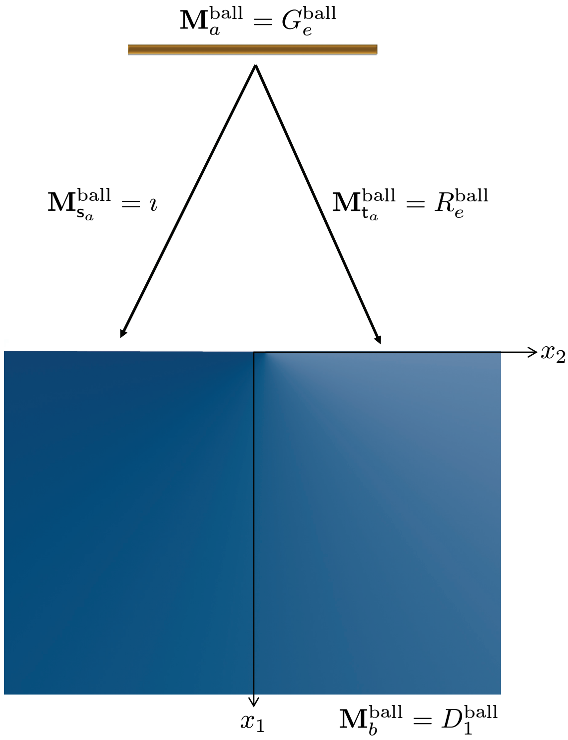


Figure 2.4: The hybrid manifold for the bouncing ball.

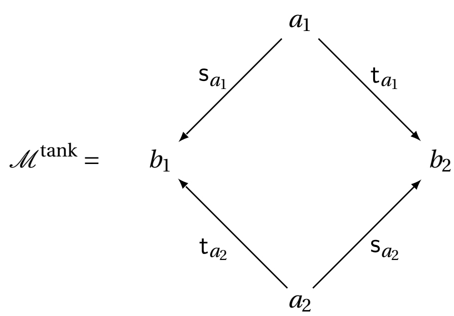
A graphical representation of this hybrid manifold can be seen in Figure 2.4

We now see the original motivation for considering D-categories; the edge sets of these categories serve the purpose of "pulling out the guard." The claim is that small categories of any other shape would not allow for the representation of hybrid systems as functors in such a clear fashion.

Example 2.4. The hybrid space for the water tank system is given by:

$$\mathbb{H}^{\text{tank}} = (\Gamma^{\text{tank}}, D^{\text{tank}}, G^{\text{tank}}, R^{\text{tank}}).$$

We will construct the associated hybrid object $(\mathscr{M}^{\text{tank}}, \mathbf{M}^{\text{tank}})$. The D-category associated with Γ^{tank} is given by



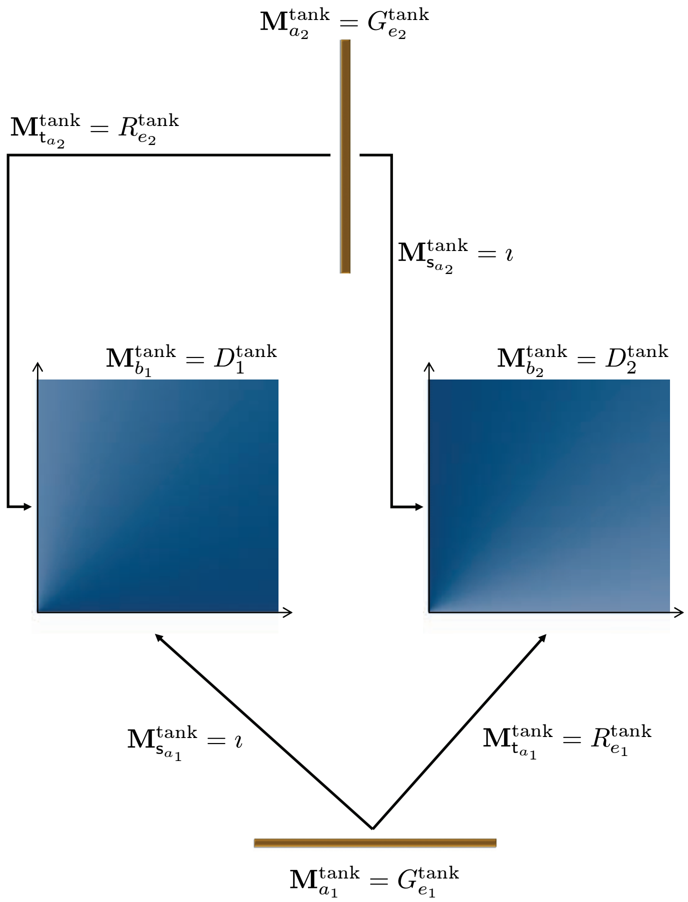


Figure 2.5: The hybrid manifold for the water tank system.

together with the identity morphisms on each object. The functor

$$\mathbf{M}^{\text{tank}} : \mathscr{M}^{\text{tank}} \to \mathsf{Man}$$

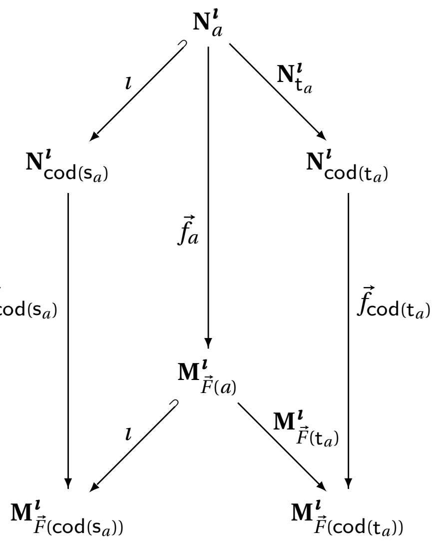
takes the following values:

$$\mathbf{M}^{\text{tank}}(\mathscr{M}^{\text{tank}}) = \begin{array}{ccc} & \mathbf{M}^{\text{tank}}_{a_1} = G^{\text{tank}}_{e_1} & \\ \mathbf{M}^{\text{tank}}_{\mathsf{s}_{a_1}} = \imath \swarrow & & \searrow \mathbf{M}^{\text{tank}}_{\mathsf{t}_{a_1}} = R^{\text{tank}}_{e_1} \\ \mathbf{M}^{\text{tank}}_{b_1} = D^{\text{tank}}_{1} & & \mathbf{M}^{\text{tank}}_{b_2} = D^{\text{tank}}_{2} \\ \mathbf{M}^{\text{tank}}_{\mathsf{t}_{a_2}} = R^{\text{tank}}_{e_2} \nwarrow & & \nearrow \mathbf{M}^{\text{tank}}_{\mathsf{s}_{a_2}} = \imath \\ & \mathbf{M}^{\text{tank}}_{a_2} = G^{\text{tank}}_{e_2} & \end{array}$$

which completes the description of $(\mathscr{M}^{\text{tank}}, \mathbf{M}^{\text{tank}})$. This hybrid manifold can be seen in Figure 2.5

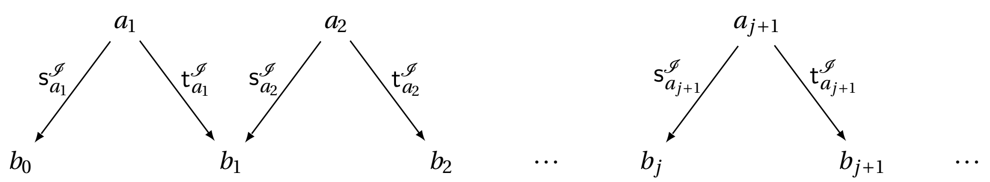
2.1.3 Morphisms of hybrid manifolds. Morphisms between hybrid manifolds are just the standard morphisms in the category of hybrid objects over Man. That is, for two hybrid manifolds $(\mathscr{N},\mathbf{N})$ and $(\mathscr{M},\mathbf{M})$, a morphism is just a pair $(\vec{F},\vec{f}):(\mathscr{N},\mathbf{N})\to(\mathscr{M},\mathbf{M})$ where $\vec{F}:\mathscr{N}\to\mathscr{M}$ and $\vec{f}:\mathbf{N}\dot{\to}\vec{F}^*(\mathbf{M})$. This defines the category of hybrid manifolds $\mathsf{Hy}(\mathsf{Man})$.

When considering hybrid manifolds of the form $\mathbf{N}^{\iota}:\mathscr{N}\to\mathsf{Man}$ and $\mathbf{M}^{\iota}:\mathscr{M}\to\mathsf{Man}$, the requirement that $\vec{f}:\mathbf{N}^{\iota}\dot{\to}\vec{F}^*(\mathbf{M}^{\iota})$ implies that, for every $a\in\mathsf{E}(\mathscr{N})$, there must be a commuting diagram:

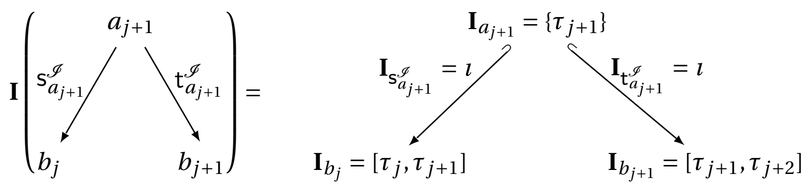


Using this, one could define the category of hybrid spaces utilizing the "classical" notation.

Example 2.5. To provide a non-trivial example of a morphism of hybrid manifolds, let us consider a *hybrid path* for the bouncing ball hybrid manifold $(\mathscr{M}^{\mathrm{ball}},\mathbf{M}^{\mathrm{ball}})$. One begins with a *hybrid interval* $(\mathscr{I},\mathbf{I})$ where $\mathbf{I}:\mathscr{I}\to\mathsf{Interval}(\mathsf{Man})$; the notion of a hybrid interval will be formally introduced in Section 2.3. For the time being, it suffices to let $\mathscr{I}$ be the D-category defined by the following (infinite) diagram



and $\mathbf{I}$ be a functor such that

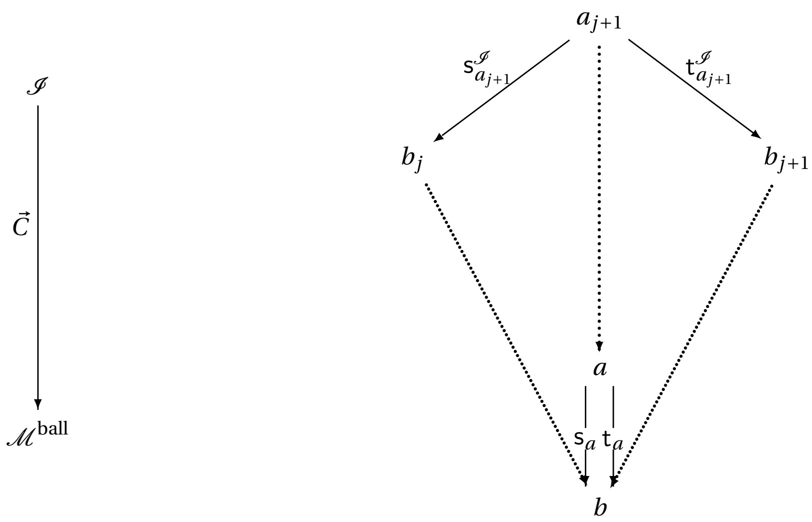


for all $j\in\mathbb{N}$ and some $\tau_j,\tau_{j+1},\tau_{j+2}\in\mathbb{R}$ such that $\tau_j\le\tau_{j+1}\le\tau_{j+2}$.

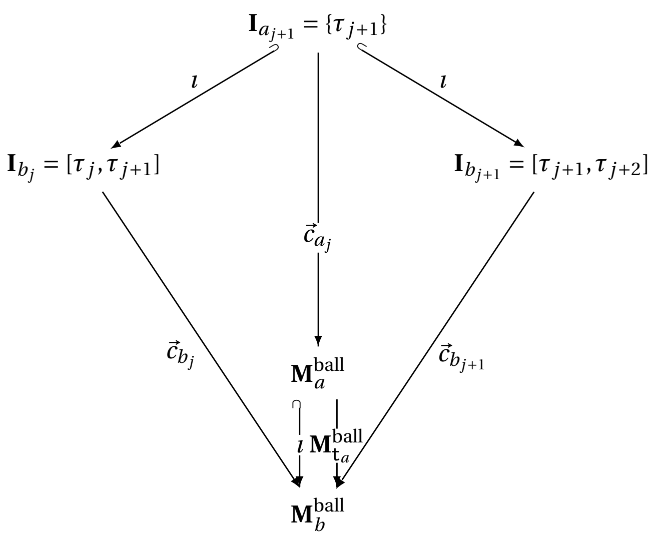
Now a hybrid path is a morphism of hybrid manifolds:

$$(\vec{C},\vec{c}):(\mathscr{I},\mathbf{I})\to(\mathscr{M}^{\mathrm{ball}},\mathbf{M}^{\mathrm{ball}}),$$

where $\vec{C} : \mathscr{I} \to \mathscr{M}^{\text{ball}}$ and $\vec{c} : \mathbf{I} \stackrel{\cdot}{\to} \vec{C}^*(\mathbf{M}^{\text{ball}})$. To better understand such a morphism, note that the conditions on morphisms of D-categories imply that $\vec{C}$ *must* take the following values on objects: $\vec{C}(a_j) \equiv a$ and $\vec{C}(b_j) \equiv b$, where in this case $\{a\} = \mathsf{E}(\mathscr{M}^{\text{ball}})$ and $\{b\} = \mathsf{V}(\mathscr{M}^{\text{ball}})$, and the symbol $\equiv$ denotes "identically equal to." In addition, it must be orientation preserving, i.e., if s and t are the orientation functions for $\mathscr{M}^{\text{ball}}$, the morphism $\vec{C}$ must take the following values on morphisms: $\vec{C}(\mathsf{s}^{\mathscr{I}}_{a_j}) \equiv \mathsf{s}_a$ and $\vec{C}(\mathsf{t}^{\mathscr{I}}_{a_j}) \equiv \mathsf{t}_a$. The morphism $\vec{C}$ can be visualized as follows:



The natural transformation $\vec{c}$ can be thought of as a collection of paths (in the "non-hybrid" sense of the word) on $\mathbf{M}^{\text{ball}}_b$ such that they satisfy certain "discrete" consistency conditions. Specifically, the natural transformation can be visualized in the following diagram



where the condition that this diagram commutes is equivalent to requiring that transitions occur when the guard is reached.

2.1.4 The category of dynamical systems. Let M be a manifold and let $X : M \to TM$ be a smooth vector field, i.e., a section of the tangent bundle. Objects of this form are termed *dynamical systems*. The category of dynamical systems, Dyn, has as

Objects: Dynamical systems, i.e., pairs (M, X), where M is a manifold and X is a vector field on that manifold.

Morphisms: Smooth maps of manifolds $f : N \to M$ such that the following diagram

$$\begin{array}{ccc} TN & \xrightarrow{Tf} & TM \\ Y \uparrow & & \uparrow X \\ N & \xrightarrow{f} & M \end{array} \tag{2.3}$$

commutes; here Tf is the pushforward of f.

Remark 2.1. One might be tempted to define a (categorical) hybrid dynamical system as an object over the category of dynamical systems, i.e., a functor: $\mathbf{D} : \mathscr{D} \to \mathsf{Dyn}$. While this is a hybrid system (in the classical sense), it is not an especially interesting one.

Definition 2.2. A **(categorical) hybrid system** is a tuple[1]

$$\mathfrak{H} = (\mathscr{M}, \mathbf{M}, \mathbf{X}),$$

where

- $(\mathscr{M}, \mathbf{M})$ is a hybrid manifold,
- $\mathbf{X} = \{\mathbf{X}_b\}_{b \in \bigvee(\mathscr{M})}$ is a *collection of vector fields* with $\mathbf{X}_b : \mathbf{M}_b \to T\mathbf{M}_b$ a smooth vector field on $\mathbf{M}_b$, i.e., $(\mathbf{M}_b, \mathbf{X}_b)$ is an object of Dyn for all $b \in \bigvee(\mathscr{M})$.

Remark 2.2. The categorical definition of a hybrid system nicely parallels the classical definition of a dynamical system. A dynamical system consists of a pair (M, X), where M is a manifold and X is a vector field on that manifold. Similarly, a hybrid system is a tuple $(\mathscr{M}, \mathbf{M}, \mathbf{X})$, where $(\mathscr{M}, \mathbf{M})$ is a hybrid manifold and $\mathbf{X}$ is a collection of vector fields "on" that hybrid manifold.

Proposition 2.2. *There is a bijective correspondence:*

$$\begin{gathered} \{\text{Classical Hybrid Systems}, \ (\Gamma, D, G, R, X)\} \\ \updownarrow \\ \{\text{Categorical Hybrid Systems}, \ (\mathscr{M}, \mathbf{M}^{\imath}, \mathbf{X})\}. \end{gathered}$$

Proof. Beginning with a classical hybrid system (Γ, D, G, R, X), from the hybrid space associated to this hybrid system, $\mathbb{H} = (\Gamma, D, G, R)$, we obtain a hybrid manifold $(\mathscr{M}_\Gamma, \mathbf{M}^{(D,G,R)})$. Define the collection of vector fields on this hybrid manifold by:

$$\mathbf{X}^X = \{X_i\}_{i \in Q = \bigvee(\mathscr{M}_\Gamma)}.$$

[1] We denote both "classical" and "categorical" hybrid systems by the symbol $\mathfrak{H}$; the reason for this will soon become transparent.

The tuple $(\mathscr{M}_\Gamma, \mathbf{M}^{(D,G,R)}, \mathbf{X}^X)$ is clearly a categorical hybrid system.

Conversely, for a categorical hybrid system $(\mathscr{M}, \mathbf{M}^{\iota}, \mathbf{X})$, from $(\mathscr{M}, \mathbf{M}^{\iota})$ we obtain a hybrid space:

$$\mathbb{H}_{(\mathscr{M},\mathbf{M}^{\iota})} = (\Gamma_{\mathscr{M}} = (Q_{\mathscr{M}}, E_{\mathscr{M}}), D_{\mathbf{M}^{\iota}}, G_{\mathbf{M}^{\iota}}, R_{\mathbf{M}^{\iota}}).$$

Defining

$$X_{\mathbf{X}} = \{\mathbf{X}_b\}_{b \in Q_{\mathscr{M}} = \bigvee(\mathscr{M})},$$

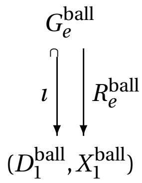
the end result is a classical hybrid system $(\Gamma_{\mathscr{M}}, D_{\mathbf{M}^{\iota}}, G_{\mathbf{M}^{\iota}}, R_{\mathbf{M}^{\iota}}, X_{\mathbf{X}})$. □

Example 2.6. From the bouncing ball hybrid system $\mathfrak{H}^{\text{ball}} = (\Gamma^{\text{ball}}, D^{\text{ball}}, G^{\text{ball}}, R^{\text{ball}}, X^{\text{ball}})$ introduced in Example 2.1, we obtain a categorical hybrid system:

$$(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}, \mathbf{X}^{\text{ball}}),$$

where $(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}})$ is the hybrid manifold associated to the hybrid space of the bouncing ball as introduced in Example 2.3 and $\mathbf{X}^{\text{ball}} = \{X_1^{\text{ball}}\}$. This categorical hybrid system can be visualized graphically using the original data defining the system $\mathfrak{H}^{\text{ball}}$ as follows[2]

$(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}, \mathbf{X}^{\text{ball}}) =$

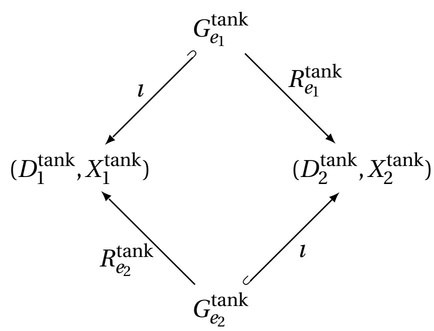


Example 2.7. For the water tank hybrid system $\mathfrak{H}^{\text{tank}} = (\Gamma^{\text{tank}}, D^{\text{tank}}, G^{\text{tank}}, R^{\text{tank}}, X^{\text{tank}})$ introduced in Example 2.2 we obtain a categorical hybrid system:

$$(\mathscr{M}^{\text{tank}}, \mathbf{M}^{\text{tank}}, \mathbf{X}^{\text{tank}}),$$

where $(\mathscr{M}^{\text{tank}}, \mathbf{M}^{\text{tank}})$ is the hybrid manifold associated to the hybrid space of the bouncing ball as introduced in Example 2.4, and $\mathbf{X}^{\text{tank}} = \{X_b^{\text{tank}}\}_{b \in \bigvee(\mathscr{M}^{\text{tank}}) = Q^{\text{tank}}} = \{X_1^{\text{tank}}, X_2^{\text{tank}}\}$. This categorical hybrid system can be visualized graphically using the original data defining the system $\mathfrak{H}^{\text{tank}}$ as follows

$(\mathscr{M}^{\text{tank}}, \mathbf{M}^{\text{tank}}, \mathbf{X}^{\text{tank}}) =$



[2] Note that this is not a diagram in a category, but rather a convenient way for representing the data defining a categorical hybrid system.

2.1.5 The category of hybrid systems. With the categorical formulation of hybrid systems, we can form the category of hybrid systems, HySys, with

Objects: Hybrid systems, $(\mathscr{M}, \mathbf{M}, \mathbf{X})$,
Morphisms: Pairs $(\vec{F}, \vec{f}) : (\mathscr{N}, \mathbf{N}, \mathbf{Y}) \to (\mathscr{M}, \mathbf{M}, \mathbf{X})$, where $(\vec{F}, \vec{f}) : (\mathscr{N}, \mathbf{N}) \to (\mathscr{M}, \mathbf{M})$ is a morphism in Hy(Man) such that there is a commuting diagram:

$$\begin{array}{ccc} T\mathbf{N}_b & \xrightarrow{T\vec{f}_b} & T\mathbf{M}_{\vec{F}(b)} \\ {\scriptstyle \mathbf{Y}_b}\uparrow & & \uparrow{\scriptstyle \mathbf{X}_{\vec{F}(b)}} \\ \mathbf{N}_b & \xrightarrow{\vec{f}_b} & \mathbf{M}_{\vec{F}(b)} \end{array}$$

for all $b \in \mathsf{V}(\mathscr{N})$. That is, for all $b \in \mathsf{V}(\mathscr{N})$, $\vec{f}_b : (\mathbf{N}_b, \mathbf{Y}_b) \to (\mathbf{M}_{\vec{F}(b)}, \mathbf{X}_{\vec{F}(b)})$ is a morphism in Dyn.

Remark 2.3. The definition of the category of hybrid systems again nicely parallels the definition of the category of dynamical systems. In the latter case, morphisms are morphisms of manifolds such that the vector fields on each manifold are f-related. Similarly, a morphism of hybrid systems is a morphism of hybrid manifolds such that the collections of vector fields are "hybrid $(\vec{F}, \vec{f})$-related."

2.2 Hybrid Intervals

This section begins with the introduction of the "standard" hybrid interval (much in the spirit of [68, 69, 82, 83, 118, 119]). We then associate to a hybrid interval its categorical counterpart, i.e., a categorical hybrid interval. In order to do so, we introduce *interval subcategories* of some categories of interest:

$$\mathsf{Grph}, \qquad \mathsf{Dcat}, \qquad \mathsf{Man}, \qquad \mathsf{Hy(Man)}. \tag{2.4}$$

These interval subcategories can be thought of as a generalization of the standard interval (for example, in $\mathbb{R}$) and, as such, will be instrumental in defining trajectories of hybrid systems. For example, a trajectory of an object in any of the categories given in (2.4) is just a morphism from an object of the interval subcategory to this object.

2.2.1 Hybrid intervals. Hybrid intervals can be thought of as the "time domain" for trajectories of hybrid systems. A *hybrid interval* is a pair (Λ, I), where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is a finite or infinite indexing set,
- $I = \{I_i\}_{i \in \Lambda}$ where for each $i \in \Lambda$, I_i is defined as follows:

$$\begin{array}{lll} I_i = [\tau_i, \tau_{i+1}] & \text{if} & i, i+1 \in \Lambda \\ I_{N-1} = [\tau_{N-1}, \tau_N] \text{ or } [\tau_{N-1}, \tau_N) \text{ or } [\tau_{N-1}, \infty) & \text{if} & |\Lambda| = N, \ N \text{ finite.} \end{array}$$

Here, for all $i, i+1 \in \Lambda$, $\tau_i \le \tau_{i+1}$ with $\tau_i, \tau_{i+1} \in \mathbb{R}$, and $\tau_{N-1} \le \tau_N$ with $\tau_{N-1}, \tau_N \in \mathbb{R}$.

Remark 2.4. It is sometimes notationally convenient to utilize a set of *switching times* instead of intervals. Specifically, we can specify a hybrid interval equivalently as a pair $(\Lambda, \{\tau_i\}_{i\in\Lambda})$ such that $\tau_i \le \tau_{i+1}$. From this we obtain intervals $I_i = [\tau_i, \tau_{i+1}]$ when $i, i+1 \in \Lambda$. If $|\Lambda| = N$ is finite, we can specify a final interval I_{N-1} as above.

Example 2.8. To provide a simple example, let $\Lambda = \{0,1,2\}$. Associated to this indexing set there are three intervals, e.g.,

$$I_0 = [\tau_0, \tau_1], \qquad I_1 = [\tau_1, \tau_2], \qquad I_2 = [\tau_2, \infty).$$

Example 2.9. To provide a much more interesting example, let $\{ar^n\}_{n\in\mathbb{N}}$ be a geometric sequence. Then to this geometric sequence, we have an associated hybrid interval $(\mathbb{N}, I^{(a,r)})$, termed the *geometric hybrid interval*, where

$$I_i^{(a,r)} = [\tau_i^{(a,r)}, \tau_{i+1}^{(a,r)}],$$

with $\tau_0^{(a,r)} = 0$ and

$$\tau_{i+1}^{(a,r)} = \tau_i^{(a,r)} + ar^i = \sum_{n=0}^{i} ar^n.$$

Hybrid intervals of this form, as we will see, naturally arise in Zeno hybrid systems.

We would like to understand hybrid intervals categorically. In order to do so, we will introduce the notion of an interval subcategory of a category. We first introduce intervals in the category of graphs, which define intervals in the category of D-categories. Finally, we use intervals in the category of manifolds to define hybrid intervals categorically.

2.2.2 Intervals in Grph.

For a finite or infinite indexing set $\Lambda = \{0,1,2,\dots\} \subseteq \mathbb{N}$ we have an associated a graph $\Gamma_\Lambda = (Q_\Lambda, E_\Lambda)$, where $Q_\Lambda = \Lambda$ and E_Λ is the set of pairs $\eta_{j+1} = (j, j+1)$ such that $j, j+1 \in \Lambda$. Define Interval(Grph) as the full subcategory of Grph with objects graphs of this form, i.e., graphs obtained from indexing sets. That is, Interval(Grph) consists of graphs of the form:

$$0 \xrightarrow{\eta_1} 1 \xrightarrow{\eta_2} 2 \quad \cdots \quad j \xrightarrow{\eta_{j+1}} j+1 \quad \cdots \tag{2.5}$$

Example 2.10. For the simple indexing set $\Lambda = \{0,1,2\}$, Γ_Λ is the graph:

$$0 \xrightarrow{\eta_1} 1 \xrightarrow{\eta_2} 2.$$

For an indexing set $\Lambda = \mathbb{N}$, the graph is of the form given in (2.5).

2.2.3 Intervals in Dcat.

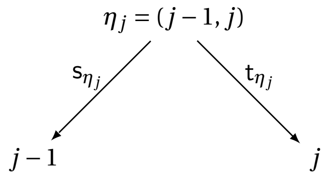
Intervals in D-categories are obtained from the intervals in Grph, i.e., using the isomorphism of categories given in Theorem 1.1, we define:

$$\mathsf{Interval(Dcat)} := \mathsf{dcat(Interval(Grph))}.$$

To be slightly more explicit, from a graph Γ_Λ obtained from a finite or infinite indexing set Λ we obtain a D-category:

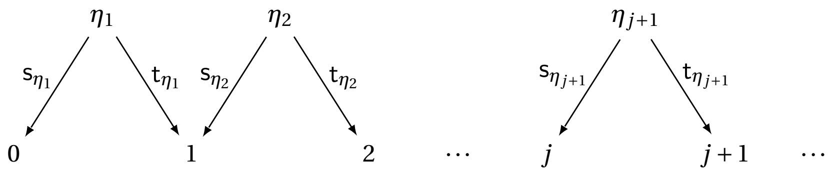
$$\mathscr{I}_\Lambda = \mathrm{dcat}(\Gamma_\Lambda),$$

which implies that $\mathsf{V}(\mathscr{I}_\Lambda) = Q_\Lambda = \Lambda$ and $\mathsf{E}(\mathscr{I}_\Lambda) = E_\Lambda$. Therefore, every diagram $\bullet \longleftarrow \bullet \longrightarrow \bullet$ in this D-category must have the form:



(2.6)

That is, the D-categories in $\mathsf{Interval(Dcat)}$ have the form:

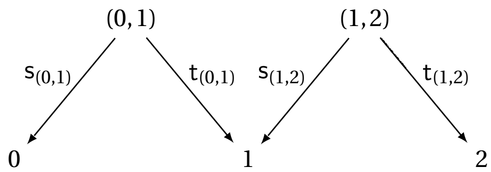


(2.7)

This particular D-category is the one obtained from the graph given in (2.5).

The category $\mathsf{Interval(Dcat)}$ is, therefore, the full subcategory of Dcat consisting of all D-categories obtained from graphs of this form.

Example 2.11. For the simple indexing set $\Lambda = \{0,1,2\}$, $\mathscr{I}_\Lambda$ is the D-category:



(2.8)

For an indexing set $\Lambda = \mathbb{N}$, the associated D-category is of the form given in (2.7).

2.2.4 Intervals in Man**.** To study trajectories (of both dynamical and hybrid systems), we must first understand intervals in the category Man. For $t, t' \in \mathbb{R} \cup \{\infty\}$, $t \le t'$, an interval in Man is given by any of the following sets:

$$I = [t,t'],\ (t,t'],\ [t,t'),\ (t,t'),\ \{t\}. \tag{2.9}$$

where $[t,t']$ is a manifold with boundary (and so is $(t,t']$ and $[t,t')$) and $\{t\}$ is a zero-dimensional manifold consisting of the single point t (which is trivially a smooth manifold).

We can form the full subcategory of Man, $\mathsf{Interval(Man)}$, with objects intervals, i.e., manifolds of the form (2.9), and morphisms smooth maps (note that any smooth map from a zero-dimensional manifold is automatically smooth).

Definition 2.3. An **interval in** Hy(Man) is a pair $(\mathscr{I},\mathbf{I})$, where

$$\mathbf{I}:\mathscr{I}\to \mathsf{Interval}(\mathsf{Man}),$$

which must satisfy:

- ◇ $\mathscr{I}$ is an object of Interval(Dcat).
- ◇ For all $i\in \mathsf{V}(\mathscr{I})$,

$$\begin{array}{lll} \mathbf{I}_i = [\tau_i,\tau_{i+1}] & \text{if} & i,i+1\in \mathsf{V}(\mathscr{I}) \\ \mathbf{I}_{N-1} = [\tau_{N-1},\tau_N] \text{ or } [\tau_{N-1},\tau_N) \text{ or } [\tau_{N-1},\infty) & \text{if} & |\mathsf{V}(\mathscr{I})| = N,\ N \text{ finite.} \end{array}$$

- ◇ For every $\eta\in \mathsf{E}(\mathscr{I})$, there is the associated diagram in $\mathscr{I}$:

$$\begin{array}{ccccc} & & \mathbf{I}_\eta = \mathbf{I}_{\mathsf{cod}(\mathsf{s}_\eta)}\cap \mathbf{I}_{\mathsf{cod}(\mathsf{t}_\eta)} & & \\ & \swarrow^{\mathbf{I}_{\mathsf{s}_\eta}=\iota} & & \searrow^{\mathbf{I}_{\mathsf{t}_\eta}=\iota} & \\ \mathbf{I}_{\mathsf{cod}(\mathsf{s}_\eta)} & & & & \mathbf{I}_{\mathsf{cod}(\mathsf{t}_\eta)}. \end{array}$$

2.2.5 Intervals in Hy(Man)**.** For every $\eta\in \mathsf{E}(\mathscr{I})$, $\eta=(i,i+1)$ for $i,i+1\in \mathsf{V}(\mathscr{I})$, Definition 2.3 implies that for every such edge there exist $\tau,\tau',\tau''\in\mathbb{R}$, with $\tau\le\tau'\le\tau''$, such that

$$\begin{array}{ccccc} & & \mathbf{I}_{(i,i+1)} = \{\tau'\} & & \\ & \swarrow^{\mathbf{I}_{\mathsf{s}_{(i,i+1)}}=\iota} & & \searrow^{\mathbf{I}_{\mathsf{t}_{(i,i+1)}}=\iota} & \\ \mathbf{I}_i = [\tau,\tau'] & & & & \mathbf{I}_{i+1} = [\tau',\tau''] \text{ or } [\tau',\tau'') \text{ or } [\tau',\infty). \end{array}$$

If $\mathscr{I}$ is the D-category in (2.7), an example of an interval in Man is given by $\mathbf{I}:\mathscr{I}\to\mathsf{Interval}(\mathsf{Man})$ where

$$\mathbf{I}(\mathscr{I}) = \begin{array}{ccccccccccc} & \{\tau_1\} & & \{\tau_2\} & & & & \{\tau_{j+1}\} & & \\ \swarrow & & \searrow\ \swarrow & & \searrow & & \swarrow & & \searrow & \\ [\tau_0,\tau_1] & & [\tau_1,\tau_2] & & [\tau_2,\tau_3] & \cdots & [\tau_j,\tau_{j+1}] & & [\tau_{j+1},\tau_{j+2}] & \cdots \end{array} \tag{2.10}$$

with $\tau_1,\tau_2,\ldots,\tau_{j+1},\ldots$ the set of switching times.

Let Interval(Hy(Man)) be the full subcategory of Hy(Man) with objects all intervals in Hy(Man). The importance of intervals in Hy(Man) is that to every hybrid interval (as introduced in Paragraph 2.2.1), we can associate an object of Interval(Hy(Man)), and vice versa.

Remark 2.5. One can now define paths as in Example 2.5. A *hybrid path* of a hybrid manifold $(\mathscr{M},\mathbf{M})$ is an object $(\mathscr{I},\mathbf{I})$ of Interval(Hy(Man)) together with a morphism of hybrid manifolds: $(\vec{C},\vec{c}):(\mathscr{I},\mathbf{I})\to(\mathscr{M},\mathbf{M})$. Paths can also be considered in Hy(Top) and other related categories of hybrid objects.

Proposition 2.3. *There is a bijective correspondence:*

$$\{\text{Hybrid Intervals, } (\Lambda, I)\} \leftrightarrow \{\text{Intervals in } \mathsf{Hy}(\mathsf{Man}),\ \mathbf{I}:\mathscr{I}\to\mathsf{Interval}(\mathsf{Man})\}.$$

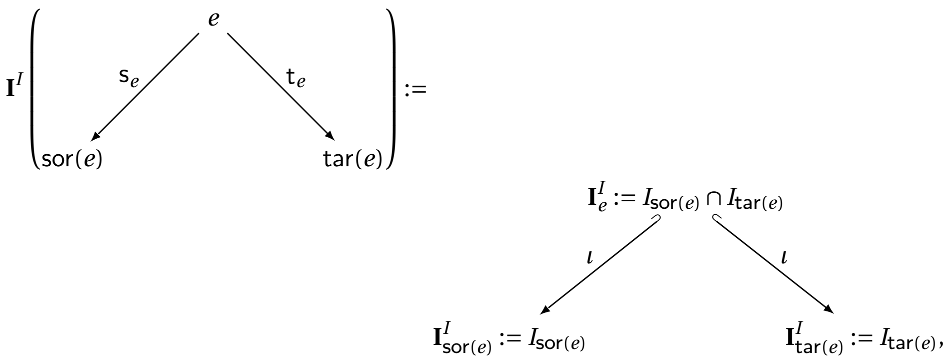
Proof. Consider a hybrid interval (Λ, I) with $I = \{I_i\}_{i\in\Lambda}$. We have the following associations:

$$\Lambda \quad \mapsto \quad \Gamma_\Lambda \in \mathsf{Ob}(\mathsf{Interval}(\mathsf{Grph})) \quad \mapsto \quad \mathscr{I}_\Lambda \in \mathsf{Ob}(\mathsf{Interval}(\mathsf{Dcat})).$$

Therefore, to the pair (Λ, I) we have the associated functor and D-category:

$$\mathbf{I}^I : \mathscr{I}_\Lambda \to \mathsf{Interval}(\mathsf{Man})$$

where



for every $e \in \mathsf{E}(\mathscr{I}_\Lambda) = E_\Lambda$.

Conversely, given an object $(\mathscr{I}, \mathbf{I})$ of $\mathsf{Interval}(\mathsf{Hy}(\mathsf{Man}))$, we have an associated hybrid interval:

$$(\mathsf{V}(\mathscr{I}), I^{\mathbf{I}} = \{\mathbf{I}_i\}_{i\in\mathsf{V}(\mathscr{I})}).$$

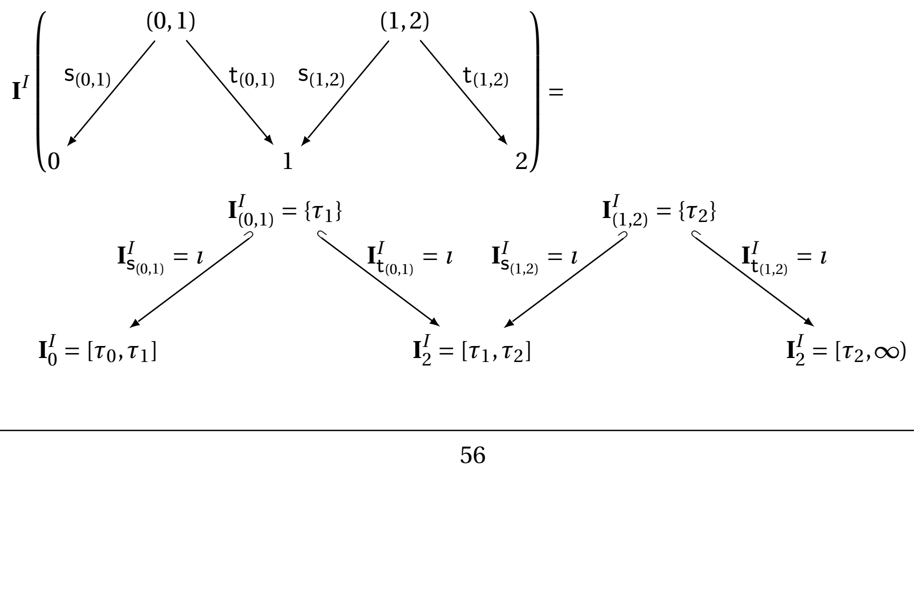
The definition of intervals in $\mathsf{Hy}(\mathsf{Man})$ imply that this is a hybrid interval. □

Notation 2.1. As a result of Proposition 2.3, we will refer to objects of $\mathsf{Interval}(\mathsf{Hy}(\mathsf{Man}))$ as categorical hybrid intervals or just hybrid intervals.

Example 2.12. For the simple hybrid interval (Λ, I) introduced in Example 2.8, the associated categorical interval is given by:

$$\mathbf{I}^I : \mathscr{I}_\Lambda \to \mathsf{Interval}(\mathsf{Man}),$$

where $\mathscr{I}_\Lambda$ is the D-category given in (2.8) and $\mathbf{I}^I$ is defined by:



Example 2.13. For the geometric hybrid interval $(\mathbb{N}, I^{(a,r)})$ introduced in Example 2.9, the associated categorical interval is given by:

$$\mathbf{I}^{(a,r)} : \mathscr{I}_{\mathbb{N}} \to \mathsf{Interval}(\mathsf{Man}),$$

where $\mathbf{I}^{(a,r)}$ takes the values indicated in the following diagram:

$$\mathbf{I}^{(a,r)}\left(\begin{array}{ccccccccc} & \eta_1 & & \eta_2 & & & \eta_{j+1} & & \\ \mathsf{s}_{\eta_1}\swarrow & & \searrow\mathsf{t}_{\eta_1}\ \mathsf{s}_{\eta_2}\swarrow & & \searrow\mathsf{t}_{\eta_2} & & \mathsf{s}_{\eta_{j+1}}\swarrow & & \searrow\mathsf{t}_{\eta_{j+1}} \\ 0 & & 1 & & 2 & \cdots & j & & j+1 \quad \cdots \end{array}\right) =$$

$$\begin{array}{ccccc} & \mathbf{I}^{(a,r)}_{\eta_1} = \{a\} & & \mathbf{I}^{(a,r)}_{\eta_2} = \{a(1+r)\} & \\ \iota\swarrow & & \searrow\iota \quad \iota\swarrow & & \searrow\iota \\ \mathbf{I}^{(a,r)}_0 = [0, a] & & \mathbf{I}^{(a,r)}_1 = [a, a(1+r)] & & \mathbf{I}^{(a,r)}_2 = [a(1+r), a(1+r+r^2)] \quad \cdots \end{array}$$

$$\begin{array}{ccc} & \mathbf{I}^{(a,r)}_{\eta_{j+1}} = \left\{\sum_{n=0}^{j} ar^n\right\} & \\ \iota\swarrow & & \searrow\iota \\ \mathbf{I}^{(a,r)}_j = \left[\sum_{n=0}^{j-1} ar^n, \sum_{n=0}^{j} ar^n\right] & & \mathbf{I}^{(a,r)}_{j+1} = \left[\sum_{n=0}^{j} ar^n, \sum_{n=0}^{j+1} ar^n\right] \quad \cdots \end{array}$$

2.3 Hybrid Trajectories

This goal of this section is to introduce trajectories of hybrid systems in the context of our categorical formulation of hybrid system. Utilizing (categorical) hybrid intervals it is a simple matter to define trajectories of hybrid systems. We define the interval subcategory of Dyn, which is used to construct the interval subcategory of HySys. A trajectory of a hybrid system is just a morphism from an object in the interval subcategory of HySys to the hybrid system. We conclude by demonstrating that this formulation is equivalent to the "standard" notion of an execution.

The constructions presented in this section are motivated by similar ideas that appeared in the study of bisimulation relations; see [56] and the references therein.

2.3.1 Intervals in Dyn**.** Trajectories of dynamical systems are morphisms in Dyn whose domain is an interval in Man together with a vector field that is a "clock". Specifically, let I be an interval in Man, M a smooth manifold, and X a vector field on that manifold. Consider a smooth map $c : I \to M$. This map is said to be a *trajectory* (or *flow* or *integral curve*) of X on M if

$$\dot{c}(t) = X(c(t)) \tag{2.11}$$

for $t \in I$. This is equivalent to requiring that the following diagram

$$\begin{array}{ccc} TI & \xrightarrow{Tc} & TM \\ \uparrow{\scriptstyle \frac{\mathrm{d}}{\mathrm{dt}}} & & \uparrow{\scriptstyle X} \\ I & \xrightarrow{c} & M \end{array} \tag{2.12}$$

commute. That is, for $t' \in I$,

$$\dot{c}(t') = T_{t'}c\left(\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t'}\right) \in T_{c(t')}M$$

So the commutativity of (2.12) enforces condition 2.11, or just $\dot{c} = X(c)$, which is the usual requirement on a trajectory.

Therefore, we define the category Interval(Dyn) to be the full subcategory of Dyn consisting of all objects of the form $(I, \mathrm{d}/\mathrm{dt})$ where I is an object in Interval(Man); here, the vector field d/dt can be thought of as a unit clock, i.e., the vector field $\dot{x} = 1$.

Definition 2.4. A **trajectory** of a dynamical system (M, X) is a morphism in Dyn:

$$c : (I, \mathrm{d}/\mathrm{dt}) \to (M, X),$$

where $(I, \mathrm{d}/\mathrm{dt})$ is an object of Interval(Dyn).

Some further explanation of trajectories of dynamical systems is needed because of the specific domains for these curves that we are considering; that is, I may be a closed interval or a point.

In the case when $I = [t_0, t_1]$ with $t_1 > t_0$, we view I as a manifold with boundary $\partial I = \{t_0, t_1\}$. We define $\dot{c}$ at these endpoints by:

$$\dot{c}(t_0) := \lim_{t \to t_0^+} \dot{c}(t), \qquad \dot{c}(t_1) := \lim_{t \to t_1^-} \dot{c}(t).$$

Similar definitions hold when $I = [t_0, t_1)$ and $I = (t_0, t_1]$. In either of these cases, we again consider the dynamical system $(I, \mathrm{d}/\mathrm{dt})$, and say that $c : I \to M$ is an integral curve of X if $c : (I, \mathrm{d}/\mathrm{dt}) \to (M, X)$ is a morphism in Dyn, i.e., if (2.12) commutes.

When $I = \{t_0\}$ *every map* $c : I = \{t_0\} \to M$ is an integral curve of X, and we define $\dot{c}(t_0) := X(c(t_0))$. There are ways to justify this by (for example, when $\dim M > 0$ and $c(t_0)$ is not on the boundary of M) considering an open interval containing t_0. By slight abuse of notation, we still write $(\{t_0\}, \mathrm{d}/\mathrm{dt})$ in this case; although, it should be understood that what is meant by this is (2.12) trivially commutes, since the diagram commuting amounts to evaluating X at $c(t_0)$.

Example 2.14. For the "dynamical system" portion of the bouncing ball hybrid system as introduced in Example 2.1, $(D_1^{\text{ball}}, X_1^{\text{ball}})$, for an interval $[t_0, t_1]$ and an initial condition $x = (x_1, x_2)^T$ we obtain a trajectory for this dynamical system $c : ([t_0, t_1], \mathrm{d}/\mathrm{dt}) \to (D_1^{\text{ball}}, X_1^{\text{ball}})$ given by:

$$c(t) = \begin{pmatrix} -\frac{g(t-t_0)^2}{2} + x_2(t - t_0) + x_1 \\ -g(t - t_0) + x_2 \end{pmatrix}$$

Clearly, $\dot{c} = X_1^{\text{ball}}(c)$.

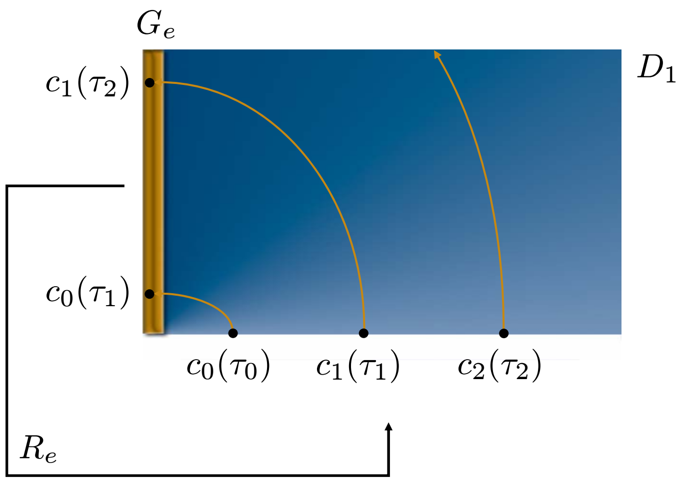


Figure 2.6: A graphical illustration of an execution.

Definition 2.5. An **execution** of a hybrid system $\mathfrak{H} = (\Gamma, D, G, R, X)$ is a tuple:

$$\epsilon = (\Lambda, I, \rho, C)$$

where

- (Λ, I) is a hybrid interval.
- $\rho : \Lambda \to Q$ is a map such that for all $i, i+1 \in \Lambda$, $e_i := (\rho(i), \rho(i+1)) \in E$. This is the *discrete evolution* of the execution.
- $C = \{c_i\}_{i \in \Lambda}$ is a set of *continuous trajectories* such that $c_i : (I_i, \mathrm{d/dt}) \to (D_{\rho(i)}, X_{\rho(i)})$ for all $i \in \Lambda$, i.e., $c_i : I_i \to D_{\rho(i)}$ is a trajectory of $X_{\rho(i)}$ and thus satisfies $\dot{c}_i(t) = X_{\rho(i)}(c_i(t))$ for $t \in I_i$.

We require that for all $i, i+1 \in \Lambda$,

$$\begin{array}{ll} (1) & c_i(\tau_{i+1}) \in G_{e_i} \\ (2) & R_{e_i}(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1}). \end{array}$$

The continuous initial condition of an execution ϵ is, when $I_0 = [\tau_0, \tau_1]$, given by $c_0(\tau_0) \in D_{\rho(0)}$. The discrete initial condition is given by $\rho(0)$. Note that given an initial condition, it is by no means assumed that there is a unique execution with this initial condition. We never concern ourselves with uniqueness because the results presented never need to make this assumption in order to be valid.

Remark 2.6. In the definition of an execution we did not specify when the switching should occur once the guard is reached, i.e., we did not specify whether we are considering as-is (forced) or enabling semantics (see [109] for more on the semantics of hybrid systems). Again, the theorems introduced do not depend on this choice, so we opted for simplicity by not making a specific choice regarding transition semantics. In all of the examples, as-is semantics are used.

Example 2.15. An graphical illustration of an execution with the same hybrid interval as the one given in Example 2.8 can be seen in Figure 2.6. In this case, the hybrid system consists of a single domain, guard and reset map; therefore, the discrete evolution $\rho(i) = 1$ for $i = 1, 2, 3$. Hybrid systems of this form are termed *simple hybrid systems* and will be discussed in detail in Chapter 3.

Example 2.16. The bouncing ball hybrid system allows for the unique luxury of explicitly solving for its executions. That is, given an initial condition, we can explicitly produce a corresponding execution.

Starting at the initial condition $x = (x_1, x_2)^T \in D_1^{\text{ball}}$ at time τ_0, the system evolves according to the dynamics X_1^{ball},

$$c_0(t) = \begin{pmatrix} -\frac{g(t-\tau_0)^2}{2} + x_2(t-\tau_0) + x_1 \\ -g(t-\tau_0) + x_2 \end{pmatrix}$$

until the guard G_e^{ball} is reached, which occurs in time and space at:

$$\begin{aligned} \tau_1 &= \tau_0 + \frac{x_2 + \sqrt{2gx_1 + x_2^2}}{g} \\ c_0(\tau_1) &= \begin{pmatrix} 0 \\ -\sqrt{2gx_1 + x_2^2} \end{pmatrix} \end{aligned}$$

Applying the reset map R_e^{ball} to $c_0(\tau_1)$ yields the initial condition to subsequent trajectory $c_1(t)$ on D_1^{ball} which starts at time τ_1, i.e.,

$$c_1(t) = \begin{pmatrix} -\frac{g(t-\tau_1)^2}{2} + r(t-\tau_1)\sqrt{2gx_1 + x_2^2} \\ -g(t-\tau_1) + r\sqrt{2gx_1 + x_2^2} \end{pmatrix}$$

wherein we can again determine the subsequent time and point in which the guard is reached:

$$\begin{aligned} \tau_2 &= \tau_1 + 2r\frac{\sqrt{2gx_1 + x_2^2}}{g} \\ c_0(\tau_1) &= \begin{pmatrix} 0 \\ -r\sqrt{2gx_1 + x_2^2} \end{pmatrix} \end{aligned}$$

Repeating this process iteratively yields an execution:

$$\epsilon = (\Lambda, I, \rho, C).$$

Here $\Lambda = \mathbb{N}$, $I = \{I_i\}_{i\in\mathbb{N}}$ where $I_i = [\tau_i, \tau_{i+1}]$ with

$$\begin{aligned} \tau_1 &= \tau_0 + \frac{x_2 + \sqrt{2gx_1 + x_2^2}}{g}, \\ \tau_{i+1} &= \tau_i + 2r^i\frac{\sqrt{2gx_1 + x_2^2}}{g}, \qquad i \geq 1. \end{aligned}$$

Since there is only one domain, $\rho(i) \equiv 1$. Finally, $C = \{c_i\}_{i\in\mathbb{N}}$ with

$$c_i(t) = \begin{pmatrix} -\frac{g(t-\tau_i)^2}{2} + r^i(t-\tau_i)\sqrt{2gx_1 + x_2^2} \\ -g(t-\tau_i) + r^i\sqrt{2gx_1 + x_2^2} \end{pmatrix}.$$

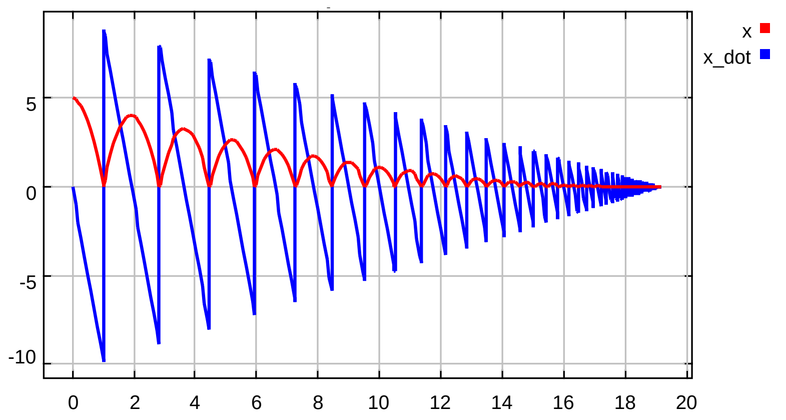


Figure 2.7: Positions and velocities over time of an execution of the bouncing ball hybrid system.

An execution of the bouncing ball can be seen in Figure 2.7.

2.3.2 Zeno executions.

A particular class of executions of hybrid systems that will be of particular interest are *Zeno executions*.

Definition 2.6. An execution $\epsilon = (\Lambda, I, \rho, C)$ of $\mathfrak{H}$ is **Zeno** if $\Lambda = \mathbb{N}$ and

$$\sum_{i=0}^{\infty}(\tau_{i+1}-\tau_i) = \lim_{i\to\infty}\tau_i = \tau_\infty$$

for some finite constant τ_∞, termed the *Zeno time*.

Example 2.17. Using the constructed executions for the bouncing ball hybrid system, we can verify that it is Zeno. That is,

$$\sum_{i=0}^{\infty}(\tau_{i+1}-\tau_i) = \frac{x_2+(1-2g)\sqrt{2gx_1+x_2^2}}{g} + \sum_{i=0}^{\infty} 2\frac{\sqrt{2gx_1+x_2^2}}{g}r^i$$

where the series on the right is a geometric series, and converges if $0 \le r < 1$. What this says physically is if the ball looses energy on each bounce, then it will eventually stop bouncing; moreover, it will do so in finite time. We give conditions (in Chapter 5) on the Zenoness of the bouncing ball *without* solving for the vector fields.

Consider again the geometric hybrid interval, $(\mathbb{N}, I^{(a,r)})$. It is easy to see that

$$\sum_{i=0}^{\infty}(\tau_{i+1}^{(a,r)}-\tau_i^{(a,r)}) = \sum_{i=0}^{\infty} ar^i.$$

Now, picking initial conditions $x_1 = a^2g/8$ and $x_2 = 0$ for an execution of the bouncing ball yields

$$\sum_{i=0}^{\infty}(\tau_{i+1}-\tau_i) = \pm\left(\frac{1}{2}a(1-2g)+\sum_{i=0}^{\infty} ar^i\right),$$

where the expression on the right is positive if a is positive and negative if a is negative. Therefore, the hybrid interval for the bouncing ball is an example of a geometric hybrid interval.

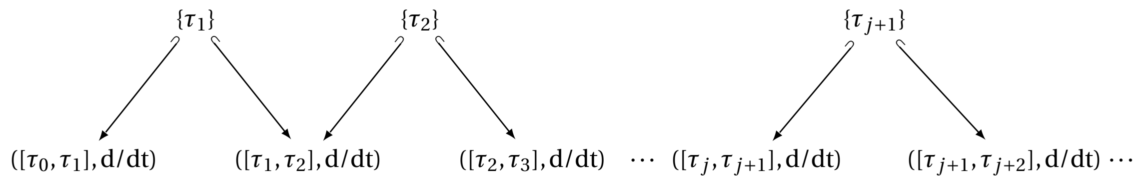
2.3.3 Intervals in HySys**.** The interval category of HySys, denoted by Interval(HySys), is the full subcategory of HySys with objects consisting of hybrid systems of the form:

$$(\mathscr{I}, \mathbf{I}, \mathbf{d/dt}),$$

where

- $(\mathscr{I}, \mathbf{I})$ is a hybrid interval, i.e., an object of Interval(Hy(Man)),
- $\mathbf{d/dt}_j = \mathrm{d/dt}$ for all $j \in \mathsf{Ob}(\mathscr{I})$.

For example, if $(\mathscr{I}, \mathbf{I})$ is the hybrid interval given in (2.10), the corresponding object $(\mathscr{I}, \mathbf{I}, \mathbf{d/dt})$ of Interval(HySys) can be visualized graphically as follows:



One of the many benefits of defining intervals in HySys is the way in which they allow us to parallel the definition of trajectories of dynamical systems as given in Definition 2.4.

Definition 2.7. A **trajectory** of a hybrid system $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ is a morphism in HySys:

$$(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}, \mathbf{X}),$$

where $(\mathscr{I}, \mathbf{I}, \mathbf{d/dt})$ is an object of Interval(HySys).

Note that the functor $\vec{C}$ corresponds to the "discrete" portion of the trajectory, while the natural transformation $\vec{c}$ corresponds to the "continuous" portion. In particular, it follows that

$$\dot{\vec{c}}_j(t) = \mathbf{X}_{\vec{C}(j)}(\vec{c}_j(t))$$

for every object $j \in \mathsf{V}(\mathscr{I})$.

The discrete initial condition is given by $\vec{C}(0)$ and the continuous initial condition is given by $\vec{c}_0(\tau_0) \in \mathbf{M}_{\vec{C}(0)}$ with τ_0 the right endpoint of $\mathbf{I}_0$, i.e., the initial condition to the trajectory is $(\vec{C}(0), \vec{c}_0(\tau_0))$.

Proposition 2.4. *There is a bijective correspondence:*

$$\{\text{Executions of } \mathfrak{H}\} \leftrightarrow \{\text{Trajectories of } (\mathscr{M}, \mathbf{M}^{\imath}, \mathbf{X})\}.$$

Proof. If $\epsilon = (\Lambda, I, \rho, C)$ is an execution of $\mathfrak{H} = (\Gamma, D, G, R, X)$, to the pair (Λ, I) we have an associated object of Interval(Hy(Man)):

$$\mathbf{I}^I : \mathscr{I}_\Lambda \to \mathsf{Interval(Man)}.$$

From this, we obtain an object of Interval(HySys):

$$(\mathscr{I}_\Lambda, \mathbf{I}^I, \mathbf{d/dt}).$$

The remaining data defining the execution $\epsilon = (\Lambda, I, \rho, C)$, i.e., the discrete evolution ρ and the continuous evolution C, allows us to define a morphism:

$$(\vec{C}_\rho, \vec{c}_C) : (\mathscr{I}_\Lambda, \mathbf{I}^I, \mathbf{d/dt}) \to (\mathscr{M}_\Gamma, \mathbf{M}^{(D,G,R)}, \mathbf{X}^X),$$

where $(\mathscr{M}_\Gamma, \mathbf{M}^{(D,G,R)}, \mathbf{X}^X)$ is the categorical hybrid system obtained from the classical hybrid system $\mathfrak{H}$ via Proposition 2.2. The morphism of D-categories $\vec{C}_\rho : \mathscr{I}_\Lambda \to \mathscr{M}_\Gamma$ is defined on objects of $\mathscr{I}_\Lambda$ by

$$\vec{C}_\rho(a) = \begin{cases} e_j = (\rho(j), \rho(j+1)) & \text{if} \quad a = (j, j+1) \in \mathsf{E}(\mathscr{I}_\Lambda) \\ \rho(j) & \text{if} \quad a = j \in \mathsf{V}(\mathscr{I}_\Lambda) \end{cases}$$

and on morphisms in the obvious way:

$$\vec{C}_\rho(\mathsf{s}_a^{\mathscr{I}_\Lambda}) := \mathsf{s}_{\vec{C}_\rho(a)}^{\mathscr{M}_\Gamma}, \qquad \vec{C}_\rho(\mathsf{t}_a^{\mathscr{I}_\Lambda}) := \mathsf{t}_{\vec{C}_\rho(a)}^{\mathscr{M}_\Gamma},$$

with $(\mathsf{s}^{\mathscr{I}_\Lambda}, \mathsf{t}^{\mathscr{I}_\Lambda})$ the orientation for $\mathscr{I}_\Lambda$ and $(\mathsf{s}^{\mathscr{M}_\Gamma}, \mathsf{t}^{\mathscr{M}_\Gamma})$ the orientation for $\mathscr{M}_\Gamma$. It follows that this is a valid morphism of D-categories. Finally, define the natural transformation

$$\vec{c}_C : \mathbf{I}^I \xrightarrow{\cdot} \vec{C}_\rho^*(\mathbf{M}^{(D,G,R)})$$

on objects of $\mathscr{I}_\Lambda$ as follows:

$$(\vec{c}_C)_a = \begin{cases} c_j|_{I_j \cap I_{j+1}} & \text{if} \quad a = (j, j+1) \in \mathsf{E}(\mathscr{I}_\Lambda) \\ c_j & \text{if} \quad a = j \in \mathsf{V}(\mathscr{I}_\Lambda). \end{cases}$$

To verify that this is a natural transformation, we need only verify that the following diagram

$$\begin{array}{ccccc}
\mathbf{I}^I_j = I_j & \hookleftarrow & \mathbf{I}^I_{(j,j+1)} = I_j \cap I_{j+1} & \hookrightarrow & \mathbf{I}^I_{j+1} = I_{j+1} \\
\downarrow {\scriptstyle (\vec{c}_C)_j = c_j} & & \downarrow {\scriptstyle (\vec{c}_C)_{(j,j+1)} = c_j|_{I_j \cap I_{j+1}}} & & \downarrow {\scriptstyle (\vec{c}_C)_{j+1} = c_{j+1}} \\
\mathbf{M}^{(D,G,R)}_{\vec{C}_\rho(j)} = D_{\rho(j)} & \xhookleftarrow{\iota} & \mathbf{M}^{(D,G,R)}_{\vec{C}_\rho(j,j+1)} = G_{e_j} & \xrightarrow{\mathbf{M}^{(D,G,R)}_{\vec{C}_\rho(\mathsf{s}^{\mathscr{I}_\Lambda}_{(j,j+1)})} = R_{e_j}} & \mathbf{M}^{(D,G,R)}_{\vec{C}_\rho(j+1)} = D_{\rho(j+1)}
\end{array}$$

commutes for every $(j, j+1) \in \mathsf{E}(\mathscr{I})$. But this happens exactly when, for $I_j = [\tau_j, \tau_{j+1}]$,

$$c_j(\tau_{j+1}) \in G_{e_j}, \qquad c_{j+1}(\tau_{j+1}) = R_{e_j}(c_j(\tau_{j+1})),$$

which are just requirements (1) and (2) given in Definition 2.5.

The converse direction proceeds in much the same manner, so we will be brief. Let $(\mathscr{M}, \mathbf{M}^\imath, \mathbf{X})$ be a categorical hybrid system for which we have an associated hybrid system $(\Gamma_{\mathscr{M}}, D_{\mathbf{M}^\imath}, G_{\mathbf{M}^\imath}, R_{\mathbf{M}^\imath}, X_{\mathbf{X}})$ as given in Proposition 2.2. For a trajectory $(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}^\imath, \mathbf{X})$, we define a corresponding execution as

$$(\mathsf{V}(\mathscr{I}), I^{\mathbf{I}}, \vec{C}|_{\mathsf{V}(\mathscr{I})}, \{\vec{c}_j\}_{j \in \mathsf{V}(\mathscr{I})}),$$

where $(\mathsf{V}(\mathscr{I}), I^{\mathbf{I}})$ is the hybrid interval obtained from $\mathbf{I} : \mathscr{I} \to \mathsf{Interval}(\mathsf{Man})$ via Proposition 2.3 and $\vec{C}|_{\mathsf{V}(\mathscr{I})}$ is the object function of the functor $\vec{C}$ restricted to the elements of $\mathsf{V}(\mathscr{I})$. It is easy to verify that this is a valid execution. □

Example 2.18. Consider the categorical bouncing ball hybrid system $(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}, \mathbf{X}^{\text{ball}})$ and a trajectory

$$(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}, \mathbf{X}^{\text{ball}}),$$

which, for example, can be obtained from the executions of $\mathfrak{H}^{\text{ball}}$ introduced in Example 2.16. The condition that this is a trajectory implies that $(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}) \to (\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}})$ in $\mathsf{Hy}(\mathsf{Man})$ is a hybrid path as discussed in Example 2.5. The additional condition, and the one that makes $(\vec{C}, \vec{c})$ a trajectory, is that the collection of paths defined by $\vec{c}$ satisfy the ordinary differential equation $\mathbf{X}_b^{\text{ball}}$.

Another of the many benefits obtained from defining trajectories of hybrid systems categorically is that morphisms of hybrid systems carry trajectories of one hybrid system to trajectories of another hybrid system.

Lemma 2.1. *If $(\vec{F}, \vec{f}) : (\mathscr{N}, \mathbf{N}, \mathbf{Y}) \to (\mathscr{M}, \mathbf{M}, \mathbf{X})$ is a morphism of hybrid systems, and $(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{N}, \mathbf{N}, \mathbf{Y})$ is a trajectory of $(\mathscr{N}, \mathbf{N}, \mathbf{Y})$, then*

$$(\vec{F}, \vec{f}) \circ (\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}, \mathbf{X})$$

is a trajectory of $(\mathscr{M}, \mathbf{M}, \mathbf{X})$.

The proof of this statement is immediately obvious using the categorical framework for hybrid systems, while it is not immediately clear if one were to consider the standard notion of an execution.

Part II

Hybrid Systems

Chapter 3

Simple Hybrid Reduction & Bipedal Robotic Walking

Hybrid systems introduce a level of complexity not found in their continuous and discrete counterparts, e.g., simulating hybrid systems is orders of magnitude more difficult than simulating their continuous counterparts due to the presence of state dependent events. This added complexity makes the dimensionality of hybrid systems a critical factor in understanding, analyzing, verifying and controlling these systems and motivates the importance of understanding geometric reduction in a hybrid setting. Since it is not possible to directly apply classical (continuous) reduction to hybrid systems—the nonsmooth nature of these systems inherently violates the assumptions needed to perform this type of reduction—we would like to answer the question:

> *If it is possible to reduce the continuous components of a hybrid system, when is it possible to reduce the entire hybrid system?*

This chapter addresses this question in the context of mechanical systems undergoing impacts; we will address this question for general hybrid system in Chapter 4. The simple structure of mechanical systems undergoing impacts allows us to derive explicit conditions on when continuous reduction can be applied to the continuous component of these systems so as to be consistent with the discrete component, i.e., when hybrid reduction can be performed.

History. Lagrangians and Hamiltonians provide the basic elements for describing the behavior of physical systems. For mechanical systems, one begins with a configuration space Q and a Lagrangian $L: TQ \to \mathbb{R}$ or a Hamiltonian $H: T^*Q \to \mathbb{R}$ given in coordinates by[1]

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \qquad H(q,p) = \frac{1}{2}p^T M(q)^{-1} p + V(q).$$

[1] We slightly abuse notation here by using "matrix" notation; this is done only in the introduction because of its (presumed) familiarity.

It is possible to reduce the dimensionality of systems of this form when they have symmetries through reduction; this process plays a fundamental role in understanding the many important and interesting properties of these systems.

The first form of reduction was discovered by Routh in 1860 [100] (see [87] for a more modern account)—now understood to be an abelian form of Lagrangian reduction—which is concerned with *cyclic Lagrangians*, i.e., Lagrangians that are independent of certain "cyclic" variables. The symmetries of these systems are generated by these cyclic variables. Given a cyclic Lagrangian, the phase space of the system (the tangent bundle of the configuration space) can be reduced, flows of the original system can be projected down to flows of the reduced system, and flows of the reduced system can be used to reconstruct flows of the full-order system.

Lagrangian reduction provides the first step toward a more general type of reduction: Hamiltonian (or symplectic) reduction. This form of reduction begins with a Lie group together with an action of this Lie group on the phase space (a symplectic manifold); this makes explicit the symmetries of the system. One then looks for a momentum map from the phase space to the dual of the Lie algebra of the Lie group; this makes explicit the conserved quantities of the system. The combination of this data defines a Hamiltonian G-space, which provides the ingredients necessary for classical Hamiltonian reduction. The classical reduction theorem, first established by Marsden and Weinstein [88], says that when the Hamiltonian G-space satisfies certain conditions, the phase space can be reduced to a new space which is also a symplectic manifold, with a symplectic structure induced from the one on the phase space. Moreover, given a G-invariant Hamiltonian on the phase space, the corresponding trajectories of the associated Hamiltonian vector field reduce to trajectories on the reduced phase space.

Simple hybrid reduction. In this chapter, we begin by considering a class of mechanical systems with unilateral constraints (usually physical in nature) on the configuration space, i.e., there is a function $h : Q \to \mathbb{R}$ describing the admissible configurations of the system: $Q|_{\{h(q)\geq 0\}}$. When considering Lagrangians, a unilateral constraint function defines a *hybrid Lagrangian*, which is a tuple $\mathbf{L} = (Q, L, h)$. In the case of Hamiltonians, *simple hybrid mechanical systems (HMS's)*, $\mathbf{H} = (Q, H, h)$, are considered. In both cases, the term "hybrid" is used because the constraints on the configuration space result in discontinuities in the vector field describing the evolution of the mechanical system. Therefore, we can explicitly associate hybrid systems, $\mathfrak{H}_{\mathbf{L}}$ and $\mathfrak{H}_{\mathbf{H}}$, to hybrid Lagrangians and HMS's, respectively. We provide conditions on when it is possible to reduce hybrid systems of this form, along with more general simple hybrid systems (hybrid systems consisting of one domain and one edge) whose dynamics are dictated by a Hamiltonian system.

A *cyclic hybrid Lagrangian* is a hybrid Lagrangian in which L is cyclic (coupled with the cyclicity of the unilateral constraint function h); we demonstrate explicitly how a Lagrangian hybrid system, $\mathfrak{H}_{\mathbf{L}}$, can be obtained from a hybrid Lagrangian, $\mathbf{L}$, with dynamics dictated by the Euler-Lagrange equations of a Lagrangian. From a cyclic hybrid Lagrangian we obtain a hybrid Routhian $\mathbf{L}_\mu = (Q_\mu, L_\mu, h_\mu)$, which is also a hybrid Lagrangian, and so has an associated hybrid system, $\mathfrak{H}_{\mathbf{L}_\mu}$, with dynamics dictated by the

Routhian. We prove that hybrid flows (or executions) of $\mathfrak{H}_{\mathbf{L}}$ project to hybrid flows of $\mathfrak{H}_{\mathbf{L}_\mu}$, and that hybrid flows of $\mathfrak{H}_{\mathbf{L}}$ can be reconstructed from hybrid flows of $\mathfrak{H}_{\mathbf{L}_\mu}$. These results motivate the consideration of a more general form of hybrid reduction: hybrid Hamiltonian (or symplectic) reduction.

In order to perform Hamiltonian reduction, we begin by considering a Hamiltonian G-space and give conditions on the elements of this G-space so that it defines a hybrid Hamiltonian G-space. Explicitly, this involves defining the notion of a hybrid group action and a hybrid momentum map, which is first done in the general setting of simple hybrid systems, followed by the special case of HMS's. Using these general notions, conditions are obtained on when a simple hybrid system, $\mathfrak{H}$, can be reduced; the result is a simple hybrid system $\mathfrak{H}_\mu$.

Bipedal Walking. A very interesting and promising application of hybrid reduction is bipedal robotic walking since bipedal walkers are naturally modeled by hybrid systems—the continuous component consists of the dynamics dictated by the Lagrangian modeling this system, and the discrete component consists of the impact equations which instantaneously change the velocity of the system when the foot contacts the ground. In order to apply reduction to systems of this form in a useful manner, we introduce a variation of classical Routhian reduction, *functional Routhian reduction*, which can be extended to a hybrid setting in a manner analogous to the extension of classical Routhian reduction.

In classical geometric reduction the conserved quantities used to reduce and reconstruct systems are constants; this indicates that the "cyclic" variables eliminated when passing to the reduced phase space are typically uncontrolled. Yet it is often the case that these variables are the ones of interest—it may be desirable to *control* the cyclic variables while not affecting the reduced order system. This motivates an extension of Routhian reduction to the case when the conserved quantities are functions of the cyclic variables instead of constants.

These concepts motivate our main goal:

Develop a feedback control law that results in walking gaits on flat ground for a three-dimensional bipedal robotic walker given walking gaits for a two-dimensional bipedal robotic walker.

In order to achieve this goal, we begin by considering Lagrangians that are cyclic except for an additional non-cyclic term in the potential energy, i.e., *almost-cyclic* Lagrangians. When Routhian reduction is performed with a function (of the cyclic variables) the result is a Lagrangian on the reduced phase-space: the *functional Routhian.* We are able to show that the dynamics of an almost-cyclic Lagrangian satisfying certain initial conditions project to dynamics of the corresponding functional Routhian, and dynamics of the functional Routhian can be used to reconstruct dynamics of the full-order system. In order to use this result to develop control strategies for bipedal walkers, it first must be generalized to a hybrid setting. That is, after discussing how to explicitly obtain a hybrid system model of a bipedal walker, we generalize functional Routhian reduction to a hybrid setting, demonstrating that hybrid flows of the reduced and full order system are related in a way analogous to the continuous result.

We then proceed to consider two-dimensional (2D) bipedal walkers. It is well-known that 2D bipedal walkers can walk down shallow slopes without actuation (cf. [90], [54]). [107] used this observation to develop a positional feedback control strategy that allows for walking on flat ground. We use these results to obtain a hybrid system, $\mathfrak{H}^{s}_{2D}$, modeling a 2D bipedal robot that walks on flat ground.

We conclude by considering three-dimensional (3D) bipedal walkers. Our main result is a positional feedback control law that produces walking gaits in three-dimensions. To obtain this controller we shape the potential energy of the Lagrangian describing the dynamics of the 3D bipedal walker so that it becomes an almost-cyclic Lagrangian, where the cyclic variable is the roll (the unstable component) of the walker. We are able to control the roll through our choice of a non-cyclic term in the potential energy. Since the *functional Routhian* hybrid system obtained by reducing this system is $\mathfrak{H}^{s}_{2D}$, by picking the "correct" function of the roll, we can force the roll to go to zero for certain initial conditions. That is, we obtain a non-trivial set of initial conditions that provably result in three-dimensional walking.

Related work. The simple hybrid mechanical systems considered have been well-studied in the literature under many names and incarnations (cf. [36] and the more than 1000 references therein). Amazingly, the authors are unaware of any results regarding the reduction of systems of this form. General hybrid systems have been studied extensively; especially relevant are [38] which studies hybrid mechanical systems of a more general form than considered here, and [66] which considers hybrid systems with symmetries. Again, the authors are unaware of any results regarding the reduction of these systems; although, the "non-geometric" reduction of these systems in the context of abstraction and bisimulation relations [96] has been well-studied and proven useful for verification techniques such as reachability analysis [57].

Classical reduction has developed into a mature area of study over the last forty years (see [89] for a nice overview of the history of the subject). We will assume that the reader is at least tentatively familiar with classical reduction, although we briefly review crucial prerequisite material. We refer the reader to [4, 79, 86, 87, 88] for any necessary background material not covered. Although never explicitly mentioned, the literature on classical reduction has touched upon issues relating to hybrid reduction. In [86] a form of discrete reduction is discussed where the assumptions needed to perform this form of reduction are very similar to conditions enumerated later. Similarly, the reduction of continuous systems with constraints has been studied in [85] and related references therein. Therefore, the results proven can be viewed as the next logical step in understanding how to reduce the dimensionality of systems with symmetry.

The results presented in this chapter have appeared, or will appear, in the following papers: [8, 15, 17].

3.1 Simple Hybrid Lagrangians & Simple Hybrid Mechanical Systems

In this section, we introduce the notion of simple hybrid Lagrangians and simple hybrid mechanical systems. While introducing these definitions, we simultaneously review their continuous "non-hybrid" counterparts. This is done both to introduce the reader to the notation and simultaneously to make explicit the natural way that these "hybrid objects" relate to fundamental objects in classical mechanics. This relationship is further explored by introducing examples.

3.1.a Simple Hybrid Lagrangians

We begin by considering simple hybrid Lagrangians.

3.1.1 Lagrangians. Let Q be the *configuration space*, assumed to be a smooth manifold, and TQ the tangent bundle of Q (the velocity phase space). Suppose $L : TQ \to \mathbb{R}$ is a *hyperregular* Lagrangian (cf. [4, 87]). In this case, there is a Lagrangian vector field X_L on TQ, $X_L : TQ \to T(TQ)$, associated to L; that is, there is a dynamical system (TQ, X_L) associated to the Lagrangian. For $t \in [t_0, t_1]$, we say that $c(t) = (q(t), \dot{q}(t))$ is an integral curve of X_L with initial condition $c(t_0) = x_0$ if

$$c : ([t_0, t_1], \mathrm{d}/\mathrm{dt}) \to (TQ, X_L)$$

is a morphism in Dyn, i.e., if

$$\dot{c}(t) = X_L(c(t)).$$

This is equivalent to the curve $q(t)$ satisfying the classical Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) - \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) = 0. \tag{3.1}$$

We will consider primarily Lagrangians describing mechanical, or robotic, systems; that is, Lagrangians given in coordinates by

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \tag{3.2}$$

where $M(q)$ is the inertial matrix, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy and $V(q)$ is the potential energy. In this case, the Euler-Lagrange equations yield the equations of motion for the system:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = 0, \tag{3.3}$$

where $C(q, \dot{q})$ is the *Coriolis matrix* (cf. [93]) and $N(q) = \frac{\partial V}{\partial q}(q)$. Setting $x = (q, \dot{q})$, the Lagrangian vector field, X_L, associated to L takes the familiar form.

$$\dot{x} = X_L(x) = \left(\dot{q}, M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q))\right). \tag{3.4}$$

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

Remark 3.1. It is common for other authors to write x as a single vector, i.e., $x = (q^T, \dot{q}^T)^T$, rather than a pair of vectors; we opt for the latter notation in order to avoid the proliferation of transposes. Also, our notation is supported by the fact that an element of TQ is typically denoted by a pair $(q, \dot{q})$ with $q \in Q$ and $\dot{q} \in T_q Q$.

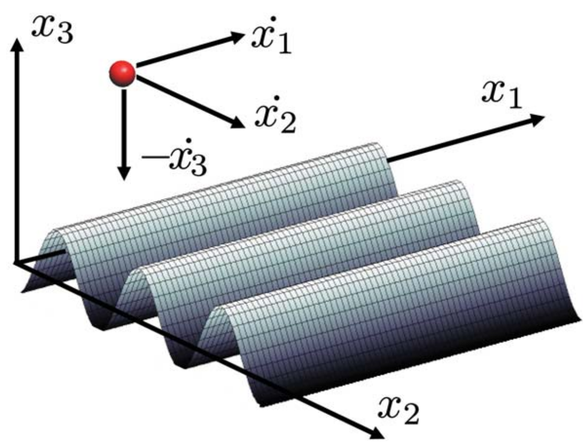


Figure 3.1: Ball bouncing on a sinusoidal surface.

Definition 3.1. A **simple hybrid Lagrangian** is defined to be a tuple

$$\mathbf{L} = (Q, L, h),$$

where

- Q is the configuration space,
- $L : TQ \to \mathbb{R}$ is a hyperregular Lagrangian,
- $h : Q \to \mathbb{R}$ is a smooth function providing unilateral constraints on the configuration space; we assume that $h^{-1}(0)$ is a manifold.

Example 3.1. Our first running example of this chapter is a ball bouncing on a sinusoidal surface (cf. Figure 3.1). In this case

$$\mathbf{B} = (Q_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}}),$$

where $Q_{\mathbf{B}} = \mathbb{R}^3$, and for $x = (x_1, x_2, x_3)^T$,

$$L_{\mathbf{B}}(x, \dot{x}) = \frac{1}{2} m \|\dot{x}\|^2 - mgx_3.$$

Finally, we make the problem interesting by considering the sinusoidal constraint function

$$h_{\mathbf{B}}(x_1, x_2, x_3) = x_3 - \sin(x_2).$$

For this example there are trivial dynamics and a nontrivial constraint function. Note that this system certainly is more complex then the simple one-dimensional bouncing ball introduced in Example 2.1

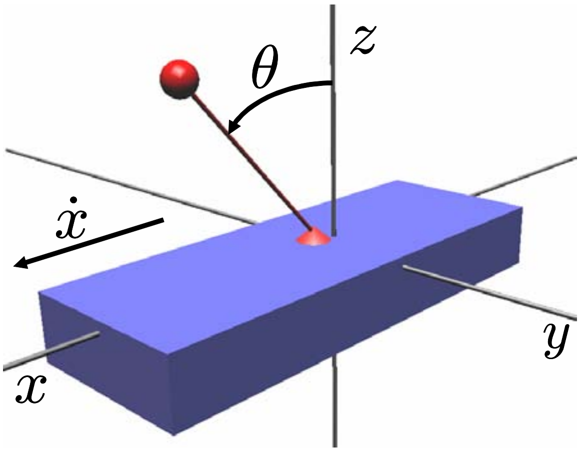


Figure 3.2: Pendulum on a cart.

Example 3.2. Our second running example is a constrained pendulum on a cart (cf. Figure 3.2); this is a variation on the classical pendulum on a cart, where the pendulum is not allowed to "pass through" the cart, i.e., the cart gives physical constraints on the configuration space. In this case

$$\mathbf{C} = (Q_{\mathbf{C}}, L_{\mathbf{C}}, h_{\mathbf{C}}),$$

where $Q_{\mathbf{C}} = \mathbb{S}^1 \times \mathbb{R}$, $q = (\theta, x)^T$, and

$$L_{\mathbf{C}}(\theta, \dot\theta, x, \dot x) = \frac{1}{2}\begin{pmatrix} \dot\theta & \dot x \end{pmatrix}\begin{pmatrix} mR^2 & mR\cos(\theta) \\ mR\cos(\theta) & M+m \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot x \end{pmatrix} - mgR\cos(\theta).$$

Finally, the constraint that the pendulum in not allowed to pass through the cart is manifested in the constraint function

$$h_{\mathbf{C}}(\theta, x) = \cos(\theta).$$

3.1.b Simple Hybrid Mechanical Systems

We now turn our attention toward Hamiltonians and their hybrid analogues.

3.1.2 Hamiltonians. The starting point for simple mechanical systems is a configuration space Q. Let $T^\star Q$ be the cotangent bundle of Q (the momentum phase space). We denote the pairing between the vector spaces[2] $T^\star_q Q$ and $T_q Q$ by

$$\langle \cdot, \cdot \rangle : T^\star_q Q \times T_q Q \to \mathbb{R},$$

which for $(p, v) \in T^\star_q Q \times T_q Q$ is given in coordinates by $\langle p, v\rangle := \sum_{i=1}^n p_i v^i$, with $n = \dim(Q)$.

[2]We later will use the same notation to denote the pairing between a Lie algebra and its dual.

Let $M(q)$ be the inertial matrix for a mechanical system and $K(q) = M(q)^{-1}$. For each $q \in Q$, we consider the $K(q)$-inner product on the vector space $T_q^\star Q$ given in coordinates by[3]

$$\langle\langle p, p' \rangle\rangle_q = p^T K(q) p' := \sum_{i,j=1}^{n} p_i p'_j K_{ij}(q)$$

for $p, p' \in T_q^\star Q$; we use $||\cdot||_q$ to denote the corresponding norm on $T_q^\star Q$. This induces (or is obtained from, depending on the perspective taken) an inner product on $T_q Q$ (the $M(q)$-inner product, which defines a Riemannian metric on Q) via the Legendre transformation: $\mathbb{F}L\colon TQ \to T^\star Q$, where $\mathbb{F}L(q,\dot{q}) = (q, M(q)\dot{q})$.

A Hamiltonian is a map $H : T^\star Q \to \mathbb{R}$. We suppose that the Hamiltonian H describes a mechanical system, i.e., that it has the following form

$$H(q,p) = \frac{1}{2}||p||_q^2 + V(q), \tag{3.5}$$

where $\frac{1}{2}||p||_q^2$ is the kinetic energy and $V(q)$ is the potential energy. Note that this Hamiltonian is obtained from a Lagrangian of the form given in (3.2) via the Legendre transformation.

The cotangent bundle, $T^\star Q$, is a symplectic manifold with its symplectic structure obtained from the canonical symplectic form given in coordinates by:

$$\omega_{\text{canonical}} = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

With this symplectic form, we obtain a vector field on $T^\star Q$ from a Hamiltonian, $X_H : T^\star Q \to T(T^\star Q)$, by requiring that it satisfies:

$$d(H) = \iota_{X_H} \omega_{\text{canonical}}.$$

In coordinates, this yields the classical Hamiltonian equations

$$(\dot{q}, \dot{p}) = X_H(q,p) = \left(\frac{\partial H}{\partial p}(q,p), -\frac{\partial H}{\partial q}(q,p) \right). \tag{3.6}$$

We refer the reader to [4, 27] and [87] for more details.

Definition 3.2. A **simple hybrid mechanical system** (HMS) is defined to be a tuple:

$$\mathbf{H} = (Q, H, h),$$

where H is defined as in (3.5), and $h : Q \to \mathbb{R}$ is a smooth function that defines constraints on the configuration of the system; again, $h^{-1}(0)$ is assumed to be a manifold.

Utilizing the Legendre transformation, to a hybrid Lagrangian we obtain an associated hybrid mechanical system. That is, the classical relationship between Lagrangians and Hamiltonians can be extended to the hybrid setting considered here. This relationship implies that in considering hybrid Lagrangians, one automatically is considering hybrid mechanical systems and vice versa (since we are assuming that L is a hyperregular Lagrangian). Therefore, our later results on hybrid Hamiltonian reduction

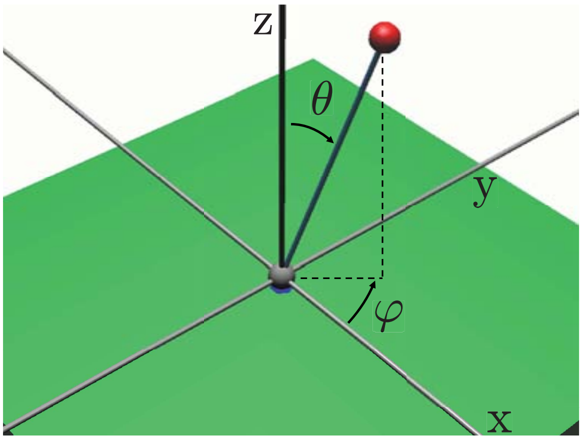


Figure 3.3: Spherical pendulum mounted on the floor.

(which are more general then the earlier results on hybrid Lagrangian reduction) are automatically applicable to hybrid Lagrangians.

Example 3.3. The running example of a HMS will be a spherical pendulum mounted on the floor (Figure 3.3). Here $Q_{\mathbf{P}} = \mathbb{S}^2$ and, using the standard spherical coordinates, we denote an element $q \in \mathbb{S}^2$ by $q = (\theta, \varphi)^T$ and we denote an element $p \in T_q^\star \mathbb{S}^2$ by $p = (p_\theta, p_\varphi)^T$. For this example, the Hamiltonian $H_{\mathbf{P}}$ is given by

$$H_{\mathbf{P}}(q,p) = \frac{1}{2mR^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2(\theta)}\right) - mgR\cos(\theta).$$

Finally, $h_{\mathbf{P}}$ is the height function $h_{\mathbf{P}}(\theta,\varphi) = \cos(\theta)$, i.e., we have a simple hybrid mechanical system given by $\mathbf{P} = (Q_{\mathbf{P}}, H_{\mathbf{P}}, h_{\mathbf{P}})$.

3.2 Simple Hybrid Systems

In this section, we introduce *simple hybrid systems*, and show explicitly how to associate to hybrid Lagrangians and HMS's simple hybrid systems. This association is achieved through the use of *Newtonian impact equations*, which provide a method for describing the behavior of a mechanical system undergoing impacts. It is important to note that this construction has support in the literature (cf. [36], [38], [49]), and hybrid systems of this form have the ability to model a large class of physical systems.

Definition 3.3. A **simple hybrid system**[4] is a tuple:

$$\mathfrak{H} = (D, S, R, X),$$

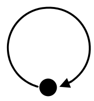
[3] We use the notation "$p^T K(q) p'$" so as to relate "matrix" notation (which is more common when discussing Lagrangians) with summation notation (which is more common when discussing Hamiltonians and reduction in general). Typically, We will only use "matrix notation" when discussing Lagrangians.

[4] So named because of its connection with simple HMS's, coupled with its "simple" structure.

where

- ⋄ D is a smooth manifold called the *domain*,
- ⋄ S is an embedded submanifold of D called the *switching surface* or *guard*,
- ⋄ $R : S \to D$ is a smooth map called the *reset map*,
- ⋄ X is a vector field on D.

In this case, the hybrid manifold associated to this hybrid system is a tuple $\mathbf{D}^{\mathfrak{H}} = (D, S, R)$ with elements as defined above. That is, since the graph Γ for a simple hybrid system consists of a single edge and vertex, i.e., it is of the form:



the associated D-category $\mathscr{D}_\Gamma$ is given by:

$$a \underset{\mathsf{t}_a}{\overset{\mathsf{s}_a}{\rightrightarrows}} b$$

and $\mathbf{D}^{\mathfrak{H}}$ is a functor $\mathbf{D}^{\mathfrak{H}} : \mathscr{D}_\Gamma \to \mathsf{Man}$ defined by:

$$\mathbf{D}^{\mathfrak{H}}\left(a \underset{\mathsf{t}_a}{\overset{\mathsf{s}_a}{\rightrightarrows}} b \right) = S \underset{R}{\overset{\iota}{\rightrightarrows}} D$$

where ι is the natural inclusion.

Remark 3.2. Note that simple hybrid systems are clearly just a special case of the notion of a hybrid system introduced in Definition 2.1, although we have opted to shift notation slightly. The first noticeable difference is that there is no reference to an indexing graph Γ. This is because simple hybrid systems always have as an indexing graph a graph with a single edge and vertex. This also explains why we do not index the domains, guards, reset maps and vector fields—there is only one of each. Finally, for simple hybrid systems, we denote the guard by "S" instead of "G". The motivation for this is that we will use the symbol "G" to denote groups in this chapter.

Executions for simple hybrid systems are of a somewhat simpler form than the executions for general hybrid systems as introduced in Definition 2.5. That is, since there is only one domain, the discrete evolution $\rho : \Lambda \to Q$ must take a single value, and so need not be mentioned. To make this explicit, we restate the definition of an execution in the context of simple hybrid systems. Moreover, to highlight the difference between executions of hybrid systems and executions of simple hybrid systems, we refer to the later as *hybrid flows*.

3.2.1 Hybrid flows. A *hybrid flow* is a tuple:

$$\epsilon^{\mathfrak{H}} = (\Lambda, I, C),$$

where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is a finite or infinite indexing set.
- $I = \{I_i\}_{i\in\Lambda}$ is a collection of intervals where $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, N finite; here $\tau_i, \tau_{i+1}, \tau_N \in \mathbb{R}$ and $\tau_i \leq \tau_{i+1}$,
- $C = \{c_i\}_{i\in\Lambda}$ is a collection of integral curves of X, i.e., $\dot{c}_i(t) = X(c_i(t))$ for all $i \in \Lambda$.

In addition, we require that the following conditions hold for every $i, i+1 \in \Lambda$,

$$\begin{array}{ll} \text{(i)} & c_i(\tau_{i+1}) \in S, \\ \text{(ii)} & R(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1}). \end{array}$$

The initial condition for the execution is $x_0 = c_0(\tau_0)$. When we wish to make explicit the initial condition of $\epsilon^{\mathfrak{H}}$ we write $\epsilon^{\mathfrak{H}}(x_0)$.

Example 3.4. A graphical illustration of a hybrid flow can be seen Figure 2.6. The reader should make the appropriate changes of notion in this figure as discussed in Remark 3.2.

3.2.a Lagrangian Hybrid Systems

We now discuss how to obtain simple hybrid systems from simple hybrid Lagrangians.

3.2.2 Domains from constraints. Given a smooth (constraint) function $h : Q \to \mathbb{R}$ on a configuration space Q such that $h^{-1}(0)$ is a smooth manifold, i.e., 0 is a regular value of h, we can construct a domain and a guard explicitly. To this constraint function we have an associated domain, D_h, defined to be the manifold (with boundary):

$$D_h = \{(q, \dot{q}) \in TQ : h(q) \geq 0\}.$$

Similarly, we have an associated guard, S_h, defined as the following submanifold of D_h:

$$S_h = \{(q, \dot{q}) \in TQ : h(q) = 0 \text{ and } dh_q \dot{q} \leq 0\},$$

where in coordinates

$$dh_q = \begin{pmatrix} \frac{\partial h}{\partial q_1}(q) & \cdots & \frac{\partial h}{\partial q_n}(q) \end{pmatrix}.$$

These constructions will be utilized throughout this chapter.

Definition 3.4. A hybrid system is said to be a **Lagrangian hybrid system** with respect to a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ if it is of the form:

$$\mathfrak{H} = (D_h, S_h, R, X_L),$$

where D_h and S_h are the domain and guard associated to h and X_L is the vector field associated to L.

3.2.3 Special Lagrangian hybrid systems. There is a class of Lagrangian hybrid systems that are of special interest; these model unilaterally constrained systems undergoing impacts [36], and so have reset maps obtained from *Newtonian impact equations*.

Given a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$, the *Lagrangian hybrid system associated to* $\mathbf{L}$ is the hybrid system

$$\mathfrak{H}_{\mathbf{L}} = (D_{\mathbf{L}}, S_{\mathbf{L}}, R_{\mathbf{L}}, X_{\mathbf{L}}),$$

where $D_{\mathbf{L}} = D_h$, $S_{\mathbf{L}} = G_h$, $X_{\mathbf{L}} = X_L$ and

$$R_{\mathbf{L}}(q, \dot{q}) = (q, P_{\mathbf{L}}(q, \dot{q})),$$

with $P_{\mathbf{L}}$ given in coordinates by:

$$P_{\mathbf{L}}(q, \dot{q}) = \dot{q} - (1 + e) \frac{dh_q \dot{q}}{dh_q M(q)^{-1} dh_q^T} M(q)^{-1} dh_q^T, \tag{3.7}$$

where $0 \leq e \leq 1$ is the *coefficient of restitution*, e.g., for a perfectly elastic impact $e = 1$, and for a perfectly plastic impact $e = 0$.

Of course, the Lagrangian hybrid system associated to a hybrid Lagrangian is a Lagrangian hybrid system w.r.t. this hybrid Lagrangian. The converse statement is not true. General Lagrangian hybrid systems, as introduced in Definition 3.4, describe a much larger class of systems, e.g., it is not assumed that the reset map is continuous in the configuration variables. An important class of systems that general Lagrangian hybrid systems describe are bipedal robotic walkers (cf. [55, 107, 116]). In fact, general Lagrangian hybrid systems and the hybrid reduction thereof are used in Section 3.5 to reduce the dimensionality of bipedal walkers. It is then possible to use results relating to two-dimensional bipedal walkers to allow three-dimensional bipedal walkers to walk while stabilizing to the upright position.

Example 3.5. The Lagrangian hybrid system for the bouncing ball on a sinusoidal surface is given by

$$\mathfrak{H}_{\mathbf{B}} = (D_{\mathbf{B}}, S_{\mathbf{B}}, R_{\mathbf{B}}, X_{\mathbf{B}}).$$

First, we define

$$\begin{aligned} D_{\mathbf{B}} &= \{(x, \dot{x}) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_3 - \sin(x_2) \geq 0\}, \\ S_{\mathbf{B}} &= \{(x, \dot{x}) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_3 = \sin(x_2) \text{ and } \dot{x}_3 - \cos(x_2)\dot{x}_2 \leq 0\}, \end{aligned}$$

and $R_{\mathbf{B}}(x, \dot{x}) = (x, P_{\mathbf{B}}(x, \dot{x}))$, where

$$P_{\mathbf{B}}(x, \dot{x}) = \begin{pmatrix} \dot{x}_1 \\ \frac{(1 - e\cos(x_2)^2)\dot{x}_2 + (1+e)\cos(x_2)\dot{x}_3}{1+\cos(x_2)^2} \\ \frac{(1+e)\cos(x_2)\dot{x}_2 + (-e+\cos(x_2)^2)\dot{x}_3}{1+\cos(x_2)^2} \end{pmatrix}$$

with $0 \leq e \leq 1$ the coefficient of restitution. Finally,

$$X_{\mathbf{B}}(x, \dot{x}) = \left(\dot{x}, \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \right).$$

Example 3.6. For the pendulum on a cart example:

$$\mathfrak{H}_{\mathbf{C}} = (D_{\mathbf{C}}, S_{\mathbf{C}}, R_{\mathbf{C}}, X_{\mathbf{C}}),$$

where $q = (\theta, x)^T$, $\dot{q} = (\dot{\theta}, \dot{x})^T$,

$$\begin{aligned} D_{\mathbf{C}} &= \{(q,\dot{q}) \in (\mathbb{S}^1 \times \mathbb{R}) \times \mathbb{R}^2 : \cos(\theta) \geq 0\}, \\ S_{\mathbf{C}} &= \{(q,\dot{q}) \in (\mathbb{S}^1 \times \mathbb{R}) \times \mathbb{R}^2 : \cos(\theta) = 0 \text{ and } \sin(\theta)\dot{\theta} \geq 0\}, \end{aligned}$$

and $R_{\mathbf{C}}(q,\dot{q}) = (q, P_{\mathbf{C}}(q,\dot{q}))$, where

$$P_{\mathbf{C}}(q,\dot{q}) = \begin{pmatrix} -e\dot{\theta} \\ \dot{x} \end{pmatrix}$$

with $0 \leq e \leq 1$ the coefficient of restitution. Finally,

$$X_{\mathbf{C}}(q,\dot{q}) = \left(\dot{q}, \begin{pmatrix} \frac{\sin(\theta)(-g(m+M)+mR\cos(\theta)\dot{\theta}^2)}{-(m+M)R+mR\cos(\theta)^2} \\ -\frac{m\sin(\theta)(g\cos(\theta)-R\dot{\theta}^2)}{m+M-m\cos(\theta)^2} \end{pmatrix}\right)$$

3.2.b Hamiltonian Hybrid Systems

We now discuss how to obtain simple hybrid systems from simple hybrid mechanical systems.

Definition 3.5. We say that $\mathfrak{H} = (D, S, R, X)$ is a **Hamiltonian hybrid system** with respect to a Hamiltonian H if there exists a symplectic form ω on D such that (D, ω, X) is a Hamiltonian system with respect to the Hamiltonian H, i.e., $d(H) = \iota_X \omega$.

3.2.4 Hybrid manifolds from HMS's. In order to construct a hybrid system from a HMS, we begin by constructing the hybrid manfold $\mathbf{D}^{\mathfrak{H}}_{\mathbf{H}} = (D_{\mathbf{H}}, S_{\mathbf{H}}, R_{\mathbf{H}})$ from a HMS $\mathbf{H} = (Q, H, h)$. First, $D_{\mathbf{H}}$ and $S_{\mathbf{H}}$ are given as follows:

$$\begin{aligned} D_{\mathbf{H}} &= \{(q,p) \in T^\star Q : h(q) \geq 0\}, \\ S_{\mathbf{H}} &= \{(q,p) \in T^\star Q : h(q) = 0 \text{ and } \langle\langle p, dh_q \rangle\rangle_q \leq 0\}. \end{aligned}$$

This is exactly the set up in mechanical systems with unilateral constraints. With this in mind, we can define a reset map $R_{\mathbf{H}}$ by

$$R_{\mathbf{H}}(q,p) = (q, P_q(p)), \tag{3.8}$$

where $P_q : T^\star_q Q \to T^\star_q Q$ is given by

$$P_q(p) = p - (1+e)\frac{\langle\langle p, dh_q \rangle\rangle_q}{||dh_q||^2_q} dh_q \tag{3.9}$$

with $0 \leq e \leq 1$ is the coefficient of restitution.

3.2.5 Hybrid systems from HMS's. We already have constructed a hybrid manifold $\mathbf{D}^{\mathfrak{H}}_{\mathbf{H}} = (D_{\mathbf{H}}, S_{\mathbf{H}}, R_{\mathbf{H}})$ from the hybrid mechanical system $\mathbf{H}$, so it only remains to define the vector field $X_{\mathbf{H}}$. Using the canonical symplectic form, $\omega_{\text{canonical}}$, we define $X_{\mathbf{H}} = X_H$ as given in (3.6). Finally, $\mathfrak{H}_{\mathbf{H}} = (D_{\mathbf{H}}, S_{\mathbf{H}}, R_{\mathbf{H}}, X_{\mathbf{H}})$. Therefore, for $\mathbf{H} = (Q, H, h)$, the hybrid system $\mathfrak{H}_{\mathbf{H}}$ is a Hamiltonian hybrid system w.r.t. the Hamiltonian H.

Example 3.7. The hybrid manifold for the spherical pendulum (Example 3.3),

$$\mathbf{D}^{\mathfrak{H}_{\mathbf{P}}}_{\mathbf{P}} = (D_{\mathbf{P}}, S_{\mathbf{P}}, R_{\mathbf{P}}),$$

is given by

$$\begin{aligned} D_{\mathbf{P}} &= \{(q,p) \in \mathbb{S}^2 \times \mathbb{R}^2 : \cos(\theta) \geq 0\}, \\ S_{\mathbf{P}} &= \{(q,p) \in \mathbb{S}^2 \times \mathbb{R}^2 : \cos(\theta) = 0 \text{ and } p_\theta \geq 0\}, \end{aligned}$$

and

$$R_{\mathbf{P}}(q,p) = \left(q, \begin{pmatrix} -e p_\theta \\ p_\varphi \end{pmatrix} \right).$$

Finally, the vector field is given by

$$X_{\mathbf{P}}(q,p) = \left(\begin{pmatrix} \frac{p_\theta}{mR^2} \\ \frac{p_\varphi}{mR^2 \sin^2(\theta)} \end{pmatrix}, \begin{pmatrix} \frac{-p_\varphi^2}{mR^2 \cos(\theta) \sin^2(\theta)} - mgR\sin(\theta) \\ 0 \end{pmatrix} \right)$$

and $\mathfrak{H}_{\mathbf{P}} = (D_{\mathbf{P}}, S_{\mathbf{P}}, R_{\mathbf{P}}, X_{\mathbf{P}})$.

3.3 Hybrid Routhian Reduction

In this section, we begin by reviewing classical (or "non-hybrid") Routhian reduction (cf. [87] and the references to the subject therein). The motivation for this is that the hybrid version of Routhian reduction nicely mirrors the classical version, and the construction and definitions needed for classical Routhian reduction are also needed for hybrid Routhian reduction.

We then proceed to generalize Routhian reduction to a hybrid setting, first for Lagrangian hybrid systems associated to hybrid Lagrangians, and then for general Lagrangian hybrid systems. In both cases we derive conditions on when "hybrid" Routhian reduction can be carried out. In the first case, these conditions are concrete and easily verifiable, and in the later case, they are more general but also more abstract. Finally, Routhian hybrid systems are related to Lagrangian hybrid systems obtained from hybrid Routhians.

3.3.a A Review of Routhian Reduction

Consider the Lie group

$$\mathbb{G} = \underbrace{(\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1)}_{m-\text{times}} \times \mathbb{R}^p \tag{3.10}$$

with $k = m + p = \dim(\mathbb{G})$; here $\mathbb{S}^1$ is the circle. The starting point for classical Routhian reduction is a configuration space of the form[5]

$$Q = Q_\mu \times \mathbb{G},$$

where Q_μ is called the *shape space*; we denote an element $q \in Q$ by $q = (\theta, \varphi)$ where $\theta \in Q_\mu$ and $\varphi \in \mathbb{G}$. Note that $\mathbb{G}$ is an abelian Lie group, with Lie algebra $\mathfrak{g} \cong \mathbb{R}^k$; this observation relates Routhian reduction to more general "non-abelian" forms of reduction (cf. [4, 86, 87]) that will be discussed in Chapter 4.

3.3.1 Cyclic Lagrangians. If $L : TQ \to \mathbb{R}$ is a Lagrangian—as given in (3.2)—then in order to carry out Routhian reduction, we must assume that L is *cyclic*, that is, independent of φ:

$$\frac{\partial L}{\partial \varphi} = 0.$$

This implies that we can write L in coordinates as[6]

$$\begin{aligned} L(\theta,\dot\theta,\varphi,\dot\varphi) &= \frac{1}{2}\begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix}^T \begin{pmatrix} M_\theta(\theta) & M_{\varphi,\theta}(\theta)^T \\ M_{\varphi,\theta}(\theta) & M_\varphi(\theta) \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix} - V(\theta) \\ &= \frac{1}{2}\left(\dot\theta^T M_\theta(\theta)\dot\theta + \dot\varphi^T M_\varphi(\theta)\dot\varphi\right) + \dot\varphi^T M_{\varphi,\theta}(\theta)\dot\theta - V(\theta). \end{aligned} \tag{3.11}$$

Here $M_\theta(\theta) \in \mathbb{R}^{n\times n}$ and $M_\varphi(\theta) \in \mathbb{R}^{k\times k}$ are both symmetric positive definite matrices and $M_{\varphi,\theta}(\theta) \in \mathbb{R}^{k\times n}$ with $n = \dim(Q_\mu)$ and $k = \dim(\mathbb{G})$.

3.3.2 Routhians. Fundamental to reduction is the notion of a momentum map $J : TQ \to \mathfrak{g}^\star \cong \mathbb{R}^k$, which makes explicit the conserved quantities in the system. In the framework we are considering here,

$$J(\theta,\dot\theta,\varphi,\dot\varphi) = \frac{\partial L}{\partial \dot\varphi}(\theta,\dot\theta,\varphi,\dot\varphi) = M_{\varphi,\theta}(\theta)\dot\theta + M_\varphi(\theta)\dot\varphi. \tag{3.12}$$

The *Routhian* $L_\mu : TQ_\mu \to \mathbb{R}$ is given by, for $\mu \in \mathbb{R}^k$,

$$L_\mu(\theta,\dot\theta) = \left[L(\theta,\dot\theta,\varphi,\dot\varphi) - \mu^T\dot\varphi\right]\Big|_{J^{-1}(\mu)}. \tag{3.13}$$

Because

$$J(\theta,\dot\theta,\varphi,\dot\varphi) = \mu \qquad \Rightarrow \qquad \dot\varphi = M_\varphi(\theta)^{-1}(\mu - M_{\varphi,\theta}(\theta)\dot\theta), \tag{3.14}$$

by direct calculation, the Routhian is given by

$$\begin{aligned} L_\mu(\theta,\dot\theta) &= \frac{1}{2}\dot\theta^T\left(M_\theta(\theta) - M_{\varphi,\theta}(\theta)^T M_\varphi(\theta)^{-1} M_{\varphi,\theta}(\theta)\right)\dot\theta + \mu^T M_\varphi(\theta)^{-1} M_{\varphi,\theta}(\theta)\dot\theta - V_\mu(\theta) \\ &:= \frac{1}{2}\dot\theta^T M_\mu(\theta)\dot\theta + \mu^T A(\theta)\dot\theta - V_\mu(\theta), \end{aligned} \tag{3.15}$$

where

$$V_\mu(\theta) = V(\theta) + \frac{1}{2}\mu^T M_\varphi(\theta)^{-1}\mu$$

is the *amended potential*.

[5] The shape space Q_μ is often denoted by "S" is the literature. We use the former notation rather than the later because we are reserving the symbol S for the switching surface of a hybrid system.

[6] Throughout the rest of this section, we will work in coordinates.

3.3.3 Reduction. From the vector field X_L on TQ, we obtain a vector field X_{L_μ} on TQ_μ from the Routhian via the Euler-Lagrange equations (3.1); see [87] for more details.

Note that we have a projection map

$$\begin{array}{rcl} \pi : TQ & \to & TQ_\mu \\ (\theta, \dot{\theta}, \varphi, \dot{\varphi}) & \mapsto & (\theta, \dot{\theta}). \end{array}$$

The main result of Routhian reduction is that flows of X_L project to flows of X_{L_μ}, i.e., there is the following well-known proposition (see [87]).

Proposition 3.1. *Let L be a cyclic Lagrangian, L_μ the associated Routhian, with X_L and X_{L_μ} the associated Lagrangian vector fields. If $c(t)$ is a flow of the X_L such that $c(t_0) \in J^{-1}(\mu)$, then $\pi(c(t))$ is a flow of X_{L_μ} with initial condition $\pi(c(t_0))$.*

3.3.b Hybrid Routhians

We now proceed to generalize Routhian reduction to a hybrid setting by considering *cyclic* hybrid Lagrangians. The intuition gained from considering these systems will be vital; for example, the similarity between the diagram in (3.17) and the diagram in (3.27) is not coincidental.

Definition 3.6. A **cyclic hybrid Lagrangian** is a hybrid Lagrangian, $\mathbf{L} = (Q, L, h)$, such that $Q = Q_\mu \times \mathbb{G}$, L is a cyclic Lagrangian and h is cyclic, i.e.,

$$\frac{\partial h}{\partial \varphi} = 0. \tag{3.16}$$

3.3.4 Hybrid Routhians. For a cyclic hybrid Lagrangian, $\mathbf{L} = (Q = Q_\mu \times \mathbb{G}, L, h)$, we obtain a reduced constraint function $h_\mu : Q_\mu \to \mathbb{R}$, where h_μ is the function h viewed as a function on S; this makes sense because h is assumed to be cyclic. From the cyclic Lagrangian $\mathbf{L}$, define the corresponding *hybrid Routhian* by:

$$\mathbf{L}_\mu = (Q_\mu, L_\mu, h_\mu),$$

which is again a hybrid Lagranigan. From this hybrid Routhian, we obtain a Routhian hybrid system associated to the hybrid Routhian $\mathbf{L}_\mu$:

$$\mathfrak{H}_{\mathbf{L}_\mu} = (D_{\mathbf{L}_\mu}, S_{\mathbf{L}_\mu}, R_{\mathbf{L}_\mu}, X_{\mathbf{L}_\mu}),$$

with $D_{\mathbf{L}_\mu} = D_{h_\mu}$, $S_{\mathbf{L}_\mu} = S_{h_\mu}$, $X_{\mathbf{L}_\mu} = X_{L_\mu}$ and

$$R_{\mathbf{L}_\mu}(\theta, \dot{\theta}) = (\theta, P_\mu(\theta, \dot{\theta})),$$

where

$$P_\mu(\theta, \dot{\theta}) = \dot{\theta} - (1+e)\frac{d(h_\mu)_\theta \dot{\theta}}{d(h_\mu)_\theta M_\mu(\theta)^{-1} d(h_\mu)_\theta^T} M_\mu(\theta)^{-1} d(h_\mu)_\theta^T.$$

Here $M_\mu(\theta)$ is defined as in (3.15).

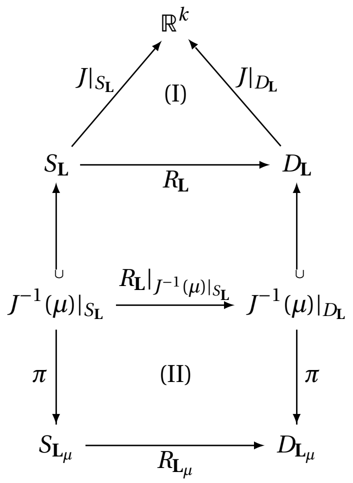
Theorem 3.1. *Let* $\mathbf{L}$ *be a cyclic hybrid Lagrangian,* $\mathbf{L}_\mu$ *the associated hybrid Routhian, with* $\mathfrak{H}_{\mathbf{L}}$ *and* $\mathfrak{H}_{\mathbf{L}_\mu}$ *the associated Lagrangian hybrid systems. If* $\epsilon^{\mathfrak{H}_{\mathbf{L}}}(x_0) = (\Lambda, I, C)$ *is a hybrid flow of* $\mathfrak{H}_{\mathbf{L}}$ *with* $x_0 \in J^{-1}(\mu)$*, then*

$$\epsilon^{\mathfrak{H}_{\mathbf{L}_\mu}}(\pi(x_0)) = (\Lambda, I, \pi(C))$$

is a hybrid flow of $\mathfrak{H}_{\mathbf{L}_\mu}$*, where* $\pi(C) = \{\pi(c_i) : c_i \in C\}$.

Before proving this theorem, we establish the following proposition which says that the conserved quantities are preserved by the reset map when they are obtained from cyclic hybrid Lagrangians.

Proposition 3.2. *If* $\mathbf{L}$ *is cyclic, then the following diagram*



(3.17)

commutes for all $\mu \in \mathfrak{g}^\star \cong \mathbb{R}^k$.

Proof. To show the commutativity of (3.17), we need to show that (I) and (II) commute. Before doing this, some set-up is required, i.e., we will first find an explicit formulation for $P_{\mathbf{L}}$, as defined in (3.7), based on the assumption that $\mathbf{L}$ is cyclic.

By (3.11) and block diagonal matrix inversion,

$$M(\theta,\varphi)^{-1} = \begin{pmatrix} M_\mu(\theta)^{-1} & -M_\mu(\theta)^{-1}A(\theta)^T \\ -A(\theta)M_\mu(\theta)^{-1} & M_\varphi(\theta)^{-1} + A(\theta)M_\mu(\theta)^{-1}A(\theta)^T \end{pmatrix}.$$

By (3.16),

$$dh_{(\theta,\varphi)} = \begin{pmatrix} d(h_\mu)_\theta & 0 \end{pmatrix}. \tag{3.18}$$

Combining these two equations implies that in the case when $\mathbf{L}$ is cyclic

$$P_{\mathbf{L}}(\theta,\varphi,\dot\theta,\dot\varphi) = \\ \begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix} - (1+e)\frac{d(h_\mu)_\theta\dot\theta}{d(h_\mu)_\theta M_\mu(\theta)^{-1}d(h_\mu)_\theta^T}\begin{pmatrix} M_\mu(\theta)^{-1}d(h_\mu)_\theta^T \\ -A(\theta)M_\mu(\theta)^{-1}d(h_\mu)_\theta^T \end{pmatrix}. \tag{3.19}$$

Using this, we demonstrate the commutativity of (I) and (II) in turn.

Commutativity of (I): Because of (3.12), to show the commutativity of (I) we need to show that

$$\begin{pmatrix} M_{\varphi,\theta}(\theta) & M_{\varphi}(\theta) \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix} = \begin{pmatrix} M_{\varphi,\theta}(\theta) & M_{\varphi}(\theta) \end{pmatrix} P_{\mathbf{L}}(\theta,\varphi,\dot\theta,\dot\varphi).$$

By (3.19), this is equivalent to showing that

$$\begin{pmatrix} M_{\varphi,\theta}(\theta) & M_{\varphi}(\theta) \end{pmatrix} \begin{pmatrix} M_{\mu}(\theta)^{-1} d(h_{\mu})_{\theta}^{T} \\ -A(\theta) M_{\mu}(\theta)^{-1} d(h_{\mu})_{\theta}^{T} \end{pmatrix} = 0,$$

which follows from the fact that

$$\begin{aligned} M_{\varphi,\theta}(\theta) M_{\mu}(\theta)^{-1} d(h_{\mu})_{\theta}^{T} &= M_{\varphi}(\theta) M_{\varphi}(\theta)^{-1} M_{\varphi,\theta}(\theta) M_{\mu}(\theta)^{-1} d(h_{\mu})_{\theta}^{T} \\ &= M_{\varphi}(\theta) A(\theta) M_{\mu}(\theta)^{-1} d(h_{\mu})_{\theta}^{T}. \end{aligned}$$

Commutativity of (II): First note that this diagram is well-defined (the codomain of π is $S_{\mathbf{L}_{\mu}}$) because

$$\begin{aligned} h(\theta,\varphi) = 0 \quad &\Rightarrow \quad h_{\mu}(\theta) = 0, \\ dh_{(\theta,\varphi)} \begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix} \leq 0 \quad &\Rightarrow \quad d(h_{\mu})_{\theta}\dot\theta \leq 0, \end{aligned}$$

by (3.18). Now, to show the commutativity of (II), we need only show that

$$\pi_{\dot\theta}\left(P_{\mathbf{L}}(\theta,\varphi,\dot\theta, M_{\varphi}(\theta)^{-1}(\mu - M_{\varphi,\theta}(\theta)\dot\theta))\right) = P_{\mathbf{L}_{\mu}}(\theta,\dot\theta),$$

where $\pi_{\dot\theta}$ is the projection onto the $\dot\theta$-component of $P_{\mathbf{L}}$ and

$$P_{\mathbf{L}_{\mu}}(\theta,\dot\theta) = \dot\theta - (1+e)\frac{d(h_{\mu})_{\theta}\dot\theta}{d(h_{\mu})_{\theta} M_{\mu}(\theta)^{-1} d(h_{\mu})_{\theta}^{T}} M_{\mu}(\theta)^{-1} d(h_{\mu})_{\theta}^{T}.$$

But, this can be seen by directly inspecting (3.19). □

Proof of Theorem 3.1. Let $c_i^{\mu}(t) = \pi(c_i(t))$. We need only show that

$$\begin{aligned} &(\mathrm{i}\mu) && c_i^{\mu}(\tau_{i+1}) \in S_{\mathbf{L}_{\mu}}, \\ &(\mathrm{ii}\mu) && R_{\mathbf{L}_{\mu}}(c_i^{\mu}(\tau_{i+1})) = c_{i+1}^{\mu}(\tau_{i+1}), \\ &(\mathrm{iii}\mu) && \dot{c}^{\mu}{}_i(t) = X_{\mathbf{L}_{\mu}}(c_i^{\mu}(t)) = X_{L_{\mu}}(c_i^{\mu}(t)). \end{aligned}$$

First, consider the case when $i = 0$. Since by assumption $x_0 = c_0(\tau_0) \in J^{-1}(\mu)$, we know that $c_0(t) \in J^{-1}(\mu)$ for all $t \in [\tau_0, \tau_1]$; since $c_0(\tau_1) \in S_{\mathbf{L}}$, this implies that $c_0(\tau_1) \in J^{-1}(\mu)|_{S_{\mathbf{L}}}$.

Condition (iμ)*:* Follows from the fact that

$$c_0(\tau_1) \in J^{-1}(\mu)|_{S_{\mathbf{L}}} \quad \Rightarrow \quad \pi(c_0(\tau_1)) = c_0^{\mu}(\tau_1) \in S_{\mathbf{L}_{\mu}}.$$

Condition (iiμ)*:* Follows from the commutativity of (II) in (3.17) since it implies that

$$R_{\mathbf{L}_{\mu}}(c_0^{\mu}(\tau_1)) = R_{\mathbf{L}_{\mu}}(\pi(c_0(\tau_1))) = \pi(R_{\mathbf{L}}(c_0(\tau_1))) = \pi(c_1(\tau_1)) = c_1^{\mu}(\tau_1).$$

Condition (iiiμ)*:* Follows from Proposition 3.1.

Finally, the commutativity of (I) in (3.17) implies that $c_1(\tau_1) \in J^{-1}(\mu)$. Therefore, the result follows by induction on i, i.e., the same argument that was utilized for $i = 0$ can be applied to any i such that $i, i+1 \in \Lambda$ together with the assumption that $c_i(\tau_i) \in J^{-1}(\mu)$. □

3.3.c Routhian Hybrid Systems

Definition 3.7. A Lagrangian hybrid system $\mathfrak{H} = (D_h, S_h, R, X_L)$ w.r.t. a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ is a **cyclic Lagrangian hybrid system** if $\mathbf{L}$ is a cyclic hybrid Lagrangian and the following diagram

$$\begin{array}{ccccc} & & \mathbb{R}^k & & \\ & \overset{J|_{S_h}}{\nearrow} & & \overset{J|_{D_h}}{\nwarrow} & \\ S_h & & \xrightarrow[R]{} & & D_h \end{array} \tag{3.20}$$

commutes.

3.3.5 Routhian hybrid systems. From a cyclic Lagrangian hybrid system, $\mathfrak{H}$, we can construct a *Routhian hybrid system*, $\mathfrak{H}_\mu$, which is a Lagrangian hybrid system with respect to the hybrid Routhian $\mathbf{L}_\mu$. We define this hybrid system as follows:

$$\mathfrak{H}_\mu = (D_\mu, S_\mu, R_\mu, X_\mu) = (D_{h_\mu}, S_{h_\mu}, R_\mu, X_{L_\mu}),$$

where $R_\mu : S_{h_\mu} \to D_{h_\mu}$ (possibly dependent on μ) is the induced map defined by the requirement that it make the following diagram

$$\begin{array}{ccc} J^{-1}(\mu)|_{S_h} & \xrightarrow{R|_{J^{-1}(\mu)|_{S_h}}} & J^{-1}(\mu)|_{D_h} \\ \pi \downarrow & & \downarrow \pi \\ S_{h_\mu} & \xrightarrow{R_\mu} & D_{h_\mu} \end{array} \tag{3.21}$$

commute for all $\mu \in \mathbb{R}^k$.

Theorem 3.2. *Let $\mathfrak{H}$ be a cyclic Lagrangian hybrid system, and $\mathfrak{H}_\mu$ the associated Routhian hybrid system. If $\epsilon^{\mathfrak{H}}(x_0) = (\Lambda, I, C)$ is a hybrid flow of $\mathfrak{H}$ with $x_0 \in J^{-1}(\mu)$, then*

$$\epsilon^{\mathfrak{H}_\mu}(\pi(x_0)) = (\Lambda, I, \pi(C))$$

is a hybrid flow of $\mathfrak{H}_\mu$, where $\pi(C) = \{\pi(c_i) : c_i \in C\}$.

Proof. The proof of this theorem is analogous to the proof of Theorem 3.1. □

It follows from Proposition 3.2, and specifically from the fact that the commutativity of (3.17) implies the commutativity (3.20) and (3.21), that the operation of "reduction" commutes.

Proposition 3.3. *Let $\mathfrak{H}_{\mathbf{L}}$ be the Lagrangian hybrid system associated to a cyclic hybrid Lagrangian $\mathbf{L}$, then $\mathfrak{H}_{\mathbf{L}}$ is a cyclic Lagrangian hybrid system and*

$$(\mathfrak{H}_{\mathbf{L}})_\mu = \mathfrak{H}_{\mathbf{L}_\mu}$$

where $\mathfrak{H}_{\mathbf{L}_\mu}$ is the Routhian hybrid system associated to the hybrid Routhian $\mathbf{L}_\mu$.

3.3.6 Commutativity of reduction. In the case when $\mathbf{L}$ is cyclic, $\mathfrak{H}_{\mathbf{L}}$ is cyclic and we can carry out Routhian reduction on this hybrid system to obtain a Routhian hybrid system $(\mathfrak{H}_{\mathbf{L}})_\mu$. This process is described graphically by the following diagram:

$$\mathbf{L} \xrightarrow{\text{association}} \mathfrak{H}_{\mathbf{L}} \xrightarrow{\text{reduction}} (\mathfrak{H}_{\mathbf{L}})_\mu.$$

Alternately, a cyclic hybrid Lagrangian can be reduced to obtain a hybrid Routhian $\mathbf{L}_\mu$, and to this hybrid Routhian we can associate a Lagrangian hybrid system $\mathfrak{H}_{\mathbf{L}_\mu}$; this again is described graphically by

$$\mathbf{L} \xrightarrow{\text{reduction}} \mathbf{L}_\mu \xrightarrow{\text{association}} \mathfrak{H}_{\mathbf{L}_\mu}.$$

Proposition 3.3 implies that the processes of “association” and “reduction” commute, i.e., the order in which they are taken is irrelevant. This can be visualized in a commuting diagram of the form:

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{\text{association}} & \mathfrak{H}_{\mathbf{L}} \\ \text{reduction} \downarrow & & \downarrow \text{reduction} \\ \mathbf{L}_\mu & \xrightarrow{\text{association}} & (\mathfrak{H}_{\mathbf{L}})_\mu = \mathfrak{H}_{\mathbf{L}_\mu} \end{array}$$

This result yields an explicit method for computing Routhian hybrid systems from cyclic hybrid Lagrangians.

3.3.7 Hybrid reconstruction. Suppose that $\epsilon^{\mathfrak{H}_{\mathbf{L}_\mu}}(c_0^\mu(\tau_0)) = (\Lambda, I, C_\mu)$ is a hybrid flow of $\mathfrak{H}_{\mathbf{L}_\mu}$. Then we can construct a hybrid flow $\epsilon^{\mathfrak{H}_{\mathbf{L}}}(c_0(\tau_0)) = (\Lambda, I, C)$ of $\mathfrak{H}_{\mathbf{L}}$ by reconstructing the flow recursively. Writing $c_i^\mu(t) = (\theta_i(t), \dot\theta_i(t))$, we define

$$c_i(t) = (\theta_i(t), \dot\theta_i(t), \varphi_i(t), \dot\varphi_i(t))$$

recursively to be:

$$\begin{aligned} \dot\varphi_i(t) &= M_\varphi(\theta_i(t))^{-1}(\mu - M_{\varphi,\theta}(\theta_i(t))\dot\theta_i(t)), \\ \varphi_i(t) &= \pi_\varphi(R(c_{i-1}(\tau_i))) + \int_{\tau_i}^{t-\tau_i} \dot\varphi_i(s)\,ds, \end{aligned}$$

where $t \in [\tau_i, \tau_{i+1}]$ and $\pi_\varphi(R(c_{i-1}(\tau_i)))$ is the φ-component of $R(c_{i-1}(\tau_i))$.

Example 3.8. For the ball bouncing on a sinusoidal surface (Example 3.1 and Example 3.5), the Lagrangian $L_{\mathbf{B}}$ has two cyclic variables: x_1 and x_2. Since $h_{\mathbf{B}}$ is only independent of one of these variables, the

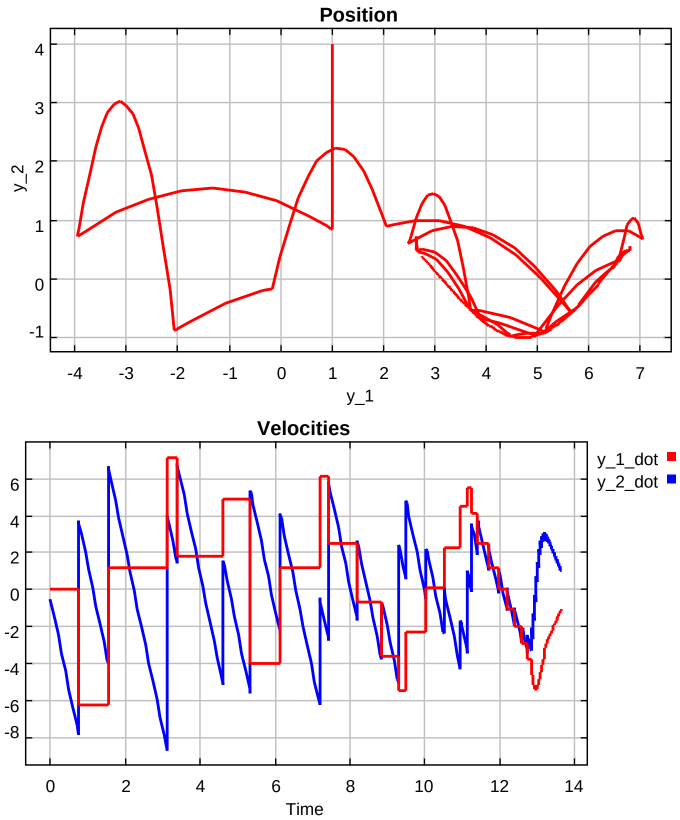


Figure 3.4: Positions y_1 vs. y_2 and velocities over time of $\mathfrak{H}_{\mathbf{B}_\mu}$.

only "hybrid" cyclic variable is x_1. That is, through continuous reduction we could reduce the dimensionality of the phase space by four, while through hybrid reduction we can only reduce the dimensionality of the phase space by two. Therefore, we will carry out hybrid Routhian reduction on the system with $\mathbb{G} = \mathbb{R}$.

Specifically, our hybrid Routhian is given by

$$\mathbf{B}_\mu = (Q_{\mathbf{B}_\mu}, L_{\mathbf{B}_\mu}, h_{\mathbf{B}_\mu}),$$

where $Q_{\mathbf{B}_\mu} = \mathbb{R}^2$, and for $y = (y_1, y_2)^T$,

$$L_{\mathbf{B}_\mu}(y, \dot{y}) = \frac{1}{2} m \|\dot{y}\|^2 - m g y_2 - \frac{1}{2}\frac{\mu^2}{m}.$$

Finally, $h_{\mathbf{B}_\mu}(y_1, y_2) = y_2 - \sin(y_1)$.

The Routhian hybrid system for the bouncing ball on a sinusoidal surface, as obtained from $\mathbf{B}_\mu$, is given by

$$\mathfrak{H}_{\mathbf{B}_\mu} = (D_{\mathbf{B}_\mu}, S_{\mathbf{B}_\mu}, R_{\mathbf{B}_\mu}, X_{\mathbf{B}_\mu}).$$

First, we define

$$\begin{aligned} D_{\mathbf{B}_\mu} &= \{(y,\dot{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : y_2 - \sin(y_1) \geq 0\}, \\ S_{\mathbf{B}_\mu} &= \{(y,\dot{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : y_2 = \sin(y_1) \text{ and } \dot{y}_2 - \cos(y_1)\dot{y}_1 \leq 0\}, \end{aligned}$$

and $R_{\mathbf{B}_\mu}(y,\dot{y}) = (y, P_{\mathbf{B}_\mu}(y,\dot{y}))$, where

$$P_{\mathbf{B}_\mu}(y,\dot{y}) = \begin{pmatrix} \frac{(1-e\cos(y_1)^2)\dot{y}_1 + (1+e)\cos(x_1)\dot{y}_2}{1+\cos(y_1)^2} \\ \frac{(1+e)\cos(y_1)\dot{y}_1 + (-e+\cos(y_1)^2)\dot{y}_2}{1+\cos(y_1)^2} \end{pmatrix}$$

with $0 \leq e \leq 1$ the coefficient of restitution. Finally,

$$X_{\mathbf{B}_\mu}(y,\dot{y}) = \left(\dot{y}, \begin{pmatrix} 0 \\ -g \end{pmatrix}\right).$$

A simulation of the reduced system $\mathfrak{H}_{\mathbf{B}_\mu}$ can be seen in Figure 3.4. Note that this system is Zeno (both the reduced and full-order system display Zeno behavior). This type of behavior will be discussed in detail in Chapter 5; in fact, Section 5.5 of this chapter discusses how to extend the hybrid flows of hybrid Lagrangians—a process which is illustrated on this example.

Example 3.9. For the pendulum on a cart (Example 3.2 and Example 3.6), the x variable is a cyclic variable for both the Lagrangian $L_{\mathbf{C}}$ and the hybrid Lagrangian $\mathbf{C}$. Therefore, we can carry out Routhian reduction with $\mathbb{G} = \mathbb{R}$. In this case

$$\mathbf{C}_\mu = (Q_{\mathbf{C}_\mu}, L_{\mathbf{C}_\mu}, h_{\mathbf{C}_\mu}),$$

where $Q_{\mathbf{C}_\mu} = \mathbb{S}^1$, and

$$J(\theta,\dot{\theta},x,\dot{x}) = mR\cos(\theta)\dot{\theta} + (M+m)\dot{x}.$$

So

$$L_{\mathbf{C}_\mu}(\theta,\dot{\theta}) = \frac{1}{2}\left(mR^2 - \frac{m^2R^2\cos(\theta)^2}{M+m}\right)\dot{\theta}^2 + \mu\left(\frac{mR\cos(\theta)}{M+m}\right)\dot{\theta} - V_{\mathbf{C}_\mu}(\theta),$$

with

$$V_{\mathbf{C}_\mu}(\theta) = mgR\cos(\theta) + \frac{\mu^2}{2(M+m)}$$

the amended potential. Finally, $h_{\mathbf{C}_\mu}(\theta,) = \cos(\theta)$.

The Routhian hybrid system for the pendulum on a cart example is given by:

$$\mathfrak{H}_{\mathbf{C}_\mu} = (D_{\mathbf{C}_\mu}, S_{\mathbf{C}_\mu}, R_{\mathbf{C}_\mu}, X_{\mathbf{C}_\mu}),$$

where

$$\begin{aligned} D_{\mathbf{C}_\mu} &= \{(\theta,\dot{\theta}) \in \mathbb{S}^1 \times \mathbb{R} : \cos(\theta) \geq 0\}, \\ S_{\mathbf{C}_\mu} &= \{(\theta,\dot{\theta}) \in \mathbb{S}^1 \times \mathbb{R} : \cos(\theta) = 0 \text{ and } \sin(\theta)\dot{\theta} \geq 0\}, \end{aligned}$$

and $R_{\mathbf{C}_\mu}(\theta,\dot{\theta}) = (\theta, P_{\mathbf{C}_\mu}(\theta,\dot{\theta}))$, where

$$P_{\mathbf{C}_\mu}(\theta,\dot{\theta}) = \begin{pmatrix} -e\dot{\theta} \end{pmatrix}$$

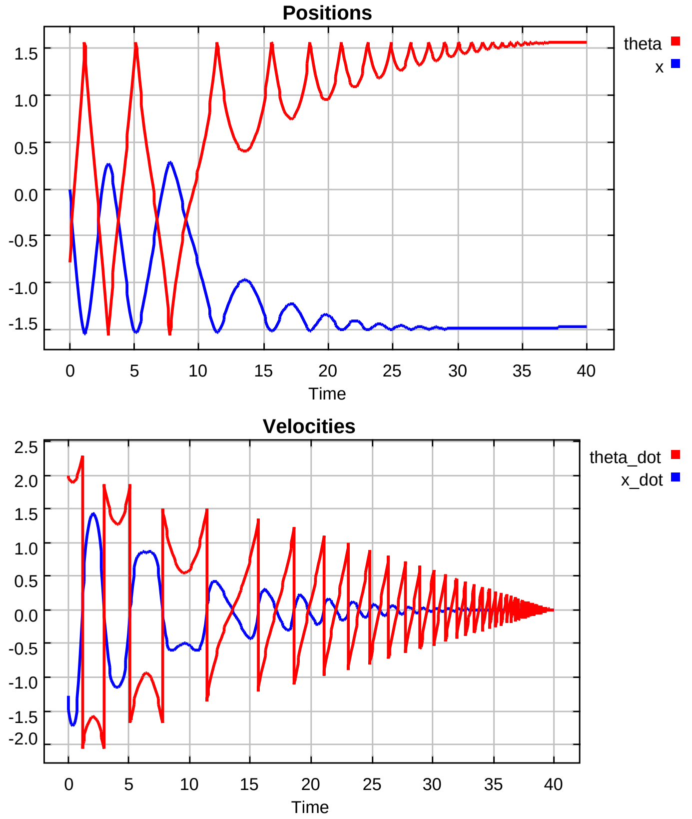


Figure 3.5: Positions and velocities over time, as reconstructed from the reduced system $\mathfrak{H}_{\mathbf{C}_\mu}$.

with $0 \leq e \leq 1$ the coefficient of restitution. Finally,

$$X_{\mathbf{C}_\mu}(\theta, \dot{\theta}) = \left(\dot{\theta}, \frac{\sin(\theta)(-g(m+M) + mR\cos(\theta)\dot{\theta}^2)}{R(-(m+M) + m\cos(\theta)^2)} \right).$$

The positions and velocities of the full-order system, as reconstructed from the reduced systems, can be see in Figure 3.5; in this simulation $m = 5, M = 50, R = 10, e = 0.9$ and $\mu = 0.1$. In this example, both the reduced and full order model are Zeno; again, Section 5.5 of Chapter 5 discusses how to extend the hybrid flow of this system past the Zeno point.

3.4 Simple Hybrid Reduction

We now turn our attention toward hybrid Hamiltonian reduction. Doing so necessarily requires the basic ingredients needed for classical reduction to be understood in a hybrid setting. As Hamiltonian reduction is more general than Routhian reduction, the ingredients necessary to perform this type of reduction are necessarily more sophisticated. We refer the reader to [4, 27, 79]-[88] for the prerequisite background material.

The main result of this section is the generalization of the classical reduction theorem [88] (as stated in Theorem 4.1) to simple hybrid Hamiltonian systems.

Remark 3.3. The hybrid objects studied in this section are studied in their more general form in Chapter 4. For example, here we consider hybrid Hamiltonian G-spaces which are a special case of the more general notion of a hybrid Hamiltonian **G**-space. The motivation for considering these "simple" hybrid objects is that they motivate the concepts introduced in Chapter 4, while simultaneously allowing us to better understand simple hybrid mechanical systems.

3.4.a Hybrid Hamiltonian G-spaces.

We begin by introducing the notion of a hybrid Hamiltonian G-space, the starting point for which is a Hamiltonian G-space (see Paragraph 4.4.1) with respect to the continuous portion of $\mathfrak{H}$. We discuss hybrid Hamiltonian G-spaces in the context of both simple hybrid systems and HMS's; in the later case, explicit constructions are carried out.

3.4.1 Hybrid group actions. Let $\mathfrak{H} = (D, S, R, X)$ be a hybrid system. Consider an action $\Phi : G \times D \to D$ of a Lie group G on D. We say that this is a *hybrid action* if $\Phi|_S$ is an action of G on S and for all $g \in G$

$$R \circ \Phi_g|_S = \Phi_g \circ R.$$

That is, for all $g \in G$ we have a commuting diagram:

$$\begin{array}{ccc} S & \xrightarrow{R} & D \\ \Big\downarrow{\scriptstyle \Phi_g|_S} & & \Big\downarrow{\scriptstyle \Phi_g} \\ S & \xrightarrow{R} & D \end{array} \tag{3.22}$$

Or in other words, R is equivariant with respect to the actions Φ and $\Phi|_S$. We say that Φ is a free and proper hybrid action, if Φ is a free and proper action that is hybrid. Similarly, Φ is a symplectic hybrid action if it is both symplectic and hybrid.

3.4.2 Hybrid orbit spaces. For the hybrid manifold $\mathbf{D}^{\mathfrak{H}} = (D, S, R)$, a Lie group G, and a hybrid action Φ, we define the hybrid orbit space as a tuple:

$$\mathbf{D}^{\mathfrak{H}}/G = (D/G, S/G, \widehat{R}),$$

where D/G and S/G are the orbit spaces of Φ and $\Phi|_S$, respectively, and $\widehat{R} : D/G \to S/G$ is the induced map. Specifically, the map $\pi : D \to D/G$, $\pi(x) = [x]$, is given by sending x to the Φ-orbit containing x: $x \sim \Phi_g(x)$ for all $g \in G$. The map $\widehat{R}$ is defined by requiring that $\widehat{R}([x]) = [R(x)]$; it is well-defined because of the commutativity of the diagram (3.22).

We would like to give conditions on the hybrid action Φ so that $\mathbf{D}^{\mathfrak{H}}/G$ is a hybrid manifold, i.e., such that we have a diagram

$$D/G \xleftarrow{\ \iota\ } S/G \xrightarrow{\ \widehat{R}\ } D/G$$

in which D/G is a smooth manifold, S/G is an embedded submanifold and $\widehat{R}$ is a smooth map. In fact, conditions for when these occur are well-known (cf. [4]).

Proposition 3.4. *If $\Phi : G \times D \to D$ is a free and proper hybrid action, then $\mathbf{D}^{\mathfrak{H}}/G$ is a hybrid manifold. Moreover, there is a submersion $\pi : D \to D/G$ such that the following diagram*

$$\begin{array}{ccccc} D & \xleftarrow{\ \iota\ } & S & \xrightarrow{\ R\ } & D \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi|_S} & & \downarrow{\scriptstyle \pi} \\ D/G & \xleftarrow{\ \iota\ } & S/G & \xrightarrow{\ \widehat{R}\ } & D/G \end{array}$$

commutes and $\pi|_S$ is a submersion.

Definition 3.8. An $\mathrm{Ad}^\star$-equivariant momentum map $J : D \to \mathfrak{g}^\star$ is said to be a **hybrid** $\mathrm{Ad}^\star$-equivariant momentum map if the following diagram

$$\begin{array}{ccccc} & & \mathfrak{g}^\star & & \\ & {\scriptstyle J}\nearrow & \uparrow{\scriptstyle J|_S} & \nwarrow{\scriptstyle J} & \\ D & \xleftarrow{\ \iota\ } & S & \xrightarrow{\ R\ } & D \end{array} \tag{3.23}$$

commutes.

Definition 3.9. A **hybrid Hamiltonian G-space** is defined to be a tuple

$$(\mathbf{D}^{\mathfrak{H}}, \omega, \Phi, J)$$

such that (D, ω) is a symplectic manifold, Φ is a symplectic hybrid action, and J is a hybrid Ad^*-equivariant momentum map.

3.4.3 Lifted group actions.

For a hybrid mechanical system, $\mathbf{H} = (Q, H, h)$,

$$D_{\mathbf{H}} = T^\star Q|_{\{h(q)\geq 0\}}.$$

Therefore, it is natural to consider actions on $T^\star Q$ that are obtained by lifting an action on Q. Specifically, for an action $\Psi : G \times Q \to Q$, we obtain an action of G on $T^\star Q$ by cotangent lifts, i.e., we obtain an action $\Psi^{T^\star} : G \times T^\star Q \to T^\star Q$ by defining

$$\Psi^{T^\star}(g, (q, p)) := T^\star \Psi_{g^{-1}}(q, p) = (\Psi_g(q), \Psi^\star_{g^{-1}}(p)).$$

It is possible to give conditions on when this action is a hybrid action by considering the constraint function h, the potential energy V, and H.

Definition 3.10. A simple hybrid mechanical system $\mathbf{H} = (Q, H, h)$ is said to be **G-invariant** if there is an action Ψ of G on Q such that h, the potential energy V, and H are G-invariant:

$$h(\Psi_g(q)) = h(q), \qquad V(\Psi_g(q)) = V(q), \qquad H(\Psi_g^{T^\star}(q,p)) = H(q,p),$$

for all $g \in G$ and $q \in Q$; note that the last assumption says that H is G-invariant under the lifted action, which holds iff

$$\langle\langle \Psi^\star_{g^{-1}}(\cdot), \Psi^\star_{g^{-1}}(\cdot)\rangle\rangle_{\Psi_g(q)} = \langle\langle \cdot,\cdot \rangle\rangle_q, \tag{3.24}$$

when coupled with the assumption on the G-invariance of the potential energy.

Proposition 3.5. *If* $\mathbf{H} = (Q, H, h)$ *is G-invariant, then the lifted action* $\Psi^{T^\star}$ *of* G *on* $D_{\mathbf{H}}$ *is a hybrid action.*

Proof. We need to show that for all $g \in G$ we have a commuting diagram

$$\begin{array}{ccc} S_{\mathbf{H}} & \xrightarrow{R_{\mathbf{H}}} & D_{\mathbf{H}} \\ {\scriptstyle \Psi_g^{T^\star}|_{S_{\mathbf{H}}}}\downarrow & & \downarrow{\scriptstyle \Psi_g^{T^\star}} \\ S_{\mathbf{H}} & \xrightarrow{R_{\mathbf{H}}} & D_{\mathbf{H}} \end{array}$$

where $R_{\mathbf{H}}$ is given in (3.9). Because of the special form of $R_{\mathbf{H}}$, this is equivalent to showing that the following diagram commutes

$$\begin{array}{ccc} T_q^\star Q & \xrightarrow{P_q} & T_q^\star Q \\ {\scriptstyle \Psi^\star_{g^{-1}}}\downarrow & & \downarrow{\scriptstyle \Psi^\star_{g^{-1}}} \\ T^\star_{\Psi_g(q)} Q & \xrightarrow{P_{\Psi_g(q)}} & T^\star_{\Psi_g(q)} Q \end{array}$$

for all $g \in G$ and $q \in h^{-1}(0)$. First, note that by the G-invariance of h, $\Psi^\star_{g^{-1}}(dh_q) = dh_{\Psi_g(q)}$. This, coupled with our assumption on the invariance of the inner-product under the lifted action (3.24), implies that

$$\begin{aligned}
\Psi^\star_{g^{-1}} \circ P_q(p) &= \Psi^\star_{g^{-1}}(p) + \Psi^\star_{g^{-1}}\Big(-(1+e)\frac{\langle\langle p, dh_q\rangle\rangle_q}{||dh_q||_q^2} dh_q\Big) \\
&= \Psi^\star_{g^{-1}}(p) - (1+e)\frac{\langle\langle p, dh_q\rangle\rangle_q}{||dh_q||_q^2} \Psi^\star_{g^{-1}}(dh_q) \\
&= \Psi^\star_{g^{-1}}(p) - (1+e)\frac{\langle\langle p, dh_q\rangle\rangle_q}{||dh_q||_q^2} dh_{\Psi_g(q)}. \\
&= \Psi^\star_{g^{-1}}(p) - (1+e)\frac{\langle\langle \Psi^\star_{g^{-1}}(p), \Psi^\star_{g^{-1}}(dh_q)\rangle\rangle_{\Psi_g(q)}}{||\Psi^\star_{g^{-1}}(dh_q)||_{\Psi_g(q)}^2} dh_{\Psi_g(q)}. \\
&= \Psi^\star_{g^{-1}}(p) - (1+e)\frac{\langle\langle \Psi^\star_{g^{-1}}(p), dh_{\Psi_g(q)}\rangle\rangle_{\Psi_g(q)}}{||dh_{\Psi_g(q)}||_{\Psi_g(q)}^2} dh_{\Psi_g(q)}. \\
&= P_{\Psi_g(q)} \circ \Psi^\star_{g^{-1}}(p)
\end{aligned}$$

as desired. □

3.4.4 Momentum maps for HMS's. For simple mechanical systems, there is an explicit definition of an $\mathrm{Ad}^\star$-equivariant momentum map $J_{\mathbf{H}}$. Let Ψ be the action of G on Q, and define a vector field on Q by

$$\xi_Q(q) = \frac{\mathrm{d}}{\mathrm{dt}}\Psi(\exp(t\xi), q)\Big|_{t=0} \in T_q Q, \tag{3.25}$$

for $\xi \in T_e G \cong \mathfrak{g}$. Using this, we can define $J_{\mathbf{H}}$, and prove that it is a hybrid $\mathrm{Ad}^\star$-equivariant momentum map under easily verifiable conditions.

Proposition 3.6. *For* $\mathbf{H} = (Q, H, h)$*, if* h *is G-invariant, then* $J_{\mathbf{H}} : D_{\mathbf{H}} \to \mathfrak{g}^\star$ *defined by*

$$\langle J_{\mathbf{H}}(q,p), \xi\rangle = \langle p, \xi_Q(q)\rangle,$$

is a hybrid $\mathrm{Ad}^\star$*-equivariant momentum map.*

Proof. We are assuming that $h^{-1}(0)$ is a manifold. For $q \in h^{-1}(0)$, the first step is to show that $\xi_Q(q) \in T_q h^{-1}(0)$, but this follows from the assumption that h is G-invariant, i.e., $h(\Psi(\exp(t\xi), q)) = h(q) = 0$ for all t. Therefore, for $q \in h^{-1}(0)$,

$$\langle dh_q, \xi_Q(q)\rangle = 0.$$

Now for $(q,p) \in S_{\mathbf{H}}$, (wherein it follows that $h(q) = 0$),

$$\begin{aligned}
\langle J_{\mathbf{H}}(R_{\mathbf{H}}(q,p)), \xi\rangle &= \langle p - (1+e)\frac{\langle\langle p, dh_q\rangle\rangle_q}{||dh_q||_q^2} dh_q, \xi_Q(q)\rangle \\
&= \langle p, \xi_Q(q)\rangle - (1+e)\frac{\langle\langle p, dh_q\rangle\rangle_q}{||dh_q||_q^2}\langle dh_q, \xi_Q(q)\rangle \\
&= \langle J_{\mathbf{H}}(q,p), \xi\rangle.
\end{aligned}$$

This, coupled with the fact that $J_{\mathbf{H}}$ is $\mathrm{Ad}^\star$-equivariant (cf. [4]), yields the desired result. □

Combining the results from Propositions 3.5 and 3.6, we have the following theorem that provides easily verifiable conditions on when a specific Hamiltonian G-space associated to a HMS is a hybrid Hamiltonian G-space.

Theorem 3.3. *If* $\mathbf{H} = (Q, H, h)$ *is G-invariant, then*

$$(\mathbf{D}_{\mathbf{H}}^{\mathfrak{H}}, \omega, \Psi^{T^\star}, J_{\mathbf{H}})$$

is a hybrid Hamiltonian G-space.

Example 3.10. For the spherical pendulum mounted on the ground (Example 3.3 and Example 3.7), $G_{\mathbf{P}} = \mathbb{S}^1$, which acts by rotations about the vertical axis, i.e., $\Psi_{\mathbf{P}} : \mathbb{S}^1 \times Q_{\mathbf{P}} \to Q_{\mathbf{P}}$ is given by

$$\Psi_{\mathbf{P}}(\psi, (\theta, \varphi)) = \begin{pmatrix} \theta \\ \varphi + \psi \end{pmatrix},$$

and the lifted action on $D_{\mathbf{P}}$ is given by

$$\Psi^{T^\star}_{\mathbf{P}}(\psi,(\theta,\varphi,p_\theta,p_\varphi)) = \left(\left(\begin{array}{c}\theta\\ \varphi+\psi\end{array}\right),\left(\begin{array}{c}p_\theta\\ p_\varphi\end{array}\right)\right),$$

which is clearly a hybrid action by Proposition 3.5. Now for $\xi \in \mathfrak{g}_{\mathbf{P}} \cong \mathbb{R}$,

$$\xi_{Q_{\mathbf{P}}}(\theta,\varphi) = \left(\begin{array}{c}0\\ \xi\end{array}\right) \in T_{(\theta,\varphi)}Q_{\mathbf{P}}$$

so the momentum map is given by

$$J_{\mathbf{P}}(\theta,\varphi,p_\theta,p_\varphi) = p_\varphi,$$

which is a hybrid momentum map by Proposition 3.6. Finally, it follows from Theorem 3.3 that

$$(\mathbf{D}^{\mathfrak{H}_{\mathbf{P}}}_{\mathbf{P}}, \omega_{\text{canonical}}, \Psi^{T^\star}_{\mathbf{P}}, J_{\mathbf{P}})$$

is a hybrid Hamiltonian G-space.

3.4.b Simple Hybrid System Reduction

We use the classical reduction theorem (see Theorem 4.1) to prove the existence of a reduced Hamiltonian hybrid system given a Hamiltonian hybrid system together with a hybrid Hamiltonian G-space. Moreover, we are able to prove a relationship between the hybrid flows of these two systems—a result that is very similar to the classical trajectory reduction theorem.

3.4.5 The reduced phase space. Let (D,ω,Φ,J) be a Hamiltonian G-space, and assume that $\mu \in \mathfrak{g}^\star$ is a regular value of J. If

$$G_\mu = \{g \in G : \mathrm{Ad}^\star_{g^{-1}}(\mu) = \mu\}$$

is the isotropy subgroup of G (see Paragraph 4.2.6 and Paragraph 4.2.7), then the action Φ of G on D restricts to an action of G_μ on $J^{-1}(\mu)$,

$$\Phi : G_\mu \times J^{-1}(\mu) \to J^{-1}(\mu)$$

because of the $\mathrm{Ad}^\star$-equivariance of J. Moreover, if the action of G_μ on $J^{-1}(\mu)$ is free and proper, then $D_\mu = J^{-1}(\mu)/G_\mu$ is a manifold, referred to as the reduced phase space, and there is a submersion $\pi_\mu : J^{-1}(\mu) \to D_\mu$. Finally, the main theorem of [88] (see Theorem 4.1 for a formal statement of this theorem) says that D_μ has a unique symplectic form ω_μ with the property

$$\Omega^2(\pi_\mu)(\omega_\mu) = \Omega^2(\iota_\mu)(\omega)$$

where $\iota_\mu : J^{-1}(\mu) \to D$ is the inclusion and Ω^2 is the 2-form functor (see Paragraph 4.1.3).

3.4.6 Hybrid regular values. Let $\mathfrak{H} = (D, S, R, X)$ be a hybrid system. Suppose that μ is a regular value of $J : D \to \mathfrak{g}^\star$. We say that this is a *hybrid regular value* if it is also a regular value of $J|_S$. This implies, when coupled with the commuting diagram (3.23), that the following diagram

$$\begin{array}{ccccc} J^{-1}(\mu) & \xleftarrow{\iota} & J|_S^{-1}(\mu) & \xrightarrow{R|_{J|_S^{-1}(\mu)}} & J^{-1}(\mu) \\ \downarrow & & \downarrow & & \downarrow \\ D & \xleftarrow{\iota} & S & \xrightarrow{R} & D \end{array} \tag{3.26}$$

commutes, where $J^{-1}(\mu)$ and $J|_S^{-1}(\mu)$ are embedded submanifolds.

Theorem 3.4. *Let $(\mathbf{D}^{\mathfrak{H}}, \omega, \Phi, J)$ be a hybrid Hamiltonian G-space. Assume $\mu \in \mathfrak{g}^\star$ is a hybrid regular value of a hybrid $\mathrm{Ad}^\star$-equivariant momentum map J and that the action of G_μ on $J^{-1}(\mu)$ is free, proper and hybrid. Then*

$$\begin{aligned} \mathbf{D}^{\mathfrak{H}}_\mu &= (D_\mu, S_\mu, R_\mu) \\ &:= \left(J^{-1}(\mu)/G_\mu, J|_S^{-1}(\mu)/G_\mu, \widehat{R|_{J|_S^{-1}(\mu)}}\right) \end{aligned}$$

is a hybrid manifold.

Proof. We need to show that a hybrid action Φ of G on D restricts to a hybrid action of G_μ on $J^{-1}(\mu)$, and then the result follows from Proposition 3.4.

Because μ is a hybrid regular value, $J|_S^{-1}(\mu)$ is a manifold that is clearly a submanifold of $J^{-1}(\mu)$. We are assuming that Φ is a hybrid action, so by the $\mathrm{Ad}^\star$-equivariance of J, $\Phi|_S$ restricts to an action of G_μ on $J|_S^{-1}(\mu)$:

$$\Phi|_S : G_\mu \times J|_S^{-1}(\mu) \to J|_S^{-1}(\mu).$$

To see this, note that for $x \in J|_S^{-1}(\mu)$, i.e., $x \in S$ such that $J(x) = \mu$, and for $g \in G_\mu$, $\Phi_g(x) \in S$ and

$$J(\Phi_g(x)) = \mathrm{Ad}^\star_{g^{-1}}(\mu) = \mu,$$

so $\Phi_g(x) \in J|_S^{-1}(\mu)$.

Therefore, to complete the proof we must show that $R|_{J|_S^{-1}(\mu)}$ is equivariant with respect to the action of G_μ on $J|_S^{-1}(\mu)$ and the action of G_μ on $J^{-1}(\mu)$, i.e., we must show that the following diagram

$$\begin{array}{ccc} J|_S^{-1}(\mu) & \xrightarrow{R|_{J|_S^{-1}(\mu)}} & J^{-1}(\mu) \\ \Phi_g|_S \downarrow & & \downarrow \Phi_g \\ J|_S^{-1}(\mu) & \xrightarrow{R|_{J|_S^{-1}(\mu)}} & J^{-1}(\mu) \end{array}$$

commutes for $g \in G_\mu$. This follows from the equivariance of R. □

3.4.7 The reduced hybrid phase space. The hybrid manifold introduced in the above theorem is referred to as the *reduced hybrid phase space*. To better understand this hybrid manifold, note that the submersion π_μ together with (3.23) and (3.26) yields the following commuting diagram

$$
\begin{array}{ccccc}
 & & \mathfrak{g}^\star & & \\
 & \nearrow^{J} & \uparrow {\scriptstyle J|_S} & \nwarrow^{J} & \\
D & \xleftarrow{\ \iota\ } & S & \xrightarrow{\ R\ } & D \\
\uparrow & & \uparrow & & \uparrow \\
J^{-1}(\mu) & \xleftarrow{\ \iota\ } & J|_S^{-1}(\mu) & \xrightarrow{R|_{J|_S^{-1}(\mu)}} & J^{-1}(\mu) \\
\downarrow {\scriptstyle \pi_\mu} & & \downarrow {\scriptstyle \pi_\mu|_{J|_S^{-1}(\mu)}} & & \downarrow {\scriptstyle \pi_\mu} \\
D_\mu = J^{-1}(\mu)/G_\mu & \xleftarrow{\ \iota\ } & S_\mu = J|_S^{-1}(\mu)/G_\mu & \xrightarrow{\ R_\mu\ } & D_\mu = J^{-1}(\mu)/G_\mu
\end{array} \tag{3.27}
$$

where $\pi_\mu|_{J|_S^{-1}(\mu)}$ is also a submersion; this implies that R_μ is defined by requiring that the bottom right square in this diagram commute. Note the similarity between this diagram and the one given in Proposition 3.2.

3.4.8 Reduced Hamiltonians. We refer the reader to Section 4.5 for the definition of G-invariant Hamiltonians and Hamiltonian systems. If H is a G-invariant Hamiltonian on D, then the *reduced Hamiltonian* H_μ on D_μ is defined uniquely by requiring that

$$H_\mu \circ \pi_\mu = H \circ \iota_\mu. \tag{3.28}$$

If (D, ω, X_H) is a Hamiltonian system for the Hamiltonian H, then the classical reduction theorem of [88] says that there is an associated reduced Hamiltonian system $(D_\mu, \omega_\mu, X_{H_\mu})$ for the Hamiltonian H_μ. Moreover, these two Hamiltonian Systems are related to each other in a way analogous to the relationship given in Proposition 3.1. If $c(t)$ is the flow of X_H with initial condition $c(t_0) \in J^{-1}(\mu)$, then $\pi_\mu(c(t))$ is a flow of X_{H_μ} with initial condition $\pi_\mu(c(t_0))$. This fact will be used to prove a result similar to Theorem 3.1, but first we give conditions on when a reduced Hamiltonian hybrid system can be obtained.

Theorem 3.5. *Given a Hamiltonian hybrid system $\mathfrak{H} = (D, S, R, X)$ w.r.t. a G-invariant Hamiltonian H, and an associated hybrid Hamiltonian G-space satisfying the assumptions of Theorem 3.4, then there is a reduced Hamiltonian hybrid system w.r.t. H_μ,*

$$\mathfrak{H}_\mu = (D_\mu, S_\mu, R_\mu, X_\mu),$$

where $\mathbf{D}_\mu^{\mathfrak{H}} = (D_\mu, S_\mu, R_\mu)$ is defined as in Theorem 3.4, and X_μ is defined by $d(H_\mu) = \iota_{X_\mu}\omega_\mu$.

Proof. Follows from Theorem 3.4, coupled with the classical reduction results. □

Theorem 3.6. *With $\mathfrak{H}$ and $\mathfrak{H}_\mu$ as in Theorem 3.5, if $\epsilon^{\mathfrak{H}}(x_0)$ is a hybrid flow of $\mathfrak{H}$ with $x_0 \in J^{-1}(\mu)$, then there is a corresponding hybrid flow $\epsilon^{\mathfrak{H}_\mu}$ of $\mathfrak{H}_\mu$ defined by*

$$\epsilon^{\mathfrak{H}_\mu}(\pi_\mu(x_0)) = (\Lambda, I, \pi_\mu(C)),$$

where $\pi_\mu(C) := \{\pi_\mu(c_i) : c_i \in C\}$.

Proof. We need only show that

$$\begin{array}{lc} (\mathrm{i}\mu) & c_i^\mu(\tau_{i+1}) \in S_\mu, \\ (\mathrm{ii}\mu) & R_\mu(c_i^\mu(\tau_{i+1})) = c_{i+1}^\mu(\tau_{i+1}), \\ (\mathrm{iii}\mu) & \dot{c}_i^\mu(t) = X_\mu(c_i^\mu(t)), \end{array}$$

where $c_i^\mu(t) = \pi_\mu(c_i(t))$. Using arguments completely analogous to those given in the proof of Theorem 3.1, the commutativity of (3.27) and the classical trajectory reduction theorem, it is easy to see that conditions (iμ)-(iiiμ) are satisfied. □

The hybrid reduction result given in Theorem 3.5 only provides, to quote [102], "soft" information about the reduced Hamiltonian hybrid system in that it does not yield a method for explicitly constructing this system. There are more concrete methods for computing the reduced system by using methods from classical mechanics which allow for the explicit reduction of Hamiltonians (see [4, 86, 102]). The end result is two methods for reducing a hybrid system associated to a HMS, described graphically by:

$$\mathbf{H} \xrightarrow{\text{association}} \mathfrak{H}_{\mathbf{H}} \xrightarrow{\text{reduction}} (\mathfrak{H}_{\mathbf{H}})_\mu$$

$$\mathbf{H} \xrightarrow{\text{reduction}} \mathbf{H}_\mu \xrightarrow{\text{association}} \mathfrak{H}_{\mathbf{H}_\mu}$$

It is possible to show that the processes of "association" and "reduction" commute, i.e., the order in which they are taken is irrelevant as was the case with Routhian reduction (see Proposition 3.3). This can be visualized in a commuting diagram of the form:

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{\text{association}} & \mathfrak{H}_{\mathbf{H}} \\ \text{reduction} \big\downarrow & & \big\downarrow \text{reduction} \\ \mathbf{H}_{\vec{\mu}} & \xrightarrow{\text{association}} & (\mathfrak{H}_{\mathbf{H}})_\mu = \mathfrak{H}_{\mathbf{H}_\mu} \end{array}$$

This result yields a method for computing reduced hybrid systems obtained from HMS's.

Example 3.11. Returning to the spherical pendulum mounted on the ground, we can explicitly calculate the reduced hybrid system for this example. We first compute the associated reduced HMS $\mathbf{P}_\mu = (Q_{\mathbf{P}_\mu}, H_{\mathbf{P}_\mu}, h_{\mathbf{P}_\mu})$ and then associate to the system a simple hybrid system using the techniques outlined in

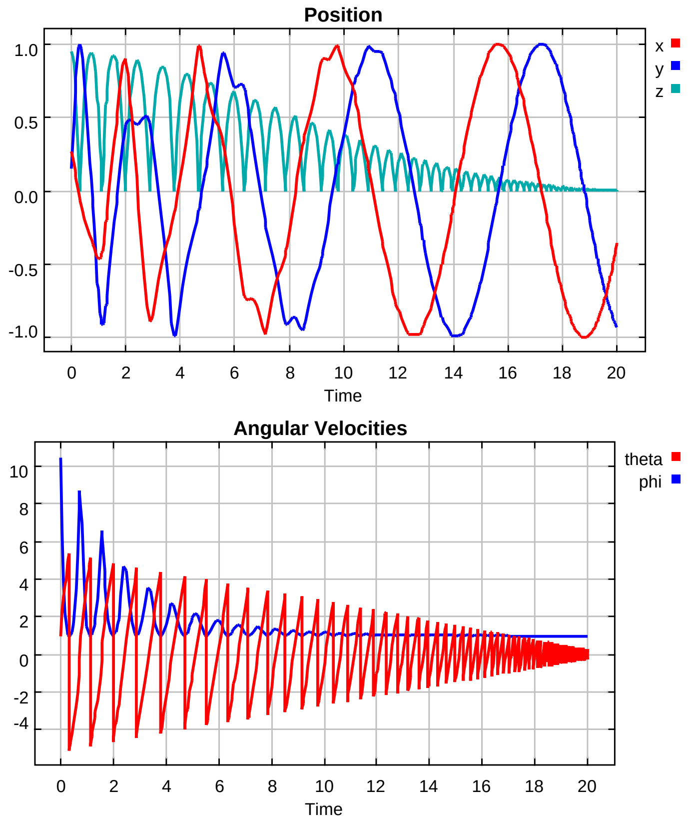


Figure 3.6: Reconstruction of the reduced spherical pendulum: position of the mass and angular velocities over time.

Section 3.2. The continuous portion of the reduction follows from [86]. In this example, $T^\star(Q_\mathbf{P}/G_\mathbf{P})$ is identified with $T^\star(S^1/\mathbb{Z}_2)$, i.e., $Q_{\mathbf{P}_\mu} = S^1/\mathbb{Z}_2$. The reduced Hamiltonian $H_{\mathbf{P}_\mu} : T^\star(Q_{\mathbf{P}_\mu}) = T^\star(S^1/\mathbb{Z}_2) \to \mathbb{R}$ is given by

$$H_{\mathbf{P}_\mu}(\theta, p_\theta) = \frac{1}{2}\frac{p_\theta^2}{mR^2} + mgR\cos(\theta) + \frac{1}{2}\frac{\mu^2}{mR^2\sin^2(\theta)}.$$

Finally, we have $h_{\mathbf{P}_\mu}(\theta) = \cos(\theta)$.

The hybrid manifold for the reduced spherical pendulum $\mathbf{D}_{\mathbf{P}_\mu}^{\mathfrak{H}_{\mathbf{P}_\mu}} = (D_{\mathbf{P}_\mu}, S_{\mathbf{P}_\mu}, R_{\mathbf{P}_\mu})$ is given by

$$\begin{aligned} D_{\mathbf{P}_\mu} &= \{(\theta, p_\theta) \in T^\star(Q_{\mathbf{P}_\mu}) : \cos(\theta) \geq 0\}, \\ S_{\mathbf{P}_\mu} &= \{(\theta, p_\theta) \in T^\star(Q_{\mathbf{P}_\mu}) : \cos(\theta) = 0 \text{ and } p_\theta \geq 0\}, \end{aligned}$$

and

$$R_{\mathbf{P}_\mu}(\theta, p_\theta) = (\theta, -e p_\theta).$$

Finally, the vector field is given by

$$X_{\mathbf{P}_\mu}(\theta, p_\theta) = \left(\frac{p_\theta}{mR^2}, mgR\sin(\theta) + \frac{\mu^2\cos(\theta)}{mR^2\sin^3(\theta)}\right)$$

and $\mathfrak{H}_{\mathbf{P}_\mu} = (D_{\mathbf{P}_\mu}, S_{\mathbf{P}_\mu}, R_{\mathbf{P}_\mu}, X_{\mathbf{P}_\mu})$. It can be verified by direct inspection that in fact this hybrid system is the reduced hybrid system associated to $\mathfrak{H}_{\mathbf{P}}$ as given in Theorem 3.5 as it makes the diagram in (3.27) commute.

Note that in this example it is easy to reconstruct the trajectories of the full-order pendulum from the reduced pendulum through integration. A trajectory of the full-order pendulum mounted on the ground, as reconstructed from the reduced system, can be found in Figure 3.6; here $e = .95$, $R = 1$ and $m = 1$. As in the previous examples, both the full-order pendulum and the reduced pendulum are Zeno with these parameters. Section 5.5 of Chapter 5 discusses how to extend the hybrid flow of this system past the Zeno point.

3.5 Bipedal Robotic Walking

The purpose of this section is to apply methods from geometric mechanics to the analysis and control of bipedal robotic walkers. We begin by introducing a generalization of Routhian reduction, *functional* Routhian Reduction, which allows for the conserved quantities to be functions of the cyclic variables rather than constants. Since bipedal robotic walkers are naturally modeled as hybrid systems, which are inherently nonsmooth, in order to apply this framework to these systems it is necessary to first extend functional Routhian reduction to a hybrid setting. We apply this extension, along with potential shaping and controlled symmetries, to derive a feedback control law that provably results in walking gaits on flat ground for a three-dimensional bipedal walker given walking gaits in two-dimensions.

3.5.a Controlled Lagrangians

In order to discuss how to control bipedal walkers, we must discuss how to model them as control systems. So far, we have only introduced "passive" Lagrangian models; we now introduce their "controlled" analogues.

3.5.1 Controlled Lagrangians. Controlled Lagrangians will now be of interest. As in Paragraph 3.1.1, we begin with a Lagrangian $L\colon TQ \to \mathbb{R}$ given in coordinates[7] by

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q). \tag{3.29}$$

The *controlled Euler-Lagrange* equations yield the equations of motion:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = Bu,$$

where we assume that B is an invertible matrix. The result is a control system of the form:

$$\begin{aligned}(\dot{q},\ddot{q}) &= X_L(q,\dot{q},u)\\ &= \left(\dot{q}, M(q)^{-1}(-C(q,\dot{q})\dot{q} - N(q) + Bu)\right).\end{aligned}$$

[7] In this section, we will again only work in coordinates.

In the future, it will be clear from context whether, for a Lagrangian L, we are dealing with a corresponding vector field $(\dot{q},\ddot{q}) = X_L(q,\dot{q})$ (as given in (3.4)) or a control system $(\dot{q},\ddot{q}) = X_L(q,\dot{q},u)$.

If $\mathfrak{H} = (D_h, S_h, R, X_L)$ is a hybrid system w.r.t. a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ in which X_L is a control system we call $\mathfrak{H}$ a *controlled hybrid system* or a *hybrid control system.*

3.5.2 Impact equations. In Paragraph 3.2.2 it was shown how to associate to a unilateral constraint function a domain and guard. This motivated the definition of a Lagrangian hybrid system w.r.t. a hybrid lagrangian (Definition 3.4). For such a hybrid system, $\mathfrak{H} = (D_h, S_h, R, X_L)$, we have still yet to specify the reset map. In order to determine this map in the context of bipedal robots, we will utilize an additional constraint function.

A *kinematic* constraint function is a smooth function $\Upsilon : Q \to \mathbb{R}^{\upsilon}$ ($\upsilon \geq 1$); this function usually describes the position of the end-effector of a kinematic chain, e.g., in the case of bipedal robots, this is the position of the swing foot.

For a unilateral constraint function $h : Q \to \mathbb{R}$ and a kinematic constraint function $\Upsilon : Q \to \mathbb{R}^{\upsilon}$, we define a corresponding map:

$$R_\Upsilon : S_h \to D_h$$

where $R_\Upsilon(q,\dot{q}) = (q, P_\Upsilon(\dot{q}))$, with

$$P_\Upsilon(\dot{q}) = \dot{q} - M(q)^{-1} d\Upsilon_q^T (d\Upsilon_q M(q)^{-1} d\Upsilon_q^T)^{-1} d\Upsilon_q \dot{q}. \tag{3.30}$$

This reset map models a perfectly plastic impact without slipping and was derived using the set-up in [55] together with block-diagonal matrix inversion.

Note that for a bipedal walker, to compute such a map one must use a coordinate system including the position of the stance foot. In reality, after computing the reset map using this full-order coordinate system, one can assume that the foot is located at origin. This allows for the construction of a reduced coordinate system. In the reduced system, the reset map is obtained from the one determined in the full-order coordinate system, although it will no longer be constant for the configuration variables. For further details, we refer the reader to [55].

3.5.b Functional Routhian Reduction

We now introduce a variation of classical Routhian reduction termed *functional Routhian reduction.* The main differences between these two types of reduction are that we allow the original Lagrangian to have a non-cyclic term in the potential energy, and we allow the conserved quantities to be functions (of the cyclic variable) rather than constants. The author is unaware of similar procedures in the literature.

Notation 3.1. In the rest of this section, the notation utilized in Section 3.3 is in force.

3.5.3 Almost-cyclic Lagrangians. We will be interested (in the context of bipedal walking) in Lagrangians of a very special form. We say that a Lagrangian $L_\lambda : TQ_\mu \times T\mathbb{G} \to \mathbb{R}$ is *almost-cyclic* if it has the form:

$$L_\lambda(\theta,\dot\theta,\varphi,\dot\varphi) = \frac{1}{2}\begin{pmatrix}\dot\theta\\ \dot\varphi\end{pmatrix}^T \begin{pmatrix} M_\theta(\theta) & 0\\ 0 & M_\varphi(\theta)\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\varphi\end{pmatrix} - V_\lambda(\theta,\varphi), \tag{3.31}$$

where

$$V_\lambda(\theta,\varphi) = \widetilde{V}(\theta) - \frac{1}{2}\lambda(\varphi)^T M_\varphi(\theta)^{-1}\lambda(\varphi)$$

for some function $\lambda : \mathbb{G} \to \mathbb{R}^k$ such that

$$\left(\frac{\partial\lambda}{\partial\varphi}\right)^T = \frac{\partial\lambda}{\partial\varphi}. \tag{3.32}$$

Here $\mathbb{G}$ is the abelian Lie group given in (3.10), $M_\theta(\theta) \in \mathbb{R}^{n\times n}$ and $M_\varphi(\theta) \in \mathbb{R}^{k\times k}$ are both symmetric positive definite matrices. The most important difference between a cyclic Lagrangian (as introduced in Paragraph 3.3.1) and an almost-cyclic Lagrangian is the presence of a non-cyclic term in the potential energy.

3.5.4 Functional momentum maps. In the framework we are considering here, the momentum map $J : TQ \to \mathfrak{g}^\star \cong \mathbb{R}^k$, takes the form

$$J(\theta,\dot\theta,\varphi,\dot\varphi) = \frac{\partial L_\lambda}{\partial\dot\varphi}(\theta,\dot\theta,\varphi,\dot\varphi) = M_\varphi(\theta)\dot\varphi.$$

As we have seen in (3.13), one typically sets the momentum map equal to a constant $\mu \in \mathbb{R}^k$; this defines the conserved quantities of the system. In our framework, we will breach this convention and set J equal to a function: this motivates the name *functional Routhian reduction*.

3.5.5 Functional Routhians. For an almost-cyclic Lagrangian L_λ as given in (3.31), define the corresponding *functional Routhian* $\widetilde{L} : TQ_\mu \to \mathbb{R}$ by

$$\widetilde{L}(\theta,\dot\theta) = \left[L_\lambda(\theta,\dot\theta,\varphi,\dot\varphi) - \lambda(\varphi)^T\dot\varphi\right]\Big|_{J(\theta,\dot\theta,\varphi,\dot\varphi)=\lambda(\varphi)}$$

Because $J(\theta,\dot\theta,\varphi,\dot\varphi) = \lambda(\varphi)$ implies that

$$\dot\varphi = M_\varphi(\theta)^{-1}\lambda(\varphi), \tag{3.33}$$

and so by direct calculation the functional Routhian is given by

$$\widetilde{L}(\theta,\dot\theta) = \frac{1}{2}\dot\theta^T M_\theta(\theta)\dot\theta - \widetilde{V}(\theta).$$

That is, any Lagrangian of the form given in (3.29) is the functional Routhian of an almost-cyclic Lagrangian.

The goal is to relate the flows of the Lagrangian vector field $X_{\widetilde{L}}$ to the flows of the Lagrangian vector field X_{L_λ} and vice versa in a way analogous to the classical Routhian reduction result given in Proposition 3.1. This relationship is made clear in the following proposition.

Proposition 3.7. *Let L_λ be an almost-cyclic Lagrangian, and $\widetilde{L}$ the corresponding functional Routhian. Then $(\theta(t), \dot\theta(t), \varphi(t), \dot\varphi(t))$ is a flow of the vector field X_{L_λ} on $[t_0, t_F]$ with*

$$\dot\varphi(t_0) = M_\varphi(\theta(t_0))^{-1}\lambda(\varphi(t_0)),$$

if and only if $(\theta(t), \dot\theta(t))$ is a flow of the vector field $X_{\widetilde{L}}$ on $[t_0, t_F]$ and $(\varphi(t), \dot\varphi(t))$ satisfies:

$$\dot\varphi(t) = M_\varphi(\theta(t))^{-1}\lambda(\varphi(t)).$$

Proof. We begin by noting that

$$\begin{aligned} L_\lambda(\theta,\dot\theta,\varphi,\dot\varphi) &= \frac{1}{2}\dot\theta^T M_\theta(\theta)\dot\theta + \frac{1}{2}\dot\varphi^T M_\varphi(\theta)\dot\varphi - V_\lambda(\theta,\varphi) \\ &= \frac{1}{2}\dot\theta^T M_\theta(\theta)\dot\theta + \frac{1}{2}\dot\varphi^T M_\varphi(\theta)\dot\varphi - \widetilde{V}(\theta) + \frac{1}{2}\lambda(\varphi)^T M_\varphi(\theta)^{-1}\lambda(\varphi) \\ &= \widetilde{L}(\theta,\dot\theta) + \frac{1}{2}\dot\varphi^T M_\varphi(\theta)\dot\varphi + \frac{1}{2}\lambda(\varphi)^T M_\varphi(\theta)^{-1}\lambda(\varphi). \end{aligned}$$

Let

$$\mathrm{Rem}(\theta,\varphi,\dot\varphi) := \frac{1}{2}\dot\varphi^T M_\varphi(\theta)\dot\varphi + \frac{1}{2}\lambda(\varphi)^T M_\varphi(\theta)^{-1}\lambda(\varphi),$$

in which case

$$L_\lambda(\theta,\dot\theta,\varphi,\dot\varphi) = \widetilde{L}(\theta,\dot\theta) + \mathrm{Rem}(\theta,\varphi,\dot\varphi).$$

With this notation, the Euler-Lagrange equations for L_λ become:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L_\lambda}{\partial\dot\theta} - \frac{\partial L_\lambda}{\partial\theta} &= \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \widetilde{L}}{\partial\dot\theta} - \frac{\partial \widetilde{L}}{\partial\theta} - \frac{\partial \mathrm{Rem}}{\partial\theta} \\ \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L_\lambda}{\partial\dot\varphi} - \frac{\partial L_\lambda}{\partial\varphi} &= \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \mathrm{Rem}}{\partial\dot\varphi} - \frac{\partial \mathrm{Rem}}{\partial\varphi} \end{aligned}$$

By direct calculation, we have that

$$\begin{aligned} \frac{\partial \mathrm{Rem}}{\partial\theta} &= \frac{1}{2}\dot\varphi^T\frac{\partial}{\partial\theta}\left(M_\varphi(\theta)\right)\dot\varphi + \frac{1}{2}\lambda(\varphi)^T\frac{\partial}{\partial\theta}\left(M_\varphi(\theta)^{-1}\right)\lambda(\varphi) \\ &= \frac{1}{2}\dot\varphi^T\frac{\partial}{\partial\theta}\left(M_\varphi(\theta)\right)\dot\varphi - \frac{1}{2}\lambda(\varphi)^T M_\varphi(\theta)^{-1}\frac{\partial}{\partial\theta}\left(M_\varphi(\theta)\right)M_\varphi(\theta)^{-1}\lambda(\varphi) \qquad (3.34) \\ \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \mathrm{Rem}}{\partial\dot\varphi} &= \frac{\mathrm{d}}{\mathrm{dt}}\left(M_\varphi(\theta)\dot\varphi\right) \\ &= \frac{\mathrm{d}}{\mathrm{dt}}\left(M_\varphi(\theta)\right)\dot\varphi + M_\varphi(\theta)\ddot\varphi \\ \frac{\partial \mathrm{Rem}}{\partial\varphi} &= \frac{\partial}{\partial\varphi}\left(\frac{1}{2}\lambda(\varphi)^T M_\varphi(\theta)^{-1}\lambda(\varphi)\right) \\ &= \frac{1}{2}\frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right)^T M_\varphi(\theta)^{-1}\lambda(\varphi)^T + \frac{1}{2}\lambda(\varphi)M_\varphi(\theta)^{-1}\frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right) \\ &= \frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right)M_\varphi(\theta)^{-1}\lambda(\varphi). \end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \mathrm{Rem}}{\partial\dot\varphi} - \frac{\partial \mathrm{Rem}}{\partial\varphi} = \frac{\mathrm{d}}{\mathrm{dt}}\left(M_\varphi(\theta)\right)\dot\varphi + M_\varphi(\theta)\ddot\varphi - \frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right)M_\varphi(\theta)^{-1}\lambda(\varphi). \qquad (3.35)$$

Now, in the case when $\dot{\varphi} = M_\varphi(\theta)^{-1}\lambda(\varphi)$, we have that:

$$\begin{aligned}
\ddot{\varphi} &= \frac{\mathrm{d}}{\mathrm{dt}}\left(M_\varphi(\theta)^{-1}\right)\lambda(\varphi) + M_\varphi(\theta)^{-1}\frac{\mathrm{d}}{\mathrm{dt}}\left(\lambda(\varphi)\right) \\
&= -M_\varphi(\theta)^{-1}\frac{\mathrm{d}}{\mathrm{dt}}\left(M_\varphi(\theta)\right)M_\varphi(\theta)^{-1}\lambda(\varphi) + M_\varphi(\theta)^{-1}\frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right)^T\dot{\varphi} \\
&= M_\varphi(\theta)^{-1}\left(\frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right)^T - \frac{\mathrm{d}}{\mathrm{dt}}\left(M_\varphi(\theta)\right)\right)M_\varphi(\theta)^{-1}\lambda(\varphi) \\
&= M_\varphi(\theta)^{-1}\left(\frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right) - \frac{\mathrm{d}}{\mathrm{dt}}\left(M_\varphi(\theta)\right)\right)M_\varphi(\theta)^{-1}\lambda(\varphi), \qquad (3.36)
\end{aligned}$$

where the last equality follows from (3.32), i.e., we assumed that λ was a function such that:

$$\frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right)^T = \frac{\partial}{\partial\varphi}\left(\lambda(\varphi)\right).$$

From (3.34), (3.35) and (3.36) we conclude that:

$$\begin{aligned}
\frac{\partial \mathrm{Rem}}{\partial\theta}(\theta,\varphi,M_\varphi(\theta)^{-1}\lambda(\varphi)) &= 0 \qquad (3.37)\\
\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \mathrm{Rem}}{\partial\dot{\varphi}}(\theta,\varphi,M_\varphi(\theta)^{-1}\lambda(\varphi)) - \frac{\partial \mathrm{Rem}}{\partial\varphi}(\theta,\varphi,M_\varphi(\theta)^{-1}\lambda(\varphi)) &= 0 \qquad (3.38)
\end{aligned}$$

Using this, we establish necessity and sufficiency.

($\Rightarrow$) Let $(\theta(t),\dot{\theta}(t),\varphi(t),\dot{\varphi}(t))$ is a flow of the vector field X_{L_λ} on $[t_0,t_F]$ with

$$\dot{\varphi}(t_0) = M_\varphi(\theta(t_0))^{-1}\lambda(\varphi(t_0)),$$

and let $(\overline{\theta}(t),\dot{\overline{\theta}}(t))$ be a flow of the vector field $X_{\tilde{L}}$ on $[t_0,t_F]$ with $\overline{\theta}(t_0)=\theta(t_0)$ and $\dot{\overline{\theta}}(t_0)=\dot{\theta}(t_0)$. In addition, let $\overline{\varphi}(t)$ be a curve satisfying

$$\overline{\varphi}(t_0) = \varphi(t_0), \qquad \dot{\overline{\varphi}}(t) = M_\varphi(\overline{\theta}(t))^{-1}\lambda(\overline{\varphi}(t)).$$

By (3.37) and (3.38) it follows that:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L_\lambda}{\partial\dot{\theta}}(\overline{\theta}(t),\dot{\overline{\theta}}(t),\overline{\varphi}(t),\dot{\overline{\varphi}}(t)) - \frac{\partial L_\lambda}{\partial\theta}(\overline{\theta}(t),\dot{\overline{\theta}}(t),\overline{\varphi}(t),\dot{\overline{\varphi}}(t)) &= 0\\
\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L_\lambda}{\partial\dot{\varphi}}(\overline{\theta}(t),\dot{\overline{\theta}}(t),\overline{\varphi}(t),\dot{\overline{\varphi}}(t)) - \frac{\partial L_\lambda}{\partial\varphi}(\overline{\theta}(t),\dot{\overline{\theta}}(t),\overline{\varphi}(t),\dot{\overline{\varphi}}(t)) &= 0.
\end{aligned}$$

Therefore, $(\overline{\theta}(t),\dot{\overline{\theta}}(t),\overline{\varphi}(t),\dot{\overline{\varphi}}(t))$ is a flow of the vector field X_{L_λ} on $[t_0,t_F]$. Moreover, since

$$(\overline{\theta}(t_0),\dot{\overline{\theta}}(t_0),\overline{\varphi}(t_0),\dot{\overline{\varphi}}(t_0)) = (\theta(t_0),\dot{\theta}(t_0),\varphi(t_0),\dot{\varphi}(t_0))$$

and by the uniqueness of solutions of X_{L_λ} (to flows with the same initial condition must be the same), it follows that

$$(\overline{\theta}(t),\dot{\overline{\theta}}(t),\overline{\varphi}(t),\dot{\overline{\varphi}}(t)) = (\theta(t),\dot{\theta}(t),\varphi(t),\dot{\varphi}(t))$$

or $(\theta(t),\dot{\theta}(t))$ is a flow of the vector field $X_{\tilde{L}}$ and $(\varphi(t),\dot{\varphi}(t))$ satisfies:

$$\dot{\varphi}(t) = M_\varphi(\theta(t))^{-1}\lambda(\varphi(t)).$$

(⇐) Let $(\theta(t),\dot\theta(t))$ be a flow of the vector field $X_{\widetilde L}$ on $[t_0,t_F]$ and $(\varphi(t),\dot\varphi(t))$ a pair satisfying

$$\dot\varphi(t) = M_\varphi(\theta(t))^{-1}\lambda(\varphi(t)).$$

We need only show that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L_\lambda}{\partial\dot\theta}(\theta(t),\dot\theta(t),\varphi(t),\dot\varphi(t)) - \frac{\partial L_\lambda}{\partial\theta}(\theta(t),\dot\theta(t),\varphi(t),\dot\varphi(t)) &= 0\\
\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L_\lambda}{\partial\dot\varphi}(\theta(t),\dot\theta(t),\varphi(t),\dot\varphi(t)) - \frac{\partial L_\lambda}{\partial\varphi}(\theta(t),\dot\theta(t),\varphi(t),\dot\varphi(t)) &= 0.
\end{aligned}$$

This follows from (3.37) and (3.38) together with the fact that, since $(\theta(t),\dot\theta(t))$ is a flow of $X_{\widetilde L}$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\widetilde L}{\partial\dot\theta}(\theta(t),\dot\theta(t)) - \frac{\partial\widetilde L}{\partial\theta}(\theta(t),\dot\theta(t)) = 0$$

□

Note that Proposition 3.7 has some interesting implications.

- It implies that flows of X_{L_λ} can be determined solving 2n+k ordinary differential equations rather than 2(n+k) ordinary differential equations; this has important ramifications in the context of numerical integration (and errors thereof).
- It implies that the $(\theta(t),\dot\theta(t))$ component of flows of X_{L_λ} with certain initial conditions can be effectively decoupled from the $(\varphi(t),\dot\varphi(t))$ component of the solution; this will have important ramifications in the context of bipedal walking.

We now have the necessary background material needed to introduce our framework for hybrid functional Routhian reduction. We will first define the notion of an *almost-cyclic* Lagrangian hybrid system and then introduce the hybrid functional Routhian reduction theorem which is analogous to Theorem 3.1. It is important to note that this definition is not the most general one, but provides sufficient generality for the systems under consideration, i.e., bipedal walkers.

Definition 3.11. If $\mathfrak{H}_\lambda = (D_h, S_h, R, X_{L_\lambda})$ is a Lagrangian hybrid system with respect to the hybrid Lagrangian $\mathbf{L}_\lambda = (Q, L_\lambda, h)$, then $\mathfrak{H}_\lambda$ is **almost-cyclic** if the following conditions hold:

- $Q = Q_\mu \times \mathbb{G}$
- $h : Q_\mu \times \mathbb{G} \to \mathbb{R}$ is cyclic,
$$\frac{\partial h}{\partial\varphi} = 0,$$
and so can be viewed as a function $h_\mu : Q_\mu \to \mathbb{R}$.
- $L_\lambda : TQ_\mu \times T\mathbb{G} \to \mathbb{R}$ is almost-cyclic,
- $\pi_\varphi(R(\theta,\dot\theta,\varphi,\dot\varphi)) = \varphi$, where $\pi_\varphi(R(\theta,\dot\theta,\varphi,\dot\varphi))$ is the φ-component of $R(\theta,\dot\theta,\varphi,\dot\varphi)$,

⋄ The following diagram commutes:

$$\begin{array}{ccc} & \mathbb{R}^k & \\ {}^{J|_{S_h}}\nearrow & \text{(I)} & \nwarrow^{J|_{D_h}} \\ S_h & \xrightarrow{\quad R \quad} & D_h \\ \pi\big\downarrow & \text{(II)} & \big\downarrow\pi \\ S_{h_\mu} & \xrightarrow{\quad \widetilde{R} \quad} & D_{h_\mu}\end{array} \tag{3.39}$$

for some map $\widetilde{R} : S_{h_\mu} \to D_{h_\mu}$.

3.5.6 Hybrid functional Routhian. If $\mathfrak{H}_\lambda = (D_h, S_h, R, X_{L_\lambda})$ is an almost-cyclic Lagrangian hybrid system, we can associate to this hybrid system a reduced hybrid system, termed a *functional Routhian* hybrid system, denoted by $\widetilde{\mathfrak{H}}$ and defined by:

$$\widetilde{\mathfrak{H}} := (D_{h_\mu}, S_{h_\mu}, \widetilde{R}, X_{\widetilde{L}}).$$

The following theorem quantifies the relationship between $\mathfrak{H}_\lambda$ and $\widetilde{\mathfrak{H}}$.

Theorem 3.7. *Let $\mathfrak{H}_\lambda$ be an almost-cyclic Lagrangian hybrid system, and $\widetilde{\mathfrak{H}}$ the associated functional Routhian hybrid system. Then*

$$\epsilon^{\mathfrak{H}_\lambda} = (\Lambda, I, \{(\theta_i, \dot{\theta}_i, \varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda})$$

is a hybrid flow of $\mathfrak{H}_\lambda$ with

$$\dot{\varphi}_0(\tau_0) = M_\varphi(\theta_0(\tau_0))^{-1}\lambda(\varphi_0(\tau_0)),$$

if and only if

$$\epsilon^{\widetilde{\mathfrak{H}}} = (\Lambda, I, \{\theta_i, \dot{\theta}_i\}_{i\in\Lambda})$$

is a hybrid flow of $\widetilde{\mathfrak{H}}$ and $\{(\varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda}$ satisfies:

$$\dot{\varphi}_i(t) = M_\varphi(\theta_i(t))^{-1}\lambda(\varphi_i(t)), \qquad \varphi_{i+1}(\tau_{i+1}) = \varphi_i(\tau_{i+1}).$$

Proof. (⇒) By Proposition 3.7, we need only show that the following conditions hold:

(i) $$(\theta_i(\tau_{i+1}), \dot{\theta}_i(\tau_{i+1})) \in S_{h_\mu},$$

(ii) $$\widetilde{R}(\theta_i(\tau_{i+1}), \dot{\theta}_i(\tau_{i+1})) = (\theta_{i+1}(\tau_{i+1}), \dot{\theta}_{i+1}(\tau_{i+1})),$$

(iii) $$\dot{\varphi}_0(\tau_0) = M_\varphi(\theta_0(\tau_0))^{-1}\lambda(\varphi_0(\tau_0)) \quad \Rightarrow \quad \dot{\varphi}_i(\tau_i) = M_\varphi(\theta_i(\tau_i))^{-1}\lambda(\varphi_i(\tau_i)).$$

Condition (i) follows from the cyclicity of h which implies that the codomain of π is S_{h_μ} (see the proof of Proposition 3.2). For

$$(\theta_i(\tau_{i+1}), \dot{\theta}_i(\tau_{i+1}), \varphi_i(\tau_{i+1}), \dot{\varphi}_i(\tau_{i+1})) \in S_h$$

it follows by the commutativity of (3.39), and specifically (II), that

$$\begin{aligned}
\widetilde{R}(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1})) &= \widetilde{R}(\pi(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot\varphi_i(\tau_{i+1}))) \\
&= \pi(R(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot\varphi_i(\tau_{i+1}))) \\
&= \pi(\theta_{i+1}(\tau_{i+1}),\dot\theta_{i+1}(\tau_{i+1}),\varphi_{i+1}(\tau_{i+1}),\dot\varphi_{i+1}(\tau_{i+1})) \\
&= (\theta_{i+1}(\tau_{i+1}),\dot\theta_{i+1}(\tau_{i+1})).
\end{aligned}$$

Therefore, condition (ii) holds. Finally, we show condition (iii) through induction. It holds by assumption for $i = 0$, therefore, assume that

$$\dot\varphi_i(\tau_i) = M_\varphi(\theta_i(\tau_i))^{-1}\lambda(\varphi_i(\tau_i)).$$

Then, Proposition 3.7 implies that

$$\dot\varphi_i(t) = M_\varphi(\theta_i(t))^{-1}\lambda(\varphi_i(t)).$$

for all $t \in I_i$. Specifically,

$$\dot\varphi_i(\tau_{i+1}) = M_\varphi(\theta_i(\tau_{i+1}))^{-1}\lambda(\varphi_i(\tau_{i+1})),$$

or

$$J(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot\varphi_i(\tau_{i+1})) = \lambda(\varphi_i(\tau_{i+1})).$$

By the commutativity of (3.39), and specifically (I), it follows that

$$J(\theta_{i+1}(\tau_{i+1}),\dot\theta_{i+1}(\tau_{i+1}),\varphi_{i+1}(\tau_{i+1}),\dot\varphi_{i+1}(\tau_{i+1})) = \lambda(\varphi_i(\tau_{i+1})).$$

To complete the induction step, we note that:

$$\begin{aligned}
\pi_\varphi(R(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot\varphi_i(\tau_{i+1}))) &= \pi_\varphi(\theta_{i+1}(\tau_{i+1}),\dot\theta_{i+1}(\tau_{i+1}),\varphi_{i+1}(\tau_{i+1}),\dot\varphi_{i+1}(\tau_{i+1})) \\
&= \varphi_{i+1}(\tau_{i+1}).
\end{aligned}$$

Therefore, by the assumption that $\pi_\varphi(R(\theta,\dot\theta,\varphi,\dot\varphi)) = \varphi$,

$$\lambda(\varphi_{i+1}(\tau_{i+1})) = \lambda(\pi_\varphi(R(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot\varphi_i(\tau_{i+1})))) = \lambda(\varphi_i(\tau_{i+1})),$$

and so

$$J(\theta_{i+1}(\tau_{i+1}),\dot\theta_{i+1}(\tau_{i+1}),\varphi_{i+1}(\tau_{i+1}),\dot\varphi_{i+1}(\tau_{i+1})) = \lambda(\varphi_{i+1}(\tau_{i+1})),$$

or

$$\dot\varphi_{i+1}(\tau_{i+1}) = M_\varphi(\theta_{i+1}(\tau_{i+1}))^{-1}\lambda(\varphi_{i+1}(\tau_{i+1})),$$

as desired.

($\Leftarrow$) By Proposition 3.7, we need only show that the following conditions hold:

(i) $$(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot\varphi_i(\tau_{i+1})) \in S_h,$$

(ii) $$R(\theta_i(\tau_{i+1}),\dot\theta_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot\varphi_i(\tau_{i+1})) = (\theta_{i+1}(\tau_{i+1}),\dot\theta_{i+1}(\tau_{i+1}),\varphi_{i+1}(\tau_{i+1}),\dot\varphi_{i+1}(\tau_{i+1})),$$

where φ_i and $\dot{\varphi}_i$ satisfy:

$$\dot{\varphi}_i(t) = M_\varphi(\theta_i(t))^{-1}\lambda(\varphi_i(t)).$$

for all $t \in I_i$. Condition (i) follows from the cyclicity of h. Since by assumption

$$\pi_\varphi(R(\theta,\dot{\theta},\varphi,\dot{\varphi})) = \varphi,$$

and by the commutativity of (3.39), it follows that

$$M_\varphi(\theta)\dot{\varphi} = M_\varphi(\pi_\theta(\widetilde{R}(\theta,\dot{\theta})))\pi_{\dot{\varphi}}(R(\theta,\dot{\theta},\varphi,\dot{\varphi}))$$

for all $(\theta,\dot{\theta},\varphi,\dot{\varphi}) \in S_h$. By the commutativity of the same diagram we have that:

$$\begin{aligned}
\dot{\varphi}_{i+1}(\tau_{i+1}) &= M_\varphi(\theta_{i+1}(\tau_{i+1}))^{-1}\lambda(\varphi_{i+1}(\tau_{i+1})) \\
&= M_\varphi(\pi_\theta(\widetilde{R}(\theta_i(\tau_{i+1}),\dot{\theta}_i(\tau_{i+1}))))^{-1}\lambda(\varphi_{i+1}(\tau_{i+1})) \\
&= M_\varphi(\pi_\theta(\widetilde{R}(\theta_i(\tau_{i+1}),\dot{\theta}_i(\tau_{i+1}))))^{-1}\lambda(\varphi_i(\tau_{i+1})) \\
&= M_\varphi(\pi_\theta(\widetilde{R}(\theta_i(\tau_{i+1}),\dot{\theta}_i(\tau_{i+1}))))^{-1}M_\varphi(\theta_i(\tau_{i+1}))\dot{\varphi}_i(\tau_{i+1}) \\
&= \pi_{\dot{\varphi}}(R(\theta_i(\tau_{i+1}),\dot{\theta}_i(\tau_{i+1}),\varphi_i(\tau_{i+1}),\dot{\varphi}_i(\tau_{i+1})).
\end{aligned}$$

Thus we have shown that condition (ii) holds. □

3.5.c 2D Bipedal Walkers

We now turn our attention toward the standard model of a two-dimensional bipedal robotic walker walking down a slope; walkers of this form have been well-studied by [106], [90] and [54], to name a few. We then use controlled symmetries to shape the potential energy of the Lagrangian describing this model so that it can walk stably on flat ground. The result is two important hybrid systems; they will be used to construct a control law for a three-dimensional walker that results in a walking gait on flat ground.

3.5.7 2D biped model. We begin by introducing a model describing a controlled bipedal robot walking in two-dimensions, walking down a slope of γ degrees; see Figure 3.7. That is, we explicitly construct the controlled hybrid system

$$\mathfrak{H}^\gamma_{2D} = (D^\gamma_{2D}, S^\gamma_{2D}, R_{2D}, X_{2D})$$

which describes this robotic system.

The configuration space for the 2D biped is $Q_{2D} = \mathbb{T}^2$, the 2-torus, and the Lagrangian describing this system is given by:

$$L_{2D}(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T M_{2D}(\theta)\dot{\theta} - V_{2D}(\theta),$$

where $\theta = (\theta_{ns},\theta_s)^T$ with $M_{2D}(\theta)$ and $V_{2D}(\theta)$ given in Table 3.1.

Using the controlled Euler-Lagrange equations, the dynamics for the walker are given by

$$M_{2D}(\theta)\ddot{\theta} + C_{2D}(\theta,\dot{\theta})\dot{\theta} + N_{2D}(\theta) = B_{2D}u.$$

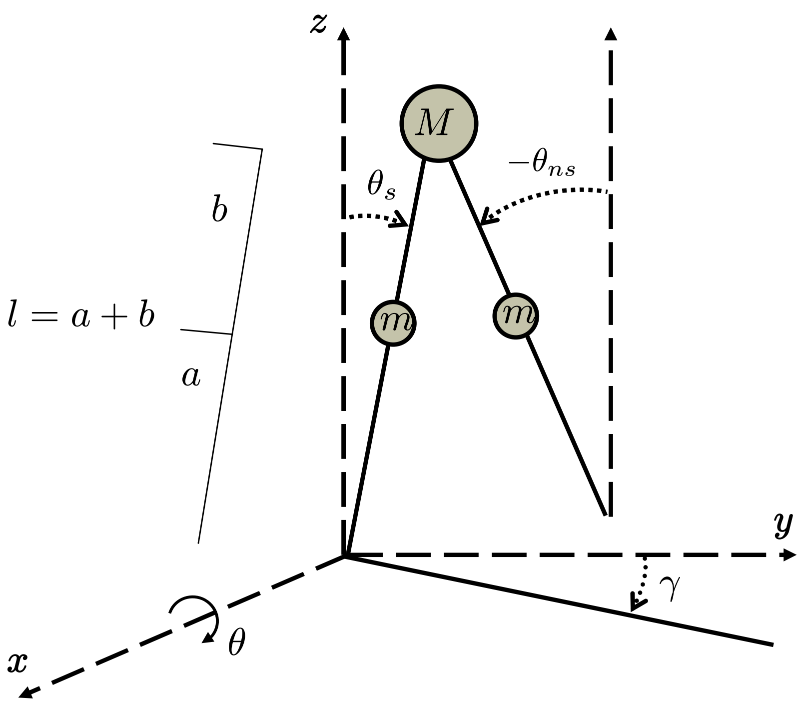


Figure 3.7: Two-dimensional bipedal robot.

These equations yield the control system:

$$(\dot{\theta}, \ddot{\theta}) = X_{2D}(\theta, \dot{\theta}, u) := X_{L_{2D}}(\theta, \dot{\theta}, u).$$

We construct D^{γ}_{2D} and S^{γ}_{2D} by using the unilateral constraint function

$$h^{\gamma}_{2D}(\theta) = \cos(\theta_s) - \cos(\theta_{ns}) + (\sin(\theta_s) - \sin(\theta_{ns}))\tan(\gamma),$$

which gives the height of the foot of the walker above the slope with normalized unit leg length.

Finally, the reset map R_{2D} is given by

$$R_{2D}(\theta, \dot{\theta}) = \left(T_{2D}\theta, P_{2D}(\theta)\dot{\theta}\right),$$

where T_{2D} and $P_{2D}(\theta)$ are given in Table 3.1. Note that this reset map was computed using (3.30) coupled with the condition that the stance foot is fixed (see [55] for more details).

Setting the control $u = 0$ yields the standard model of a 2D passive bipedal robot walking down a slope. For such a model, it has been well-established (for example, in [54]) that for certain γ, $\mathfrak{H}^{\gamma}_{2D}$ has a walking gait. For the rest of the paper we pick, once and for all, such a γ.

3.5.8 Controlled symmetries. Controlled symmetries were introduced in [106] and later in [107] in order to shape the potential energy of bipedal robotic walkers to allow for stable walking on flat ground

Additional equations for $\mathfrak{H}_{2D}$:

$$\begin{aligned}
M_{2D}(\theta) &= \begin{pmatrix} \frac{l^2 m}{4} & -\frac{l^2 m\cos(\theta_s-\theta_{ns})}{2} \\ -\frac{l^2 m\cos(\theta_s-\theta_{ns})}{2} & \frac{l^2 m}{4} + l^2(m+M) \end{pmatrix} \\
V_{2D}(\theta) &= \frac{1}{2} g l((3m+2M)\cos(\theta_s) - m\cos(\theta_{ns})) \\
B_{2D} &= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \\
T_{2D} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
P_{2D}(\theta) &= \frac{1}{-3m-4M+2m\cos(2(\theta_s-\theta_{ns}))} \\
&\qquad \begin{pmatrix} 2m\cos(\theta_{ns}-\theta_s) & m-4(m+M)\cos(2(\theta_{ns}-\theta_s)) \\ m & -2(m+2M)\cos(\theta_{ns}-\theta_s) \end{pmatrix}
\end{aligned}$$

Additional equations for $\mathfrak{H}_{3D}$:

$$\begin{aligned}
m_{3D}(\theta) &= \frac{1}{8}(l^2(6m+4M) + l^2(m\cos(2\theta_{ns}) \\
&\qquad -8m\cos(\theta_{ns})\cos(\theta_s) + (5m+4M)\cos(2\theta_s)) \\
V_{3D}(\theta,\varphi) &= V_{2D}(\theta)\cos(\varphi) \\
p_{3D}(\theta) &= \frac{-m\cos(2\theta_{ns}) + 8(m+M)\cos(\theta_{ns})\cos(\theta_s) - m(2+\cos(2\theta_s))}{6m+4M+(5m+4M)\cos(2\theta_{ns}) - 8m\cos(\theta_{ns})\cos(\theta_s) + m\cos(2\theta_s)}
\end{aligned}$$

Table 3.1: Additional equations for $\mathfrak{H}_{2D}$ and $\mathfrak{H}_{3D}$.

based on stable walking down a slope. We will briefly apply the results of this work to derive a feedback control law that yields a hybrid system, $\mathfrak{H}^s_{2D}$, with stable walking gaits on flat ground.

The main idea of [107] is that inherent symmetries in $\mathfrak{H}^\gamma_{2D}$ can be used to "rotate the world" (via a group action) to allow for walking on flat ground. Specifically, we have a group action $\Phi : \mathbb{S}^1 \times Q_{2D} \to Q_{2D}$ denoted by:

$$\Phi(\gamma,\theta) := \begin{pmatrix} \theta_{ns} - \gamma \\ \theta_s - \gamma \end{pmatrix},$$

for $\gamma \in \mathbb{S}^1$. Using this, define the following feedback control law:

$$u = K^\gamma_{2D}(\theta) = B^{-1}_{2D}\frac{\partial}{\partial\theta}\left(V_{2D}(\theta) - V_{2D}(\Phi(\gamma,\theta))\right).$$

Applying this control law to the control system $(\dot{q}, \ddot{q}) = X_{2D}(\theta, \dot{\theta}, u)$ yields the dynamical system:

$$(\dot{\theta}, \ddot{\theta}) = X^{\gamma}_{2D}(\theta, \dot{\theta}) := X_{2D}(\theta, \dot{\theta}, K^{\gamma}_{2D}(\theta))$$

which is just the vector field associated to the Lagrangian

$$L^{\gamma}_{2D}(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M_{2D}(\theta)\dot{\theta} - V^{\gamma}_{2D}(\theta),$$

where $V^{\gamma}_{2D}(\theta) := V_{2D}(\Phi(\gamma, \theta))$. That is, $X^{\gamma}_{2D} = X_{L^{\gamma}_{2D}}$.

Now define, for some γ that results in stable passive walking for $\mathfrak{H}^{\gamma}_{2D}$,

$$\mathfrak{H}^{s}_{2D} := (D^0_{2D}, S^0_{2D}, R_{2D}, X^{\gamma}_{2D}),$$

which is a Lagrangian hybrid system. In particular, it is related to $\mathfrak{H}^{\gamma}_{2D}$ via the main result of [107] as follows:

Theorem 3.8. *Let*

$$\epsilon^{\mathfrak{H}^{\gamma}_{2D}} = (\Lambda, I, \{(\theta_i, \dot{\theta}_i)\}_{i \in \Lambda})$$

be a hybrid flow of $\mathfrak{H}^{\gamma}_{2D}$ *(with* $u = 0$*), then*

$$\epsilon^{\mathfrak{H}^{s}_{2D}} = (\Lambda, I, \{(\Phi(\gamma, \theta_i), \dot{\theta}_i)\}_{i \in \Lambda})$$

is a hybrid flow of $\mathfrak{H}^{s}_{2D}$.

Theorem 3.8 implies that if $\mathfrak{H}^{\gamma}_{2D}$ walks (stably) on a slope, then $\mathfrak{H}^{s}_{2D}$ walks (stably) on flat ground.

3.5.d Functional Routhian Reduction Applied to 3D Bipedal Walkers

In this section we construct a control law that results in stable walking for a simple model of a three-dimensional bipedal robotic walker. In order to achieve this goal, we shape the potential energy of this model via feedback control so that when hybrid functional Routhian reduction is carried out, the result is the stable 2D walker introduced in the previous section. We utilize Theorem 3.7 to demonstrate that this implies that the 3D walker has a walking gait on flat ground (in three dimensions).

3.5.9 3D biped model. We now introduce the model describing a controlled bipedal robot walking in three-dimensions on flat ground, i.e., we will explicitly construct the controlled hybrid system describing this system:

$$\mathfrak{H}_{3D} = (D_{3D}, S_{3D}, R_{3D}, X_{3D}).$$

The configuration space for the 3D biped is $Q_{3D} = \mathbb{T}^2 \times \mathbb{S}$ and the Lagrangian describing this system is given by:

$$L_{3D}(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = \frac{1}{2}\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix}^T \begin{pmatrix} M_{2D}(\theta) & 0 \\ 0 & m_{3D}(\theta) \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} - V_{3D}(\theta, \varphi), \tag{3.40}$$

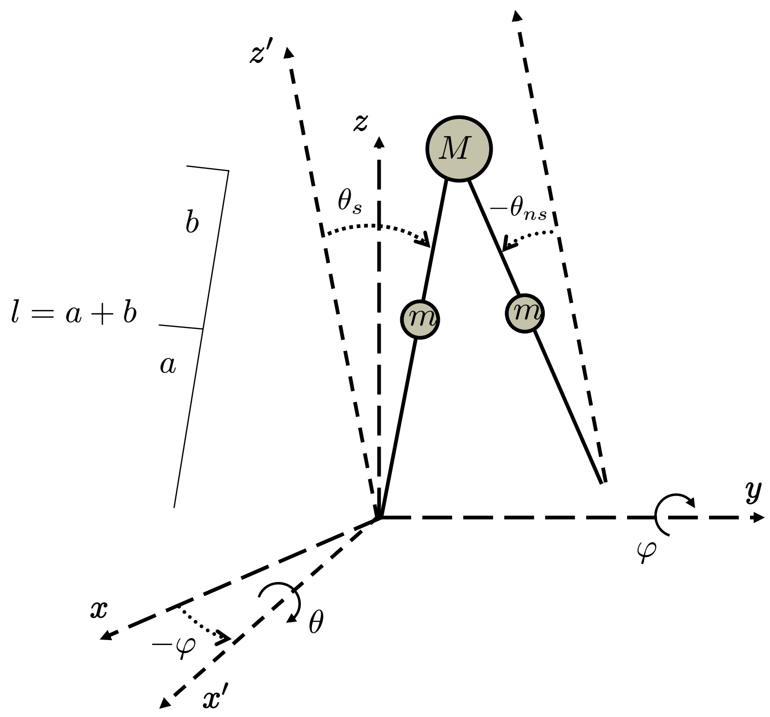


Figure 3.8: Three-dimensional bipedal robot.

where $m_{3D}(\theta)$ and $V_{3D}(\theta, \varphi)$ are given in the Table 3.1. Note that, referring to the notation introduced in (3.31), $M_\theta(\theta) = M_{2D}(\theta)$ and $M_\varphi(\theta) = m_{3D}(\theta)$. Also note that L_{3D} is nearly cyclic; it is only the potential energy that prevents its cyclicity. This will motivate the use of a control law that shapes this potential energy.

Using the controlled Euler-Lagrange equations, the dynamics for the walker are given by

$$M_{3D}(q)\ddot{q} + C_{3D}(q, \dot{q})\dot{q} + N_{3D}(q) = B_{3D}u$$

with $q = (\theta, \varphi)$ and

$$B_{3D} = \begin{pmatrix} B_{2D} & 0 \\ 0 & 1 \end{pmatrix}.$$

These equations yield the control system:

$$(\dot{q}, \ddot{q}) = X_{3D}(q, \dot{q}, u) := X_{L_{3D}}(q, \dot{q}, u).$$

We construct D_{3D} and S_{3D} by utilizing the unilateral constraint function

$$h_{3D}(\theta, \varphi) = h^0_{2D}(\theta) = \cos(\theta_s) - \cos(\theta_{ns}).$$

This function gives the normalized height of the foot of the walker above the ground with the implicit assumption that $\varphi \in (-\pi/2, \pi/2)$ (which allows us to disregard the scaling factor $\cos(\varphi)$ that would have been present). The result is that h_{3D} is cyclic.

Finally, the reset map R_{3D} is given by

$$R_{3D}(\theta,\dot{\theta},\varphi,\dot{\varphi}) = \left(T_{2D}\theta, P_{2D}(\theta)\dot{\theta}, \varphi, p_{3D}(\theta)\dot{\varphi}\right)$$

where $p_{3D}(\theta)$ is given in Table 3.1. Note that this map was again computed using (3.30) coupled with the condition that the stance foot is fixed.

3.5.10 Control law construction. We now proceed to construct a feedback control law for $\mathfrak{H}_{3D}$ that makes this hybrid system an almost-cyclic Lagrangian hybrid system, $\mathfrak{H}^{\alpha}_{3D}$. We will then demonstrate, using Theorem 3.7, that $\mathfrak{H}^{\alpha}_{3D}$ has a walking gait by relating it to $\mathfrak{H}^{s}_{2D}$.

Define the feedback control law parameterized by $\alpha \in \mathbb{R}$:

$$u = K^{\alpha}_{3D}(q) = B^{-1}_{3D}\frac{\partial}{\partial q}\left(V_{3D}(q) - V^{\gamma}_{2D}(\theta) + \frac{1}{2}\frac{\alpha^2\varphi^2}{m_{3D}(\theta)}\right).$$

Applying this control law to the control system $(\dot{q},\ddot{q}) = X_{3D}(q,\dot{q},u)$ yields the dynamical system:

$$(\dot{q},\ddot{q}) = X^{\alpha}_{3D}(q,\dot{q}) := X_{3D}(q,\dot{q},K^{\alpha}_{3D}(q)),$$

which is just the vector field associated to the almost-cyclic Lagrangian

$$L^{\alpha}_{3D}(\theta,\dot{\theta},\varphi,\dot{\varphi}) = \frac{1}{2}\begin{pmatrix}\dot{\theta}\\ \dot{\varphi}\end{pmatrix}^T\begin{pmatrix}M_{2D}(\theta) & 0\\ 0 & m_{3D}(\theta)\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\varphi}\end{pmatrix} - V^{\alpha}_{3D}(\theta,\varphi), \qquad (3.41)$$

where

$$V^{\alpha}_{3D}(\theta,\varphi) = V^{\gamma}_{2D}(\theta) - \frac{1}{2}\frac{\alpha^2\varphi^2}{m_{3D}(\theta)}.$$

That is, $X^{\alpha}_{3D} = X_{L^{\alpha}_{3D}}$.

We now define

$$\mathfrak{H}^{\alpha}_{3D} := (D_{3D}, S_{3D}, R_{3D}, X^{\alpha}_{3D}),$$

which is a Lagrangian hybrid system.

3.5.11 Applying hybrid functional Routhian reduction. Using the methods outlined in Subsection 3.5.b, there is a momentum map $J_{3D} : TQ_{3D} \to \mathbb{R}$ given by

$$J_{3D}(\theta,\dot{\theta},\varphi,\dot{\varphi}) = m_{3D}(\theta)\dot{\varphi},$$

and so setting $J_{3D}(\theta,\dot{\theta},\varphi,\dot{\varphi}) = \lambda(\varphi) = -\alpha\varphi$ implies that

$$\dot{\varphi} = -\frac{\alpha\varphi}{m_{3D}(\theta)}.$$

The importance of $\mathfrak{H}^{\alpha}_{3D}$ is illustrated by the following theorem.

Theorem 3.9. *$\mathfrak{H}^{\alpha}_{3D}$ is an almost-cyclic Lagrangian hybrid system. Moreover, the following diagram commutes:*

$$\begin{array}{ccccc} & & \mathbb{R}^k & & \\ & \overset{J_{3D}|_{S_{3D}}}{\nearrow} & & \overset{J_{3D}|_{D_{3D}}}{\nwarrow} & \\ S_{3D} & & \xrightarrow{R_{3D}} & & D_{3D} \\ \downarrow \pi & & & & \downarrow \pi \\ S_{2D} & & \xrightarrow{R_{2D}} & & D_{2D} \end{array}$$

Therefore, $\mathfrak{H}^{s}_{2D}$ is the functional Routhian hybrid system associated with $\mathfrak{H}^{\alpha}_{3D}$.

Proof. By the construction of $\mathfrak{H}^{\alpha}_{3D}$, one need only show that:

$$m_{3D}(\theta) = m_{3D}(T_{2D}\theta)p_{3D}(\theta),$$

which follows by direct calculation. □

This result allows us to prove—using Theorem 3.7—that the control law used to construct $\mathfrak{H}^{\alpha}_{3D}$ in fact results in walking in three-dimensions when $\alpha > 0$.

Theorem 3.10.

$$\epsilon^{\mathfrak{H}^{\alpha}_{3D}} = (\Lambda, I, \{(\theta_i, \dot{\theta}_i, \varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda})$$

is a hybrid flow of $\mathfrak{H}^{\alpha}_{3D}$ with

$$\dot{\varphi}_0(\tau_0) = -\frac{\alpha\varphi_0(\tau_0)}{m_{3D}(\theta_0(\tau_0))}, \tag{3.42}$$

if and only if

$$\epsilon^{\mathfrak{H}^{s}_{2D}} = (\Lambda, I, \{\theta_i, \dot{\theta}_i\}_{i\in\Lambda})$$

is a hybrid flow of $\mathfrak{H}^{s}_{2D}$ and $\{(\varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda}$ satisfies:

$$\dot{\varphi}_i(t) = -\frac{\alpha\varphi_i(t)}{m_{3D}(\theta_i(t))}, \qquad \varphi_{i+1}(\tau_{i+1}) = \varphi_i(\tau_{i+1}). \tag{3.43}$$

Proof. Follows from Theorem 3.7 and Theorem 3.9. □

3.5.12 Simulation results. We conclude this Chapter by discussing the implications of Theorem 3.10. Moreover, we demonstrate the usefulness of this result by showing through simulation that it does result in walking the three-dimensions. To better visualize the following discussion, refer to Figure 3.11 for an initial configuration of the robot and Figure 3.12 for a walking sequence of the robot.

Suppose that $\epsilon^{\mathfrak{H}^{\alpha}_{3D}} = (\Lambda, I, \{(\theta_i, \dot{\theta}_i, \varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda})$ is a hybrid flow of $\mathfrak{H}^{\alpha}_{3D}$. If this hybrid flow has an initial condition satisfying (3.42) with $\alpha > 0$ and the corresponding hybrid flow, $\epsilon^{\mathfrak{H}^{s}_{2D}} = (\Lambda, I, \{\theta_i, \dot{\theta}_i\}_{i\in\Lambda})$, of $\mathfrak{H}^{s}_{2D}$ is a walking gait in 2D:

$$\Lambda = \mathbb{N}, \qquad \lim_{i\to\infty} \tau_i = \infty, \qquad \theta_i(\tau_i) = \theta_{i+1}(\tau_{i+1}),$$

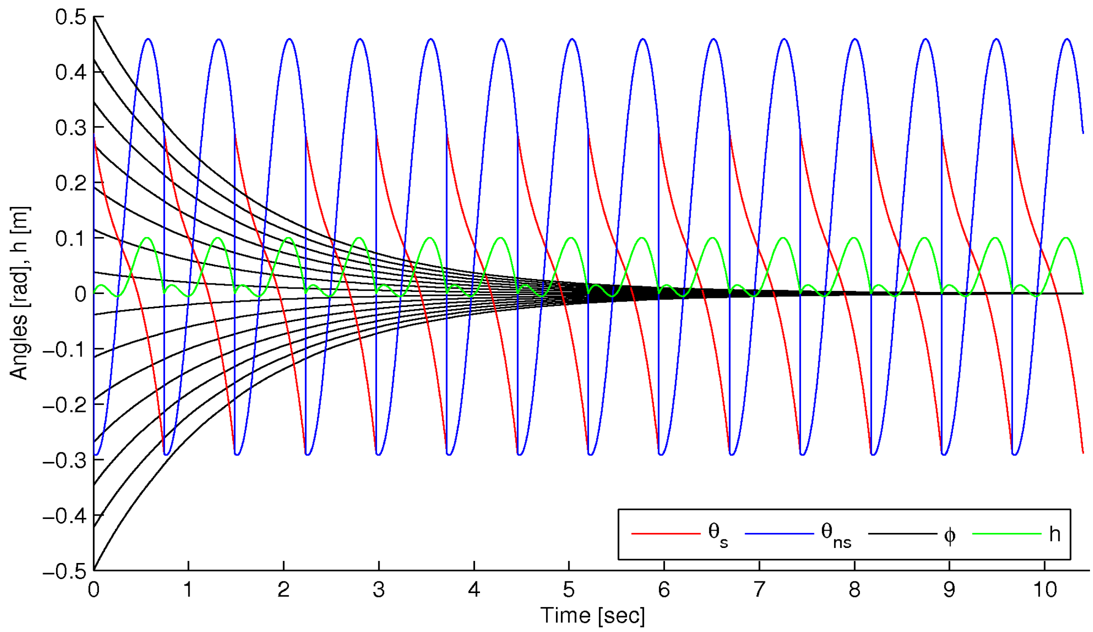


Figure 3.9: θ_{ns}, θ_{s} and φ over time for a stable walking gait and different initial values of φ.

then the result is walking in three-dimensions. This follows from the fact that θ and $\dot{\theta}$ will have the same behavior over time for the full-order system—the bipedal robot will walk. Moreover, since Theorem 3.10 implies that (3.43) holds, the walker stabilizes to the "upright" position. This follows from the fact that the roll, φ, will tend to zero as time goes to infinity since (3.43) essentially defines a stable linear system $\dot{\varphi} = -\alpha\varphi$ (because $m_{3\text{D}}(\theta_i(t)) > 0$ and $\alpha > 0$), which controls the behavior of φ when (3.42) is satisfied. This convergence can be seen in Figure 3.9.

Theorem 3.10 only implies that the 3D biped has walking gaits for hybrid flows with initial conditions that satisfy (3.42); the set of all such initial conditions defines a region that is stable to the origin $(\varphi, \dot{\varphi}) = (0, 0)$, which corresponds to "upright" walking (see Figure 3.10). This illustrates that our control law for the 3D biped is not a locally stabilizing controller (as would be the case if we were to linearize, see [73]) but rather stabilizes a nonlinear subset of the initial conditions. It is possible to extend this region of convergence by stabilizing to the manifold defined by

$$\dot{\varphi} + \frac{\alpha\varphi}{m_{3\text{D}}(\theta)} = 0.$$

This indicates that hybrid reduction can be used to stabilize a three-dimensional walker from a large set of initial conditions.

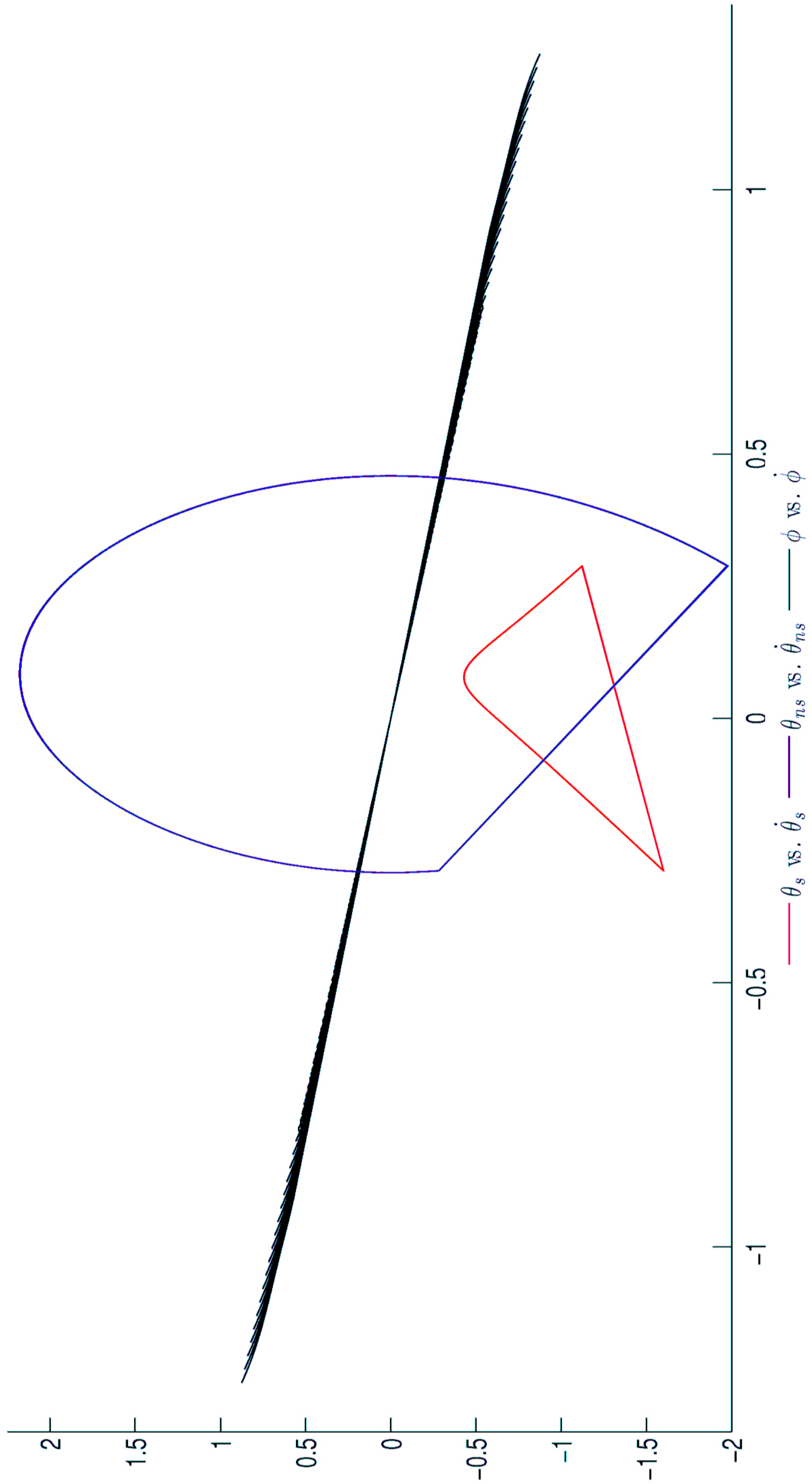


Figure 3.10: Phase portraits for a stable walking gait. The black region shows the initial conditions of φ and $\dot{\varphi}$ that converge to the origin.

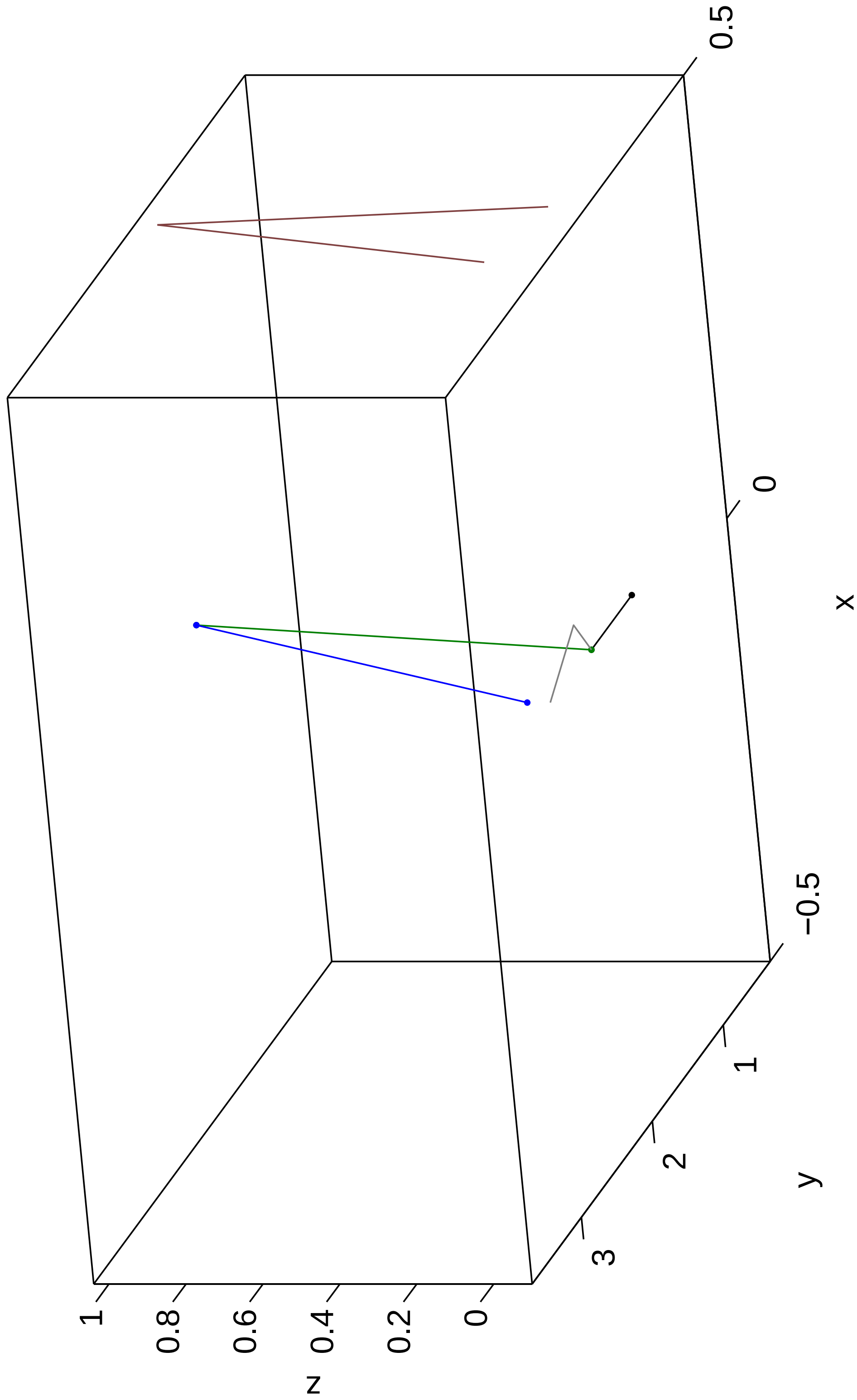


Figure 3.11: The initial configuration of the bipedal robot.

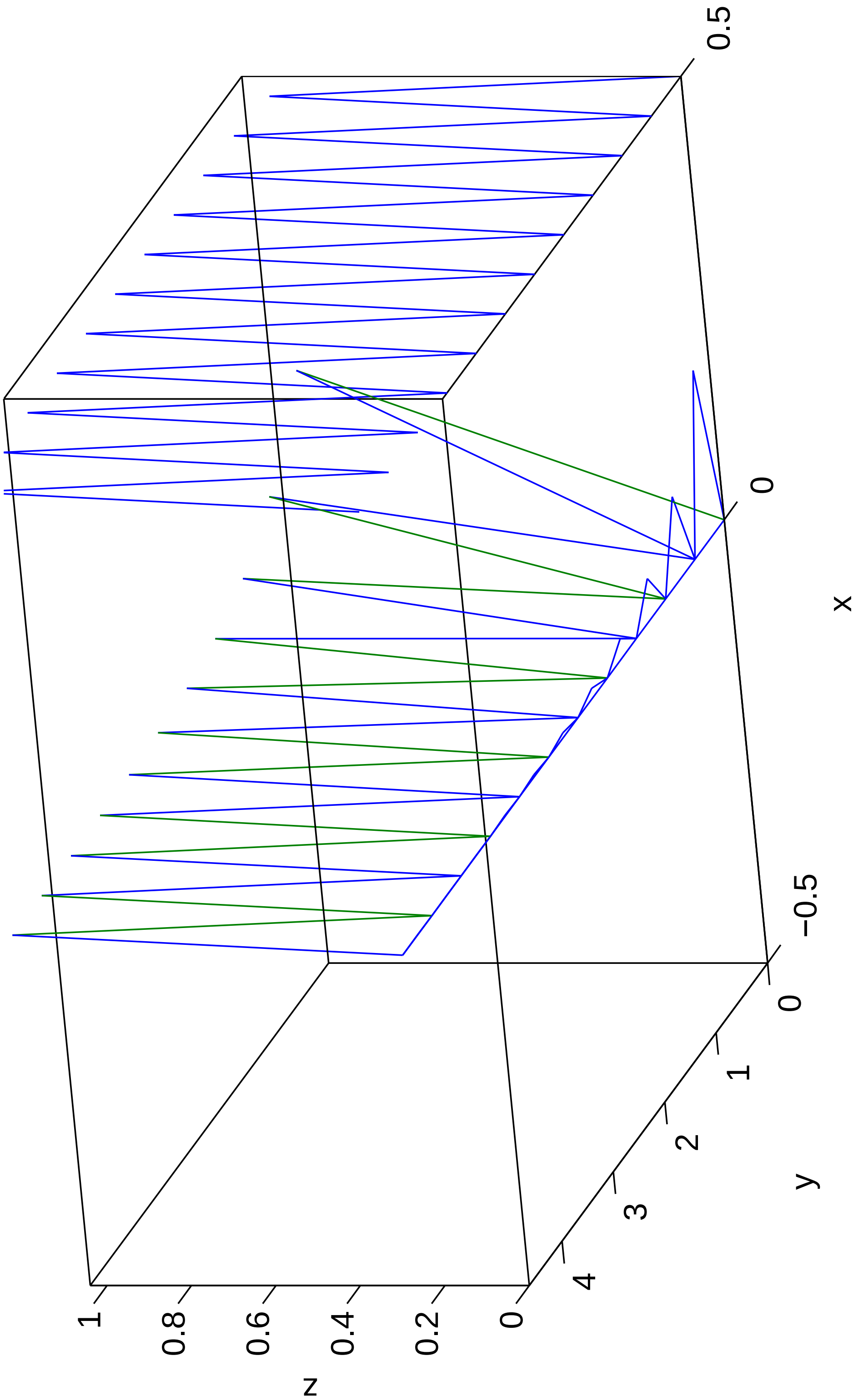


Figure 3.12: A walking sequence for the bipedal robot.

Chapter 4

Hybrid Geometric Mechanics

Mechanics, and the geometry thereof, plays a fundamental role in engineering. This chapter gives a general outline on how to extend classical ideas from geometry and mechanics to a hybrid setting through the use of hybrid objects.

The objects of study in geometry display the fundamental property of being *categorical*, i.e., they reside in certain categories. Collections of geometric objects in a category relate to one another *naturally*, i.e., morphisms between geometric objects in two diagrams extend naturally to morphisms between diagrams. The relationship between different classes of geometric objects is *functorial*, i.e. one can translate from one class of geometric objects to another through the use of functors. Therefore, using the categorical, natural and functorial nature of geometric objects, one can hybridize these objects. Specifically, given a category C consisting of the geometric objects of interest, e.g., manifolds, Lie groups, Lie algebras, etc., one can form the "hybrid version" of these objects:

$$\mathbf{A} : \mathscr{A} \to \mathsf{C},$$

with $\mathbf{A}$ either covariant or contravariant, i.e., $(\mathscr{A}, \mathbf{A})$ is a hybrid or cohybrid object. We thus form the category of hybrid or cohybrid objects, $\mathsf{Hy}(\mathsf{C})$ or $\mathsf{CoHy}(\mathsf{C})$, depending on the contravariance or covariance of $\mathbf{A}$. Using the functorial relationship between different categories of geometric objects, e.g., the functor that associated to a Lie group its Lie algebra, we obtain functors between the categories of hybrid objects of interest. Some of the hybrid objects, hybrid morphisms and functors between categories of hybrid objects that will be introduced in this chapter can be seen in Table 4.1.

Recall from Chapter 3 that we were interested in answering the following question:

> *If it is possible to reduce the continuous components of a hybrid system, when is it possible to reduce the entire hybrid system?*

In that chapter, we were only able to answer this question for *simple* hybrid systems. Using the framework established by hybrid geometry, we will be able to answer this question for general hybrid systems. First, we recall the classical reduction theorem.

Important hybrid objects:			
Hybrid vector space:	$\mathbf{V} : \mathscr{V}$	$\to$	$\mathsf{Vect}_{\mathbb{R}}$
Hybrid manifold:	$\mathbf{M} : \mathscr{M}$	$\to$	Man
Hybrid tanget bundle:	$T_*(\mathbf{M}) : \mathscr{M}$	$\to$	$\mathsf{VectBund}_{\mathbb{R}}$
Hybrid differential k – forms:	$\Omega^k_*(\mathbf{M}) : \mathscr{M}$	$\to$	$\mathsf{Vect}_{\mathbb{R}}$
Hybrid de Rham cohomology:	$H^n_{\mathrm{dR}*}(\mathbf{M}) : \mathscr{M}$	$\to$	$\mathsf{Vect}_{\mathbb{R}}$
Hybrid Lie group:	$\mathbf{G} : \mathscr{G}$	$\to$	LieGrp
Hybrid Lie algebra:	$\mathfrak{g} : \mathscr{G}$	$\to$	LieAlg
Dual hybrid Lie algebra:	$\mathfrak{g}^\star : \mathscr{G}$	$\to$	$\mathsf{Vect}_{\mathbb{R}}$
Hybrid isotropy group:	$\mathbf{G}_{\vec{\mu}} : \mathscr{G}$	$\to$	LieGrp
Hybrid orbit space:	$\mathbf{M}/\mathbf{G} : \mathscr{M}$	$\to$	Top
Reduced hybrid phase space:	$\mathbf{M}_{\vec{\mu}} : \mathscr{M}$	$\to$	Man
Important hybrid morphisms:			
Hybrid exterior derivative:	$\vec{d} : \Omega^k_*(\mathbf{M})$	$\dot{\to}$	$\Omega^{k+1}_*(\mathbf{M})$
Hybrid wedge product:	$- \vec{\wedge} - : \Omega^k_*(\mathbf{M}) \times \Omega^l_*(\mathbf{M})$	$\dot{\to}$	$\Omega^{k+l}_*(\mathbf{M})$
Hybrid conjunction map:	$I_{\vec{g}} : \mathbf{G}$	$\dot{\to}$	$\mathbf{G}$
Hybrid adjoint action:	$\mathrm{Ad}_{\vec{g}} : \mathfrak{g}$	$\dot{\to}$	$\mathfrak{g}$
Hybrid group action:	$\vec{\Phi} : \mathbf{G} \times \mathbf{M}$	$\dot{\to}$	$\mathbf{M}$
Hybrid infintesimal generator:	$\vec{\xi}_{\mathbf{M}} : \mathbf{M}$	$\dot{\to}$	$T_*(\mathbf{M})$
Hybrid momentum map:	$\vec{J} : \mathbf{M}$	$\dot{\to}$	$\mathfrak{g}^\star$
Important functors:			
Hybrid tangent bundle functor:	$\mathsf{Hy}(T) : \mathsf{Hy}(\mathsf{Man})$	$\to$	$\mathsf{Hy}(\mathsf{VectBund}_{\mathbb{R}})$
Hybrid k – form functor:	$\mathsf{Hy}(\Omega^k) : \mathsf{Hy}(\mathsf{Man})$	$\to$	$\mathsf{CoHy}(\mathsf{Vect}_{\mathbb{R}})$
Hybrid Lie functor:	$\mathsf{Hy}(\mathsf{Lie}) : \mathsf{Hy}(\mathsf{LieGrp})$	$\to$	$\mathsf{Hy}(\mathsf{LieAlg})$
Dual vector space functor:	$\mathsf{Hy}((-)^\star) : \mathsf{Hy}(\mathsf{Vect}_{\mathbb{R}})$	$\to$	$\mathsf{CoHy}(\mathsf{Vect}_{\mathbb{R}})$

Table 4.1: Important hybrid objects, morphisms between hybrid objects and functors between categories of hybrid objects.

Marsden mathematically describes, [86], the reduction theorem for classical mechanics as follows:

> *Given a symplectic manifold M (the phase space), there exists a symplectic manifold M_μ such that "M_μ inherits the symplectic structure from that of M, so it can be used as a new phase space. Also, dynamical trajectories of the Hamiltonian H on M determine corresponding trajectories on the reduced space."*

Upon inspection, it is clear that a rather copious mathematical framework is needed to perform reduction: there must be a Lie group G acting on the symplectic manifold M, and an Ad^*-equivariant momentum map $J : M \to \mathfrak{g}^*$, and the Hamiltonian H must be G-invariant.

Hybrid objects will allow us to generalize all of the ingredients necessary for reduction to a hybrid setting. The main result is the following hybrid reduction theorem:

> *Given a hybrid symplectic manifold* $\mathbf{M}$ *(the hybrid phase space), there exists a hybrid symplectic manifold* $\mathbf{M}_{\vec{\mu}}$ *such that* $\mathbf{M}_{\vec{\mu}}$ *inherits the hybrid symplectic structure from that of* $\mathbf{M}$*, so it can be used as a new hybrid phase space. Also, dynamical trajectories of the hybrid Hamiltonian* $\mathbf{H}$ *on* $\mathbf{M}$ *determine corresponding trajectories on the reduced hybrid space.*

That is, if $(\mathbf{M},\vec{\omega},\mathbf{H})$ is a *hybrid Hamiltonian system* ($\mathbf{M}$ is a hybrid manifold, $\vec{\omega}$ is a hybrid symplectic form on $\mathbf{M}$ and $\mathbf{H}$ is a hybrid Hamiltonian), then this theorem says that under certain conditions we can reduce this hybrid Hamiltonian system to obtain a *reduced* hybrid Hamiltonian system $(\mathbf{M}_{\vec{\mu}},\vec{\omega}_{\vec{\mu}},\mathbf{H}_{\vec{\mu}})$.

The hybrid reduction theorem can be used to explicitly reduce hybrid systems since we can associate to a hybrid Hamiltonian system $(\mathbf{M}^{\iota},\vec{\omega},\mathbf{H})$ a "classical" hybrid system $\mathfrak{H}_{(\mathbf{M}^{\iota},\vec{\omega},\mathbf{H})}$ (see Paragraph 2.1.2, Proposition 2.1 and Proposition 2.2). Therefore, the ability to reduce hybrid Hamiltonian systems yields a method for reducing hybrid systems; graphically, the operation of "hybrid system reduction" is defined by requiring the following diagram to commute:

$$\begin{array}{ccc} (\mathbf{M}^{\iota},\vec{\omega},\mathbf{H}) & \xrightarrow{\text{reduction}} & (\mathbf{M}^{\iota}_{\vec{\mu}},\vec{\omega}_{\vec{\mu}},\mathbf{H}_{\vec{\mu}}) \\ \text{association}\Big\downarrow & & \Big\downarrow\text{association} \\ \mathfrak{H}_{(\mathbf{M}^{\iota},\vec{\omega},\mathbf{H})} & \xrightarrow{\text{reduction}} & \mathfrak{H}_{(\mathbf{M}^{\iota}_{\vec{\mu}},\vec{\omega}_{\vec{\mu}},\mathbf{H}_{\vec{\mu}})}. \end{array}$$

Moreover, since this association is constructive in nature, the result is a concrete method for reducing hybrid systems. Finally, the hybrid reduction theorem proven in this chapter can be used to show that trajectories (or executions) of $\mathfrak{H}_{(\mathbf{M}^{\iota},\vec{\omega},\mathbf{H})}$ determine corresponding trajectories of $\mathfrak{H}_{(\mathbf{M}^{\iota}_{\vec{\mu}},\vec{\omega}_{\vec{\mu}},\mathbf{H}_{\vec{\mu}})}$.

The work on which this chapter builds is the same as that of Chapter 3; see the related work paragraph in that chapter.

Examples will be sparse in this chapter as we believe that we have more than motivated the importance of reduction in Chapter 3, especially given the application of reduction to bipedal walking. We will utilize a simple example throughout the chapter in order to illustrate the concepts involved.

Running example. Throughout this chapter, we will consider a ball bouncing in *two-dimensions*; this differs from the one-dimensional version introduced earlier as well as the three-dimensional version ball

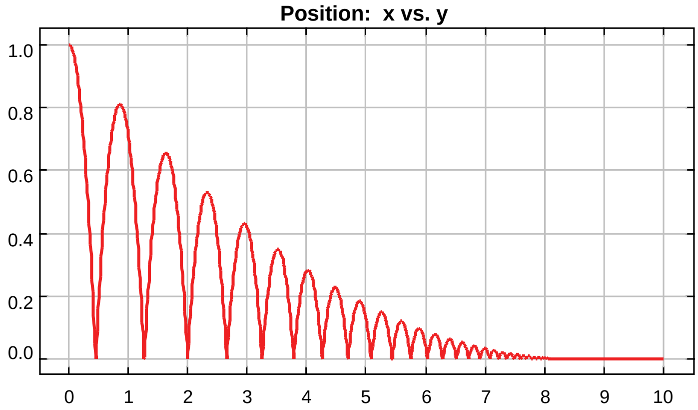


Figure 4.1: A trajectory of the two-dimensional (completed) bouncing ball model.

bouncing on a sinusoidal surface. We will *not* use the standard model of this system (which has a single domain) but rather the completed version of the system (obtained using the procedure outlined in Section 5.5 of Chapter 5) which allows the ball to "stop bouncing" after the Zeno point is reached; see Figure 4.1.

We introduce the model of this system directly as a categorical hybrid system rather than translating the definition from the standard definition as has been our custom. Since the goal is to define a hybrid system, we first define its discrete structure.

Define $\mathscr{M}^{\text{ball}}$ as the D-category given by the following diagram:

$$\mathscr{M}^{\text{ball}} = \begin{array}{ccccc} a_b & & & & \\ \mathsf{s}_{a_b} \downarrow\downarrow \mathsf{t}_{a_b} & & & & \\ b & \xleftarrow{\mathsf{s}_{a_s}} & a_s & \xrightarrow{\mathsf{t}_{a_s}} & s. \end{array}$$

where "b" will correspond to the state where the ball is "bouncing" and "s" will correspond to the state where the ball is "sliding."

Define the hybrid manifold:

$$\mathbf{M}^{\text{ball}} : \mathscr{M}^{\text{ball}} \to \mathsf{Man}$$

by:

$$\mathbf{M}^{\text{ball}}\left(\mathscr{M}^{\text{ball}}\right) = \begin{array}{ccccc} \mathbf{M}^{\text{ball}}_{a_b} & & & & \\ \mathbf{M}^{\text{ball}}_{\mathsf{s}_{a_b}} = \iota \downarrow\downarrow \mathbf{M}^{\text{ball}}_{\mathsf{t}_{a_b}} & & & & \\ \mathbf{M}^{\text{ball}}_{b} = \mathbb{R}^4 & \xleftarrow{\mathbf{M}^{\text{ball}}_{\mathsf{s}_{a_s}} = \iota} & \mathbf{M}^{\text{ball}}_{a_s} = \mathbb{R}^2 & \xrightarrow{\mathbf{M}^{\text{ball}}_{\mathsf{t}_{a_s}} = \text{id}} & \mathbf{M}^{\text{ball}}_{s} = \mathbb{R}^2 \end{array}$$

where here the coordinates on $\mathbb{R}^4$ are $(x, y, p_x, p_y)^T$, the coordinates on $\mathbb{R}^2$ are $(x, p_x)^T$, with $\mathbf{M}^{\text{ball}}_{a_b}$ and $\mathbf{M}^{\text{ball}}_{\mathsf{t}_{a_b}}$

given by:

$$\begin{aligned}
\mathbf{M}^{\text{ball}}_{a_b} &= \left\{ \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} \in \mathbb{R}^4 : y = 0 \text{ and } p_y \leq 0 \right\}, \\
\mathbf{M}^{\text{ball}}_{\mathfrak{t}_{a_b}}(x, 0, p_x, p_y) &= \begin{pmatrix} x \\ 0 \\ p_x \\ -e p_y \end{pmatrix},
\end{aligned}$$

with $0 \leq e \leq 1$ the coefficient of restitution.

Finally, the hybrid system is given by $(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}, \mathbf{X}^{\text{ball}})$ where

$$\mathbf{X}^{\text{ball}} = \{\mathbf{X}^{\text{ball}}_b, \mathbf{X}^{\text{ball}}_s\},$$

with:

$$\begin{aligned}
\mathbf{X}^{\text{ball}}_b(x, y, p_x, p_y) &= \frac{1}{m} \begin{pmatrix} p_x \\ p_y \\ 0 \\ -m^2 g \end{pmatrix}, \\
\mathbf{X}^{\text{ball}}_s(x, p_x) &= \frac{1}{m} \begin{pmatrix} p_x \\ 0 \end{pmatrix}.
\end{aligned}$$

and m the mass of the ball.

4.1 Hybrid Differential Forms

In order to discuss the general geometric reduction of hybrid systems, we introduce hybrid differential forms. The framework of hybrid objects makes this a relatively easy task, although it is nontrivial as the constructions are not always the obvious ones. Note that we could first build up the general framework of tensor bundles, etc., but since the construction is essentially the same, we proceed directly to the notion of differential forms which is the main concept of interest.

Note that when dealing with hybrid differential forms, we are forced to deal with both covariant and contravariant functors. To avoid confusion, we will explicitly state which type of functor we are considering when necessary.

We refer the reader to [79] for any necessary background material.

4.1.1 Hybrid tangent bundles. We begin by discussing how one associates to a hybrid manifold its hybrid tangent bundle. This will be useful for understanding later constructions.

The process of associating a tangent bundle to a manifold defines a functor:

$$T : \mathsf{Man} \to \mathsf{VectBund}_{\mathbb{R}}$$

where $\mathsf{VectBund}_{\mathbb{R}}$ is the category of (real) vector bundles. Specifically, we have $T(M) = TM$, where TM is the tangent bundle of M; implicit in this notation is the canonical projection $\pi : TM \to M$ that makes TM into a vector bundle. In addition, for a morphism f of manifolds Tf is the pushfoward of this function.

The functor T induces a functor:

$$\mathsf{Hy}(T) : \mathsf{Hy}(\mathsf{Man}) \to \mathsf{Hy}(\mathsf{VectBund}_{\mathbb{R}}).$$

That is, for a hybrid manifold $(\mathscr{M}, \mathbf{M})$, we can associate to this hybrid manifold its *hybrid tangent bundle*

$$(\mathscr{M}, T_*(\mathbf{M})) := \mathsf{Hy}(T)(\mathscr{M}, \mathbf{M}),$$

where

$$T_*(\mathbf{M}) : \mathscr{M} \to \mathsf{VectBund}_{\mathbb{R}}$$

is given by, for every $a \xrightarrow{\alpha} b$ in $\mathscr{M}$,

$$T_*(\mathbf{M})_a := T\mathbf{M}_a \xrightarrow{T_*(\mathbf{M})_\alpha := T\mathbf{M}_\alpha} T_*(\mathbf{M})_b := T\mathbf{M}_b.$$

Here $T\mathbf{M}_\alpha$ is the pushforward of $\mathbf{M}_\alpha$,

$$T\mathbf{M}_\alpha(p, X) = (\mathbf{M}_\alpha(p), T_p\mathbf{M}_\alpha(X)),$$

for $(p, X) \in T\mathbf{M}_a$, i.e., for $X \in T_p\mathbf{M}_a$ and $p \in \mathbf{M}_a$.

If $(\vec{F}, \vec{f}) : (\mathscr{M}, \mathbf{M}) \to (\mathscr{N}, \mathbf{N})$ is a morphism of hybrid manifolds, then there is an induced morphism:

$$\mathsf{Hy}(T)(\vec{F}, \vec{f}) = (\vec{F}, T_*(\vec{f})) : \mathsf{Hy}(T)(\mathscr{M}, \mathbf{M}) = (\mathscr{M}, T_*(\mathbf{M})) \to (\mathscr{N}, T_*(\mathbf{N}))$$

between the hybrid tangent bundles of these hybrid manifolds as outlined in Paragraph 1.3.7.

4.1.2 Hybrid sections. Note that we have a natural transformation $\vec{\pi} : T_*(\mathbf{M}) \stackrel{\cdot}{\to} \mathbf{M}$, i.e., a hybrid morphism:

$$(\vec{\mathrm{Id}}_{\mathscr{M}}, \vec{\pi}) : (\mathscr{M}, T_*(\mathbf{M})) \to (\mathscr{M}, \mathbf{M}),$$

called the canonical *hybrid projection map*, and given objectwise by the natural projection, i.e., $\vec{\pi}_a : T_*(\mathbf{M})_a = T\mathbf{M}_a \to \mathbf{M}_a$.

We can consider sections of the hybrid tangent bundle of a hybrid manifold:

$$\Gamma(\mathbf{M}) := \{\vec{X} : \mathbf{M} \stackrel{\cdot}{\to} T_*(\mathbf{M}) : \vec{\pi} \bullet \vec{X} = \vec{\mathrm{id}}_{\mathbf{M}}\},$$

which in fact defines a collection of vector fields (of a very special form) on the hybrid manifold $(\mathscr{M}, \mathbf{M})$, i.e., associated to the hybrid section, $\vec{X}$, we have the collection of vector fields $\mathbf{X} := \{\vec{X}_b\}_{b \in \mathsf{V}(\mathscr{M})}$. That being said, hybrid sections are typically not of interest as they are too restrictive.

4.1.3 Differential forms. Let M be a manifold and let $\Lambda^k(M)$ be the vector bundle consisting of all alternating tensors, i.e., we have a canonical projection map $\pi : \Lambda^k(M) \to M$. A section of this vector bundle

$$\omega : M \to \Lambda^k(M) \quad \text{s.t.} \quad \pi \circ \omega = \mathrm{id}_M$$

is a *differential k-form* or just a *k-form*. The set of all differential k-forms is denoted by:

$$\Omega^k(M) = \{\omega : M \to \Lambda^k(M) : \pi \circ \omega = \mathrm{id}_M\}.$$

The process of associating to a manifold its differential k-forms induces a contravariant functor:

$$\Omega^k : \mathsf{Man} \to \mathsf{Vect}_{\mathbb{R}},$$

where for $f : M \to N$,

$$\Omega^k(f) : \Omega^k(N) \to \Omega^k(M)$$

is the pullback of f given by, for $\omega \in \Omega^k(N)$, $p \in M$ and $X_1, \ldots X_k \in T_pM$,

$$\Omega^k(f)(\omega)_p(X_1, \ldots, X_k) := \omega_{f(p)}(T_pf(X_1), \ldots, T_pf(X_k)),$$

where $T_pf(X_1), \ldots, T_pf(X_k) \in T_{f(p)}N$.

Remark 4.1. Note that $\Omega^k(f)$, termed the pullback of the function f, is typically denoted by f^*. We opt for the non-standard notation because it demonstrates that the pullback of a function is *functorally* obtained from the original function. In addition, it avoids the proliferation of $*$'s that would be inevitable due to the notation utilized to denote the pushforward of a functor.

4.1.4 Hybrid differential forms. The contravariant functor Ω^k induces a contravariant functor:

$$\mathsf{Hy}(\Omega^k) : \mathsf{Hy}(\mathsf{Man}) \to \mathsf{CoHy}(\mathsf{Vect}_{\mathbb{R}}).$$

For a hybrid manifold $(\mathscr{M}, \mathbf{M})$,

$$\mathsf{Hy}(\Omega^k)(\mathscr{M}, \mathbf{M}) := (\mathscr{M}, \Omega^k_*(\mathbf{M})),$$

with $\Omega^k_*(\mathbf{M}) : \mathscr{M} \to \mathsf{Vect}_{\mathbb{R}}$ a contravariant functor given on objects by $\Omega^k_*(\mathbf{M})_a = \Omega^k(\mathbf{M}_a)$ and on morphisms $\alpha : a \to b$ in $\mathscr{M}$ by

$$\Omega^k_*(\mathbf{M})_\alpha = \Omega^k(\mathbf{M}_\alpha) : \Omega^k_*(\mathbf{M})_b \to \Omega^k_*(\mathbf{M})_a.$$

For a morphism $(\vec{F}, \vec{f}) : (\mathscr{M}, \mathbf{M}) \to (\mathscr{N}, \mathbf{N})$ in $\mathsf{Hy}(\mathsf{Man})$, we have the corresponding morphism in $\mathsf{CoHy}(\mathsf{Vect}_{\mathbb{R}})$:

$$\mathsf{Hy}(\Omega^k)(\vec{F}, \vec{f}) := (\vec{F}^{\mathsf{op}}, \Omega^k_*(\vec{f})) : (\mathscr{N}, \Omega^k_*(\mathbf{N})) \to (\mathscr{M}, \Omega^k_*(\mathbf{M})),$$

where $\vec{F}^{\mathsf{op}} : \mathscr{N} \to \mathscr{M}$ is the morphism in $\mathsf{Dcat}^{\mathsf{op}}$ corresponding to $\vec{F}$ and $\Omega^k_*(\vec{f}) : \vec{F}^*(\Omega^k_*(\mathbf{N})) \stackrel{.}{\to} \Omega^k_*(\mathbf{M})$ in ${}^{\mathscr{M}}\mathsf{Vect}_{\mathbb{R}}$. In particular, there is the following relationship

$$\begin{array}{ccc} \mathbf{M}_a & \xrightarrow{\vec{f}_a} & \mathbf{N}_{\vec{F}(a)} \\ {\scriptstyle \mathbf{M}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathbf{N}_{\vec{F}(\alpha)}} \\ \mathbf{M}_b & \xrightarrow{\vec{f}_b} & \mathbf{N}_{\vec{F}(b)} \end{array} \quad \mapsto \quad \begin{array}{ccc} \Omega^k(\mathbf{M}_a) & \xleftarrow{\Omega^k(\vec{f}_a)} & \Omega^k(\mathbf{N}_{\vec{F}(a)}) \\ {\scriptstyle \Omega^k(\mathbf{M}_\alpha)}\uparrow & & \uparrow{\scriptstyle \Omega^k(\mathbf{N}_{\vec{F}(\alpha)})} \\ \Omega^k(\mathbf{M}_b) & \xleftarrow{\Omega^k(\vec{f}_b)} & \Omega^k(\mathbf{N}_{\vec{F}(b)}) \end{array}$$

in Man and $\text{Vect}_{\mathbb{R}}$, respectively, for every $\alpha : a \to b$ in $\mathscr{M}$.

We are especially interested in elements of the cohybrid object $(\mathscr{M}, \Omega^k_*(\mathbf{M}))$.

Definition 4.1. Let $(\mathscr{M}, \mathbf{M})$ be a hybrid manifold. A **hybrid differential k-form** is an element of the cohybrid object $(\mathscr{M}, \Omega^k_*(\mathbf{M}))$, i.e.,

$$\vec{\omega} \in (\mathscr{M}, \Omega^k_*(\mathbf{M})).$$

Therefore, a hybrid differential k-form must satisfy:

- $\vec{\omega}_a \in \Omega^k(\mathbf{M}_a)$, i.e, $\vec{\omega}_a$ is a differential k-form,
- $\Omega^k(\mathbf{M}_\alpha)(\vec{\omega}_b) = \vec{\omega}_a$ for all $\alpha : a \to b$ in $\mathscr{M}$.

Notation 4.1. To simplify notation, when referring to elements of $(\mathscr{M}, \Omega^k_*(\mathbf{M}))$ we will often write $\vec{\omega} \in \Omega^k_*(\mathbf{M})$ since we will always be considering the same D-category $\mathscr{M}$.

4.1.5 Hybrid symplectic manifolds.

The formulation of hybrid differential forms allows us to define hybrid symplectic manifolds. Note that the definition of a hybrid symplectic manifold is not the most obvious one—we do not require a hybrid differential 2-form to be objectwise a symplectic form.

Definition 4.2. A **hybrid symplectic manifold** is a hybrid manifold $(\mathscr{M}, \mathbf{M})$ together with a hybrid 2-form $\vec{\omega} \in \Omega^2(\mathbf{M})$ such that $\vec{\omega}_b$ is smooth, closed and nondegenerate, i.e., a symplectic form, for all $b \in \bigvee(\mathscr{M})$.

Example 4.1. On the hybrid manifold $(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}})$, we will consider the canonical hybrid symplectic form, i.e., $\vec{\omega}^{\text{ball}}$ is specified by $\vec{\omega}^{\text{ball}}_b$ and $\vec{\omega}^{\text{ball}}_s$, which are the standard symplectic forms on $\mathbb{R}^4$ and $\mathbb{R}^2$, respectively, with

$$\vec{\omega}^{\text{ball}}_{a_s} := \vec{\omega}^{\text{ball}}_s, \qquad \vec{\omega}^{\text{ball}}_{a_b} := \Omega^2(\imath)(\vec{\omega}^{\text{ball}}_b).$$

A simple calculation verifies that:

$$\Omega^2(\imath)(\vec{\omega}^{\text{ball}}_b) = \vec{\omega}^{\text{ball}}_{a_s}, \qquad \vec{\omega}^{\text{ball}}_{a_b} = \Omega^2(\mathbf{M}^{\text{ball}}_{\mathsf{t}_{a_b}})(\vec{\omega}^{\text{ball}}_b),$$

so Definition 4.2 is satisfied and $(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}, \vec{\omega}^{\text{ball}})$ is a hybrid symplectic manifold.

We are now interested in defining some elementary operations on $\Omega^k_*(\mathbf{M})$.

4.1.6 Hybrid exterior derivatives.

For a hybrid manifold $(\mathscr{M}, \mathbf{M})$, the *hybrid exterior derivative* is a natural transformation:

$$\vec{d} : \Omega^k_*(\mathbf{M}) \stackrel{\centerdot}{\to} \Omega^{k+1}_*(\mathbf{M})$$

defined for all $k \in \mathbb{N}$. It is given on objects a of $\mathscr{D}$ by:

$$\vec{d}_a : \Omega^k(\mathbf{M}_a) \stackrel{\centerdot}{\to} \Omega^{k+1}(\mathbf{M}_a),$$

where $\vec{d}_a$ is the exterior derivative on $\mathbf{M}_a$. It follows that the hybrid exterior derivative is a hybrid linear map, i.e., that it is objectwise linear. Moreover, if $\vec{\omega} \in \Omega^k_*(\mathbf{M})$, then $\vec{d}(\vec{\omega}) \in \Omega^{k+1}_*(\mathbf{M})$ where $\vec{d}(\vec{\omega})_a = \vec{d}_a(\vec{\omega}_a)$.

The hybrid exterior derivative displays the following obvious and yet important property:

Lemma 4.1. $\vec{d} \bullet \vec{d} = \vec{0}$.

In the preceding lemma, $\vec{0}$ is a natural transformation that is objectwise zero.

4.1.7 Hybrid wedge product. For a hybrid manifold $(\mathscr{M}, \mathbf{M})$, the *hybrid wedge product* or *hybrid interior product* is a natural transformation:

$$- \vec{\wedge} - : \Omega^k_*(\mathbf{M}) \times \Omega^l_*(\mathbf{M}) \dot{\to} \Omega^{k+l}_*(\mathbf{M})$$

such that for all $a \in \mathsf{Ob}(\mathscr{M})$, the following map:

$$- \vec{\wedge}_a - : \Omega^k(\mathbf{M}_a) \times \Omega^l(\mathbf{M}_a) \dot{\to} \Omega^{k+l}(\mathbf{M}_a)$$

is the wedge product. It follows that if $\vec{\omega} \in \Omega^k_*(\mathbf{M})$ and $\vec{\eta} \in \Omega^k_*(\mathbf{M})$ then $\vec{\omega}\vec{\wedge}\vec{\eta} \in \Omega^{k+l}_*(\mathbf{M})$ where $(\vec{\omega}\vec{\wedge}\vec{\eta})_a := \vec{\omega}_a \vec{\wedge}_a \vec{\eta}_a$.

Hybrid wedge products and hybrid exterior derivatives are related through the following lemma.

Lemma 4.2. *For* $\vec{\omega} \in \Omega^k_*(\mathbf{M})$ *and* $\vec{\eta} \in \Omega^l_*(\mathbf{M})$,

$$\vec{d}(\vec{\omega}\vec{\wedge}\vec{\eta}) = \vec{d}(\vec{\omega})\vec{\wedge}\vec{\eta} + (-1)^k \vec{\omega}\vec{\wedge}\vec{d}(\vec{\eta}),$$

where the addition in this expression is preformed objectwise.

To provide an example of some of the other constructions that can be carried out using categories of hybrid objects and categories of cohybrid objects, we briefly discuss de Rham cohomology.

4.1.8 Hybrid de Rham cohomology. For a smooth manifold M, the exterior derivative yields a *cochain complex* in the category of vector spaces:

$$0 \xrightarrow{\ 0\ } \Omega^0(M) \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{d} \Omega^{n+1}(M) \xrightarrow{\ d\ } \cdots$$

denoted by $(\Omega^\bullet(M), d)$. Consider the linear subspaces:

$$\begin{aligned} Z^n(M) &:= \mathrm{Ker}(d : \Omega^n(M) \to \Omega^{n+1}(M)) \\ &= \{\omega \in \Omega^n(M) : d(\omega) = 0\}, \\ B^n(M) &:= \mathrm{Im}(d : \Omega^{n-1}(M) \to \Omega^n(M))) \\ &= \{\omega \in \Omega^n(M) : \exists\, \eta \in \Omega^{n-1}(M) \text{ s.t. } d(\eta) = \omega\}. \end{aligned}$$

The n^{th} de Rham cohomology group of M is defined to be the cohomology of the cohain complex $(\Omega^\bullet(M), d)$:

$$H^n_{\mathrm{dR}}(M) = H^n(\Omega^\bullet(M)) = \frac{Z^n(M)}{B^n(M)}.$$

Note that taking the de Rham cohomology of a manifold results in a contravariant functor:

$$H^n_{\mathrm{dR}} : \mathsf{Man} \to \mathsf{Vect}_{\mathbb{R}}$$

defined for each $n \in \mathbb{N}$.

For a hybrid manifold $(\mathscr{M}, \mathbf{M})$, the de Rham cohomology functor induces a contravariant functor:

$$\mathsf{Hy}(H^n_{\mathrm{dR}}) : \mathsf{Hy}(\mathsf{Man}) \to \mathsf{CoHy}(\mathsf{Vect}_{\mathbb{R}}).$$

That is, for a hybrid manifold $(\mathscr{M}, \mathbf{M})$,

$$\mathsf{Hy}(H^n_{\mathrm{dR}})(\mathscr{M}, \mathbf{M}) := (\mathscr{M}, H^n_{\mathrm{dR}*}(\mathbf{M})),$$

is called the n^{th} hybrid de Rham cohomology group of $(\mathscr{M}, \mathbf{M})$.

Therefore, we are able to do "hybrid de Rham cohomology." More on the homology of hybrid systems can be found in [14].

4.2 Hybrid Lie Groups and Algebras

We are interested in studying the relationship between hybrid Lie groups and hybrid Lie algebras via the relationship between Lie groups and Lie algebras.

4.2.1 Hybrid Lie groups. The category of Lie groups, LieGrp, has as

Objects: Lie groups, i.e., groups that are also (smooth) manifolds such that multiplication and inversion define smooth maps,
Morphisms: Smooth maps that are also group homomorphisms.

A *hybrid Lie group* is a hybrid object over the category of Lie groups, LieGrp, i.e., a pair $(\mathscr{G}, \mathbf{G})$ where

$$\mathbf{G} : \mathscr{G} \to \mathsf{LieGrp}.$$

An element of a hybrid Lie group, $\vec{g} \in (\mathscr{G}, \mathbf{G})$, must satisfy the following properties:

- $\vec{g}_a \in \mathbf{G}_a$ for all objects a of $\mathscr{G}$,
- $\vec{g}_b = \mathbf{G}_\alpha(\vec{g}_a)$ for all $\alpha : a \to b$ in $\mathscr{G}$.

In particular, every element of $(\mathscr{G}, \mathbf{G})$ has an inverse, $\vec{g}^{-1}$, defined objectwise to be the inverse of $\vec{g}_a$, i.e.,

$$\vec{g}_a^{-1} \cdot \vec{g}_a = e_{\mathbf{G}_a} = \vec{g}_a \cdot \vec{g}_a^{-1}$$

where $e_{\mathbf{G}_a}$ is the identity element of $\mathbf{G}_a$.

4.2.2 Lie algebras. Recall that a Lie algebra $\mathfrak{g}$ is a vector space together with a binary operation:

$$\begin{aligned} [\,-\,,\,-\,] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (X, Y) &\mapsto [X, Y], \end{aligned}$$

called the *Lie bracket*. This bracket must satisfy, for all $X, Y, Z \in \mathfrak{g}$:

Bilinearity. For all $r, w \in \mathbb{R}$,

$$\begin{aligned} [rX + wY, Z] &= r[X,Z] + w[Y,Z] \\ [Z, rX + wY] &= r[Z,X] + w[Z,Y] \end{aligned}$$

Antisymmetry.

$$[X,Y] = -[Y,X]$$

Jacobi Identity.

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

A morphism of Lie algebras is a linear map $A : \mathfrak{g} \to \mathfrak{h}$ such that:

$$A([X,Y]) = [A(X), A(Y)].$$

We have thus defined the category of Lie algebras, LieAlg.

4.2.3 Hybrid Lie algebras. We could directly define a hybrid Lie algebra as a hybrid object over the category of Lie algebras. We opt for a more circuitous route in order to demonstrate that hybrid objects can often be defined using more "fundamental" information and that, in fact, the end result is the same.

A *hybrid Lie algebra* is a hybrid vector space $(\mathscr{G}, \mathbf{g})$,

$$\mathbf{g} : \mathscr{G} \to \mathsf{Vect}_{\mathbb{R}}$$

together with a natural transformation:

$$\overrightarrow{[\,-\,,\,-\,]} : \mathbf{g} \times \mathbf{g} \stackrel{.}{\to} \mathbf{g}$$

where the product is the product of functors given on objects and morphisms by $(\mathbf{g} \times \mathbf{g})_a = \mathbf{g}_a \times \mathbf{g}_a$ and $(\mathbf{g} \times \mathbf{g})_\alpha = \mathbf{g}_\alpha \times \mathbf{g}_\alpha$. That is, it is the product in $\mathsf{Vect}_{\mathbb{R}}^{\mathscr{G}}$ which we know exists by Proposition A.4. In addition, we require that for every object a of $\mathscr{G}$, the corresponding binary operation:

$$\overrightarrow{[\,-\,,\,-\,]}_a : \mathbf{g}_a \times \mathbf{g}_a \to \mathbf{g}_a$$

satisfies the bilinearity, antisymmetry and Jacobi identities, i.e., it is a Lie bracket.

The following results say that this is in fact the "correct" definition of a hybrid Lie algebra:

Proposition 4.1. *A hybrid object* $(\mathscr{G}, \mathbf{g})$ *over* $\mathsf{Vect}_{\mathbb{R}}$ *is a hybrid Lie algebra iff it is a hybrid object over* LieAlg,

$$\mathbf{g} : \mathscr{G} \to \mathsf{LieAlg}.$$

Proof. Beginning with a hybrid Lie algebra $(\mathscr{G}, \mathbf{\mathfrak{g}})$, it is clear that $\mathbf{\mathfrak{g}}_a$ is a Lie algebra. Therefore, we must only verify that $\mathbf{\mathfrak{g}}_\alpha$ is a Lie algebra homomorphism. This follows from the commutativity of the diagram:

$$\begin{array}{ccc} \mathbf{\mathfrak{g}}_a \times \mathbf{\mathfrak{g}}_a & \xrightarrow{[\,-\,,\,-\,]_a} & \mathbf{\mathfrak{g}}_a \\ {\scriptstyle \mathbf{\mathfrak{g}}_\alpha \times \mathbf{\mathfrak{g}}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathbf{\mathfrak{g}}_\alpha} \\ \mathbf{\mathfrak{g}}_b \times \mathbf{\mathfrak{g}}_b & \xrightarrow{[\,-\,,\,-\,]_b} & \mathbf{\mathfrak{g}}_b \end{array}$$

The converse direction is equally straightforward. □

4.2.4 The Lie functor. The Lie functor is the functor:

$$\mathsf{Lie} : \mathsf{LieGrp} \to \mathsf{LieAlg},$$

given on objects by associating to a Lie group G its Lie algebra:

$$\mathrm{Lie}(G) = T_{e_G} G,$$

where $T_{e_G} G$ is the tangent space at the identity element of G. Note that $\mathrm{Lie}(G)$ is isomorphic to (and could be defined as) the set of all left-invariant vector fields on G. For a morphism $f : G \to H$,

$$\mathrm{Lie}(f) := T_{e_G} f : \mathrm{Lie}(G) \to \mathrm{Lie}(H).$$

Note that $\mathrm{Lie}(G)$ is often denoted by $\mathfrak{g}$.

The Lie functor yields a functor between categories of hybrid objects:

$$\mathsf{Hy}(\mathsf{Lie}) : \mathsf{Hy}(\mathsf{LieGrp}) \to \mathsf{Hy}(\mathsf{LieAlg}).$$

For a hybrid Lie group $(\mathscr{G}, \mathbf{G})$, we will denote its corresponding hybrid Lie algebra by

$$\mathsf{Hy}(\mathsf{Lie})(\mathscr{G}, \mathbf{G}) := (\mathscr{G}, \mathbf{\mathfrak{g}}),$$

and for a morphism of hybrid Lie groups $(\vec{F}, \vec{f}) : (\mathscr{G}, \mathbf{G}) \to (\mathscr{H}, \mathbf{H})$ we obtain a morphism of hybrid Lie algebras:

$$\mathsf{Hy}(\mathsf{Lie})(\vec{F}, \vec{f}) := (\vec{F}, \mathrm{Lie}_*(\vec{f})) : (\mathscr{G}, \mathbf{\mathfrak{g}}) \to (\mathscr{H}, \mathbf{\mathfrak{h}}).$$

We know that this is a hybrid vector space, $\mathbf{\mathfrak{g}} : \mathscr{G} \to \mathsf{Vect}_{\mathbb{R}}$, and so an element of $(\mathscr{G}, \mathbf{\mathfrak{g}})$, which we denote by $\vec{\xi} \in (\mathscr{G}, \mathbf{\mathfrak{g}})$ or just $\vec{\xi} \in \mathbf{\mathfrak{g}}$ when the underlying D-category is clear from context, is a hybrid vector (as introduced in Example 1.21) and so must satisfy:

- $\vec{\xi}_a \in \mathbf{\mathfrak{g}}_a$ for all objects a of $\mathscr{G}$, i.e., $\vec{\xi}_a$ is a vector,
- $\vec{\xi}_b = \mathbf{\mathfrak{g}}_\alpha(\vec{\xi}_a)$ for all $\alpha : a \to b$ in $\mathscr{G}$.

In addition, we know from Example 1.21 that the set of elements of $(\mathscr{G}, \mathbf{\mathfrak{g}})$,

$$\mathrm{Elem}_{\mathsf{Hy}(\mathsf{LieAlg})}(\mathscr{G}, \mathbf{\mathfrak{g}})$$

form a vector space. In fact, it is clear from Proposition 4.1 that the elements of $(\mathscr{G}, \mathbf{\mathfrak{g}})$ form a Lie algebra.

4.2.5 The dual to a hybrid Lie algebra. As discussed in Example 1.22, the functor that associates to a vector space its dual induces a functor between categories of hybrid and cohybrid objects:

$$\mathsf{Hy}((\,-\,)^\star) : \mathsf{Hy}(\mathsf{Vect}_\mathbb{R}) \to \mathsf{CoHy}(\mathsf{Vect}_\mathbb{R}).$$

Through this functor we obtain the dual to a hybrid Lie algebra $(\mathscr{G}, \mathbf{\mathfrak{g}})$, which is the cohybrid object:

$$(\mathscr{G}, \mathbf{\mathfrak{g}}^\star) := \mathsf{Hy}((\,-\,)^\star)(\mathscr{G}, \mathbf{\mathfrak{g}}).$$

It follows from Example 1.23 that an element of the cohybrid object $(\mathscr{G}, \mathbf{\mathfrak{g}}^\star)$, $\vec{\mu} \in (\mathscr{G}, \mathbf{\mathfrak{g}}^\star)$ or just $\vec{\mu} \in \mathbf{\mathfrak{g}}^\star$, is a *hybrid covector* and thus must satisfy:

- $\vec{\mu}_a \in \mathbf{\mathfrak{g}}_a^\star$ for all objects a of $\mathscr{G}$, i.e., $\vec{\mu}_a : \mathbf{\mathfrak{g}}_a \to \mathbb{R}$ is a covector,
- $\mathbf{\mathfrak{g}}_\alpha^\star(\vec{\mu}_b) = \vec{\mu}_a$, i.e., $\vec{\mu}_b \circ \mathbf{\mathfrak{g}}_\alpha = \vec{\mu}_a$, for all $\alpha : a \to b$ in $\mathscr{G}$.

This implies that $\vec{\mu}$ corresponds to a natural transformation $\vec{\mu} : \mathbf{G} \stackrel{.}{\to} \Delta_\mathscr{G}(\mathbb{R})$.

Example 4.2. Returning to the bouncing ball, define a hybrid Lie group

$$\mathbf{G}^{\text{ball}} : \mathscr{M}^{\text{ball}} \to \mathsf{LieGrp}$$

by

$$\mathbf{G}^{\text{ball}}(\mathscr{M}^{\text{ball}}) = \begin{array}{ccccc} \mathbf{G}^{\text{ball}}_{a_b} = \mathbb{R} & & & & \\ \mathbf{G}^{\text{ball}}_{s_{a_b}} = \text{id} \downarrow \;\; \downarrow \mathbf{G}^{\text{ball}}_{t_{a_b}} = \text{id} & & & & \\ \mathbf{G}^{\text{ball}}_{b} = \mathbb{R} & \xleftarrow{\mathbf{G}^{\text{ball}}_{s_{a_s}} = \text{id}} & \mathbf{G}^{\text{ball}}_{a_s} = \mathbb{R} & \xrightarrow{\mathbf{G}^{\text{ball}}_{t_{a_s}} = \text{id}} & \mathbf{G}^{\text{ball}}_{s} = \mathbb{R}. \end{array}$$

That is, $\mathbf{G}^{\text{ball}} = \Delta_{\mathscr{M}^{\text{ball}}}(\mathbb{R})$. Note that in this case $\mathbf{\mathfrak{g}}^{\text{ball}} = \mathbf{G}^{\text{ball}}$ and $(\mathbf{\mathfrak{g}}^{\text{ball}})^\star = \Delta^{\mathsf{op}}_{\mathscr{M}^{\text{ball}}}(\mathbb{R})$.

4.2.6 The hybrid adjoint action. Let G be a Lie group and $g \in G$. The *conjunction map* is defined to be a map $I_g : G \to G$ with $I_g(h) = ghg^{-1}$ for $h \in G$. Utilizing the Lie functor, we obtain a Lie algebra homomorphism:

$$\text{Ad}_g := \text{Lie}(I_g) : \mathfrak{g} \to \mathfrak{g},$$

which is termed the *adjoint action*. The functor that associates to a vector space its dual, $(\,-\,)^\star$, yields a morphism of vector spaces:

$$\text{Ad}_g^\star : \mathfrak{g}^\star \to \mathfrak{g}^\star$$

termed the *coadjoint action*.

The framework of hybrid objects allows adjoint and coadjoint actions to be easily generalized to a hybrid setting. Given an element $\vec{g} \in (\mathscr{G}, \mathbf{G})$, we obtain a natural transformation:

$$I_{\vec{g}} : \mathbf{G} \stackrel{.}{\to} \mathbf{G}$$

defined objectwise by $(I_{\vec{g}})_a = I_{\vec{g}_a}$. It is easy to verify that this in fact defines a natural transformation. Utilizing the functor $\mathsf{Hy}(\mathrm{Lie})$ we obtain the hybrid adjoint action, i.e., a natural transformation:

$$\mathrm{Ad}_{\vec{g}} := \mathrm{Lie}_*(I_{\vec{g}}) : \mathfrak{g} \stackrel{\cdot}{\rightarrow} \mathfrak{g},$$

wherein it follows that $(\mathrm{Ad}_{\vec{g}})_a = \mathrm{Ad}_{\vec{g}_a}$. Finally, utilizing the functor that associates to a hybrid vector space its dual, $\mathsf{Hy}((\,-\,)^\star)$, we obtain the hybrid coadjoint action, i.e., a natural transformation:

$$\mathrm{Ad}^\star_{\vec{g}} : \mathfrak{g}^\star \stackrel{\cdot}{\rightarrow} \mathfrak{g}^\star,$$

It follows that $(\mathrm{Ad}^\star_{\vec{g}})_a = \mathrm{Ad}^\star_{\vec{g}_a}$.

4.2.7 The hybrid isotropy subgroups. For a Lie group G, the isotropy subgroup under the coadjoint action is given by, for $\mu \in \mathfrak{g}^\star$,

$$G_\mu = \{g \in G : \mathrm{Ad}^\star_{g^{-1}}(\mu) = \mu\}.$$

For $\vec{\mu} \in (\mathscr{G}, \mathfrak{g}^\star)$, define the hybrid isotropy group as the hybrid Lie group

$$\mathbf{G}_{\vec{\mu}} : \mathscr{G} \rightarrow \mathsf{LieGrp}$$

defined on objects and morphisms of $\mathscr{G}$ by:

$$(\mathbf{G}_{\vec{\mu}})_a = (\mathbf{G}_a)_{\vec{\mu}_a}, \qquad (\mathbf{G}_{\vec{\mu}})_\alpha = (\mathbf{G}_\alpha)|_{(\mathbf{G}_{\vec{\mu}})_a}.$$

We must verify that:

Proposition 4.2. *$(\mathscr{G}, \mathbf{G}_{\vec{\mu}})$ is a hybrid Lie group.*

Proof. We need to show that

$$\mathrm{Ad}^\star_{g^{-1}}(\vec{\mu}_a) = \vec{\mu}_a \qquad \Rightarrow \qquad \mathrm{Ad}^\star_{\mathbf{G}_\alpha(g^{-1})}(\vec{\mu}_b) = \vec{\mu}_b.$$

Since $\vec{\mu} \in (\mathscr{G}, \mathfrak{g}^\star)$, i.e., $\mathfrak{g}^\star_\alpha(\vec{\mu}_b) = \vec{\mu}_a$, this is equivalent to showing that:

$$\mathfrak{g}^\star_\alpha(\mathrm{Ad}^\star_{\mathbf{G}_\alpha(g^{-1})}(\vec{\mu}_b)) = \vec{\mu}_a.$$

First note that

$$\begin{aligned}
\mathrm{Ad}_{\mathbf{G}_\alpha(g^{-1})} \circ \mathfrak{g}_\alpha &= T_{e_{\mathbf{G}_b}}(I_{\mathbf{G}_\alpha(g^{-1})}) \circ T_{e_{\mathbf{G}_a}}(\mathbf{G}_\alpha) \\
&= T_{e_{\mathbf{G}_a}}(I_{\mathbf{G}_\alpha(g^{-1})} \circ \mathbf{G}_\alpha) \\
&= T_{e_{\mathbf{G}_a}}(\mathbf{G}_\alpha \circ I_{g^{-1}}) \\
&= T_{e_{\mathbf{G}_a}}(\mathbf{G}_\alpha) \circ T_{e_{\mathbf{G}_a}}(I_{g^{-1}}) \\
&= \mathfrak{g}_\alpha \circ \mathrm{Ad}_{g^{-1}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathfrak{g}_\alpha^\star(\mathrm{Ad}^\star_{\mathbf{G}_\alpha(g^{-1})}(\vec{\mu}_b)) &= \vec{\mu}_b \circ \mathrm{Ad}_{\mathbf{G}_\alpha(g^{-1})} \circ \mathfrak{g}_\alpha \\
&= \vec{\mu}_b \circ \mathfrak{g}_\alpha \circ \mathrm{Ad}_{g^{-1}} \\
&= \mathfrak{g}_\alpha^\star(\vec{\mu}_b) \circ \mathrm{Ad}_{g^{-1}} \\
&= \mathrm{Ad}^\star_{g^{-1}}(\mathfrak{g}_\alpha^\star(\vec{\mu}_b)) \\
&= \mathrm{Ad}^\star_{g^{-1}}(\vec{\mu}_a)
\end{aligned}$$

yielding the desired result. □

4.3 Hybrid Momentum Maps

We now switch our focus to hybrid reduction. Utilizing the constructions of the previous section, we generalize reduction to a hybrid setting. In order to achieve this goal, we begin by introducing hybrid momentum maps.

Momentum maps make explicit the conserved quantities of a Hamiltonian system. Hybrid momentum maps serve the same function, except that they define a set of conserved quantities. In order to introduce hybrid momentum maps, it is first necessary to introduce the notion of hybrid symmetries, i.e., a hybrid action of a hybrid Lie group on a hybrid manifold. When such hybrid symmetries exist, along with a momentum map, we are able to "divide" out by these symmetries to obtain the *reduced hybrid phase space*.

We assume a basic knowledge of classical reduction throughout the rest of this chapter. We refer the reader to the excellent reference [4] for any missing details.

Notation 4.2. We now fix a D-category $\mathscr{D}$. We will assume that all hybrid objects have $\mathscr{D}$ as their underlying discrete component *except* when discussing trajectories.

4.3.1 Hybrid group actions. For $\mathbf{G} : \mathscr{D} \to \mathsf{LieGrp}$ and $\mathbf{M} : \mathscr{D} \to \mathsf{Man}$, define the hybrid manifold $\mathbf{G} \times \mathbf{M} : \mathscr{D} \to \mathsf{Man}$ as the product of $\mathbf{G}$ and $\mathbf{M}$ in $\mathsf{Man}^{\mathscr{D}}$, i.e., on objects and morphisms:

$$(\mathbf{G} \times \mathbf{M})_a := \mathbf{G}_a \times \mathbf{M}_a, \quad (\mathbf{G} \times \mathbf{M})_\alpha := (\mathbf{G}_\alpha, \mathbf{M}_\alpha).$$

A *hybrid group action* or *hybrid action* is a natural transformation

$$\vec{\Phi} : \mathbf{G} \times \mathbf{M} \stackrel{\cdot}{\to} \mathbf{M}, \tag{4.1}$$

that is objectwise a group action:

- ⋄ For all $p \in \mathbf{M}_a$, $\vec{\Phi}_a(e_{\mathbf{G}_a}, p) = p$,
- ⋄ For every $g, h \in \mathbf{G}_a$, $\vec{\Phi}_a(g, \vec{\Phi}_a(h, p)) = \vec{\Phi}_a(gh, p)$.

We say that a hybrid group action is *free* if $\vec{\Phi}$ is objectwise free and *proper* if it is objectwise proper.

For $\vec{g} \in \mathbf{G}$, we can associate to this action a hybrid diffeomorphism (a natural isomorphism)

$$\vec{\Phi}_{\vec{g}} : \mathbf{M} \dot{\to} \mathbf{M} \tag{4.2}$$

defined objectwise, for $p \in \mathbf{M}_a$, by $(\vec{\Phi}_{\vec{g}})_a(p) := \vec{\Phi}_a(\vec{g}_a, p)$. Since $\vec{\Phi}_{\vec{g}}$ is a natural transformation, we have

$$\mathbf{M}_\alpha \circ \vec{\Phi}_a(\vec{g}_a, p) = \vec{\Phi}_b(\mathbf{G}_\alpha(\vec{g}_a), \mathbf{M}_\alpha(p)), \tag{4.3}$$

which is a form of equivariance; in the case when $\mathbf{G}_a = \mathbf{G}_b$ and $\mathbf{G}_\alpha = \mathrm{id}$, this condition says that $\mathbf{M}_\alpha$ is equivariant with respect to these actions.

Recall that in Definition 4.2 we introduced the definition of a hybrid symplectic manifold $(\mathbf{M}, \vec{\omega})$.

Definition 4.3. Let $(\mathbf{M}, \vec{\omega})$ be a hybrid symplectic manifold. A hybrid action $\vec{\Phi} : \mathbf{G} \times \mathbf{M} \dot{\to} \mathbf{M}$ is a **symplectic hybrid action** if for the hybrid diffeomorphism $\vec{\Phi}_{\vec{g}} : \mathbf{M} \dot{\to} \mathbf{M}$,

$$\Omega^2_*(\vec{\Phi}_{\vec{g}})(\vec{\omega}) = \vec{\omega}$$

for each $\vec{g} \in \mathbf{G}$.

4.3.2 Hybrid orbit spaces.

For a hybrid manifold $\mathbf{M}$ with a hybrid group $\mathbf{G}$ acting on it, let $\mathbf{M}_a/\mathbf{G}_a$ be the orbit space of the action $\vec{\Phi}_a$ of $\mathbf{G}_a$ on $\mathbf{M}_a$; if $p \in \mathbf{M}_a$, we denote the elements of this space by $[p]$. Define the hybrid topological space

$$\mathbf{M}/\mathbf{G} : \mathscr{D} \to \mathsf{Top}, \tag{4.4}$$

defined on objects and morphisms of $\mathscr{D}$ by $\mathbf{M}/\mathbf{G}_a := \mathbf{M}_a/\mathbf{G}_a$ and $\mathbf{M}/\mathbf{G}_\alpha([p]) := [\mathbf{M}_\alpha(p)]$, which is well-defined by (4.3).

Proposition 4.3. *If $\vec{\Phi} : \mathbf{G} \times \mathbf{M} \dot{\to} \mathbf{M}$ is a free and proper hybrid action, then $\mathbf{M}/\mathbf{G}$ is a hybrid manifold, i.e., $\mathbf{M}/\mathbf{G} : \mathscr{D} \to \mathsf{Man}$. Moreover, there is a hybrid submersion:*

$$\vec{\pi} : \mathbf{M} \dot{\to} \mathbf{M}/\mathbf{G}.$$

That is, $\vec{\pi}$ is a natural transformation that is objectwise a submersion.

Proof. Define a natural transformation $\vec{\pi} : \mathbf{M} \to \mathbf{M}/\mathbf{G}$ by setting $\vec{\pi}_a(p) := [p]$; this is a natural transformation since

$$\mathbf{M}/\mathbf{G}_\alpha \circ \vec{\pi}_a(p) = \mathbf{M}/\mathbf{G}_\alpha([p]) = [\mathbf{M}_\alpha(p)] = \vec{\pi}_b \circ \mathbf{M}_\alpha(p).$$

By the definition of free and proper hybrid actions—they are objectwise free and proper—it follows that $\mathbf{M}_a/\mathbf{G}_a$ is a smooth manifold for every object a of $\mathscr{D}$, and $\vec{\pi}$ is objectwise a submersion.

Therefore, we need only verify that $\mathbf{M}/\mathbf{G}_\alpha$ is smooth for all morphisms α of $\mathscr{D}$. This follows, however, from the naturality of $\vec{\pi}$, i.e., we have the following commuting diagram

$$\begin{array}{ccc} \mathbf{M}_a & \xrightarrow{\mathbf{M}_\alpha} & \mathbf{M}_b \\ \vec{\pi}_a \Big\downarrow & & \Big\downarrow \vec{\pi}_b \\ \mathbf{M}/\mathbf{G}_a & \xrightarrow{\mathbf{M}/\mathbf{G}_\alpha} & \mathbf{M}/\mathbf{G}_b \end{array}$$

and $\vec{\pi}_b \circ \mathbf{M}_\alpha$ is smooth so $\mathbf{M}/\mathbf{G}_\alpha \circ \vec{\pi}_a$ must be smooth, or $\mathbf{M}/\mathbf{G}_\alpha$ must be smooth. □

4.3.3 Hybrid infinitesimal generators of hybrid actions. Suppose there is a hybrid action $\vec{\Phi} : \mathbf{G} \times \mathbf{M} \dot{\to} \mathbf{M}$. Then we can use this hybrid action to define a hybrid section of the hybrid tangent bundle of $\mathbf{M}$ for every element $\vec{\xi} \in \mathfrak{g}$.

Define the *infinitesimal generator* of the hybrid action $\vec{\Phi}$ corresponding to $\vec{\xi} \in \mathfrak{g}$ by

$$\vec{\xi}_{\mathbf{M}} : \mathbf{M} \dot{\to} T_*(\mathbf{M}) \tag{4.5}$$

which is given objectwise by

$$(\vec{\xi}_{\mathbf{M}})_a(p) := \frac{\mathrm{d}}{\mathrm{dt}} \vec{\Phi}_a(\exp(t\vec{\xi}_a), p)\bigg|_{t=0}$$

for $p \in \mathbf{M}_a$.

Lemma 4.3. *$\vec{\xi}_{\mathbf{M}}$ is a hybrid section of the hybrid tangent bundle of* $\mathbf{M}$.

Proof. We need to show that for every diagram $\alpha : a \to b$ in $\mathscr{D}$

$$T_*(\mathbf{M})_\alpha \circ (\vec{\xi}_{\mathbf{M}})_a = (\vec{\xi}_{\mathbf{M}})_b \circ \mathbf{M}_\alpha.$$

First note that since $\vec{\xi} \in \mathfrak{g}$, i.e., $\mathfrak{g}_\alpha(\vec{\xi}_a) = \vec{\xi}_b$, we have (by the properties of the exponential map, cf. [79]),

$$\begin{aligned} \mathbf{G}_\alpha(\exp(t\vec{\xi}_a)) &= \exp(t T_{e_{\mathbf{G}_a}} \mathbf{G}_\alpha(\vec{\xi}_a)) \\ &= \exp(t\mathfrak{g}_\alpha(\vec{\xi}_a)) \\ &= \exp(t\vec{\xi}_b). \end{aligned}$$

Moreover, by (4.3),

$$\begin{aligned} \mathbf{M}_\alpha \circ \vec{\Phi}_a(\exp(t\vec{\xi}_a), p) &= \vec{\Phi}_b(\mathbf{G}_\alpha(\exp(t\vec{\xi}_a)), \mathbf{M}_\alpha(p)) \\ &= \vec{\Phi}_b(\exp(t\vec{\xi}_b), \mathbf{M}_\alpha(p)). \end{aligned}$$

Finally, for every $p \in \mathbf{M}_a$, we have

$$\begin{aligned} T_p\mathbf{M}_\alpha \circ (\vec{\xi}_{\mathbf{M}})_a(p) &= T_p\mathbf{M}_\alpha \circ \frac{\mathrm{d}}{\mathrm{dt}} \vec{\Phi}_a(\exp(t\vec{\xi}_a), p)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{dt}} \left(\mathbf{M}_\alpha \circ \vec{\Phi}_a(\exp(t\vec{\xi}_a), p)\right)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{dt}} \vec{\Phi}_b(\exp(t\vec{\xi}_b), \mathbf{M}_\alpha(p))\bigg|_{t=0} \\ &= (\vec{\xi}_{\mathbf{M}})_b(\mathbf{M}_\alpha(p)). \end{aligned}$$

□

4.3.4 Hybrid regular values. Consider a natural transformation $\vec{J} : \mathbf{M} \dot{\rightarrow} \mathfrak{g}^\star$; since $\mathbf{M}$ is covariant and $\mathfrak{g}^\star$ is contravariant, this implies (see Table 1.2) that the following diagram

$$\begin{array}{ccc} \mathbf{M}_a & \xrightarrow{\vec{J}_a} & \mathfrak{g}^\star_a \\ {\scriptstyle \mathbf{M}_\alpha}\big\downarrow & & \big\uparrow{\scriptstyle \mathfrak{g}^\star_\alpha} \\ \mathbf{M}_b & \xrightarrow{\vec{J}_b} & \mathfrak{g}^\star_b \end{array} \tag{4.6}$$

must commute for all $\alpha : a \to b$ in $\mathscr{D}$.

Definition 4.4. We say that $\vec{\mu} \in \mathfrak{g}^\star$ is a **hybrid regular value** of $\vec{J}$ if

1. $\vec{\mu}_a \in \mathfrak{g}^\star_a$ is a regular value of $\vec{J}_a : \mathbf{M}_a \to \mathfrak{g}^\star_a$ for all objects a of $\mathscr{D}$,

2. $\vec{\mu}_b = \vec{J}_b \circ \mathbf{M}_\alpha(p)$ for all $\alpha : a \to b$ in $\mathscr{D}$ and $p \in \mathbf{M}_a$ such that $\vec{J}_a(p) = \vec{\mu}_a$.

4.3.5 The hybrid manifold $\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})$. Given a hybrid regular value $\vec{\mu} \in \mathfrak{g}^\star$, define a hybrid manifold

$$\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}}) : \mathscr{D} \to \mathsf{Man} \tag{4.7}$$

given on objects and morphisms of $\mathscr{D}$ by

$$\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_a := \vec{J}_a^{-1}(\vec{\mu}_a), \qquad \mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_\alpha := \mathbf{M}_\alpha|_{\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_a}.$$

Note that there is a hybrid inclusion (a natural transformation that is objectwise an inclusion):

$$\vec{i}_{\vec{\mu}} : \mathbf{J}^{-1}(\vec{\boldsymbol{\mu}}) \dot{\rightarrow} \mathbf{M}, \tag{4.8}$$

defined to be objectwise the inclusion: $(\vec{i}_{\vec{\mu}})_a : \mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_a \hookrightarrow \mathbf{M}_a$.

Before continuing, we must verify that:

Lemma 4.4. *$\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})$ is a hybrid manifold if $\vec{\mu} \in \mathfrak{g}^\star$ is a hybrid regular value of $\vec{J}$.*

Proof. Since we are assuming that $\vec{\mu}$ is objectwise a regular value, $\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_a$ is a manifold. Since, $\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_\alpha$ is just the restriction of a smooth map to a smooth submanifold, it is also smooth. What we must verify is that the image of this map is contained in $\mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_b$, i.e., for all $p \in \mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_a$, $\mathbf{M}_\alpha(p) \in \mathbf{J}^{-1}(\vec{\boldsymbol{\mu}})_b$. That is, we need to show that for $p \in \mathbf{M}_a$,

$$\vec{J}_a(p) = \vec{\mu}_a \qquad \Rightarrow \qquad \vec{J}_b(\mathbf{M}_\alpha(p)) = \vec{\mu}_b.$$

This follows, however, from the second condition in the definition of a hybrid regular value. □

4.3.6 Induced natural transformations. Given a natural transformation $\vec{J} : \mathbf{M} \dot{\rightarrow} \mathfrak{g}^\star$, for all $\vec{\xi} \in \mathfrak{g}$ we can define a natural transformation:

$$\vec{J}^{\xi} : \mathbf{M} \dot{\rightarrow} \Delta_{\mathscr{D}}(\mathbb{R}); \tag{4.9}$$

given objectwise by

$$\vec{J}^{\xi}{}_{a}(p) := \langle \vec{J}_a(p), \vec{\xi}_a \rangle = \vec{J}_a(p)(\vec{\xi}_a),$$

for $p \in \mathbf{M}_a$.

Lemma 4.5. *$\vec{J}^{\xi}$ is a natural transformation.*

Proof. Consider a diagram of the form $\alpha : a \to b$ in $\mathscr{D}$. For $p \in \mathbf{M}_a$, we have

$$\begin{aligned}
\vec{J}^{\xi}{}_{b}(\mathbf{M}_\alpha(p)) &= \langle \vec{J}_b \circ \mathbf{M}_\alpha(p), \vec{\xi}_b \rangle \\
&= \langle \vec{J}_b \circ \mathbf{M}_\alpha(p), \mathfrak{g}_\alpha(\vec{\xi}_a) \rangle \\
&= \langle \mathfrak{g}^\star_\alpha \circ \vec{J}_b \circ \mathbf{M}_\alpha(p), \vec{\xi}_a \rangle \\
&= \langle \vec{J}_a(p), \vec{\xi}_a \rangle \\
&= \vec{J}^{\xi}{}_{a}(p).
\end{aligned}$$

□

Definition 4.5. Let $(\mathbf{M}, \vec{\omega})$ be a hybrid symplectic manifold and $\vec{\Phi} : \mathbf{G} \times \mathbf{M} \dot{\rightarrow} \mathbf{M}$ a hybrid action. Define a **hybrid momentum map** as a natural transformation

$$\vec{J} : \mathbf{M} \dot{\rightarrow} \mathfrak{g}^\star, \tag{4.10}$$

such that for every $\vec{\xi} \in \mathfrak{g}$ and object a of $\mathscr{D}$:

$$d(\vec{J}^{\xi}{}_{a}) = \iota_{(\vec{\xi}_{\mathbf{M}})_a}(\vec{\omega}_a), \tag{4.11}$$

where ι is the interior product on $\mathbf{M}_a$ and $\vec{\xi}_{\mathbf{M}}$ is the hybrid infinitesimal generator of the hybrid action corresponding to $\vec{\xi}$.

Definition 4.6. Let $\vec{\Phi} : \mathbf{G} \times \mathbf{M} \dot{\rightarrow} \mathbf{M}$ be a hybrid action of $\mathbf{G}$ on $\mathbf{M}$. A hybrid momentum map is said to be $\mathrm{Ad}^\star$**-equivariant** under this action if for every $\vec{g} \in \mathbf{G}$ the following diagram of natural transformations commutes:

$$\begin{array}{ccc}
\mathbf{M} & \xrightarrow{\vec{\Phi}_{\vec{g}}} & \mathbf{M} \\
\vec{J} \downarrow & & \downarrow \vec{J} \\
\mathfrak{g}^\star & \xrightarrow{\mathrm{Ad}^\star_{\vec{g}^{-1}}} & \mathfrak{g}^\star
\end{array}$$

4.3.7 The reduced hybrid phase space. Suppose that $\mathbf{G}$ acts on $\mathbf{M}$ through the hybrid action $\vec{\Phi}$, and let $\vec{J}:\mathbf{M}\stackrel{\bullet}{\to}\mathfrak{g}^\star$ be an $\mathrm{Ad}^\star$-equivariant hybrid momentum map. Assume that $\vec{\mu}\in\mathfrak{g}^\star$ is a hybrid regular value of $\vec{J}$; therefore, $\mathbf{J^{-1}(\vec{\mu})}:\mathscr{D}\to\mathsf{Man}$ is a hybrid manifold. The restriction of the hybrid action $\vec{\Phi}_{\vec{g}}$ to $\mathbf{J^{-1}(\vec{\mu})}$ and $\mathbf{G}_{\vec{\mu}}$ (also denoted by $\vec{\Phi}_{\vec{g}}$):

$$\vec{\Phi}_{\vec{g}}:\mathbf{J^{-1}(\vec{\mu})}\stackrel{\bullet}{\to}\mathbf{J^{-1}(\vec{\mu})},\qquad \vec{g}\in\mathbf{G}_{\vec{\mu}}, \tag{4.12}$$

is a hybrid action. In other words, if $\mathbf{G}$ acts on $\mathbf{M}$, then $\mathbf{G}_{\vec{\mu}}$ acts on $\mathbf{J^{-1}(\vec{\mu})}$.

If $\mathbf{G}_{\vec{\mu}}$ acts freely and properly on $\mathbf{J^{-1}(\vec{\mu})}$, then

$$\mathbf{M}_{\vec{\mu}}:=\mathbf{J^{-1}(\vec{\mu})/G}_{\vec{\mu}}:\mathscr{D}\to\mathsf{Man}$$

is a hybrid manifold, and the canonical hybrid projection

$$\vec{\pi}_{\vec{\mu}}:\mathbf{J^{-1}(\vec{\mu})}\stackrel{\bullet}{\to}\mathbf{M}_{\vec{\mu}}=\mathbf{J^{-1}(\vec{\mu})/G}_{\vec{\mu}}$$

is a hybrid submersion. $\mathbf{M}_{\vec{\mu}}$ is called the *reduced hybrid phase space.* In particular, for every $\alpha:a\to b$ in $\mathscr{D}$, there is a commuting diagram

$$\begin{array}{ccc}
\mathbf{M}_a & \xrightarrow{\quad\mathbf{M}_\alpha\quad} & \mathbf{M}_b \\
\Big\uparrow (\vec{i}_{\vec{\mu}})_a & & \Big\uparrow (\vec{i}_{\vec{\mu}})_b \\
\mathbf{J^{-1}(\vec{\mu})}_a & \xrightarrow{\quad\mathbf{J^{-1}(\vec{\mu})}_\alpha\quad} & \mathbf{J^{-1}(\vec{\mu})}_b \\
\Big\downarrow (\vec{\pi}_{\vec{\mu}})_a & & \Big\downarrow (\vec{\pi}_{\vec{\mu}})_b \\
(\mathbf{M}_{\vec{\mu}})_a=(\mathbf{J^{-1}(\vec{\mu})/G}_{\vec{\mu}})_a & \xrightarrow{(\mathbf{M}_{\vec{\mu}})_\alpha=(\mathbf{J^{-1}(\vec{\mu})/G}_{\vec{\mu}})_\alpha} & (\mathbf{M}_{\vec{\mu}})_b=(\mathbf{J^{-1}(\vec{\mu})/G}_{\vec{\mu}})_b
\end{array} \tag{4.13}$$

in Man.

Example 4.3. For our running example of a two-dimensional bouncing ball, we define a hybrid group action by translating in the x-direction on all domains, i.e., define

$$\vec{\Phi}^{\mathrm{ball}}:\mathbf{G}^{\mathrm{ball}}\times\mathbf{M}^{\mathrm{ball}}\to\mathbf{G}^{\mathrm{ball}}$$

by

$$\begin{aligned}
\vec{\Phi}^{\mathrm{ball}}_b(a,(x,y,p_x,p_y)) &= \begin{pmatrix} x+a \\ y \\ p_x \\ p_y \end{pmatrix}\\
\vec{\Phi}^{\mathrm{ball}}_s(b,(x,p_x)) &= \begin{pmatrix} x+b \\ p_x \end{pmatrix}\\
\vec{\Phi}^{\mathrm{ball}}_{a_b} &= \vec{\Phi}^{\mathrm{ball}}_b|_{\mathbf{M}^{\mathrm{ball}}_{a_b}}\\
\vec{\Phi}^{\mathrm{ball}}_{a_s} &= \vec{\Phi}^{\mathrm{ball}}_b|_{\mathbf{M}^{\mathrm{ball}}_{a_s}}.
\end{aligned}$$

Using the canonical construction of momentum maps on cotangent bundles, the hybrid momentum map $\vec{J}^{\text{ball}}: \mathbf{M}^{\text{ball}} \to (\mathfrak{g}^{\text{ball}})^\star = \Delta^{\mathsf{op}}_{\mathscr{M}^{\text{ball}}}(\mathbb{R})$ is given by:

$$\vec{J}_b^{\text{ball}}(x,y,p_x,p_y) = p_x \qquad \vec{J}_s^{\text{ball}}(x,p_x) = p_x$$
$$\vec{J}_{a_b}^{\text{ball}} = \vec{J}_b^{\text{ball}}|_{\mathbf{M}^{\text{ball}}_{a_b}} \qquad \vec{J}_{a_s}^{\text{ball}} = \vec{J}_b^{\text{ball}}|_{\mathbf{M}^{\text{ball}}_{a_s}}.$$

It follows that a hybrid regular value for this system is given by

$$\vec{\mu} = (\vec{\mu}_b = \mu, \vec{\mu}_{a_b} = \mu, \vec{\mu}_s = \mu, \vec{\mu}_{a_s} = \mu),$$

for some $\mu \in \mathbb{R}$.

Therefore, the reduced hybrid phase space for the bouncing ball is given by $\mathbf{M}^{\text{ball}}_{\vec{\mu}}: \mathscr{M}^{\text{ball}} \to \mathsf{Man}$, which is defined by the following diagram:

$$\mathbf{M}^{\text{ball}}_{\vec{\mu}}(\mathscr{M}^{\text{ball}}) = \begin{array}{ccccc} \left\{ \begin{pmatrix} y \\ p_y \end{pmatrix} \in \mathbb{R}^2 : y = 0 \text{ and } p_y \le 0 \right\} & & & & \\ \iota \big\downarrow \;\; \big\downarrow \begin{pmatrix} y \\ p_y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -e p_y \end{pmatrix} & & & & \\ \mathbb{R}^2 & \xleftarrow{\quad \iota \quad} & \mathbb{R}^0 & \xrightarrow{\text{id}} & \mathbb{R}^0 \end{array} \tag{4.14}$$

with $\mathbb{R}^0 = \{0\}$ a point.

4.4 Hybrid Manifold Reduction

We now introduce the main theorem on reducing a hybrid symplectic manifold. We begin by reviewing the classic non-hybrid version of this theorem, originally proven by Marsden and Weinstien [88] (also see [4, 86, 87] for a more thorough account of classical reduction), followed by a statement of the hybrid version of this theorem.

4.4.1 Classical reduction. The starting point for classical reduction is a Hamiltonian G-space,

$$(M, \omega, \Phi, J),$$

where

- (M, ω) is a symplectic manifold,
- $\Phi: G \times M \to M$ is a symplectic action of a Lie group on M,
- J is an $\text{Ad}^\star$-equivariant momentum map for this action.

Under these conditions, the classical reduction theorem [88] reads:

Theorem 4.1. *Let (M,ω,Φ,J) be a Hamiltonian G-space and $\mu\in\mathfrak{g}^\star$ be a regular value of J. If the action of G_μ on $J^{-1}(\mu)$ is free and proper, then $M_\mu = J^{-1}(\mu)/G_\mu$ has a unique symplectic form ω_μ with the property:*

$$\Omega^2(\pi_\mu)(\omega_\mu) = \Omega^2(i_\mu)(\omega),$$

where $\pi_\mu : J^{-1}(\mu)\to M_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu)\to M$ is the inclusion.

The hybrid reduction theorem will nicely mirror and utilize this theorem. There also is an intriguing connection between the classical reduction theorem and hybrid symplectic manifolds; this theorem implies that the following hybrid manifold

$$M \xleftarrow{\;i_\mu\;} J^{-1}(\mu) \xrightarrow{\;\pi_\mu\;} M_\mu$$

is a hybrid symplectic manifold.

4.4.2 Hybrid Hamiltonian G-spaces. Utilizing the framework developed thus far, we can prove a hybrid version of Theorem 4.1. First, we note that the necessary information in order to generalize this theorem is a *hybrid Hamiltonian* **G**-*space*, i.e., a tuple

$$(\mathbf{M},\vec{\omega},\vec{\Phi},\vec{J}),$$

where

- $(\mathbf{M},\vec{\omega})$ is a hybrid symplectic manifold,
- $\vec{\Phi} : \mathbf{G}\times\mathbf{M} \dot{\to} \mathbf{M}$ is a symplectic hybrid action of a Lie group on $\mathbf{M}$,
- $\vec{J}$ is an $\mathrm{Ad}^\star$-equivariant hybrid momentum map for this hybrid action.

For such a hybrid Hamiltonian **G**-space, we can reduce the dimensionality of **M** through *hybrid reduction*. This is done by utilizing the classical reduction theorem through the observation that

$$(\mathbf{M}_b,\vec{\omega}_b,\vec{\Phi}_b,\vec{J}_b)$$

is a Hamiltonian $\mathbf{G}_b$-space for every $b\in\bigvee(\mathscr{D})$.

Before stating the theorem, recall that for a morphism of hybrid manifolds $\vec{f} : \mathbf{M}\dot{\to}\mathbf{N}$,

$$\Omega^2_*(\vec{f}) : \Omega^2_*(\mathbf{N}) \dot{\to} \Omega^2_*(\mathbf{M}),$$

which is the natural transformation obtained from applying the functor $\mathsf{Hy}(\Omega^2)$ (see Paragraph 4.1.4).

Theorem 4.2. *Let $(\mathbf{M},\vec{\omega},\vec{\Phi},\vec{J})$ be a hybrid Hamiltonian **G**-space. Assume $\vec{\mu}\in\mathbf{g}^\star$ is a hybrid regular value of $\vec{J}$ and that the hybrid action of $\mathbf{G}_{\vec{\mu}}$ on $\mathbf{J^{-1}(\vec{\mu})}$ is free and proper. Then $\mathbf{M}_{\vec{\mu}}$ has a unique hybrid symplectic form $\vec{\omega}_{\vec{\mu}}$ with the property:*

$$\Omega^2_*(\vec{\pi}_{\vec{\mu}})(\vec{\omega}_{\vec{\mu}}) = \Omega^2_*(\vec{i}_{\vec{\mu}})(\vec{\omega}).$$

Proof. The goal is to define a hybrid symplectic form $\vec{\omega}_{\vec{\mu}}$ on $\mathbf{M}_{\vec{\mu}}$. To do so, we first note that it follows from the definition of $(\mathbf{M}, \vec{\omega}, \vec{\Phi}, \vec{J})$ that, for all $b \in \mathsf{V}(\mathscr{D})$,

$$(\mathbf{M}_b, \vec{\omega}_b, \vec{\Phi}_b, \vec{J}_b)$$

is a Hamiltonian $\mathbf{G}_b$-space, $\vec{\mu}_b$ is a regular value of $\vec{J}_b$, and the action of $(\mathbf{G}_{\vec{\mu}})_b$ on $\mathbf{J^{-1}(\vec{\mu})}_b$ is free and proper. Therefore, for $b \in \mathsf{V}(\mathscr{D})$, define $(\vec{\omega}_{\vec{\mu}})_b$ to be the unique symplectic form satisfying:

$$\Omega^2_*(\vec{\pi}_{\vec{\mu}})_b((\vec{\omega}_{\vec{\mu}})_b) = \Omega^2_*(\vec{i}_{\vec{\mu}})_b(\vec{\omega}_b).$$

For $a \in \mathsf{E}(\mathscr{D})$ there is a diagram of the form

$$\mathsf{cod}(\mathsf{s}_a) \xleftarrow{\mathsf{s}_a} a \xrightarrow{\mathsf{t}_a} \mathsf{cod}(\mathsf{t}_a) \tag{4.15}$$

in $\mathscr{D}$. To complete the description of $\vec{\omega}_{\vec{\mu}}$, define $(\vec{\omega}_{\vec{\mu}})_a$ by the requirement that

$$\Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{s}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}) = (\vec{\omega}_{\vec{\mu}})_a = \Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{t}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}).$$

To complete the proof, we must show that $\vec{\omega}_{\vec{\mu}}$ is well-defined and unique. Uniqueness clearly follows from the uniqueness of $(\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}$ and $(\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}$ and the definition of a hybrid symplectic form. Therefore, we must only show that it is well-defined, i.e., that

$$\Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{s}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}) = \Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{t}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)})$$

for all $a \in \mathsf{E}(\mathscr{D})$.

For the diagram (4.15) in $\mathscr{D}$, because the diagram in (4.13) commutes, we have that the diagram:

$$\begin{array}{ccccc}
\mathbf{M}_{\mathsf{cod}(\mathsf{s}_a)} & \xleftarrow{\mathbf{M}_{\mathsf{s}_a}} & \mathbf{M}_a & \xrightarrow{\mathbf{M}_{\mathsf{t}_a}} & \mathbf{M}_{\mathsf{cod}(\mathsf{t}_a)} \\
\Big\uparrow {\scriptstyle (\vec{i}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}} & & \Big\uparrow {\scriptstyle (\vec{i}_{\vec{\mu}})_a} & & \Big\uparrow {\scriptstyle (\vec{i}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}} \\
\mathbf{J^{-1}(\vec{\mu})}_{\mathsf{cod}(\mathsf{s}_a)} & \xleftarrow{\mathbf{J^{-1}(\vec{\mu})}_{\mathsf{s}_a}} & \mathbf{J^{-1}(\vec{\mu})}_a & \xrightarrow{\mathbf{J^{-1}(\vec{\mu})}_{\mathsf{t}_a}} & \mathbf{J^{-1}(\vec{\mu})}_{\mathsf{cod}(\mathsf{t}_a)} \\
\Big\downarrow {\scriptstyle (\vec{\pi}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}} & & \Big\downarrow {\scriptstyle (\vec{\pi}_{\vec{\mu}})_a} & & \Big\downarrow {\scriptstyle (\vec{\pi}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}} \\
(\mathbf{M}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)} & \xleftarrow{(\mathbf{M}_{\vec{\mu}})_{\mathsf{s}_a}} & (\mathbf{M}_{\vec{\mu}})_a & \xrightarrow{(\mathbf{M}_{\vec{\mu}})_{\mathsf{t}_a}} & (\mathbf{M}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}
\end{array}$$

commutes in Man. The commutativity of this diagram implies that the following diagram:

$$\begin{array}{ccccc}
\Omega^2_*(\mathbf{M})_{\mathsf{cod}(\mathsf{s}_a)} & \xrightarrow{\Omega^2_*(\mathbf{M})_{\mathsf{s}_a}} & \Omega^2_*(\mathbf{M})_a & \xleftarrow{\Omega^2_*(\mathbf{M})_{\mathsf{t}_a}} & \Omega^2_*(\mathbf{M})_{\mathsf{cod}(\mathsf{t}_a)} \\
\Big\downarrow{\scriptstyle \Omega^2_*(\vec{i}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}} & & \Big\downarrow{\scriptstyle \Omega^2_*(\vec{i}_{\vec{\mu}})_a} & & \Big\downarrow{\scriptstyle \Omega^2_*(\vec{i}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}} \\
\Omega^2_*(\mathbf{J}^{-1}(\boldsymbol{\vec{\mu}}))_{\mathsf{cod}(\mathsf{s}_a)} & \xrightarrow{\Omega^2_*(\mathbf{J}^{-1}(\boldsymbol{\vec{\mu}}))_{\mathsf{s}_a}} & \Omega^2_*(\mathbf{J}^{-1}(\boldsymbol{\vec{\mu}}))_a & \xleftarrow{\Omega^2_*(\mathbf{J}^{-1}(\boldsymbol{\vec{\mu}}))_{\mathsf{t}_a}} & \Omega^2_*(\mathbf{J}^{-1}(\boldsymbol{\vec{\mu}}))_{\mathsf{cod}(\mathsf{t}_a)} \\
\Big\uparrow{\scriptstyle \Omega^2_*(\vec{\pi}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}} & & \Big\uparrow{\scriptstyle \Omega^2_*(\vec{\pi}_{\vec{\mu}})_a} & & \Big\uparrow{\scriptstyle \Omega^2_*(\vec{\pi}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}} \\
\Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)} & \xrightarrow{\Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{s}_a}} & \Omega^2_*(\mathbf{M}_{\vec{\mu}})_a & \xleftarrow{\Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{t}_a}} & \Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}
\end{array}$$

commutes by the functorality of Ω^2. The commutativity of this diagram implies that for the symplectic form $(\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}$,

$$\begin{aligned}
\Omega^2_*(\vec{\pi}_{\vec{\mu}})_a \circ \Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{s}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}) &= \Omega^2_*(\mathbf{J}^{-1}(\boldsymbol{\vec{\mu}}))_{\mathsf{s}_a} \circ \Omega^2_*(\vec{\pi}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}) \\
&= \Omega^2_*(\mathbf{J}^{-1}(\boldsymbol{\vec{\mu}}))_{\mathsf{s}_a} \circ \Omega^2_*(\vec{i}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}(\vec{\omega}_{\mathsf{cod}(\mathsf{s}_a)}) \\
&= \Omega^2_*(\vec{i}_{\vec{\mu}})_a \circ \Omega^2_*(\mathbf{M})_{\mathsf{s}_a}(\vec{\omega}_{\mathsf{cod}(\mathsf{s}_a)}).
\end{aligned}$$

A similar calculation shows that for the symplectic form $(\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}$,

$$\Omega^2_*(\vec{\pi}_{\vec{\mu}})_a \circ \Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{t}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}) = \Omega^2_*(\vec{i}_{\vec{\mu}})_a \circ \Omega^2_*(\mathbf{M})_{\mathsf{t}_a}(\vec{\omega}_{\mathsf{cod}(\mathsf{t}_a)}).$$

Because $\vec{\omega}$ is a hybrid symplectic form:

$$\Omega^2_*(\mathbf{M})_{\mathsf{s}_a}(\vec{\omega}_{\mathsf{cod}(\mathsf{s}_a)}) = \Omega^2_*(\mathbf{M})_{\mathsf{t}_a}(\vec{\omega}_{\mathsf{cod}(\mathsf{t}_a)}),$$

which implies that

$$\Omega^2_*(\vec{\pi}_{\vec{\mu}})_a \circ \Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{s}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{s}_a)}) = \Omega^2_*(\vec{\pi}_{\vec{\mu}})_a \circ \Omega^2_*(\mathbf{M}_{\vec{\mu}})_{\mathsf{t}_a}((\vec{\omega}_{\vec{\mu}})_{\mathsf{cod}(\mathsf{t}_a)}).$$

By Proposition 4.3, $\vec{\pi}_{\vec{\mu}}$ is a hybrid surjective submersion, i.e., $(\vec{\pi}_{\vec{\mu}})_a$ is a surjective submersion. Therefore, the following lemma completes the proof. □

Lemma 4.6. *If $\pi : M \to N$ is a surjective submersion, then $\Omega^k(\pi)$ is injective.*

Proof. Let $\omega, \omega' \in \Omega^k(N)$, $q \in N$ and $Y_1, \cdots, Y_k \in T_qN$. Because π is surjective, there exists a $p \in M$ such that $\pi(p) = q$. Because π is a submersion, there exists $X_1, \cdots, X_k \in T_pM$ such that $T_p(X_i) = Y_i$. Therefore, if

$\Omega^k(\pi)(\omega) = \Omega^k(\pi)(\omega')$,

$$\begin{aligned} \omega_q(Y_1, \cdots, Y_n) &= \Omega^k(\pi)(\omega)_p(X_1, \cdots, X_k) \\ &= \Omega^k(\pi)(\omega')_p(X_1, \cdots, X_k) \\ &= \omega'_q(Y_1, \cdots, Y_n) \end{aligned}$$

as desired. □

4.5 Hybrid Hamiltonian Reduction

The hybrid reduction theorem (Theorem 4.2) only gave conditions on when the phase space of a hybrid system can be reduced. In practice, we are interested in reducing the dynamics of a hybrid system. That is, we want to understand how to reduce *hybrid Hamiltonians*. This yields a method for reducing hybrid systems.

4.5.1 Classical Hamiltonian reduction. Before discussing how to reduce hybrid Hamiltonians, and hence hybrid systems obtained from hybrid Hamiltonians, we review the classical Hamiltonian reduction theorem (cf. [4]). The setup for this theorem is a Hamiltonian G-space (M, ω, Φ, J) satisfying the assumptions given in Theorem 4.1.

A *Hamiltonian system* is a tuple (M, ω, H), where (M, ω) is a symplectic manifold and $H : M \to \mathbb{R}$ is a Hamiltonian. From the Hamiltonian H, we obtain a vector field X_H defined by $d(H) = \iota_{X_H}(\omega)$. That is, associated to the Hamiltonian system (M, ω, H) there is a dynamical system (M, X_H), or an object of Dyn. Recall (see Definition 2.4) that a trajectory of (M, X_H) is an object $(I, \mathrm{d}/\mathrm{dt})$ of Interval(Dyn) together with a morphism of dynamical systems:

$$c : (I, \mathrm{d}/\mathrm{dt}) \to (M, X_H).$$

In other words $\dot{c}(t) = X_H(c(t))$. The initial condition of such a trajectory is $c(t_0)$.

A Hamiltonian $H : M \to \mathbb{R}$ is said to be *G-invariant* if for the action $\Phi : G \times M \to M$,

$$H \circ \Phi(g, -) = H.$$

for all $g \in G$. From a G-invariant Hamiltonian, we obtain a Hamiltonian H_μ on M_μ defined by the requirement that it make the following diagram

$$\begin{array}{ccc} J^{-1}(\mu) & \xrightarrow{i_\mu} & M \\ \downarrow{\scriptstyle \pi_\mu} & & \downarrow{\scriptstyle H} \\ M_\mu & \xrightarrow{H_\mu} & \mathbb{R} \end{array}$$

commute. The end result is reduced Hamiltonian system $(M_\mu, \omega_\mu, H_\mu)$, for which we have an associated dynamical system (M_μ, X_{H_μ}). We denote trajectories of this dynamical system by

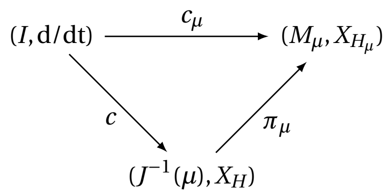
$$c_\mu : (I, \mathrm{d}/\mathrm{dt}) \to (M_\mu, X_{H_\mu}).$$

The classical Hamiltonian reduction theorem (cf. [4] and [88]) relates trajectories of (M, X_H) and trajectories of (M_μ, X_{H_μ}). We state this result in a slightly different formulism, although it is equivalent to the standard result.

Theorem 4.3. *Let (M, ω, Φ, J) be a Hamiltonian G-space satisfying the assumptions given in Theorem 4.1. If H is a G-invariant Hamiltonian and $c : (I, \mathrm{d}/\mathrm{dt}) \to (M, X_H)$ is a trajectory of (M, X_H) with $c(t_0) \in J^{-1}(\mu)$, then*

$$c : (I, \mathrm{d}/\mathrm{dt}) \to (J^{-1}(\mu), X_H)$$

and there exists a trajectory $c_\mu : (I, \mathrm{d}/\mathrm{dt}) \to (M_\mu, X_{H_\mu})$ of (M_μ, X_{H_μ}) defined by the factorization:



Remark 4.2. In Theorem 4.3, it would have been more accurate to write $(J^{-1}(\mu), X_H|_{J^{-1}(\mu)})$ instead of $(J^{-1}(\mu), X_H)$. We opted for the latter notation as it is clear from context that, in this case, X_H must be restricted to take values in $J^{-1}(\mu)$.

We now establish the necessary groundwork needed in order to establish the hybrid analogue of Theorem 4.3. We begin by defining hybrid Hamiltonians. In doing so, we again make explicit the D-category associated with hybrid objects, e.g., we will now denote the hybrid manifold $\mathbf{M}$ by $(\mathscr{M}, \mathbf{M})$. The motivation for this is that we are once again interested in trajectories.

Definition 4.7. A **hybrid Hamiltonian H** on a hybrid manifold $(\mathscr{M}, \mathbf{M})$ is defined to be a set of maps:

$$\mathbf{H} = \{\mathbf{H}_q : \mathbf{M}_q \to \mathbb{R}\}_{q \in \bigvee(\mathscr{M})}.$$

A *hybrid Hamiltonian system* is a tuple $(\mathscr{M}, \mathbf{M}, \vec{\omega}, \mathbf{H})$, where $(\mathscr{M}, \mathbf{M}, \vec{\omega})$ is a hybrid symplectic manifold and $\mathbf{H}$ is a hybrid Hamiltonian.

Example 4.4. For the bouncing ball hybrid manifold $(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}})$, consider the hybrid Hamiltonian

$$\mathbf{H}^{\text{ball}} = \{\mathbf{H}^{\text{ball}}_b, \mathbf{H}^{\text{ball}}_s\},$$

where

$$\begin{aligned} \mathbf{H}^{\text{ball}}_b(x, y, p_x, p_y) &= \frac{1}{2m}(p_x^2 + p_y^2) + mgy, \\ \mathbf{H}^{\text{ball}}_s(x, p_x) &= \frac{1}{2m} p_x^2. \end{aligned}$$

The bouncing ball hybrid Hamiltonian system is given by

$$(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}, \vec{\omega}^{\text{ball}}, \mathbf{H}^{\text{ball}}),$$

where $\vec{\omega}^{\text{ball}}$ is the hybrid symplectic form given in Example 4.1.

4.5.2 G-Invariant hybrid Hamiltonians. Let $(\mathscr{M},\mathbf{G})$ be a hybrid Lie group acting on the hybrid manifold $(\mathscr{M},\mathbf{M})$ through the hybrid action:

$$(\vec{\mathrm{Id}}_{\mathscr{M}},\vec{\Phi}):(\mathscr{M},\mathbf{G}\times\mathbf{M})\to(\mathscr{M},\mathbf{M}).$$

A hybrid Hamiltonian $\mathbf{H}$ is said to be $\mathbf{G}$-*invariant* if

$$\mathbf{H}_q\circ\vec{\Phi}_q(g,-)=\mathbf{H}_q$$

for all $g\in\mathbf{G}_q$ and $q\in\bigvee(\mathscr{M})$, i.e., $\mathbf{H}_q$ is $\mathbf{G}_q$-invariant for all $q\in\bigvee(\mathscr{M})$.

Under the assumptions of Theorem 4.2, if $\mathbf{H}$ is a $\mathbf{G}$-invariant hybrid Hamiltonian on $(\mathscr{M},\mathbf{M})$, then there is a hybrid Hamiltonian

$$\mathbf{H}_{\vec{\mu}}=\{(\mathbf{H}_{\vec{\mu}})_q:(\mathbf{M}_{\vec{\mu}})_q\to\mathbb{R}\}_{q\in\bigvee(\mathscr{M})}$$

on $\mathbf{M}_{\vec{\mu}}$ defined by requiring that the following diagram commute:

$$\begin{array}{ccc}
\mathbf{J}^{-1}(\vec{\mu})_q & \xrightarrow{(\vec{i}_{\vec{\mu}})_q} & \mathbf{M}_q \\
{\scriptstyle(\vec{\pi}_{\vec{\mu}})_q}\downarrow & & \downarrow{\scriptstyle\mathbf{H}_q} \\
(\mathbf{M}_{\vec{\mu}})_q & \xrightarrow{(\mathbf{H}_{\vec{\mu}})_q} & \mathbb{R}
\end{array}$$

for all $q\in\bigvee(\mathscr{M})$. This defines a hybrid Hamiltonian system $(\mathscr{M},\mathbf{M}_{\vec{\mu}},\vec{\omega}_{\vec{\mu}},\mathbf{H}_{\vec{\mu}})$.

4.5.3 Trajectories of hybrid Hamiltonian systems. From a hybrid Hamiltonian system $(\mathscr{M},\mathbf{M},\vec{\omega},\mathbf{H})$ we obtain a (categorical) hybrid system:

$$(\mathscr{M},\mathbf{M},\mathbf{X_H}),$$

where $\mathbf{X_H}$ is the collection of vector fields given by $\mathbf{X_H}=\{(\mathbf{X_H})_q\}_{q\in\bigvee(\mathscr{M})}$, with $(\mathbf{X_H})_q$ defined by the requirement that

$$d(\mathbf{H}_q)=\iota_{(\mathbf{X_H})_q}(\vec{\omega}_q).$$

Similarly, we obtain a hybrid system $(\mathscr{M},\mathbf{M}_{\vec{\mu}},\mathbf{X}_{\mathbf{H}_{\vec{\mu}}})$ from the hybrid Hamiltonian system $(\mathscr{M},\mathbf{M}_{\vec{\mu}},\vec{\omega}_{\vec{\mu}},\mathbf{H}_{\vec{\mu}})$.

Note that $(\mathscr{M},\mathbf{M},\mathbf{X_H})$ and $(\mathscr{M},\mathbf{M}_{\vec{\mu}},\mathbf{X}_{\mathbf{H}_{\vec{\mu}}})$ do *not* correspond to "classical" hybrid systems unless we make certain assumptions on the hybrid manifold $(\mathscr{M},\mathbf{M})$, i.e., that it is of the form $(\mathscr{M},\mathbf{M}^{\imath})$ (see Paragraph 2.1.2, Proposition 2.1 and Proposition 2.2).

Recall from Definition 2.7 that a trajectory of the hybrid system $(\mathscr{M},\mathbf{M},\mathbf{X_H})$ consists of an object $(\mathscr{I},\mathbf{I},\mathbf{d/dt})$ of $\mathsf{Interval(HySys)}$ together with a morphism of hybrid systems:

$$(\vec{C},\vec{c}):(\mathscr{I},\mathbf{I},\mathbf{d/dt})\to(\mathscr{M},\mathbf{M},\mathbf{X_H}).$$

The initial condition of such a trajectory is $(\vec{C}(0),\vec{c}_0(\tau_0))$ with $\vec{c}_0(\tau_0)\in\mathbf{M}_{\vec{C}(0)}$.

Example 4.5. For the bouncing ball, the reduced hybrid Hamiltonian is given by:

$$\mathbf{H}^{\text{ball}}_{\vec{\mu}} = \{(\mathbf{H}^{\text{ball}}_{\vec{\mu}})_b, (\mathbf{H}^{\text{ball}}_{\vec{\mu}})_s\},$$

where

$$\begin{aligned}(\mathbf{H}^{\text{ball}}_{\vec{\mu}})_b(y, p_y) &= \frac{1}{2m} p_y^2 + mgy + \frac{1}{2m}\mu^2, \\ (\mathbf{H}^{\text{ball}}_{\vec{\mu}})_s(0) &= \frac{1}{2m}\mu^2.\end{aligned}$$

From this we obtain the reduced hybrid Hamiltonian system:

$$(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}_{\vec{\mu}}, \vec{\omega}^{\text{ball}}_{\vec{\mu}}, \mathbf{H}^{\text{ball}}_{\vec{\mu}}),$$

where $\vec{\omega}^{\text{ball}}_{\vec{\mu}}$ is the "canonical" symplectic form on the reduced phase space $(\mathscr{M}, \mathbf{M}^{\text{ball}}_{\vec{\mu}})$. From the reduced hybrid Hamiltonian system, we obtain the reduced hybrid system:

$$(\mathscr{M}^{\text{ball}}, \mathbf{M}^{\text{ball}}_{\vec{\mu}}, \mathbf{X}^{\text{ball}}_{\vec{\mu}}),$$

where $\mathbf{M}^{\text{ball}}_{\vec{\mu}}$ is defined as in (4.14) and

$$\mathbf{X}^{\text{ball}}_{\vec{\mu}} = \{(\mathbf{X}^{\text{ball}}_{\vec{\mu}})_b, (\mathbf{X}^{\text{ball}}_{\vec{\mu}})_s\}$$

with

$$\begin{aligned}(\mathbf{X}^{\text{ball}}_{\vec{\mu}})_b(y, p_y) &= \frac{1}{m}\begin{pmatrix} p_y \\ -m^2 g \end{pmatrix}, \\ (\mathbf{X}^{\text{ball}}_{\vec{\mu}})_s(0) &= 0.\end{aligned}$$

Therefore, the reduced hybrid system is exactly the hybrid system modeling a one-dimensional bouncing ball which stops bouncing once the Zeno point is reached.

We now demonstrate that the "dynamics" of $\mathbf{H}$ determine the corresponding "dynamics" of $\mathbf{H}_{\vec{\mu}}$ in the hybrid analogue to Theorem 4.3.

Theorem 4.4. *Let* $(\mathbf{M}, \vec{\omega}, \vec{\Phi}, \vec{J})$ *be a hybrid Hamiltonian* $\mathbf{G}$*-space satisfying the assumptions of Theorem 4.2. If* $\mathbf{H}$ *is a* $\mathbf{G}$*-invariant Hamiltonian and* $(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}, \mathbf{X_H})$ *is a trajectory of* $(\mathscr{M}, \mathbf{M}, \mathbf{X_H})$ *with* $\vec{c}_0(\tau_0) \in \mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(0)}$*, then*

$$(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{J^{-1}(\vec{\mu})}, \mathbf{X_H})$$

and there exists a trajectory $(\vec{C}, \vec{c}_{\vec{\mu}}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}_{\vec{\mu}}, \mathbf{X}_{\mathbf{H}_{\vec{\mu}}})$ *of* $(\mathscr{M}, \mathbf{M}_{\vec{\mu}}, \mathbf{X}_{\mathbf{H}_{\vec{\mu}}})$ *defined by the factorization:*

$$\begin{array}{ccc} (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) & \xrightarrow{(\vec{C}, \vec{c}_{\vec{\mu}})} & (\mathscr{M}, \mathbf{M}_{\vec{\mu}}, \mathbf{X}_{\mathbf{H}_{\vec{\mu}}}) \\ & {\scriptstyle (\vec{C}, \vec{c})} \searrow \quad \nearrow {\scriptstyle (\vec{\mathrm{Id}}_{\mathscr{M}}, \vec{\pi}_{\vec{\mu}})} & \\ & (\mathscr{M}, \mathbf{J^{-1}(\vec{\mu})}, \mathbf{X_H}) & \end{array}$$

Proof. We must first show that for a trajectory

$$(\vec{C},\vec{c}):(\mathscr{I},\mathbf{I},\mathbf{d/dt})\to(\mathscr{M},\mathbf{M},\mathbf{X_H})$$

of $(\mathscr{M},\mathbf{M},\mathbf{X_H})$,

$$\vec{c}(\tau_0)\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(0)} \quad\Rightarrow\quad (\vec{C},\vec{c}):(\mathscr{I},\mathbf{I},\mathbf{d/dt})\to(\mathscr{M},\mathbf{J^{-1}(\vec{\mu})},\mathbf{X_H}).$$

It is enough to show that

$$\vec{c}_0(\tau_0)\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(0)} \quad\Rightarrow\quad (\vec{C},\vec{c}):(\mathscr{I},\mathbf{I})\to(\mathscr{M},\mathbf{J^{-1}(\vec{\mu})}). \tag{4.16}$$

This would imply that $\vec{c}_i(\tau_i)\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(i)}$ and by Theorem 4.3 we know that

$$\vec{c}_i(\tau_i)\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(i)} \quad\Rightarrow\quad \vec{c}_i:(\mathbf{I}_i,\mathrm{d/dt})\to(\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(i)},(\mathbf{X_H})_{\vec{C}(i)})$$

for all $i\in\bigvee(\mathscr{I})$.

To show (4.16), it is clearly sufficient to show:

$$\vec{c}_{i-1}(\tau_{i-1})\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(i-1)} \quad\Rightarrow\quad \vec{c}_i(\tau_i)\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(i)}, \tag{4.17}$$

for $i-1,i\in\bigvee(\mathscr{I})$; (4.16) would then follow from Theorem 4.3. Assuming that $\vec{c}_{i-1}(\tau_{i-1})\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(i-1)}$ (hence, assuming that $\vec{c}_{i-1}(t)\in\mathbf{J^{-1}(\vec{\mu})}_{\vec{C}(i-1)}$ for all $t\in\mathbf{I}_{i-1}$), we need to show that for every diagram of the form

$$i-1 \xleftarrow{\;\mathsf{s}_{e_i}\;} e_i=(i-1,i) \xrightarrow{\;\mathsf{t}_{e_i}\;} i$$

in $\mathscr{I}$,

$$\begin{aligned}\vec{J}_{\vec{C}(i-1)}(\vec{c}_{i-1}(\tau_i))=\vec{\mu}_{\vec{C}(i-1)} \quad&\Rightarrow\quad \vec{J}_{\vec{C}(e_i)}(\vec{c}_{e_i}(\tau_i))=\vec{\mu}_{\vec{C}(e_i)}\\ &\Rightarrow\quad \vec{J}_{\vec{C}(i)}(\vec{c}_i(\tau_i))=\vec{\mu}_{\vec{C}(i)}.\end{aligned} \tag{4.18}$$

First note that because $\vec{c}$ is a natural transformation, the diagram

$$\begin{array}{ccccc}\mathbf{I}_{i-1} & \xleftarrow{\;\mathbf{I}_{\mathsf{s}_{e_i}}=\iota\;} & \mathbf{I}_{e_i} & \xrightarrow{\;\mathbf{I}_{\mathsf{t}_{e_i}}=\iota\;} & \mathbf{I}_i\\ \Big\downarrow{\scriptstyle \vec{c}_{i-1}} & & \Big\downarrow{\scriptstyle \vec{c}_{e_i}} & & \Big\downarrow{\scriptstyle \vec{c}_i}\\ \mathbf{M}_{\vec{C}(i-1)} & \xleftarrow{\;\mathbf{M}_{\vec{C}(\mathsf{s}_{e_i})}\;} & \mathbf{M}_{\vec{C}(e_i)} & \xrightarrow{\;\mathbf{M}_{\vec{C}(\mathsf{t}_{e_i})}\;} & \mathbf{M}_{\vec{C}(i)}\end{array}$$

commutes.

Since $\vec{\mu}\in\mathfrak{g}^\star$, we know that $\mathfrak{g}^\star_{\vec{C}(\mathsf{s}_{e_i})}(\vec{\mu}_{\vec{C}(i-1)})=\vec{\mu}_{\vec{C}(e_i)}$. By the naturality of $\vec{J}$, i.e., because of the commuting diagram given in (4.6),

$$\begin{aligned}\vec{J}_{\vec{C}(e_i)}(\vec{c}_{e_i}(\tau_i)) &= \mathfrak{g}^\star_{\vec{C}(\mathsf{s}_{e_i})}(J_{\vec{C}(i-1)}(\mathbf{M}_{\vec{C}(\mathsf{s}_{e_i})}(\vec{c}_{e_i}(\tau_i))))\\ &= \mathfrak{g}^\star_{\vec{C}(\mathsf{s}_{e_i})}(J_{\vec{C}(i-1)}(\vec{c}_{i-1}(\tau_i)))\\ &= \mathfrak{g}^\star_{\vec{C}(\mathsf{s}_{e_i})}(\vec{\mu}_{\vec{C}(i-1)})\\ &= \vec{\mu}_{\vec{C}(e_i)}.\end{aligned}$$

Since we are assuming that $\vec{\mu}$ is a hybrid regular value of $\vec{J}$ (Definition 4.4), we have that:

$$\vec{J}_{\vec{C}(e_i)}(\vec{c}_{e_i}(\tau_i)) = \vec{\mu}_{\vec{C}(e_i)} \qquad \Rightarrow \qquad \vec{\mu}_{\vec{C}(i)} = \vec{J}_{\vec{C}(i)}(\mathbf{M}_{\vec{C}(\mathfrak{t}_{e_i})}(\vec{c}_{e_i}(\tau_i))) = \vec{J}_{\vec{C}(i)}(\vec{c}_i(\tau_i)).$$

Therefore, (4.18) coupled with the naturality of $\vec{c}$ implies (4.17), which in turn implies (4.16).

Now, we need to show that

$$(\vec{\mathrm{Id}}_{\mathscr{M}}, \vec{\pi}_{\vec{\mu}}) : (\mathscr{M}, \mathbf{J^{-1}(\vec{\mu})}, \mathbf{X_H}) \to (\mathscr{M}, \mathbf{M}_{\vec{\mu}}, \mathbf{X}_{\mathbf{H}_{\vec{\mu}}})$$

is a morphism of hybrid systems. Theorem 4.2 implies that

$$(\vec{\mathrm{Id}}_{\mathscr{M}}, \vec{\pi}_{\vec{\mu}}) : (\mathscr{M}, \mathbf{J^{-1}(\vec{\mu})}) \to (\mathscr{M}, \mathbf{M}_{\vec{\mu}})$$

is a morphism of hybrid manifolds, so we need only show that

$$(\vec{\pi}_{\vec{\mu}})_b : (\mathbf{J^{-1}(\vec{\mu})})_b, (\mathbf{X_H})_b) \to ((\mathbf{M}_{\vec{\mu}})_b, (\mathbf{X}_{\mathbf{H}_{\vec{\mu}}})_b)$$

is a morphism of dynamical systems for all $b \in \mathsf{V}(\mathscr{M})$, but this follows from Theorem 4.3.

Finally, it follows from Lemma 2.1 that

$$(\vec{C}, \vec{c}_{\vec{\mu}}) := (\vec{\mathrm{Id}}_{\mathscr{M}}, \vec{\pi}_{\vec{\mu}}) \circ (\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}_{\vec{\mu}}, \mathbf{X}_{\mathbf{H}_{\vec{\mu}}})$$

is a trajectory of $(\mathscr{M}, \mathbf{M}_{\vec{\mu}}, \mathbf{X}_{\mathbf{H}_{\vec{\mu}}})$. □

Chapter 5

Zeno Behavior & Hybrid Stability Theory

Zeno behavior is a phenomena that is unique to hybrid systems; it has no counterpart in discrete or continuous systems. It has remained relatively unexplored over the years—before the work of the author [18] there were no sufficient conditions for the existence of Zeno behavior in general hybrid systems. This is a byproduct of the fact that to determine whether Zeno behavior exists in a hybrid system, the vector fields on each domain must be solved for explicitly. Since this generally is not possible, finding sufficient conditions on the existence of Zeno has remained an open problem in the hybrid systems community, at least in the case when the vector fields on each domain are nontrivial, i.e., when they are not constant vector fields.

It is only through a necessary paradigm shift in the study of hybrid stability that we are able to provide a remedy. It is only through the use of categories of hybrid objects that we are able to provide sufficient conditions for a general class of hybrid systems.

Zeno behavior can be likened to stability, in that its existence implies a type of convergence; the convergence is to a set, termed a *Zeno equilibria*, that is invariant under the discrete dynamics. Superficially, this is where the similarities end, e.g., each element of the Zeno equilbria set *cannot* be a zero of its corresponding vector field. Motivated by the peculiarities of Zeno equilibria, we consider a form of asymptotic stability that is global in the continuous state, but local in the discrete state. We provide sufficient conditions for stability of these equilibria, resulting in sufficient conditions for the existence of Zeno behavior.

Regardless of, or because of, the unique nature of Zeno equilibria, they can arise in many systems of interest, e.g., mechanical systems undergoing impacts. The convergent behavior of these systems is often of interest—even if this convergence is not to "classical" notions of equilibrium points. This motivates the study of Zeno equilibria because even if the convergence is not classical, it still is important. For example, simulating trajectories of these systems is an important component in their analysis, yet this may not be possible due to the relationship between Zeno equilibria and Zeno behavior.

An equally important reason to address the stability of Zeno equilibria is to be able to assess the existence of Zeno trajectories. This behavior is infamous in the hybrid system community for its ability to halt simulations. The only way to prevent this undesirable outcome is to give *a priori* conditions on the existence of Zeno behavior. This has motivated a profuse study of Zeno hybrid systems (see [33, 59, 69, 119] to name a few) but a concrete notion of convergence (in the sense of stability) has not yet been introduced. As a result, there is a noticeable lack of sufficient conditions for the existence of Zeno behavior.

The main contribution of this chapter is sufficient conditions for the stability of Zeno equilibria. As a byproduct, we are able to give sufficient conditions for the existence of Zeno behavior. The categorical approach to hybrid systems allows us to decompose the study of stability into two manageable steps. The first step consists of identifying a sufficiently rich, yet sufficiently simple, class of hybrid systems embodying the desired stability properties: *first quadrant hybrid systems*. The second step is to understand the stability of general hybrid systems by understanding the relationships between these systems and first quadrant hybrid systems described by morphisms (in the category of hybrid systems).

In this vain, we devote some effort to the introduction of first quadrant hybrid systems, demonstrating in a step-by-step fashion how to transform these systems into categorical hybrid systems. We then study a special class of first quadrant hybrid systems, *diagonal* first quadrant hybrid systems, giving sufficient conditions for the existence of Zeno behavior in systems of this form. The techniques employed, while not immediately generalizable, indicate a fundamental connection between stability and Zeno. We then proceed to study general hybrid systems, and Zeno equilibria, through the use of categories of hybrid objects and thus solidify the connection between stability and Zeno behavior. We conclude the chapter by indicating how it is possible to "go beyond" Zeno, i.e., carry trajectories past a Zeno point, in a simple class of hybrid systems: Lagrangian hybrid systems.

Related work. There has been a rather profuse study of Zeno equilibria (see [48, 59, 68, 69, 103, 104, 118, 119], to name a few), yet a concrete notion of convergence (in the sense of stability) has not been formally introduced (except in [18], on which this chapter is based). The author has explored this relationship in some limited contexts, namely in [7] where diagonal first quadrant hybrid systems were studied, and [12] where the geometric "stability preserving" regularization of a class of hybrid systems was considered; in fact, the latter paper first introduces the notion of a Zeno equilibria, albeit a special case thereof. In addition, the author has studied the relationship between Zeno behavior and the topology of hybrid systems in [13] and [14]. Finally, methods for carrying executions past the point(s) at which Zeno behavior occurs has been studied in [19] and [121].

While the convergent properties of Zeno equilibria have not been well-studied, the stability of hybrid and switched systems has. We refer the reader to [32, 33, 34, 35, 52, 80, 81, 94, 120] for some of the approaches taken. While our approach is essentially different, there are analogies that can be drawn. For example, common to the study of stability of hybrid systems is the idea of multiple Lyapunov functions [34]. In fact, we arrive at a similar construction in Theorem 5.5, where the morphism between hybrid systems can be viewed as a "hybrid Lyapunov function" and, as such, consists of a collection of Lyapunov functions.

5.1 Zeno Behavior

This section is in many ways unorthodox in that it is more discussional in nature. We begin by reintroducing the definition of Zeno behavior, distinguishing different ways in which this phenomena can occur. We then illustrate how this Zeno behavior can occur in practice—with devastating results. The section is concluded with a discussion on the prospects of obtaining necessary, sufficient and necessary and sufficient conditions for the existence of this behavior.

5.1.1 Zeno trajectories. However one chooses to represent a hybrid system, either as a tuple

$$\mathfrak{H} = (Q, E, D, G, R, X),$$

or categorically,

$$\mathfrak{H} = (\mathscr{M}, \mathbf{M}, \mathbf{X}),$$

Zeno behavior is necessarily a factor. Since it is a property of the trajectories (or executions) of a hybrid system, pick a representation of these trajectories, i.e., either:

$$\epsilon = (\Lambda, I, \rho, C),$$

or:

$$(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{M}, \mathbf{M}, \mathbf{X}).$$

In either case, suppose that

$$\begin{aligned} I_i &= [\tau_i, \tau_{i+1}] \quad \text{if} \quad i \in \Lambda \\ \mathbf{I}_i &= [\tau_i, \tau_{i+1}] \quad \text{if} \quad i \in \bigvee(\mathscr{I}). \end{aligned}$$

Since $\bigvee(\mathscr{I}) = \Lambda$, recall that Zeno behavior is defined as follows:

Definition 5.1. A trajectory of a hybrid system $\mathfrak{H}$ is **Zeno** if $\bigvee(\mathscr{I}) = \mathbb{N}$ and

$$\sum_{i=0}^{\infty} (\tau_{i+1} - \tau_i) = \tau_\infty$$

for some finite constant τ_∞, termed the *Zeno time.*

A hybrid system is Zeno if it admits a Zeno trajectory, i.e., if there exists an trajectory ϵ that is Zeno.

5.1.2 Zeno Behavior in practice. Zeno behavior is sometimes referred to as *pathological.* While this is justified in that it does not seem to appear in nature, it certainly appears often enough in practice to warrant fervent study. We believe that it is justified to argue this point, hopefully addressing the concerns of any naysayers.

A strong motivation for studying hybrid systems is that they can greatly simplify models of complex dynamical systems; Zeno behavior can often arise out of such simplifications. It is often argued that

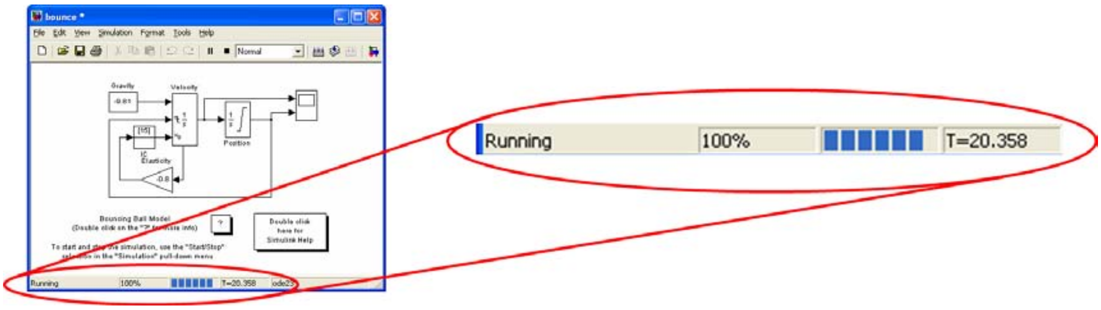


Figure 5.1: Zeno behavior that effectively makes a program (Matlab) halt.

if Zeno arises is such a case, the model is wrong—maybe a hybrid model is not the right one. Rather than "throwing the baby out with the bath water," the author believes that it is better to understand the role of Zeno behavior in hybrid systems in order to more effectively deal with the phenomena. To those who are not in agreement, a simpler rebuttal is: in order to modify Zeno models so that they are no longer Zeno, one must first *detect* that they are Zeno. Hence we are back to our initial claim: Zeno behavior needs to be properly understood.

One may now argue that Zeno behavior does not appear *often* enough to be an interesting phenomena. Yet, as was seen in Chapter 3, mechanical systems undergoing impacts are naturally modeled as hybrid systems; if systems of this form loose energy at each impact, they will tend to display Zeno behavior. If the category of all mechanical systems undergoing impacts does not provide a large enough example of systems that can display Zeno behavior, one can rest assured that there are more. For example, hybrid models of communication networks display Zeno behavior (cf. Figure 5.2), as illustrated in [2].

After hopefully convincing the reader as to the necessity of studying Zeno behavior, it is important to remark on how it manifests itself. To provide a quintessential example, if one runs the bouncing ball example in Matlab's Simulink for 25 seconds (rather then the default 20 seconds) the simulation will *never* finish; see Figure 5.1. Zeno behavior makes a simulator effectively halt. This implies that the existence of such behavior can have catastrophic effects on the simulation—hence verification—of hybrid systems. If it were possible to detect the existence or non-existence of this behavior *a priori*, it could have far-reaching effects.

5.1.3 Types of Zeno. The definition of a Zeno trajectory results in two qualitatively different types of Zeno behavior (as first introduced in [13]); they are defined as follows: a Zeno trajectory is

Chattering Zeno: If there exists a finite C such that

$$\tau_{i+1} - \tau_i = 0$$

for all $i \geq C$.

Genuinely Zeno: If

$$\tau_{i+1} - \tau_i > 0$$

for all $i \in \mathbb{N}$.

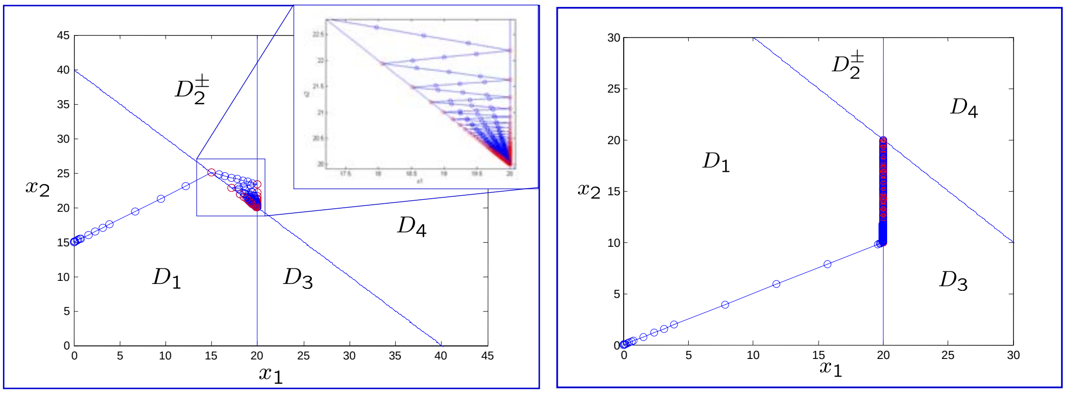


Figure 5.2: An example of Genuine Zeno behavior (left). An example of Chattering Zeno behavior (right).

There are obviously more characterizations of Zeno behavior—ones that could go beyond simple conditions on the differences between successive switching times. This will be discussed in more detail after we illustrate the differences between chattering and genuine Zeno behavior, which is especially prevalent in there detection and elimination.

Chattering Zeno trajectories (cf. Figure 5.2) result from the existence of a switching surface on which the vector fields "oppose" each other; for this reason they are easy to detect (simply look at the orientation of the vector fields along the guards). In addition, they can be eliminated in a fairly simple manner. Filippov solutions can be defined on these surfaces in order to force the flow to "slide" along the switching surface [50]. Later in this chapter we will generalize this technique to extend genuinely Zeno executions past the Zeno point.

Genuinely Zeno trajectories (cf. Figure 5.2) are much more complicated in their behavior. The only methods currently available to detect the existence of trajectories of this form can be found within this chapter. Very little has been done in the area of eliminating these executions, although there have been some results [12] and [69], again for a special class of hybrid systems.

To better understand why this is the case, recall that the bouncing ball is, in fact, globally (minus the origin) genuinely Zeno (if $0 < r < 1$), i.e., every trajectory is Zeno. Recall from Example 2.17 that the the Zeno time for the bouncing ball is given by:

$$\tau_\infty^{\text{ball}} = \sum_{i=0}^{\infty}(\tau_{i+1}^{\text{ball}} - \tau_i^{\text{ball}}) = \frac{x_2 + (1-2g)\sqrt{2gx_1 + x_2^2}}{g} + \sum_{i=0}^{\infty} 2\frac{\sqrt{2gx_1 + x_2^2}}{g} r^i.$$

For a geometric sequences $\{ar^i\}_{i\in\mathbb{N}}$, recall that picking initial conditions $x_1 = a^2 g/8$ and $x_2 = 0$ for a trajectory of the bouncing ball yields:

$$\tau_\infty^{\text{ball}} = \pm\left(\frac{1}{2}a(1-2g) + \sum_{i=0}^{\infty} ar^i\right),$$

where the expression on the right is positive if a is positive and negative if a is negative. This implies that even for one of the simplest examples of a hybrid system—the bouncing ball—the switching times effectively yield *any* geometric series.[1]

Therefore, to completely characterize the switching times of Zeno trajectories, it is reasonable to suspect that one would need to effectively characterize all series and the convergence properties thereof. This is assuming that one were able to solve for the switching times—something that is not even possible for linear hybrid systems since it involves solving transcendental equations. All of this hints at the complexity of Zeno behavior—determining conditions on the existence or non-existence of Zeno behavior is a formidable task that will take serious and concerted effort.

5.2 First Quadrant Hybrid Systems

This section is devoted to the study of first quadrant hybrid systems, categorical hybrid systems, and their interplay. We begin by defining first quadrant hybrid systems; these systems are easy to understand and analyze, but lack generality. Yet, they are very useful for understanding Zeno behavior in general hybrid systems. The connection between these two types of systems—first quadrant hybrid systems and general hybrid systems—is achieved through the use of morphisms of hybrid systems. Therefore, we begin by introducing first quadrant hybrid systems using the classical notation for hybrid systems. We then proceed to demonstrate, in a step-by-step fashion, how to obtain a categorical representation of these systems.

5.2.1 First quadrant hybrid systems. First quadrant hybrid systems are hybrid systems which can be viewed as the "simplest" hybrid systems that display Zeno behavior.

A *first quadrant hybrid system*, or just **FQ** *hybrid system*, is a tuple:

$$\mathfrak{H}_{\mathbf{FQ}} = (\Gamma, D, G, R, X),$$

where

- $\Gamma = (Q, E)$ is a directed cycle, with

$$Q = \{1, \ldots, k\}, \qquad E = \{e_1 = (1,2), e_2 = (2,3), \ldots, e_k = (k,1)\}.$$

- $D = \{D_i\}_{i \in Q}$, where for all $i \in Q$,

$$D_i = (\mathbb{R}_0^+)^2 = \left\{ \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 \geq 0 \right\},$$

  hence the name "first quadrant."

- $G = \{G_e\}_{e \in E}$, where for all $e \in E$

$$G_e = \left\{ \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \geq 0 \right\}.$$

[1] As a side note, our proof later in this chapter that the bouncing ball is Zeno does not rely on the convergence of geometric series, and so we prove, independently, that geometric series converge (when $0 \leq r < 1$).

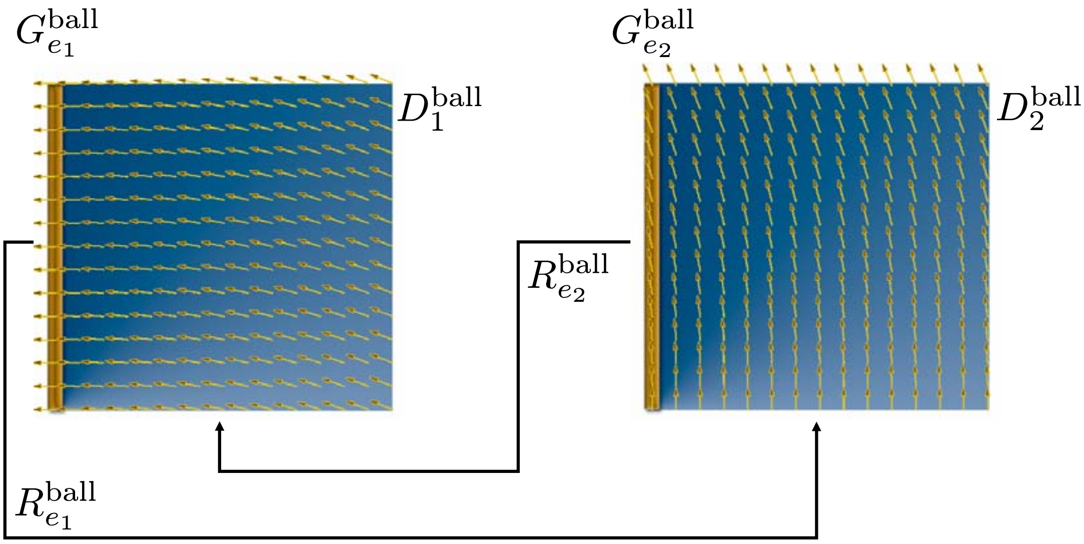


Figure 5.3: The hybrid system $\mathfrak{H}_{\mathbf{FQ}}^{\mathrm{ball}}$.

- ⋄ $R = \{R_e\}_{e\in E}$, where $R_e : G_e \to (\mathbb{R}_0^+)^2$ and for all $e \in E$ there exists a function $r_e : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$$R_e(x_1, x_2) = \begin{pmatrix} r_e(x_2) \\ x_1 \end{pmatrix}.$$

- ⋄ $X = \{X_i\}_{i\in Q}$, where X_i is a Lipshitz vector field on $(\mathbb{R}_0^+)^2$.

Remark 5.1. We could consider higher dimensional **FQ** hybrid systems, but this would add complication without generality.

Example 5.1. We will transform the bouncing ball hybrid system introduced in Example 2.1 into a first quadrant hybrid system:

$$\mathfrak{H}_{\mathbf{FQ}}^{\mathrm{ball}} = (\Gamma^{\mathrm{ball}}, D^{\mathrm{ball}}, G^{\mathrm{ball}}, R^{\mathrm{ball}}, X^{\mathrm{ball}})$$

by dividing the original domain into two components, and changing the vector fields accordingly.

We first define Γ^{ball} to be the directed cycle:

$$1 \underset{e_2}{\overset{e_1}{\rightleftarrows}} 2$$

Since $\mathfrak{H}_{\mathbf{FQ}}^{\mathrm{ball}}$ will be a first quadrant hybrid system, the domains and guards must satisfy the conditions given in Paragraph 5.2.1. In the case of the bouncing ball, the domain D_1^{ball} is obtained from the top half of the original domain for the bouncing ball by reflecting it around the line $x_1 = x_2$. The domain D_2^{ball} is obtained from the bottom half of the original domain by reflecting it around the line $x_2 = 0$. This implies that the reset maps are given by[2]

$$R_{e_1}^{\text{ball}}(x_1, x_2) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, \qquad R_{e_2}^{\text{ball}}(x_1, x_2) = \begin{pmatrix} e x_2 \\ x_1 \end{pmatrix}.$$

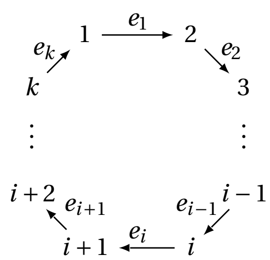
Finally, the transformed vector fields are given by

$$X_1^{\text{ball}}(x_1, x_2) = \begin{pmatrix} -g \\ x_1 \end{pmatrix}, \qquad X_2^{\text{ball}}(x_1, x_2) = \begin{pmatrix} -x_2 \\ g \end{pmatrix}.$$

A graphical representation of this system can be seen in Figure 5.3.

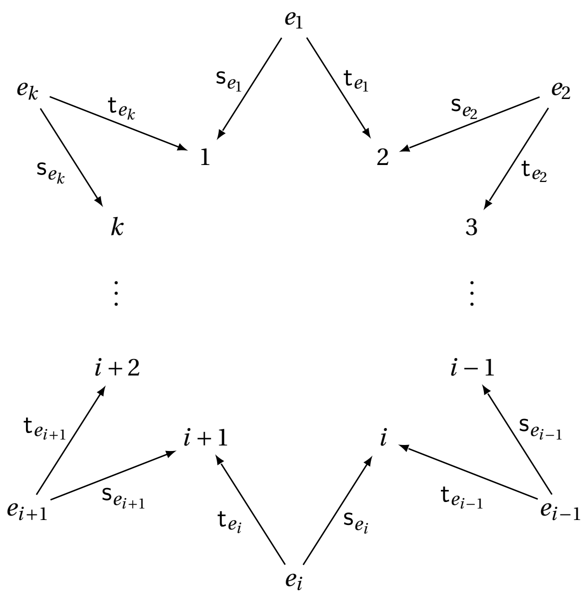
5.2.2 Categorical FQ hybrid systems. We now proceed to show, in a step-by-step fashion, how to obtain a categorical **FQ** hybrid system $(\mathscr{M}^{\text{FQ}}, \mathbf{M}^{\text{FQ}}, \mathbf{X}^{\text{FQ}})$ from a first quadrant hybrid system $\mathfrak{H}_{\mathbf{FQ}}$.

The graph for a **FQ** hybrid system is given by a directed k-cycle graph:



Therefore, the associated D-category, $\mathscr{M}^{\text{FQ}}$, is given by:

$\mathscr{M}^{\text{FQ}} =$

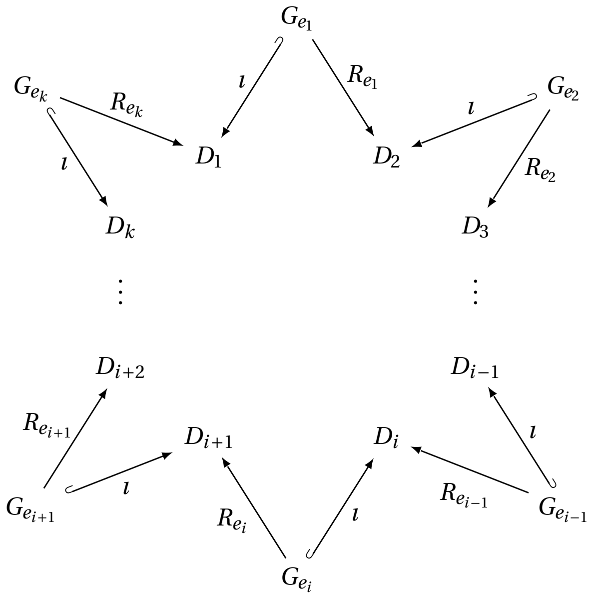


The hybrid space of $\mathfrak{H}_{\mathbf{FQ}}$ is the tuple (Γ, D, G, R). Therefore, as outlined in Proposition 2.1, the hybrid manifold associated to $\mathfrak{H}_{\mathbf{FQ}}$ is given by the pair $(\mathscr{M}^{\text{FQ}}, \mathbf{M}^{\text{FQ}})$ where $\mathbf{M}^{\text{FQ}}$ is the functor defined on

[2] Note that we do not denote the coefficient of restitution for this system by "e" and not "r" due to the notation used to define **FQ** hybrid systems.

$\mathscr{M}^{\mathrm{FQ}}$ by

$\mathbf{M}^{\mathrm{FQ}}(\mathscr{M}^{\mathrm{FQ}}) =$

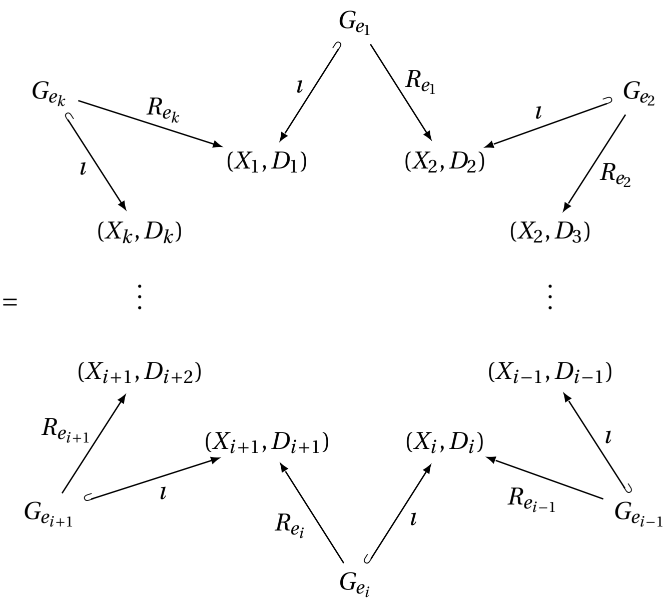


Finally, the categorical hybrid system associated to $\mathfrak{H}_{\mathbf{FQ}}$ is given by

$$(\mathscr{M}^{\mathrm{FQ}}, \mathbf{M}^{\mathrm{FQ}}, \mathbf{X}^{\mathrm{FQ}}),$$

where $\mathbf{X}^{\mathrm{FQ}} = \{X_i\}_{i \in \mathrm{V}(\mathscr{M}^{\mathrm{FQ}})=Q}$. This categorical hybrid system can be visualized graphically as follows:

$(\mathscr{M}^{\mathrm{FQ}}, \mathbf{M}^{\mathrm{FQ}}, \mathbf{X}^{\mathrm{FQ}}) =$



Example 5.2. Returning to the first quadrant bouncing ball hybrid system, $\mathfrak{H}^{\mathrm{ball}}_{\mathbf{FQ}}$, we will transform this hybrid system into a categorical hybrid system

$$(\mathscr{M}^{\mathrm{ball}}, \mathbf{M}^{\mathrm{ball}}, \mathbf{X}^{\mathrm{ball}}).$$

The D-category associated with Γ^{ball} is given by

$$\mathscr{M}^{\text{ball}} = \begin{array}{ccccc} & & e_1 & & \\ & \swarrow^{\mathsf{s}_{e_1}} & & \searrow^{\mathsf{t}_{e_1}} & \\ 1 & & & & 2 \\ & \nwarrow_{\mathsf{t}_{e_2}} & & \nearrow_{\mathsf{s}_{e_2}} & \\ & & e_2 & & \end{array}$$

together with the identity morphisms on each object. The functor

$$\mathbf{M}^{\text{ball}} : \mathscr{M}^{\text{ball}} \to \mathsf{Man}$$

takes the following values:

$$\mathbf{M}^{\text{ball}}(\mathscr{M}^{\text{ball}}) = \begin{array}{ccccc} & & \mathbf{M}^{\text{ball}}_{e_1} = G^{\text{ball}}_{e_1} & & \\ & \swarrow^{\mathbf{M}^{\text{ball}}_{\mathsf{s}_{e_1}} = \iota} & & \searrow^{\mathbf{M}^{\text{ball}}_{\mathsf{t}_{e_1}} = R^{\text{ball}}_{e_1}} & \\ \mathbf{M}^{\text{ball}}_1 = D^{\text{ball}}_1 & & & & \mathbf{M}^{\text{ball}}_2 = D^{\text{ball}}_2 \\ & \nwarrow_{\mathbf{M}^{\text{ball}}_{\mathsf{t}_{e_2}} = R^{\text{ball}}_{e_2}} & & \nearrow_{\mathbf{M}^{\text{ball}}_{\mathsf{s}_{e_2}} = \iota} & \\ & & \mathbf{M}^{\text{ball}}_{e_2} = G^{\text{ball}}_{e_2} & & \end{array}$$

Finally, the collection of vector fields $\mathbf{X}^{\text{ball}}$ is given by:

$$\mathbf{X}^{\text{ball}} = \{X^{\text{ball}}_i\}_{i \in \mathsf{V}(\mathscr{M}^{\text{ball}}) = Q^{\text{ball}}} = \{X^{\text{ball}}_1, X^{\text{ball}}_2\}.$$

5.3 Zeno Behavior in DFQ Hybrid Systems

Diagonal first quadrant (**DFQ**) hybrid systems are a special class of first quadrant hybrid systems that have diagonal affine vector fields on each domain. It is this restrictive class of hybrid systems that we will now consider. The main impetus for this is that these hybrid systems have sufficiently interesting dynamics, in that they are not trivial, while remaining amenable to analysis. Studying these systems will yield important intuition about Zeno behavior.

The main result of this section is sufficient conditions for the existence of Zeno behavior in **DFQ** hybrid systems. Given certain assumptions on a diagonal first quadrant hybrid system, we construct an infinite execution for this system. To this execution, we associate a single discrete time dynamical system that describes its continuous evolution. Therefore, we reduce the study of executions of diagonal first quadrant hybrid systems to the study of a single discrete time dynamical system. We obtain sufficient conditions for the existence of Zeno behavior by determining when this discrete time dynamical system is exponentially stable.

Definition 5.2. A **diagonal first quadrant hybrid system** (**DFQ** hybrid system) is a **FQ** hybrid system

$$\mathfrak{H}_{\mathbf{DFQ}} = (\Gamma = (Q,E), D, G, R, X)$$

such that

- $r_e = \mathrm{id}$ for all $e \in E$.
- $X = \{X_q = \Lambda_q x + a_q\}_{q \in Q}$ is a set of *diagonal affine linear systems*, i.e., $a_q \in \mathbb{R}^2$ and $\Lambda_q \in \mathbb{R}^{2\times 2}$ is a diagonal matrix for every $q \in Q$.

Notation 5.1. To avoid the proliferation of subscripts, we denote the i^{th} entry of a_q by a_q^i. Similarly, the $(i,i)^{\text{th}}$ entry of Λ_q is denoted by λ_q^i; this is just the i^{th} eigenvalue of Λ_q.

5.3.1 Trajectories. Because of the special form of the vector fields for a **DFQ** hybrid system, we can explicitly solve for trajectories. A trajectory of the dynamical systems $(\mathbb{R}^2, X_q)$ with initial condition $c(t_0)$,

$$c : (I, \mathrm{d}/\mathrm{dt}) \to (\mathbb{R}^2, X_q),$$

with t_0 the left endpoint of I, is given by:

$$c(t) = (\exp(\Lambda_q(t-t_0)) - 1)\Lambda_q^{-1} a_q + \exp(\Lambda_q(t-t_0))c(t_0),$$

which is well defined even if Λ_q has zero eigenvalues; in the case when $\Lambda_q = 0$, this expression becomes $c(t) = (t-t_0)a_q + c(t_0)$, or this is the flow of the constant system $X_q = a_q$.

5.3.a Event Detection

Discrete transitions in a hybrid system occur when there is an event—that is when the flow hits the guard. In this section we determine when an event exists for some domain and initial condition of a **DFQ** hybrid system, and we explicitly solve for the time in which this event occurs. These conditions are important because when they are satisfied, it is possible to construct an execution.

5.3.2 Existence of events. For some $x \in D_q$, we say that *there exists an event* if for a trajectory

$$c : (I, \mathrm{d}/\mathrm{dt}) \to (\mathbb{R}^2, X_q)$$

of $(\mathbb{R}^2, X_q)$ with initial condition $c(t_0) = x$, there exists a finite $\Delta t(x) \geq 0$ such that

$$\text{(i)} \qquad c_1(t_0 + \Delta t(x)) = 0$$
$$\text{(ii)} \qquad c_2(t) \geq 0 \quad \forall\ t \in [t_0, t_0 + \Delta t(x)].$$

The first condition says, in the context of **DFQ** hybrid systems, that the trajectory $c(t)$ reaches the guard of D_q at time $t_0 + \Delta t(x)$. The second condition says that

$$c : ([t_0, t_0 + \Delta t(x)], \mathrm{d}/\mathrm{dt}) \to (D_q, X_q) = ((\mathbb{R}_0^+)^2, X_q),$$

is a morphism of dynamical systems, i.e., c is a trajectory of (D_q, X_q).

In the case of **DFQ** hybrid systems, we can give conditions on when events exist. The first two components of a trajectory of $(\mathbb{R}^2, X_q)$, with initial condition $c(t_0) = (x_1, x_2)^T$, is given by

$$\begin{aligned} c_1(t) &= \frac{(\exp(\lambda_q^1(t-t_0))-1)}{\lambda_q^1} a_q^1 + \exp(\lambda_q^1(t-t_0))x_1, \\ c_2(t) &= \frac{(\exp(\lambda_q^2(t-t_0))-1)}{\lambda_q^2} a_q^2 + \exp(\lambda_q^2(t-t_0))x_2. \end{aligned}$$

There exists an event if

$$\Delta t_q(x_1) =: \frac{1}{\lambda_q^1} \log\left(\frac{a_q^1}{a_q^1 + \lambda_q^1 x_1}\right) \tag{5.1}$$

is finite and positive (possibly zero) and

$$\frac{(\exp(\lambda_q^2(t-t_0))-1)}{\lambda_q^2} a_q^2 + \exp(\lambda_q^2(t-t_0))x_2 \geq 0 \qquad \forall\ t \in [t_0, t_0 + \Delta t_q(x_1)].$$

We can make these conditions more explicit by considering initial conditions in a ball of radius $\delta > 0$ around the origin:

$$B_\delta(0) = \{x \in \mathbb{R}^n : \|x\| < \delta\}.$$

We have the following Proposition.

Proposition 5.1. *For some $\delta > 0$, there exists an event for $x \in B_\delta(0) \cap D_q$ if*

$$a_q^1 < 0 \qquad \text{and} \qquad a_q^2 \geq 0.$$

Proof. Proving this proposition amounts to first considering the inequality

$$\frac{1}{\lambda_q^1} \log\left(\frac{a_q^1}{a_q^1 + \lambda_q^1 x_1}\right) \geq 0$$

and deriving conditions on a_q^1 and λ_q^1 such that it holds for $0 \leq x_1 < \delta$ for some $\delta > 0$. It turns out that these conditions are independent of λ_q^1, i.e., we only require that $a_q^1 < 0$. Note that λ_q^1 does affect δ. Specifically, if $\lambda_q^1 \leq 0$, then $\delta = \infty$, while if $\lambda_q^1 > 0$,

$$\delta = -\frac{a_q^1}{\lambda_q^1}.$$

The second step in showing this proposition is to understand what the conditions are on a_q^2 and λ_q^2 such that

$$\frac{(\exp(\lambda_q^2(t-t_0))-1)}{\lambda_q^2} a_q^2 + \exp(\lambda_q^2(t-t_0))x_2 \geq 0$$

for $t \in [t_0, t_0 + \Delta t_q(x_1)]$. It easily can be seen that this holds as long as $a_q^2 \geq 0$, regardless of the values of x_2 and λ_q^2. □

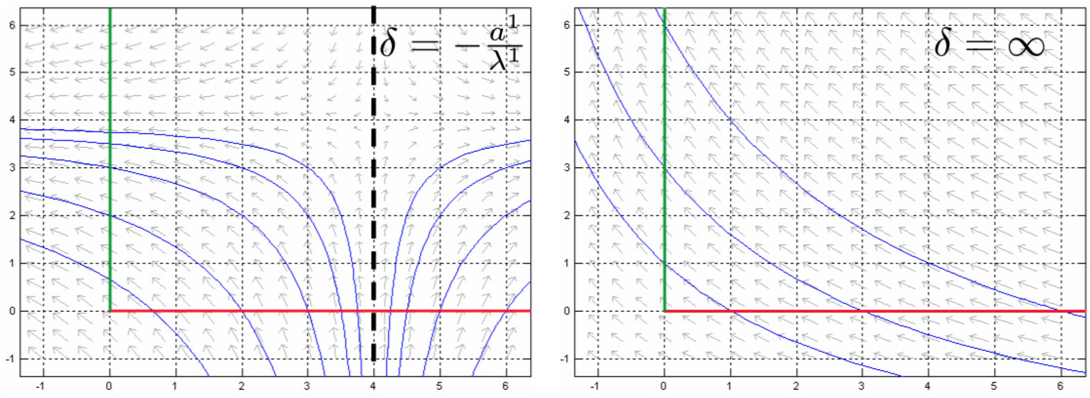


Figure 5.4: The phase space of the diagonal system given in Example 5.3 for $c = 1$ (left) and $c = -1$ (right).

Corollary 5.1. *There exists an event for all $x \in D_q$ if*

$$\lambda_q^1 \le 0 \quad \text{and} \quad a_q^1 < 0 \quad \text{and} \quad a_q^2 \ge 0.$$

Example 5.3. Consider the diagonal system given by

$$\dot{x} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} x + \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

In the case when $c = 1$, an event exists if $x_1 < 4$, and otherwise one does not exist. If $c = -1$ then an event always exists.

5.3.b Discrete Nonlinear Systems from DFQ Hybrid Systems

Using the conditions obtained in the previous section, we are able to construct an infinite execution for a **DFQ** hybrid system satisfying these conditions. From this execution, we can define a set of discrete time maps—analogous to Poincaré maps—defining the evolution of the sequence of initial conditions of this execution. Thus, studying a discrete evolution in space is equivalent to studying a set of discrete time dynamical systems. Later, we will study one of these discrete time dynamical systems, termed the discrete time dynamical system associated to a **DFQ** hybrid system, and show that its behavior in some way dictates the behavior of the other discrete time dynamical systems. Thus, we will demonstrate that studying the behavior of a hybrid system is equivalent to studying a discrete time dynamical system.

Assumption 5.1. For a **DFQ** hybrid system $\mathfrak{H}_{\mathbf{DFQ}}$, assume that for every $q \in Q$, $\Lambda_q x + a_q$ satisfies the conditions:

$$\lambda_q^1 \le 0 \quad \text{and} \quad a_q^1 < 0 < a_q^2.$$

5.3.3 Constructing an infinite execution. If the above assumption holds, we can construct an infinite execution (see Definition 2.5)

$$\epsilon = (\mathbb{N}, I, \rho, C),$$

of $\mathfrak{H}_{\mathbf{DFQ}}$. Before doing so, we introduce the set-up under consideration.

We will take the initial time for the execution to be 0, i.e., 0 will be the left endpoint of I_0. In addition, we will take the initial condition of ϵ to be $(\xi_0, 0)$ for some $\xi_0 \in \mathbb{R}$. This implies that all of the initial conditions for the trajectories $C = \{c_i\}_{i\in\mathbb{N}}$ will be of the form $c_i(\tau_i) = (\xi_i, 0)$ for τ_i the left endpoint of I_i and some $\xi_i \in \mathbb{R}$. We know that there will exist an infinite number of such trajectories because of Assumption 5.1. That is, for every $(\xi_i, 0)$ there will exist a switching time given by $\Delta t_{\rho(i)}(\xi_i)$, as defined in (5.1). The end result is a sequence:

$$\xi = \{\xi_i\}_{i\in\mathbb{N}},$$

which we can view as a *sequence of initial conditions*.

With these formulations, we define ϵ as follows:

- $I = \{I_i\}_{i\in\mathbb{N}}$ where $I_i = [\tau_i, \tau_{i+1}]$ with the switching times τ_i defined recursively by:

$$\begin{aligned} \tau_0 &= 0 \\ \tau_{i+1} &= \Delta t_{\rho(i)}(\xi_i) + \tau_i \end{aligned}$$

  for $i \geq 0$.

- $\rho : \mathbb{N} \to Q$ is defined to be

$$\rho(i) := i \bmod k + 1.$$

- $C = \{c_i\}_{i\in\mathbb{N}}$, where

$$c_i(t) = \begin{pmatrix} \frac{(\exp(\lambda^1_{\rho(i)}(t-\tau_i))-1)}{\lambda^1_{\rho(i)}} a^1_{\rho(i)} + \exp(\lambda^1_{\rho(i)}(t-\tau_i))\xi_i \\ \frac{(\exp(\lambda^2_{\rho(i)}(t-\tau_i))-1)}{\lambda^2_{\rho(i)}} a^2_{\rho(i)} \end{pmatrix}.$$

For this to be a valid execution, we require that the sequence $\xi = \{\xi_i\}_{i\in\mathbb{N}}$ satisfy:

$$\begin{aligned} \xi_{i+1} &= (c_i(\tau_{i+1}))_2 \\ &= \frac{(\exp(\lambda^2_{\rho(i)}\Delta t_{\rho(i)}(\xi_i)) - 1)}{\lambda^2_{\rho(i)}} a^2_{\rho(i)}. \end{aligned}$$

This, together with the results on the existence of events, implies that ϵ is a well-defined execution.

5.3.4 Overview of construction. From the execution given in the previous paragraph, we would like to construct a single nonlinear discrete map; this map will be used to derive sufficient conditions on the existence of Zeno behavior. This is done by first defining a map that computes this sequence ξ independently of the sequence of switching times. The next step is to define a map that computes this sequence independently of both $\{\tau_i\}_{i\in\mathbb{N}}$ and ρ. The end result is a single map that iteratively computes ξ, i.e., a discrete time dynamical system, so we can study the behavior of the sequence of initial conditions by studying the behavior of this map.

5.3.5 Step 1: Removing dependence on time. For $q \in Q$, i.e., for $q \in \{1,\ldots,k\}$, let $\Phi_q : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be given by

$$\Phi_q(x) = \frac{1}{\lambda_q^2}\left(\exp\left(\frac{\lambda_q^2}{\lambda_q^1}\log\left(\frac{a_q^1}{a_q^1+\lambda_q^1 x}\right)\right)-1\right)a_q^2.$$

Note that this function is well-defined because of Assumption 5.1. This function also has some important properties. It is a diffeomorphism and both Φ_q and its inverse satisfy the properties:

$$\Phi_q(0) = 0 \qquad \Phi_q^{-1}(0) = 0$$
$$\Phi'_q(0) = -\frac{a_q^2}{a_q^1} \qquad (\Phi_q^{-1})'(0) = -\frac{a_q^1}{a_q^2}$$

This function gives the elements in the sequence $\xi = \{\xi_i\}_{i\in\mathbb{N}}$ inductively, i.e.,

$$\xi_{i+1} = \Phi_{\rho(i)}(\xi_i).$$

So we have eliminated the dependence of ξ on $\{\tau_i\}_{i\in\mathbb{N}}$ (or the switching times).

5.3.6 Step 2: Removing dependence on the discrete evolution. The next step in defining a single nonlinear discrete map from this execution is to eliminate ρ from a subsequence of the sequence ξ that has the same limiting behavior as the original sequence. To do this, define the map $\Psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by

$$\Psi = \Phi_k \circ \Phi_{k-1} \circ \cdots \circ \Phi_1. \tag{5.2}$$

Note that this map has the following important properties:

$$\Psi(0) = 0, \qquad \Psi'(0) = \left(\prod_{q=1}^{k} -\frac{a_q^2}{a_q^1}\right).$$

It also can be verified that

$$\begin{aligned}
\xi_{ki+k} &= \Phi_{\rho(ki+k-1)}(\xi_{ki+k-1})\\
&= \Phi_{\rho(ki+k-1)} \circ \Phi_{\rho(ki+k-2)} \circ \cdots \circ \Phi_{\rho(ki)}(\xi_{ki})\\
&= \Phi_k \circ \Phi_{k-1} \circ \cdots \circ \Phi_1(\xi_{ki})\\
&= \Psi(\xi_{ki})
\end{aligned}$$

since $\rho(i) = i \bmod k + 1$. Therefore, define the following subsequence

$$z = \{z_i\}_{i\in\mathbb{N}} := \{\xi_{ki}\}_{i\in\mathbb{N}}$$

of this sequence ξ. This subsequence is important because, as we have just shown, it is defined by a discrete time dynamical system:

$$z_{i+1} = \Psi(z_i).$$

It is also important because when it converges to the origin, so does the sequence ξ; this will be demonstrated in Lemma 5.1.

5.3.7 Step 3: Removing Dependence on initial conditions. The final step in deriving a single map that describes the sequence ξ is to show that every element of ξ can be expressed in terms of the map Ψ (composed with other maps); this fact will be essential in establishing the main result of this section. Define the following subsequences of the sequence ξ,

$$\eta(J) = \{\eta(J)_i\}_{i\in\mathbb{N}} := \{\xi_{ki+J-1}\}_{i\in\mathbb{N}},$$

for $J \in \{1,\ldots,k\}$. Note that in particular $z = \eta(1)$, and it is clear that

$$\xi = \bigcup_{J=1}^{k} \eta(J).$$

Now we can relate each sequence $\eta(J)$ to the sequence z by defining the maps $\Upsilon_J : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, for $J \in \{1,\ldots,k\}$, given by

$$\begin{aligned} \Upsilon_J &= \Phi_{J-1}\circ\cdots\circ\Phi_1\circ\Phi_k\circ\Phi_{k-1}\circ\cdots\circ\Phi_J \\ &= \Phi_{J-1}\circ\cdots\circ\Phi_1\circ\Psi\circ\Phi_1^{-1}\circ\cdots\circ\Phi_{J-1}^{-1}. \end{aligned}$$

In other words, they are related to each other and Ψ by conjugation:

$$\Upsilon_1 = \Psi, \qquad \Upsilon_{J+1} = \Phi_J\circ\Upsilon_J\circ\Phi_J^{-1}.$$

These maps are important because they describe the sequences $\eta(J)$, i.e., it easily can be verified that

$$\eta(J)_{i+1} = \Upsilon_J(\eta(J)_i).$$

The maps Υ_J also have the following important properties:

$$\begin{aligned} \Upsilon_J(0) &= \Psi(0) = 0 \\ \Upsilon_J'(0) &= \Psi'(0) = \left(\prod_{q=1}^{k}\frac{a_q^2}{a_q^1}\right). \end{aligned}$$

All of the aforementioned properties can be summarized by noting that we have the following lemma.

Lemma 5.1. *If*

$$\lim_{i\to\infty} z_i = 0 \qquad \Rightarrow \qquad \lim_{i\to\infty}\eta(J)_i = 0,$$

for all $J \in \{1,\ldots,k\}$.

Proof. We will reason by induction on J. For the case when $J = 1$, by assumption:

$$\lim_{i\to\infty}\eta(1)_i = \lim_{i\to\infty} z_i = 0.$$

Now assume that $\lim_{i\to\infty} \eta(J-1)_i = 0$, and note that

$$\begin{aligned} \eta(J)_i &= \xi_{ki+J-1} \\ &= \Phi_{\rho(ki+J-2)}(\xi_{ki+J-2}) \\ &= \Phi_{J-1}(\xi_{ki+J-2}) \\ &= \Phi_{J-1}(\eta(J-1)_i). \end{aligned}$$

Therefore,

$$\lim_{i\to\infty} \eta(J)_i = \lim_{i\to\infty} \Phi_{J-1}(\eta(J-1)_i) = \Phi_{J-1}(0) = 0.$$

□

This lemma indicates that in studying the behavior of the hybrid system $\mathfrak{H}_{\mathbf{DFQ}}$, one can study the behavior of the sequence $z = \{z_i\}_{i\in\mathbb{N}}$. Moreover, analyzing the behavior of this sequence is more manageable since it is determined by a discrete time system. We thus can apply the theory of discrete time systems to hybrid systems. This motivates the following definition.

Definition 5.3. The **discrete time dynamical system associated to the hybrid system** $\mathfrak{H}_{\mathbf{DFQ}}$ is given by

$$z_{i+1} = \Psi(z_i),$$

where $\Psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is as defined in (5.2).

Note that the discrete time system given by $z_{i+1} = \Psi(z_i)$ has an isolated equilibrium point at the origin: $\Psi(0) = 0$. It also is interesting to note that this system is linear in the case when $\lambda_q^1 = \lambda_q^2 = 0$. To see this, note that in this case we have the discrete time linear system

$$z_{i+1} = \Psi(z_i) = \left(\prod_{q=1}^{k} \frac{a_q^2}{a_q^1} \right) z_i.$$

The startling fact is that the stability of the map Ψ in the general case will be directly related to the stability of this linear system. We will derive results relating the properties of this function, specifically its stability, to Zeno behavior.

5.3.c Sufficient Conditions for Zeno Behavior

Studying the discrete time dynamical system associated to a dynamical system, we are able to obtain easily verifiable conditions on the existence of Zeno behavior in **DFQ** hybrid systems.

5.3.8 Discrete time exponential stability. Recall that a discrete dynamical system, $z_{i+1} = \Psi(z_i)$, is exponentially stable at the origin if there exist constants $c > 0$ and $0 \le \alpha < 1$ such that

$$|z_i| \le c\alpha^i |z_0|.$$

We can derive conditions on when the discrete dynamical system associated to a **DFQ** hybrid system is stable—at least when it satisfies Assumption 5.1.

Theorem 5.1. *Let $\mathfrak{H}_{\mathbf{DFQ}}$ be a* **DFQ** *hybrid system and* Γ *be a cycle of the underlying graph* Γ *of this hybrid system satisfying Assumption 5.1. Then the discrete dynamical system*

$$z_{i+1} = \Psi(z_i)$$

associated to $\mathfrak{H}_{\mathbf{DFQ}}$ and Γ is exponentially stable at the origin if

$$\left| \prod_{q=1}^{k} \frac{a_q^2}{a_q^1} \right| < 1.$$

Proof. The result follows from the Hartman-Grobman theorem (cf. [101]) after suitably extending the map Ψ to the entire real numbers. □

Theorem 5.2. *Let $\mathfrak{H}_{\mathbf{DFQ}}$ be a* **DFQ** *hybrid system satisfying Assumption 5.1. Then if $\Lambda_q x + a_q$, $q \in Q$, satisfies the conditions:*

$$\left.\begin{array}{c} \lambda_q^1 \leq 0 \\ \\ a_q^1 < 0 < a_q^2 \\ \\ \left| \prod_{q=1}^{k} \frac{a_q^2}{a_q^1} \right| < 1 \end{array}\right\} \quad \Rightarrow \quad \mathfrak{H}_{\mathbf{DFQ}} \text{ is Zeno.}$$

Proof. Let $\chi = (\mathbb{N}, I, \rho, C)$ be the execution constructed in Paragraph 5.3.3. The goal is to show that the series

$$\sum_{i=0}^{\infty} (\tau_{i+1} - \tau_i)$$

converges. To do this, we will consider subsequences of the sequence $\{\tau_{i+1} - \tau_i\}_{i \in \mathbb{N}}$. Namely, recall from the definition of the execution and the sequences $\eta(J)$ that

$$\begin{aligned} \sum_{i=0}^{\infty} (\tau_{i+1} - \tau_i) &= \sum_{i=0}^{\infty} \Delta t_{\rho(i)}(\xi_i) \\ &= \sum_{i=0}^{\infty} \sum_{J=1}^{k} \Delta t_{\rho(ki+J-1)}(\xi_{ki+J-1}) \\ &= \sum_{J=1}^{k} \sum_{i=0}^{\infty} \Delta t_J(\eta(J)_i). \end{aligned}$$

Therefore, we need show only that $\sum_{i=0}^{\infty} \Delta t_J(\eta(J)_i)$ converges for each J. First, it can be seen that

$$\Delta t_J(0) = 0, \qquad \Delta t_J'(0) = \frac{-1}{a_J^1}.$$

Now our assumptions imply that the sequence $z = \{z_i\}_{i \in \mathbb{N}}$ is exponentially stable to the origin, i.e., for all $J \in \{1, \ldots, k\}$,

$$\lim_{i \to \infty} z_i = 0 \qquad \Rightarrow \qquad \lim_{i \to \infty} \eta(J)_i = 0,$$

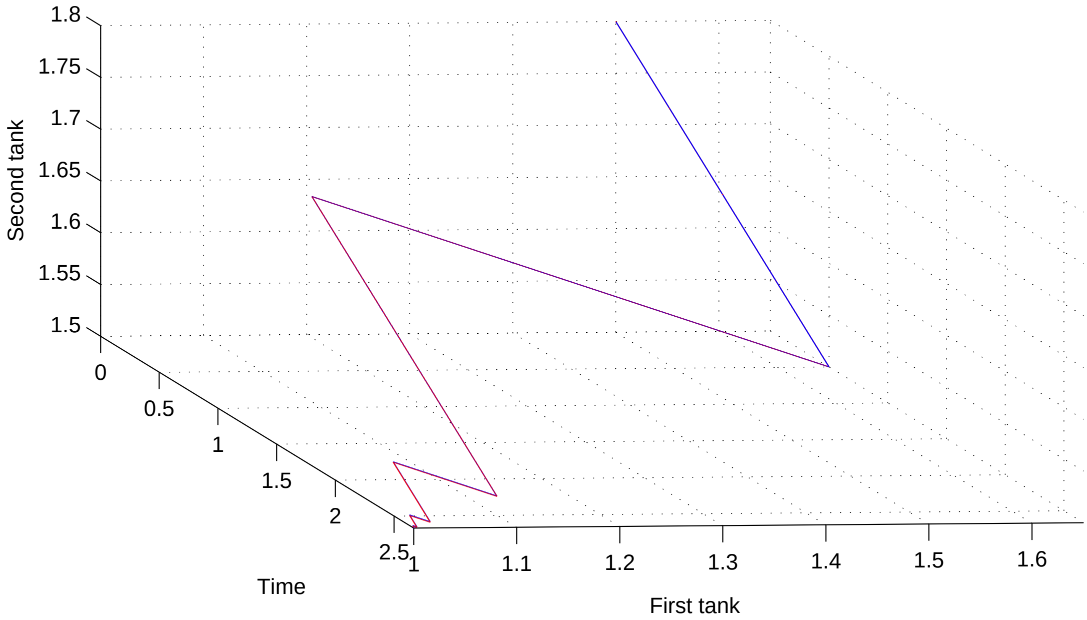


Figure 5.5: A simulated trajectory of the two tank system given in Example 5.4.

by Lemma 5.1. Applying the ratio test for each J, we have

$$\begin{aligned}
\lim_{i\to\infty}\left|\frac{\Delta t_J(\eta(J)_{i+1})}{\Delta t_J(\eta(J)_i)}\right| &= \lim_{i\to\infty}\left|\frac{\Delta t_J(\Upsilon_J(\eta(J)_i))}{\Delta t_J(\eta(J)_i)}\right| \\
&= \lim_{x\to 0}\left|\frac{\Delta t_J(\Upsilon_J(x))}{\Delta t_J(x)}\right| \\
&= \lim_{x\to 0}\left|\frac{\Delta t_J'(x)\Upsilon_J'(x)}{\Delta t_J'(x)}\right| \\
&= \left|\frac{\Delta t_J'(0)}{\Delta t_J'(0)}\Upsilon_J'(0)\right| \\
&= \left|\prod_{q=1}^{k} -\frac{a_q^2}{a_q^1}\right| < 1.
\end{aligned}$$

Or

$$\sum_{i=0}^{\infty}\Delta t_J(\eta(J)_i)$$

converges for each J and hence $\sum_{i=0}^{\infty}(\tau_{i+1}-\tau_i)$, so $\mathfrak{H}_{\mathbf{DFQ}}$ is Zeno. □

Example 5.4. The two water tanks hybrid system as introduced in Example 2.2 is a classic example of a hybrid system that displays Zeno behavior; see Figure 5.5 for a simulated trajectory of this system. We will demonstrate how the conditions above allow us to verify that this hybrid system is Zeno *without explicitly solving for the vector fields.* First, we transform the hybrid system into a **DFQ** hybrid system by “flipping" the dynamics on one of the domains. The graph and domains are the same as the ones introduced in

Example 2.2. In accordance with the fact that it should be a **DFQ** hybrid system, the guards are now the upper half of the x_2-axis. Therefore, to complete the description of this system, we need only specify the vector fields on each domain. These are given by:

$$X_1(x) = \begin{pmatrix} -v_2 \\ w - v_1 \end{pmatrix}, \qquad X_2(x) = \begin{pmatrix} -v_1 \\ w - v_2 \end{pmatrix}.$$

Here, again, $w > 0$ is the inflow of water into the system, and $v_1 > 0$ and $v_2 > 0$ are the outflows of water from each tank. Recall that we made the assumption that

$$\max\{v_1, v_2\} < w < v_1 + v_2.$$

Under these conditions, we would like to verify that this hybrid system is Zeno.

Applying Theorem 5.2 to this system we conclude that the system is Zeno because:

$$\lambda_1^1 = \lambda_1^2 = 0, \qquad -v_2, -v_1 < 0, \qquad w - v_1, w - v_2 > 0,$$

and

$$\frac{(w-v_1)(w-v_2)}{v_1 v_2} < \frac{((v_1+v_2)-v_1)((v_1+v_2)-v_2)}{v_1 v_2} = 1$$

because $w < v_1 + v_2$.

5.4 Stability of Zeno Equilibria

The purpose of this section is to study the stability of a type of equilibria that is unique to hybrid systems: Zeno equilibria. The uniqueness of these equilibria necessitates a paradigm shift in the current notions of stability, i.e., we must introduce a type of stability that is both local and global in nature and, therefore, has no direct analogue in continuous and discrete systems. The main result of this section is sufficient conditions for the stability of Zeno equilibria in general hybrid systems.

5.4.a Classical Stability: A Categorical Approach

In this section we revisit classical stability theory under a categorical light. The new perspective afforded by category theory is more than a simple exercise in abstract nonsense—it motivates the development of an analogous stability theory for hybrid systems and hybrid equilibria.

Remarkably, stability also can be described through the existence of certain morphisms. Let us first recall the definition of globally asymptotically stable equilibria; for more on the stability of dynamical systems see [70] and [101].

Notation 5.2. For all dynamical systems (M, X) considered in this section, we assume that M is a subset of $\mathbb{R}^n$. Thus we can write expressions like "$\|x - y\|$" without ambiguity. Alternatively, we could assume that M is a Riemannian manifold.

Definition 5.4. Let (M,X) be an object of Dyn. An equilibrium point $x^* \in M$ of X is said to be **globally asymptotically stable** when for any morphism $c : ([t,\infty), \mathrm{d}/\mathrm{dt}) \to (M,X)$, for any $t_1 \geq t$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying:

1. $\|c(t_1) - x^*\| < \delta \quad \Rightarrow \quad \|c(t_2) - x^*\| < \varepsilon \qquad \forall\, t_2 \geq t_1 \geq t,$
2. $\lim_{\tau\to\infty} c(\tau) = x^*$.

Consider now the full subcategory of Dyn, denoted by GasDyn, with objects $(\mathbb{R}_0^+, -\alpha)$ where α is a class $\mathcal{K}_\infty$ function, i.e., α is strictly increasing satisfying $\alpha(0) = 0$ and $\alpha(x) \to \infty$ as $x \to \infty$. Lyapunov's second method (see [70], Theorem 3.8, page 138) can then be described as follows:

Theorem 5.3. *Let (M,X) be an object of Dyn. An equilibrium point $x^* \in M$ of X is globally asymptotically stable if there exists a morphism:*

$$(M,X) \xrightarrow{v} (\mathbb{R}_0^+, -\alpha) \in \mathsf{GasDyn}$$

in Dyn satisfying:

1. *$v(x) = 0$ implies $x = x^*$,*
2. *$v : M \to \mathbb{R}_0^+$ is a proper (radially unbounded) function.*

The previous result suggests that the study of stability properties can be carried out in two steps. In the first step we identify a suitable subcategory having the desired stability properties. In the case of global asymptotic stability, this subcategory is GasDyn; for local stability we could consider the full subcategory defined by objects of the form $(\mathbb{R}_0^+, -\alpha)$ with α a non-negative definite function. The chosen category corresponds in some sense to the simplest possible objects having the desired stability properties. In the second step we show that existence of a morphism from a general object (M,X) to an object in the chosen subcategory implies that the desired stability properties also hold in (M,X). This is precisely the approach we will develop for the study of Zeno equilibria.

5.4.b Zeno Equilibria

We now proceed to study Zeno equilibria. It is important to note that we do not claim that Zeno equilibria are the most general form of equilibria corresponding to Zeno behavior. We do claim that the type of Zeno equilibria considered are general enough to cover a wide range of interesting (and somewhat peculiar) behavior, while being specific enough to allow for analysis.

Definition 5.5. Let $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ be a hybrid system. A **Zeno equilibria** is a pair $(\mathscr{Z}, \vec{z})$, where

- ⋄ $\mathscr{Z}$ is a D-subcategory of $\mathscr{M}$ such that $\mathrm{grph}(\mathscr{Z})$ is a directed cycle,
- ⋄ $\vec{z} = \{\vec{z}_a\}_{a\in \mathsf{Ob}(\mathscr{Z})}$ such that
  - – $\vec{z}_a \in \mathbf{M}_a$ for all $a \in \mathsf{Ob}(\mathscr{Z})$,

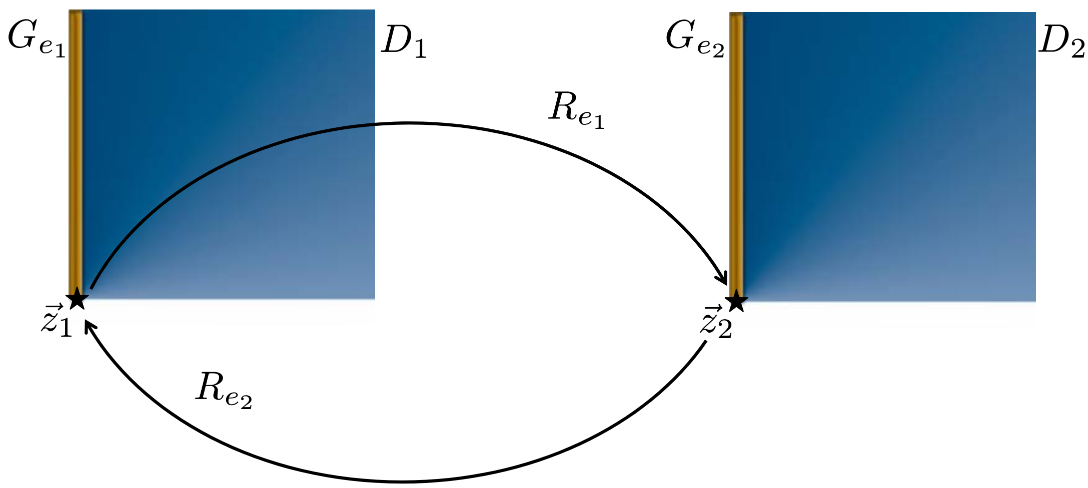


Figure 5.6: Zeno equilibria for a **FQ** hybrid system.

– $\vec{z}_b = \mathbf{M}_\alpha(\vec{z}_a)$ for all $\alpha : a \to b$ in $\mathscr{Z}$,

– $\mathbf{X}_a(\vec{z}_a) \neq 0$ for all $a \in \mathrm{Ob}(\mathscr{Z})$.

5.4.1 Another interpretation of Zeno equilibria. There is a more categorical definition of a Zeno equilibria. Starting with the one point set $*$, we obtain a hybrid manifold $(\mathscr{Z}, \Delta_{\mathscr{Z}}(*))$. Denoting by $\vec{\mathrm{In}} : \mathscr{Z} \to \mathscr{M}$ the inclusion functor, a Zeno equilibria is a morphism of hybrid manifolds:

$$(\vec{\mathrm{In}}, \vec{z}) : (\mathscr{Z}, \Delta_{\mathscr{Z}}(*)) \to (\mathscr{M}, \mathbf{M})$$

such that $\mathbf{X}_a(\vec{z}_a) \neq 0$.

To see that these definitions are equivalent, by slight abuse of notation denote $\vec{z}_a(*) := \vec{z}_a$. Now, the first condition, $\vec{z}_a \in \mathbf{M}_a$ for all $a \in \mathrm{Ob}(\mathscr{Z})$, follows trivially. The second condition is implied by the following diagram, which must commute

$$\begin{array}{ccc} & & \mathbf{M}_a \\ & \overset{\vec{z}_a}{\nearrow} & \\ * & & \downarrow \mathbf{M}_\alpha \\ & \underset{\vec{z}_b}{\searrow} & \\ & & \mathbf{M}_b \end{array}$$

for every $\alpha : a \to b$ in $\mathscr{Z}$.

Example 5.5. For the hybrid system $\mathfrak{H}_{\mathbf{FQ}}$, and since we are assuming that the underlying graph is a cycle, the conditions expressed in Definition 5.5 imply that a set $\vec{z} = \{\vec{z}_1, \ldots, \vec{z}_k\}$ is a Zeno equilibria if for all $i = 1, \ldots, k$, $\vec{z}_i \in G_{e_i}$, $X_i(\vec{z}_i) \neq 0$ and

$$R_{e_{i-1}} \circ \cdots \circ R_{e_1} \circ R_{e_k} \circ \cdots \circ R_{e_i}(\vec{z}_i) = \vec{z}_i. \tag{5.3}$$

Because of the special structure of $\mathfrak{H}_{\mathbf{FQ}}$, (5.3) holds iff $\vec{z}_i = 0$ for all i. That is, the only Zeno equilibria of $\mathfrak{H}_{\mathbf{FQ}}$ is the singleton set $\vec{z} = \{0\}$. A Zeno equilibria for a **FQ** hybrid system is pictured in Figure 5.6. Note that the formulation of Zeno equilibria for **FQ** hybrid systems implies that for $\vec{z} = \{0\}$ to be a Zeno equilibria, it must follow that $r_e(0) = 0$ for all $e \in E$ (otherwise (5.3) is not well-defined).

5.4.2 Induced hybrid subsystems. Let $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ be a hybrid system, $\mathscr{Z}$ be a D-subcategory of $\mathscr{M}$, and $\vec{\mathrm{In}} : \mathscr{Z} \to \mathscr{M}$ be the inclusion functor. In this case, there is a hybrid subsystem $(\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}})$ of $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ corresponding to this inclusion, i.e.,

$$\mathbf{M}^{\mathscr{Z}} = \vec{\mathrm{In}}^*(\mathbf{M}), \qquad \mathbf{X}^{\mathscr{Z}} = \{\mathbf{X}_a\}_{a \in \mathsf{V}(\mathscr{Z})},$$

and there is an inclusion in HySys:

$$(\vec{\mathrm{In}}, \vec{\mathrm{id}}) : (\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}}) \hookrightarrow (\mathscr{M}, \mathbf{M}, \mathbf{X})$$

where $\vec{\mathrm{id}}$ is the identity natural transformation.

Definition 5.6. A D-subcategory $\mathscr{Z}$ of a D-category $\mathscr{M}$ is said to be a **locally attracting D-cycle** if $\mathrm{grph}(\mathscr{Z})$ is a directed cycle and

$$\mathsf{cod}(\mathsf{s}_{a_1}) = \vec{\mathrm{In}}(b) = \mathsf{cod}(\mathsf{s}_{a_2}) \quad \Rightarrow \quad a_1 = a_2,$$

for all $b \in \mathsf{V}(\mathscr{Z})$.

Definition 5.7. Let $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ be a hybrid system. A Zeno equilibria $(\mathscr{Z}, \vec{z})$ of $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ is **globally asymptotically stable relative to** $(\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}})$ if $\mathscr{Z}$ is a locally attracting D-cycle and for every trajectory:

$$(\vec{C}, \vec{c}) : (\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \to (\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}}),$$

with $\Lambda = \mathbb{N}$, and for any $\varepsilon_{\vec{C}(j)} > 0$, there exists $\delta_{\vec{C}(i)} > 0$ such that:

1. If $\| \vec{c}_i(\tau_i) - \vec{z}_{\vec{C}(i)} \| < \delta_{\vec{C}(i)}$ for $i = 1, \ldots, k \in Q$ then

   $$\| \vec{c}_j(t) - \vec{z}_{\vec{C}(j)} \| < \varepsilon_{\vec{C}(j)}$$

   with $j \in \Lambda$ and $t \in \mathbf{I}_j = [\tau_j, \tau_{j+1}]$.

2. For all $a \in \mathsf{V}(\mathscr{Z})$

   $$\lim_{\substack{j \to \infty \\ \vec{C}(j) = a}} \vec{c}_j(\tau_j) = \vec{z}_a, \qquad \lim_{\substack{j \to \infty \\ \vec{C}(j) = a}} \vec{c}_j(\tau_{j+1}) = \vec{z}_a.$$

We say that a Zeno equilibria $(\mathscr{Z}, \vec{z})$ of $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ is **globally asymptotically stable** if it is globally asymptotically stable relative to $(\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}})$ and $(\mathscr{M}, \mathbf{M}, \mathbf{X}) = (\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}})$.

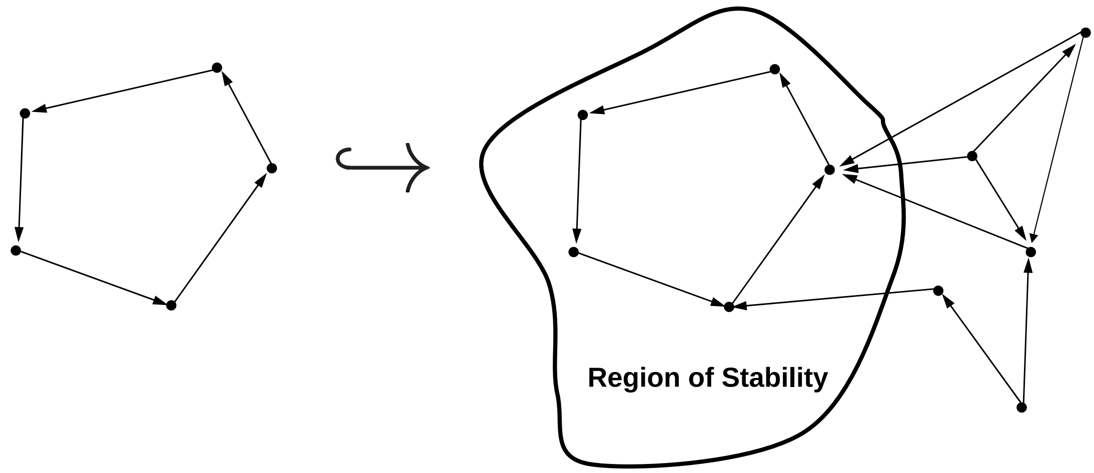


Figure 5.7: A graphical representation of the "local" nature of relatively globally asymptotically stable Zeno equilibria.

5.4.3 The global-local nature of Zeno equilibria. The definition of relative global asymptotic stability implicitly makes some very subtle points. The first is that this type of stability is both local and global in nature—hence the use of the words "global" and "relative" in the definition. While for traditional dynamical systems this would seem contradictory, the complexity of hybrid systems requires us to view stability in a much different light, i.e., we must expand the paradigm for stability.

To better explain the mixed global and local nature of relatively globally asymptotically stable Zeno equilibria, we note that the term "global" is used because the hybrid subsystem $(\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}})$ is globally stable to the Zeno equilibria; this also motivates the use of the word "relative" as $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ is stable relative to a hybrid subsystem. Finally, the local nature of this form of stability is in the discrete portion of the hybrid system, rather than the continuous one. That is, the D-subcategory $\mathscr{Z}$ can be thought of as a neighborhood inside the D-category $\mathscr{M}$ (see Figure 5.7, where the D-categories $\mathscr{Z}$ and $\mathscr{M}$ are represented by graphs in order to make their orientations explicit). The condition on the inclusion functor given in the definition is a condition that all edges (or morphisms) are pointing into the neighborhood.

5.4.4 Zeno equilibria and FQ hybrid systems. Zeno equilibria are intimately related to Zeno behavior for first quadrant hybrid systems. While this relationship is established, it is useful to have a graphical representation of the convergence to a Zeno equilibria in a **FQ** hybrid system; this can be found in Figure 5.8.

Proposition 5.2. *If a first quadrant hybrid system $\mathfrak{H}_{\mathbf{FQ}}$ is globally asymptotically stable at the Zeno equilibria $\vec{z} = \{0\}$, then every trajectory with $\Lambda = \mathbb{N}$ is Zeno.*

Proof (sketch). For simplicity, we will assume that Γ consists of a single vertex and a single edge; considering larger cycles would amount to repeating the same argument on each vertex with subsequences of the switching times. Because of asymptotic convergence of the sequence of initial conditions, $\vec{c}_i(\tau_i)$, there

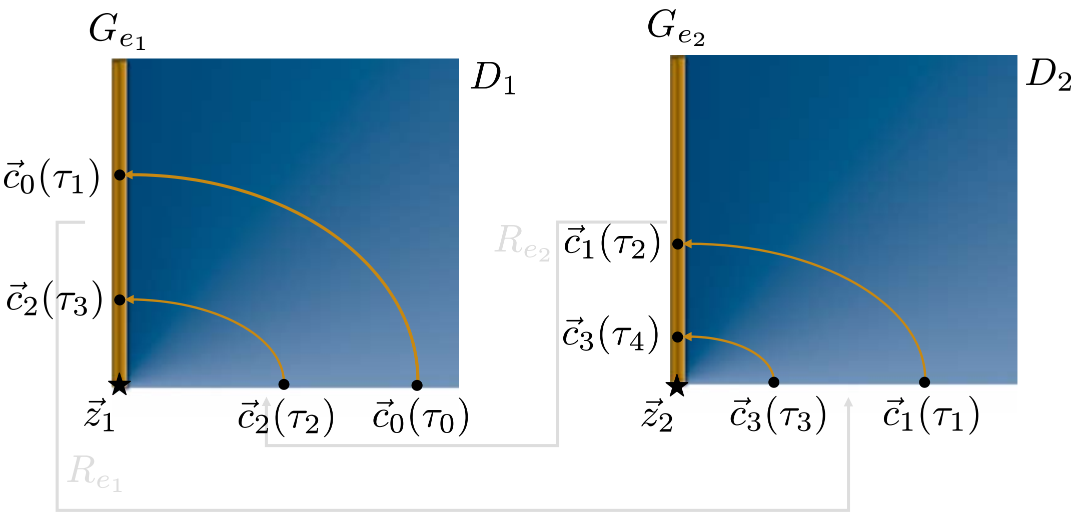


Figure 5.8: Convergence to a Zeno equilibria.

exists a neighborhood U in D_q, of the Zeno equilibria $\{0\}$, and $K \in \mathbb{N}$ such that $\vec{c}_i(t) \in U$ for all $i \geq K$ and $t \in \mathbf{I}_i$. By the assumption that $X_q(0) \neq 0$, we can apply the straightening out theorem (cf. [95]), wherein the new coordinates X_q becomes:

$$\hat{X}_q(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We now pick two rays, r_1 and r_2, emanating from the origin and such that their convex hull, $C = \text{conv}(r_1, r_2)$, contains the transformed U; if U is sufficiently small, the angle between the two rays is less than π. The time difference $\tau_{i+1} - \tau_i$, $i \geq K$, is less than the height of a vertical line, l, intersecting C and passing through the transformed $\vec{c}_i(\tau_i)$. Consider the triangle:

$$T = \text{conv}(0, r_1 \cap l, r_2 \cap l),$$

and note that:

$$\text{Area}(T) \geq \sum_{i=K}^{\infty} \tau_{i+1} - \tau_i$$

since the set of all switching time differences $\{\tau_{i+1} - \tau_i\}_{i \in \mathbb{N}}$ is not dense while the set of all vertical slices of T is (after all, they have different cardinalities).

By Cauchy's condition on the convergence of series, for any $\epsilon > 0$, we can always make U sufficiently small so that $\text{Area}(T) < \epsilon$. This implies that

$$\sum_{i=K}^{\infty} \tau_{i+1} - \tau_i < \epsilon.$$

Since the sum $\sum_{i=0}^{K-1} \tau_{i+1} - \tau_i$ is finite, we conclude that

$$\sum_{i=0}^{\infty} \tau_{i+1} - \tau_i < \infty.$$

□

5.4.c Conditions on Stability

We now give conditions on the stability of Zeno equilibria for first quadrant hybrid systems. These will be instrumental in providing conditions on the stability of Zeno equilibria for general hybrid systems.

5.4.5 Conditions for the stability of $\mathfrak{H}_{\mathbf{FQ}}$. In order to give conditions on the stability of Zeno equilibria, it is necessary to give conditions on both the continuous and discrete portions of the hybrid system. That is, the conditions on stability will relate to three aspects of the behavior of the hybrid system: the continuous portion, the existence of events and the discrete portion.

Continuous conditions: For all $i \in Q$,

(**I**) $X_i(x) \neq 0$ for all $x \in (\mathbb{R}_0^+)^2$.

(**II**) There exists a function $v_i : (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+$ of class $\mathcal{K}_\infty$ along each ray emanating from the origin in D_i and $d(v_i)_x X_i(x) \leq 0$ for all $x \in (\mathbb{R}_0^+)^2$.

Event conditions: For all $i \in Q$,

(**III**) $(X_i(x_1, 0))_2 \geq 0$.

Now consider the map ψ_i defined by requiring that:

$$\psi_i(x) = y \quad \text{if} \quad (0, y) = v_i^{-1}(v_i(x, 0)) \cap \{x_1 = 0 \text{ and } x_2 \geq 0\}$$

which is well-defined by condition (**II**). Using ψ_i we introduce the function $P_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by:

$$P_i(x) = r_{e_{i-1}} \circ \psi_{i-1} \circ \cdots \circ r_{e_1} \circ \psi_{e_1} \circ r_{e_k} \circ \psi_{e_k} \circ \cdots \circ r_{e_1} \circ \psi_1(x).$$

The map P_i can be thought of as both a Poincaré map or a discrete Lyapunov function depending on the perspective taken. The final conditions are given by:

Discrete conditions: For all $i \in Q$ and $e \in E$,

(**IV**) r_e is order preserving.

(**V**) There exists a class $\mathcal{K}_\infty$ function α such that $P_i(x) - x \leq -\alpha(x)$.

Theorem 5.4. *A first quadrant hybrid system $\mathfrak{H}_{\mathbf{FQ}}$ is globally asymptotically stable at the Zeno equilibria $\vec{z} = \{0\}$ if conditions* (**I**) – (**V**) *hold.*

Proof. Proving this result requires us to show three things: the existence of events, the boundedness of trajectories and asymptotic convergence of the sequence of initial conditions.

We begin by demonstrating the existence of events; if $c(0)$ is an initial condition of a solution $c(t)$ of $\dot{c} = X_q(c)$ on $D_q = (\mathbb{R}_0^+)^2$, then there exists an event if $c(\tau) \in G_{e_q}$ for some finite τ, i.e., if $(c(\tau))_1 = 0$ and $(c(t))_2 \geq 0$ for all $t \in [0, \tau]$. To show that there exists an event, we must rule out two other possible scenarios:

1. The trajectory $c(t)$ exits D_q in finite time without intersecting the guard. In particular, since $D_q = (\mathbb{R}_0^+)^2$, this implies that there must exist a τ such that $(c(\tau))_2 < 0$ and $(c(t))_1 > 0$ for $t \in [0,\tau]$.

2. The trajectory never exits D_q, i.e., $c(t) \in D_q \backslash G_{e_q}$ for all t.

We will show that conditions (**I**) – (**III**) exclude these scenarios.

Beginning with scenario 1, if $(c(\tau))_2 < 0$, since $(c(0))_2 \geq 0$, it follows by continuity that there exists a $t \in [0,\tau]$ such that $(c(t))_2 = 0$ and $(X_q(c(t)))_2 < 0$ for $(c(t))_1 > 0$. But these inequalities are ruled out by condition (**III**).

For scenario 2, we first note that condition (**II**) implies that

$$|v_q^{-1}(v_q(y)) \cap \{x_1 = 0 \text{ and } x_2 \geq 0\}| = 1 \tag{5.4}$$

$$|v_q^{-1}(v_q(y)) \cap \{x_2 = 0 \text{ and } x_1 \geq 0\}| = 1 \tag{5.5}$$

for all $y \in D_i$. Now define the set

$$V_q = \{x \in D_q : v_q(x) \leq v_q(c(0))\}.$$

Note that the boundary of this set consists of three smooth components, i.e., a piece of the vertical axis (the guard), a piece of the horizontal axis and a piece of the level set $v_q^{-1}(v_q(c(0)))$. We have already shown that the solution cannot exit through the horizontal axis, and it cannot exit through the piece of the level set (intersection D_q) since we are supposing that $d(v_q)_x X_q(x) \leq 0$ by condition (**II**). In view of this, if the trajectory exits the set V_q, then it must exit through the guard, i.e., there must exist an event. Therefore, we must show that the trajectory exits this set, i.e., that V_q is not an invariant set.

We are assuming that V_q is proper. Suppose, by way of contradiction, that V_q is an invariant set. By the Poincaré-Bendixson theorem, because we are assuming that V_q contains no equilibrium points (condition (**I**)), it must contain a limit cycle: $\gamma(t)$ with $\gamma(t) = \gamma(t+T)$. First suppose that $v_q(\gamma(t)) \equiv c$, then by condition (**II**) (and more specifically (5.4)), $\gamma(t)$ must intersect the vertical and horizontal axis and because this is a one-dimensional limit cycle in two dimensional space it has an orientation. Therefore, the assumption that $\gamma(t)$ is contained in V_q is violated. Alternatively, suppose that $v_q(\gamma(t)) = c$ and $v_q(\gamma(t')) = c'$ for $c > c'$ and some $t < t' < T$. By the periodicity of the $\gamma(t)$, this implies that $v_q(\gamma(t')) < v_q(\gamma(t+T)) = v_q(\gamma(t)) = c$, where $t + T > t'$, which implies that $d(v_q)_{\gamma(t)} X_q(\gamma(t)) \geq 0$ for some $t \in [t', t+T]$, which violates condition (**II**).

We conclude that events must always exist.

Let us now address the issue of asymptotic convergence of the sequence of initial conditions. Recalling the construction of $(\mathscr{M}^{\mathrm{FQ}}, \mathbf{M}^{\mathrm{FQ}}, \mathbf{X}^{\mathrm{FQ}})$ from $\mathfrak{H}_{\mathbf{FQ}}$, for a trajectory

$$(\mathscr{I}, \mathbf{I}, \mathbf{d/dt}) \xrightarrow{(\vec{C},\vec{c})} (\mathscr{M}^{\mathrm{FQ}}, \mathbf{M}^{\mathrm{FQ}}, \mathbf{X}^{\mathrm{FQ}}),$$

we need to show that for all $a \in \bigvee(\mathscr{M}^{\mathrm{FQ}})$

$$\lim_{\substack{j\to\infty \\ \vec{C}(j)=a}} \vec{c}_j(\tau_j) = 0. \tag{5.6}$$

Define $\Psi_i(x,0) = (0, \psi_i(x))^T$. Starting from $i > 1$, $\vec{c}_i(\tau_i) \in \{x_1 \geq 0 \text{ and } x_2 = 0\}$ so

$$v_{\vec{C}(i)}(\vec{c}_i(\tau_{i+1})) \leq v_{\vec{C}(i)}(\vec{c}_i(\tau_i)) = v_{\vec{C}(i)}(\Psi_{\vec{C}(i)}(\vec{c}_i(\tau_i)))$$

from condition (**II**). Also by this condition, we can take the inverse of $v_{\vec{C}(i)}$ because it is class $\mathcal{K}_\infty$ along every ray emanating from the origin (which intersects D_q) and hence along the vertical axis, which implies that

$$(\vec{c}_i(\tau_{i+1}))_2 \leq \psi_{\vec{C}(i)}((\vec{c}_i(\tau_i))_1).$$

Applying r_e to this inequality, which by condition (**IV**) is order preserving, yields

$$r_{e_i}((\vec{c}_i(\tau_{i+1}))_2) \leq r_{e_i}(\psi_{\vec{C}(i)}((\vec{c}_i(\tau_i))_1)).$$

By iterating this process k times, and noting that $\vec{C}(i+k) = \vec{C}(i)$, we have

$$(\vec{c}_{i+k}(\tau_{i+k}))_1 \leq P_{\vec{C}(i)}((\vec{c}_i(\tau_i))_1).$$

Therefore, the sequence $(\vec{c}_{i+k}(\tau_{i+k}))_1$ is bounded by the sequence $P_{\vec{C}(i)}((\vec{c}_i(\tau_i))_1)$ which converges to z_a in view of (**V**). Convergence of $\vec{c}_i(\tau_{i+1})$ now follows from convergence of $\vec{c}_i(\tau_i)$ and the fact that $\psi_i(\vec{c}_i(\tau_i)) = \vec{c}_i(\tau_{i+1})$ is an order preserving function as v_i is of class $\mathcal{K}_\infty$ when restricted to the guard of domain D_i.

Finally, boundedness of the continuous trajectories in each mode (and hence boundedness of the sequence of initial conditions) follows from the existence of the Lyapunov function $v_{\vec{C}(i)}$, i.e., by the conditions given in (**II**), and by the fact that the map $P_{\vec{C}(i)}$ satisfies condition (**V**). □

Corollary 5.2. *If $\mathfrak{H}_{\mathbf{FQ}}$ is a first quadrant hybrid system satisfying conditions* (**I**) – (**V**), *then there exist trajectories with $\Lambda = \mathbb{N}$ and every such trajectory is Zeno.*

Note that the condition that $\Lambda = \mathbb{N}$ in Proposition 5.2 and Corollary 5.2 is due to the fact that there always are trajectories with finite indexing set Λ, e.g., any trajectory with $\Lambda = \mathbb{N}$ has "sub-trajectories" with finite indexing sets. These trajectories are trivially non-Zeno, so we necessarily rule them out.

Example 5.6. To verify that $\mathfrak{H}^{\text{ball}}_{\mathbf{FQ}}$ is globally asymptotically stable at the Zeno point $\vec{z} = \{0\}$, and hence Zeno by Proposition 5.2, we need only show that conditions (**I**) – (**V**) are satisfied. It is easy to see that conditions (**I**) and (**III**) are satisfied (see Figure 5.9). Since $r_{e_1}(x) = x$ and $r_{e_2}(x) = ex$, condition (**IV**) holds. We use the original Hamiltonian

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + mgx_1,$$

suitably transformed, for the Lyapunov type functions given in (**II**), i.e., we pick:

$$v_1(x_1, x_2) = \frac{1}{2}x_1^2 + gx_2, \qquad v_2(x_1, x_2) = \frac{1}{2}x_2^2 + gx_1.$$

It is easy to see that these functions meet the specifications given in (**II**); some of the level sets of these functions can be seen in Figure 5.9. Note that the level sets on one domain increase, but this is compensated for by the decreasing level sets on the other domain. Finally, condition (**V**) is satisfied when $e < 1$ since

$$P_1(x) = P_2(x) = ex.$$

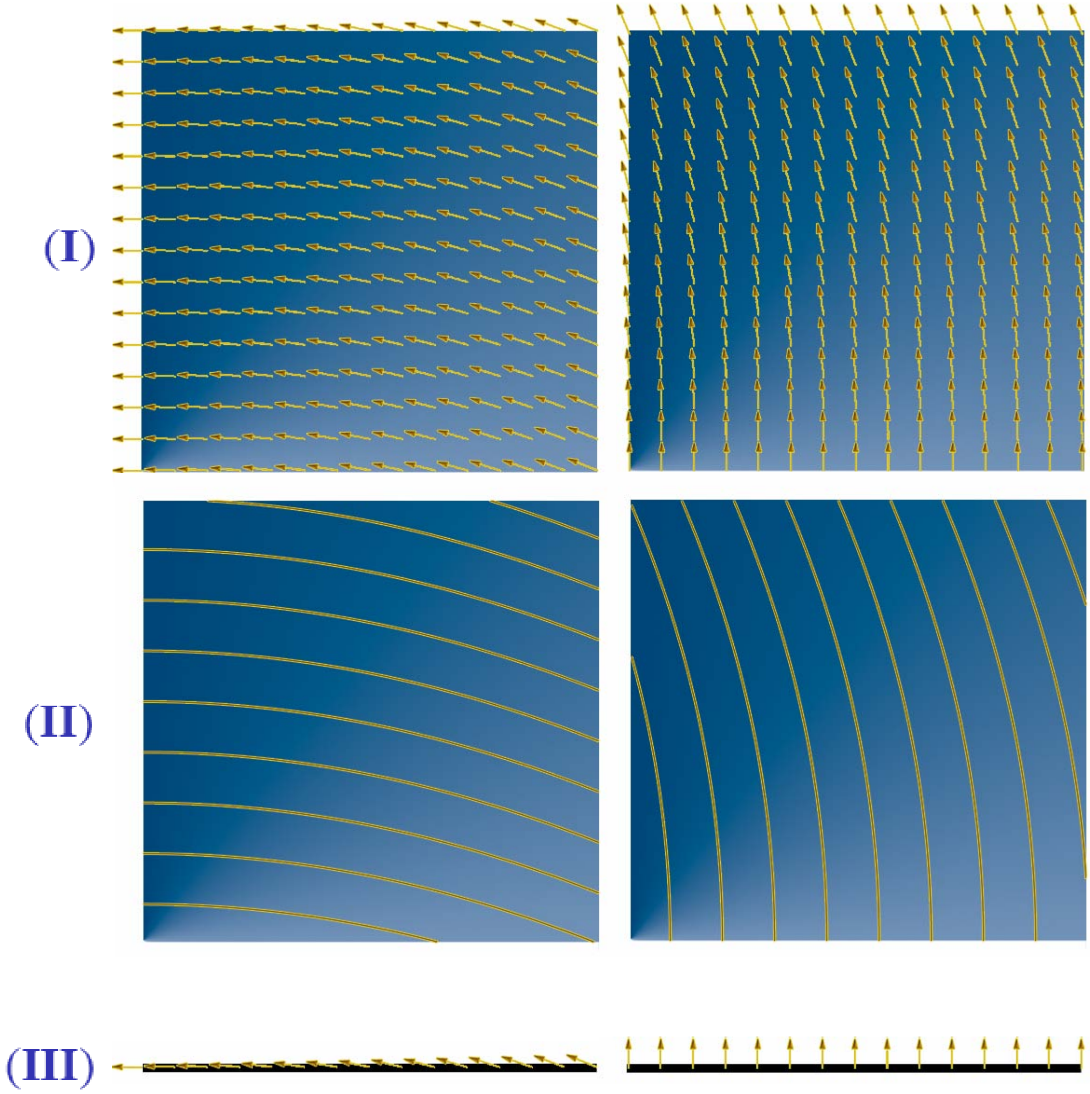


Figure 5.9: Graphical verification of the properties (**I**) – (**III**) for the bouncing ball hybrid system for the first domain (left) and the second domain (right).

A graphical representation of the maps P_1 and P_2 can be seen in Figure 5.10.

5.4.d Hybrid Stability Theory

Building upon the results of the previous section, we are able to derive sufficient conditions for the stability of general hybrid systems. Mirroring the continuous case, we simply find a morphism to the "simplest stable object," i.e., a first quadrant hybrid system. Formally:

Definition 5.8. Define GasZeno be the full subcategory of HySys consisting of first quadrant hybrid systems $(\mathscr{M}^{\mathrm{SFQ}}, \mathbf{M}^{\mathrm{SFQ}}, \mathbf{X}^{\mathrm{SFQ}})$ that are globally asymptotically stable at the Zeno equilibria $\vec{z} = \{0\}$.

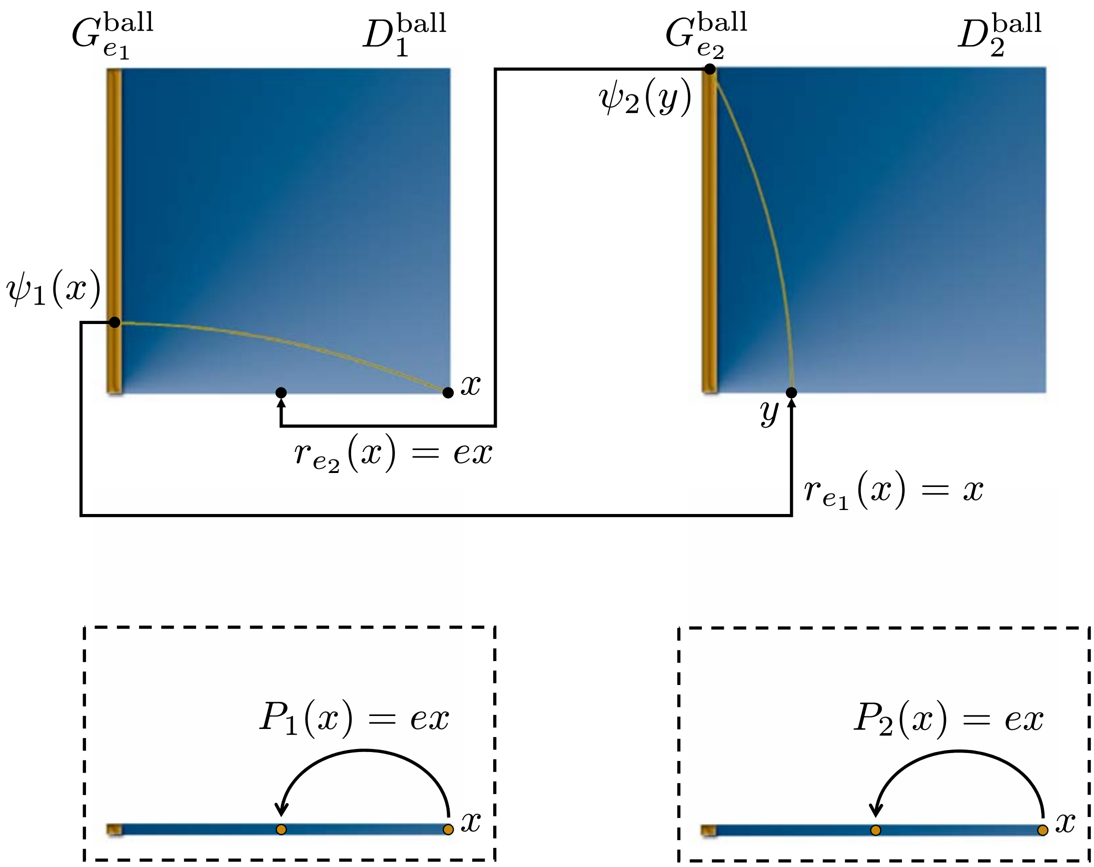


Figure 5.10: A graphical illustration of the construction of the functions P_1 and P_2 for the bouncing ball.

Theorem 5.5. *A Zeno equilibria* $(\mathscr{Z}, \vec{z})$ *of* $(\mathscr{M}, \mathbf{M}, \mathbf{X})$ *is globally asymptotically stable relative to* $(\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}})$ *if there exists a morphism of hybrid systems:*

$$(\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}}) \xrightarrow{(\vec{V}, \vec{v})} (\mathscr{M}^{\text{SFQ}}, \mathbf{M}^{\text{SFQ}}, \mathbf{X}^{\text{SFQ}}) \in \mathsf{GasZeno}$$

in HySys *satisfying, for all* $a \in \mathrm{Ob}(\mathscr{Z})$,

1. $\vec{v}_a(x) = 0$ *implies* $x = \vec{z}_a$,
2. $\vec{v}_a$ *is a proper (radially unbounded) function.*

An immediate, and very important, corollary of this theorem is that it yields sufficient conditions for the existence of Zeno behavior.

Corollary 5.3. *Under the assumptions of Theorem 5.5, there exist trajectories*

$$(\mathscr{S}, \mathbf{I}, \mathbf{d/dt}) \xrightarrow{(\vec{C}, \vec{c})} (\mathscr{Z}, \mathbf{M}^{\mathscr{Z}}, \mathbf{X}^{\mathscr{Z}})$$

with $\Lambda = \mathsf{V}(\mathscr{S}) = \mathbb{N}$ *and every such trajectory is Zeno.*

Of course, a trajectory of $(\mathscr{Z},\mathbf{M}^{\mathscr{Z}},\mathbf{X}^{\mathscr{Z}})$ immediately yields a trajectory of $(\mathscr{M},\mathbf{M},\mathbf{X})$ through composition. That is,

$$(\mathscr{I},\mathbf{I},\mathbf{d/dt}) \xrightarrow{(\vec{C},\vec{c})} (\mathscr{Z},\mathbf{M}^{\mathscr{Z}},\mathbf{X}^{\mathscr{Z}}) \xrightarrow{(\vec{\text{In}},\vec{\text{id}})} (\mathscr{M},\mathbf{M},\mathbf{X})$$

is a trajectory of $(\mathscr{M},\mathbf{M},\mathbf{X})$. Therefore, we have given conditions for the existence of Zeno trajectories for $(\mathscr{M},\mathbf{M},\mathbf{X})$.

Proof of Theorem 5.5. Let

$$(\mathscr{I},\mathbf{I},\mathbf{d/dt}) \xrightarrow{(\vec{C},\vec{c})} (\mathscr{Z},\mathbf{M}^{\mathscr{Z}},\mathbf{X}^{\mathscr{Z}})$$

be a morphism where $(\mathscr{I},\mathbf{I},\mathbf{d/dt}) \in \mathsf{Interval}(\mathsf{HySys})$ and $\Lambda = \bigvee(\mathscr{I}) = \mathbb{N}$. Assume that

$$\|\vec{c}_i(\tau_i) - \vec{z}_{\vec{C}(i)}\| \le \delta_{\vec{C}(i)}$$

for $i = 1,2,\ldots,k$. This implies that $\vec{c}_i(\tau_i)$ is contained in a compact set containing $\vec{z}_{C(i)}$. If we denote such set by $S_{\vec{C}(i)}$, it follows by continuity of $\vec{v}_{\vec{C}(i)}$ that $\vec{v}_{\vec{C}(i)}(S_{\vec{C}(i)})$ is compact and that $\vec{v}_{\vec{C}(i)}(\vec{c}_i(\tau_i)) \in \vec{v}_{\vec{C}(i)}(S_{\vec{C}(i)})$ and $\vec{v}_{\vec{C}(i)}(\vec{z}_{\vec{C}(i)}) \in \vec{v}_{\vec{C}(i)}(S_{\vec{C}(i)})$. Taking into account that $\vec{v}_{\vec{C}(i)}(\vec{z}_{\vec{C}(i)}) = 0$ we obtain:

$$\|\vec{v}_{\vec{C}(i)} \circ \vec{c}_i(\tau_i) - 0\| \le \delta'_{\vec{V}\circ\vec{C}(i)}.$$

It now follows from global asymptotic stability of the Zeno equilibria $(\mathscr{M}^{\text{SFQ}}, \vec{z} = \{0\})$ of $(\mathscr{M}^{\text{SFQ}},\mathbf{M}^{\text{SFQ}},\mathbf{X}^{\text{SFQ}})$ that:

$$\|\vec{v}_{\vec{C}(j)} \circ \vec{c}_j(t) - 0\| \le \varepsilon'_{\vec{V}\circ\vec{C}(j)}, \qquad t \in \mathbf{I}_j,\ j \in \Lambda.$$

This inequality implies that $\vec{v}_{\vec{C}(j)} \circ \vec{c}_j(t)$ belongs to a compact set for all $t \in \mathbf{I}_j$. Because $\vec{v}_j$ is assumed to be proper, there exists a $\varepsilon_{\vec{C}(j)}$ such that:

$$\|\vec{c}_j(t) - \vec{z}_{\vec{C}(j)}\| \le \varepsilon_{\vec{C}(j)}, \qquad t \in \mathbf{I}_j,\ j \in \Lambda.$$

A similar argument yields, by the continuity of $\vec{v}_{\vec{C}(j)}$, for all $a \in \bigvee(\mathscr{Z})$ with $\vec{C}(j) = a$,

$$\lim_{\substack{j\to\infty \\ \vec{C}(j)=a}} \vec{c}_j(\tau_j) = \vec{v}^{-1}_{\vec{C}(j)} \left(\lim_{\substack{j\to\infty \\ \vec{V}\circ\vec{C}(j)=\vec{V}(a)}} \vec{v}_{\vec{C}(j)} \circ \vec{c}_j(\tau_j) \right) = \vec{v}^{-1}_{\vec{C}(j)}(0) = \vec{z}_a.$$

□

Theorem 5.5 is an important result in many respects.

- ◇ It would have been very difficult to obtain without our categorical framework for hybrid systems. This indicates that categorical hybrid systems theory can yield results that are important, interesting and novel.
- ◇ It yields a general method for studying the stability of hybrid systems. That is, any other type of more "classical" stability that might be of interest—asymptotic stability, exponential stability, etc.—can be studied using the same methodology. In fact, these types of stability are *easier* to study; this is why we opted to study Zeno equilibria. For other types of stability, the "simplest" stable objects will be one-dimensional hybrid systems with the desired stability property. These extensions follow in a trivial manner from the framework established here.

5.5 Going Beyond Zeno

Motivated by the fact that for a Zeno trajectory time never progresses past a certain time (the *Zeno time*) and "point" (the *Zeno point(s)*), a natural—and intriguing—question to ask is:

What happens after a Zeno point?

Inspired by the construction of [50], we propose a method for extending Zeno executions past a Zeno point for Lagrangian hybrid systems as were introduced in Chapter 3; for the sake of simplicity and consistency, we revert to the notation utilized in that chapter.

Using the special structure of Lagrangian hybrid systems obtained from hybrid Lagrangians, we are able to demonstrate that the Zeno point must satisfy constraints imposed by the unilateral constraint function. These constraints are *holonomic* in nature, and this implies that *after* the Zeno point the hybrid system should switch to a holonomically constrained dynamical system. The resulting system obtained by "composing" the hybrid system with this dynamical system defines a *completed hybrid system*, which inherently allows an execution to continue past the Zeno point. Although we do not *prove* that this is the correct way to carry executions beyond Zeno points, we argue that our method correctly represents the physical, post-Zeno, behavior of the system being modeled. In order to substantiate this argument, we discuss how to practically implement a completed hybrid system and illustrate these concepts with a series of examples.

5.5.1 Lagrangian hybrid systems and their executions. Here we quickly recall the notation from Chapter 3, which will be in force throughout this section. First, recall from Definition 3.1 that a simple hybrid Lagrangian is given by a tuple:

$$\mathbf{L} = (Q, L, h).$$

Associated to a hybrid Lagrangian is a simple hybrid system,

$$\mathfrak{H}_{\mathbf{L}} = (D_{\mathbf{L}}, S_{\mathbf{L}}, R_{\mathbf{L}}, X_{\mathbf{L}}),$$

as constructed in Subsection 3.2.a; see Figure 5.11 for a graphical representation of a Lagrangian hybrid system.

An execution of a Lagrangian hybrid system, which we referred to as a *hybrid flow* in order to avoid confusion, is a tuple

$$\epsilon^{\mathbf{L}} = (\Lambda, I, C)$$

as introduced in Paragraph 3.2.1. We will revert to the terminology "hybrid flow" to again differentiate executions of this form from the ones considered in the rest of this chapter. In addition, we utilize the superscript "**L**" to indicate that the hybrid flow is the hybrid flow of the Lagrangian hybrid system associated to a hybrid Lagrangian.

We now introduce the notion of a Zeno point, which can be thought of as a form of Zeno equilibria for Lagrangian hybrid systems.

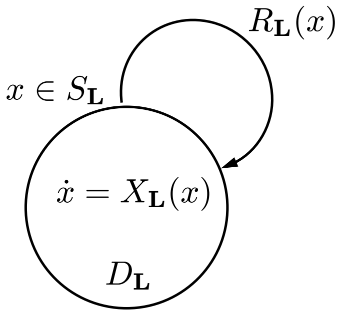


Figure 5.11: A Lagrangian hybrid system: $\mathfrak{H}_{\mathbf{L}}$.

Definition 5.9. The **Zeno point** of a Zeno hybrid flow $\epsilon^{\mathbf{L}}$ is defined to be

$$x_\infty = (q_\infty, \dot{q}_\infty) = \lim_{i\to\infty} c_i(\tau_i) = \lim_{i\to\infty} (q_i(\tau_i), \dot{q}_i(\tau_i)).$$

Here $c_i = (q_i, \dot{q}_i) \in C$, and the Zeno point is necessarily a single point because of the specific problem formulation considered in this section.

5.5.2 Zeno points. Lagrangian hybrid systems display both chattering and genuinely Zeno behavior; roughly speaking, the coefficient of restitution can be used to differentiate between these systems. Moreover, Zeno points must satisfy certain constraints based on the unilateral constraint function. In order to do so, let

$$A(q) = \begin{pmatrix} \frac{\partial h}{\partial q_1}(q) & \cdots & \frac{\partial h}{\partial q_n}(q) \end{pmatrix}.$$

Now, recalling the distinction between types of Zeno behavior made in Paragraph 5.1.3, we make the following observations:

CZ: If $\mathfrak{H}_{\mathbf{L}}$ has a chattering Zeno hybrid flow, $\epsilon^{\mathbf{L}}$, then $\tau_\infty = \tau_1 - \tau_0$ and $x_\infty = (q_1(\tau_1), \dot{q}_1(\tau_1))$ with $h(q_1(\tau_1)) = 0$ and $A(q_1(\tau_1))\dot{q}_1(\tau_1) = 0$.

GZ: If $\mathfrak{H}_{\mathbf{L}}$ has a genuinely Zeno hybrid flow, then $0 < e < 1$. Moreover, if $\epsilon^{\mathbf{L}}$ is a genuinely Zeno hybrid flow, then $x_\infty = (q_\infty, \dot{q}_\infty)$ is a point with $h(q_\infty) = 0$, and $A(q_\infty)\dot{q}_\infty = 0$.

Summarizing, our main observation is:

Main Observation. *If $\epsilon^{\mathbf{L}}$ is a Zeno hybrid flow of a Lagrangian hybrid system $\mathfrak{H}_{\mathbf{L}}$, then the Zeno point $x_\infty = (q_\infty, \dot{q}_\infty)$ is a point satisfying $h(q_\infty) = 0$ and $A(q_\infty)\dot{q}_\infty = 0$.*

This observation indicates how the system should behave *after* the Zeno point, i.e., it should satisfy a holonomic constraint. This holonomic constraint forces the system to slide along surface $h^{-1}(0) = \{q \in Q : h(q) = 0\}$. From this we argue that after the Zeno point, the hybrid system should switch to a holonomically constrained dynamical system.

5.5.3 Holonomically constrained systems. Recall that for a holonomically constrained system described by a Lagrangian, L, of the form given in (3.2), the equations of motion for the holonomically constrained system are obtained from the equations of motion for the unconstrained system (3.3); they are given by (cf. [93])

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) + A(q)^T\lambda = 0,$$

where λ is the Lagrange multiplier, which in this case is given by

$$\lambda = (A(q)M(q)^{-1}A(q)^T)^{-1}\left(\dot{A(q)}\dot{q} - A(q)M(q)^{-1}(C(q,\dot{q})\dot{q} + N(q))\right).$$

From the constrained equations of motion, for $x = (q,\dot{q})$, we get the vector field

$$\begin{aligned}\dot{x} &= X_L^{\infty}(x)\\ &= \left(\dot{q}, M(q)^{-1}(-C(q,\dot{q})\dot{q} - N(q) - A(q)^T\lambda)\right).\end{aligned}$$

Note that the X_L^{∞} defines a vector field on the manifold $TQ|_{h^{-1}(0)}$, from which we obtain the dynamical system

$$\mathfrak{D}_{\mathbf{L}}^{\infty} = (TQ|_{h^{-1}(0)}, X_L^{\infty}).$$

This, when coupled with the Main Observation, will be essential to understanding how to carry hybrid flows beyond Zeno points.

5.5.4 Completing hybrid systems. We begin by considering the case when $\mathfrak{H}_{\mathbf{L}}$ is a chattering Zeno hybrid system; in this case, the idea of carrying hybrid flows past the Zeno point has been well-studied. In [50], it is argued that once the solution hits the "switching surface" (or in our case, the guard), the solution should slide along the switching surface. In terms of Zeno points, this implies that before the Zeno point the dynamics should be dictated by X_L, while after the Zeno point the dynamics should be dictated by X_L^{∞}. We can generalize this construction to include genuinely Zeno Lagrangian hybrid systems.

Definition 5.10. If $\mathfrak{H}_{\mathbf{L}}$ is a Lagrangian hybrid system, we define the corresponding **completed hybrid system**[3] (or the *completion* of $\mathfrak{H}_{\mathbf{L}}$) as

$$\overline{\mathfrak{H}}_{\mathbf{L}} := \begin{cases} \mathfrak{D}_{\mathbf{L}}^{\infty} & \text{if} \quad h(q) = 0 \text{ and } A(q)\dot{q} = 0\\ \mathfrak{H}_{\mathbf{L}} & \qquad\qquad \text{otherwise.}\end{cases}$$

5.5.5 Completing trajectories. A completed hybrid system, as obtained from a Lagrangian hybrid system $\mathfrak{H}_{\mathbf{L}}$, can be seen in Figure 5.12. To make the definition of the completed system somewhat more transparent, some comments are in order. The Main Observation indicates that the only way for the transition to be made from the hybrid system $\mathfrak{H}_{\mathbf{L}}$ to the dynamical system $\mathfrak{D}_{\mathbf{L}}^{\infty}$ is if a specific Zeno hybrid flow reaches its Zeno point. Therefore, before the Zeno point, the Zeno system simply will be the hybrid

[3]This terminology (and notation) is borrowed from topology, where a metric space can be completed to ensure that "limits exist."

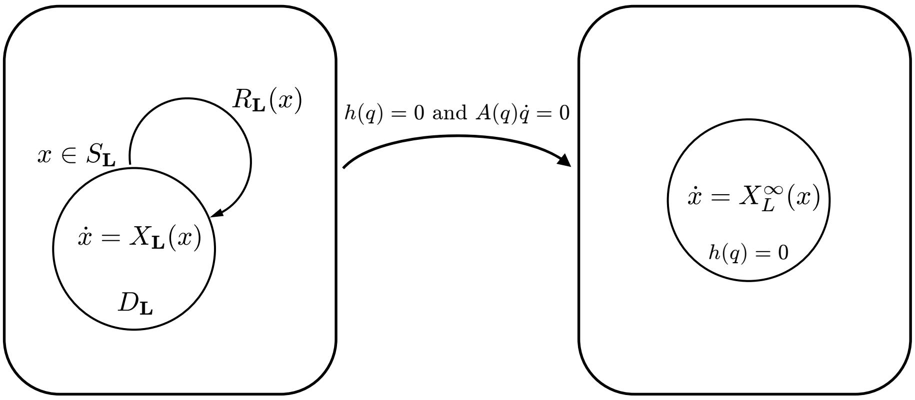


Figure 5.12: A completed hybrid system: $\overline{\mathfrak{H}}_{\mathbf{L}}$.

system $\mathfrak{H}_{\mathbf{L}}$, while after the Zeno point, the completed system will be the dynamical system $\mathfrak{D}_{\mathbf{L}}^{\infty}$. Since the dynamical system forces the system dynamics to be constrained to the manifold defined by $h^{-1}(0)$, this implies that the completed system will slide along the guard (switching surface) after the Zeno point.

This can be understood further on the level of hybrid flows. We can define a hybrid flow of the completed hybrid system $\overline{\mathfrak{H}}_{\mathbf{L}}$ by concatenating a Zeno hybrid flow of $\mathfrak{H}_{\mathbf{L}}$ with an integral curve of the dynamical system $\mathfrak{D}_{\mathbf{L}}^{\infty}$.

Specifically, let $\epsilon^{\mathbf{L}} = (\mathbb{N}, I, C)$ be a Zeno hybrid flow of $\mathfrak{H}_{\mathbf{L}}$. We obtain a hybrid flow of the completed hybrid system $\overline{\mathfrak{H}}_{\mathbf{L}}$ by defining it to be

$$\epsilon^{\overline{\mathfrak{H}}_{\mathbf{L}}} = (\mathbb{N} \cup \{\infty\}, \overline{I}, \overline{C}),$$

where

$$\overline{I} = I \cup \{I_{\infty} = [\tau_{\infty}, \infty)\}, \qquad \overline{C} = C \cup \{c_{\infty}\},$$

with $c_{\infty}(t)$ an integral curve of X_L^{∞} with initial condition the Zeno point:

$$c_{\infty}(\tau_{\infty}) = x_{\infty} = (q_{\infty}, \dot{q}_{\infty}).$$

We now discuss some practical issues related to simulating integral curves of completed systems.

5.5.6 Practical issues. We discuss two practical issues when modeling and simulating completed hybrid systems (see Figure 5.12). These issues are related to the transition from the left state $\mathfrak{H}_{\mathbf{L}}$ to the right state $\mathfrak{D}_{\mathbf{L}}^{\infty}$, and the corresponding initial conditions of $\mathfrak{D}_{\mathbf{L}}^{\infty}$. The theoretical framework established in this section allows us to justifiably surmount the practical problems introduced in simulation.

The first simulation issue is derived from the unavoidable numerical errors that result from the finite representation of values in a computer and truncation errors introduced by practical ODE solvers, i.e., a simulator produces an approximate hybrid flow $\widehat{\epsilon}^{\mathbf{L}}$ to the hybrid flow $\epsilon^{\mathbf{L}}$. Therefore, we cannot guarantee or expect an integral curve of reach the exact Zeno point $x_{\infty} = (q_{\infty}, \dot{q}_{\infty})$. Moreover, in order to reach the Zeno point, an infinite number of computation steps have to be performed (in a finite amount of time). Therefore, instead of resolving a solution that passes through the Zeno point exactly, we will compute an approximation of the Zeno solution; the approximated solution will pass through a neighborhood of the Zeno point, so we must modify the transition to the system $\mathfrak{D}^{\infty}_{\mathbf{L}}$ accordingly. Before discussing the details of the construction of the approximate solution, we first address the second modeling issue.

The other concern is the reinitialization of the new constrained system $\mathfrak{D}^{\infty}_{\mathbf{L}}$. In other words, after the transition to this system, we must give initial conditions for the constrained system. Theoretically, the initial condition is the Zeno point, but because in simulation we do not actually reach the Zeno point, an initial condition must be estimated—one that satisfies the same conditions as a Zeno point: $h(q_\infty) = 0$ and $A(q_\infty)\dot{q}_\infty = 0$.

5.5.7 Approximating completed systems. The approximation to the completed hybrid system $\overline{\mathfrak{H}}_{\mathbf{L}}$, denoted by $\overline{\mathfrak{H}}^{\delta}_{\mathbf{L}}$, is given by

$$\overline{\mathfrak{H}}^{\delta}_{\mathbf{L}} := \begin{cases} \mathfrak{D}^{\infty}_{\mathbf{L}} & \text{if} \quad \mathrm{abs}(h(q)) \leq \delta \text{ and } \mathrm{abs}(A(q)\dot{q}) \leq \delta \\ \mathfrak{H}_{\mathbf{L}} & \qquad\qquad \text{otherwise} \end{cases}$$

for some $\delta > 0$. When switching from $\mathfrak{H}_{\mathbf{L}}$ to $\mathfrak{D}^{\infty}_{\mathbf{L}}$ via the approximated guard condition, we use a map which resets the variables so that they satisfy the conditions of a Zeno point: $h(q_\infty) = 0$ and $A(q_\infty)\dot{q}_\infty = 0$. Specifically, for a point $(q, \dot{q})$ satisfying the approximate guard condition

$$\mathrm{abs}(h(q)) \leq \delta \text{ and } \mathrm{abs}(A(q)\dot{q}) \leq \delta,$$

we define a reset map R^∞ which sends $(q, \dot{q})$ to an approximate Zeno point, $(\widehat{q_\infty}, \widehat{\dot{q}_\infty}) = R^\infty(q, \dot{q})$, satisfying

$$h(\widehat{q_\infty}) = 0 \text{ and } A(\widehat{q_\infty})\widehat{\dot{q}_\infty} = 0.$$

We now briefly discuss how to construct the map R^∞ for the running examples in this section. In all of these examples, the vector fields for the constrained dynamical systems are easy to calculate.

Example 5.7. Again consider the hybrid system modeling a bouncing ball on a sinusoidal surface, $\mathfrak{H}_{\mathbf{B}}$, as first introduced in Example 3.5. In this example, the vector field for the holonomically constrained system $\mathfrak{D}^{\infty}_{\mathbf{B}}$ is given by

$$X^{\infty}_{\mathbf{B}}(q, \dot{q}) = \left(\dot{q}, \begin{pmatrix} 0 \\ \frac{2\cos(x_2)(-g+\sin(x_2)\dot{x}_2{}^2)}{3+\cos(2x_2)} \\ -\frac{2(g\cos(x_2)^2+\sin(x_2)\dot{x}_2{}^2)}{3+\cos(2x_2)} \end{pmatrix} \right).$$

Note that

$$\begin{aligned} h_{\mathbf{B}}(x) &= 0 \quad \Rightarrow \quad x_3 = \sin(x_2), \\ A_{\mathbf{B}}(x)\dot{x} &= 0 \quad \Rightarrow \quad \dot{x}_3 - \cos(x_2)\dot{x}_2 = 0. \end{aligned}$$

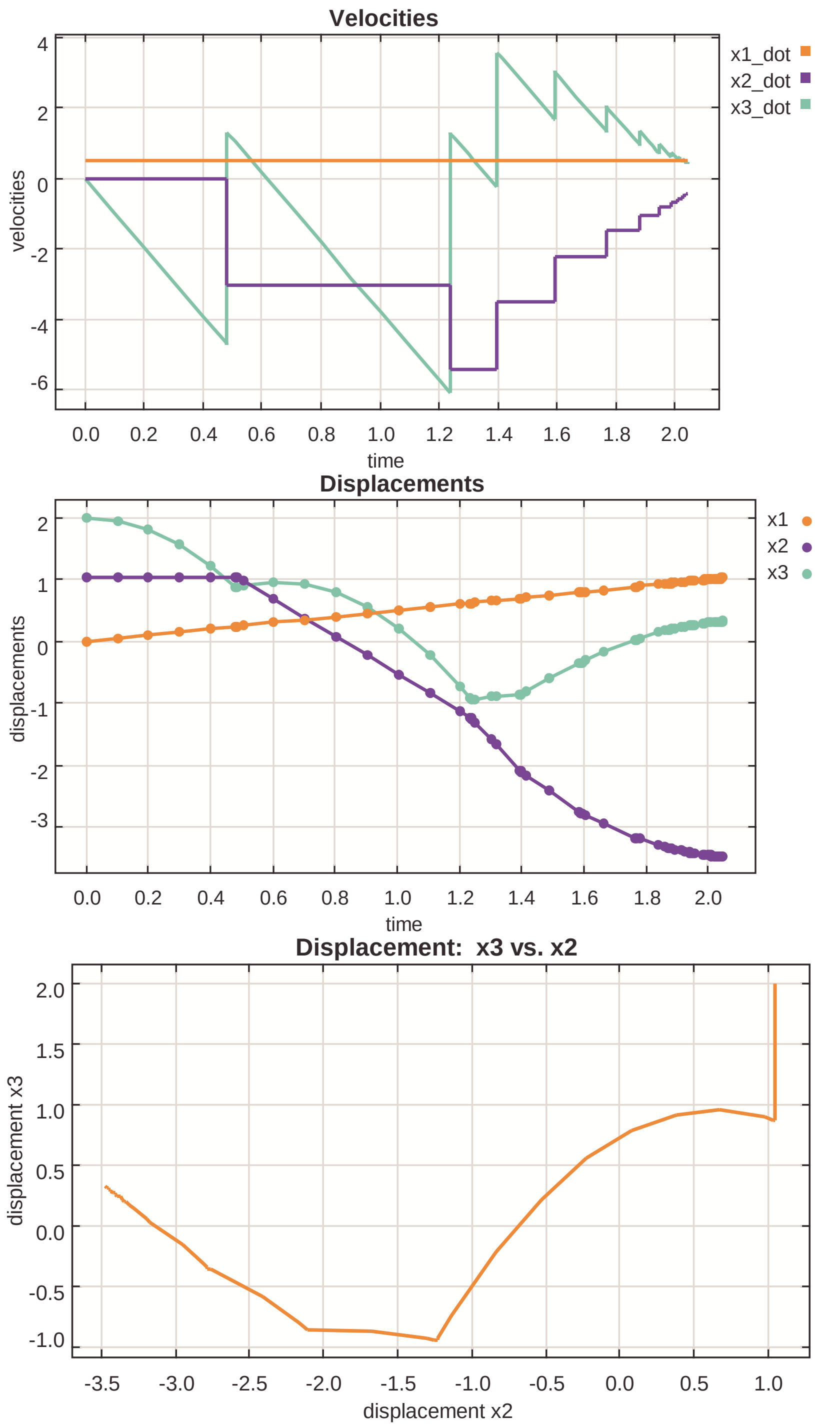


Figure 5.13: Simulation gets stuck at the Zeno point. Velocities over the time (top), displacements over the time (middle) and displacement on the x_3 direction vs. the displacement on the x_2 direction (bottom).

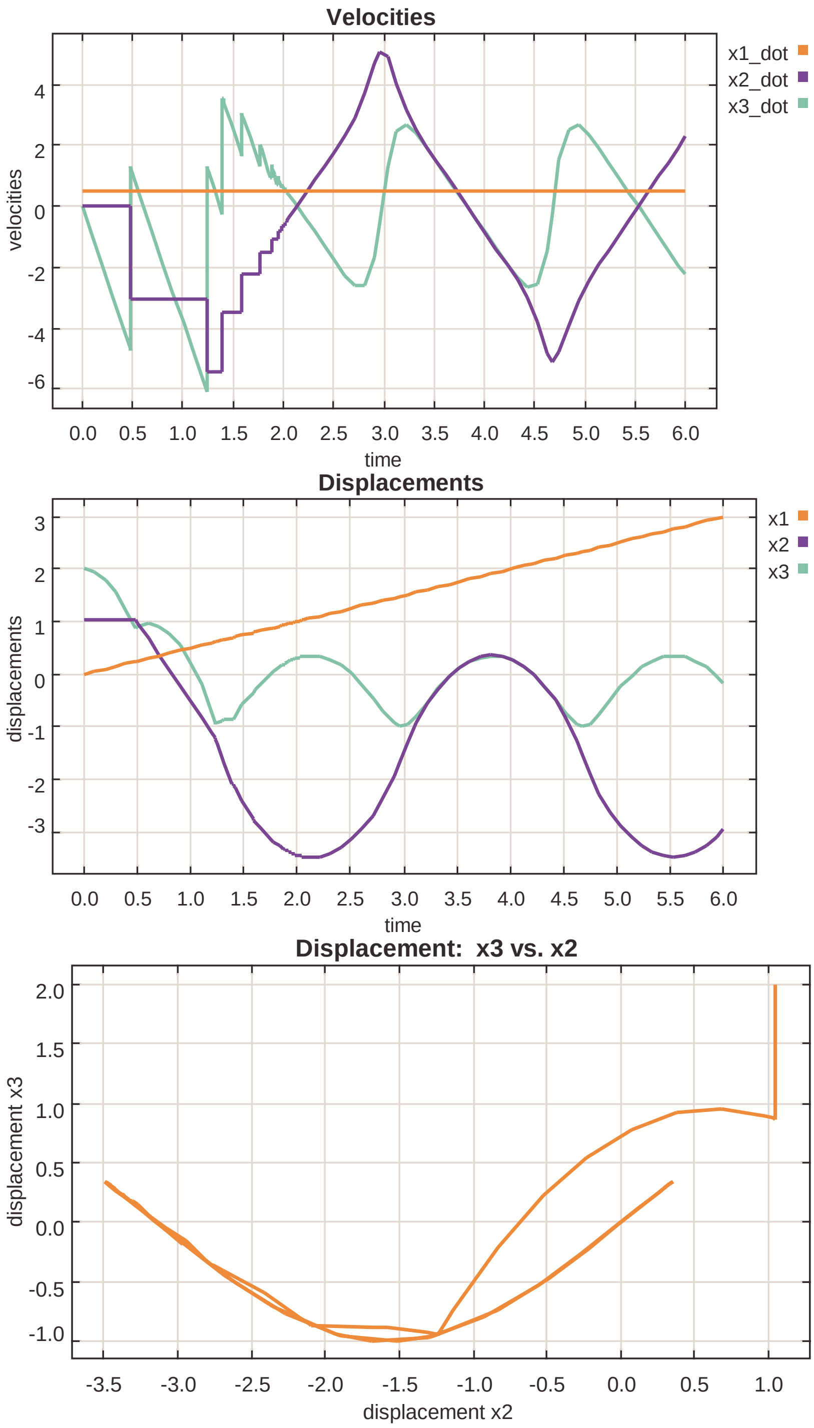


Figure 5.14: Simulation goes beyond the Zeno point. Velocities over the time (top), displacements over the time (middle) and displacement on the x_3 direction vs. the displacement on the x_2 direction (bottom).

And so we define the approximate reset map as:

$$R_{\mathbf{B}}^{\infty}(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3)=\left(\left(\begin{array}{c}x_1\\x_2\\\sin(x_2)\end{array}\right),\left(\begin{array}{c}\dot{x}_1\\\dot{x}_2\\\cos(x_2)\dot{x}_2\end{array}\right)\right).$$

We begin by simulating the "non-completed" hybrid system $\mathfrak{H}_{\mathbf{B}}$, the results of which are shown in Figure 5.13 (to which the rest of the paragraph refers). The simulation time is set to 6.0, but the simulation gets stuck at around 2.04; the bottom figure indicates that the ball tries, but is unable, to climb upwards along the surface (a sinusoidal waveform). Its velocities decrease during this process due to the energy loss through impact as can be seen in the top figure. As a consequence, more and more collisions are triggered and the time interval between two consecutive collisions keeps shrinking. The dense points near time 2.04 in the middle figure indicate that more and more computation steps are taken, which makes the simulation halt. This behavior is indicative of genuinely Zeno behavior.

Figure 5.14 shows a simulation of the completed hybrid system $\overline{\mathfrak{H}}_{\mathbf{B}}^{\delta}$ with the same initial conditions. Note that the simulation closely approaches the Zeno point before the behavior of the ball automatically switches to what $X_{\mathbf{B}}^{\infty}$ specifies, i.e., the ball oscillates along the surface (a sinusoidal waveform). Therefore, the simulation does not halt, freely moving beyond the Zeno point in a manner consistent with physical reality.

Example 5.8. Now consider the hybrid system modeling a pendulum on a cart, $\mathfrak{H}_{\mathbf{C}}$, as introduced in Example 3.6. The vector field for the holonomically constrained system $\mathfrak{D}_{\mathbf{C}}^{\infty}$ is given by

$$X_{\mathbf{C}}^{\infty}(q,\dot{q})=\left(\dot{q},\left(\begin{array}{c}-\cot(\theta)\dot{\theta}^2\\\frac{mR\csc(\theta)\dot{\theta}^2}{M+m}\end{array}\right)\right).$$

Note that

$$\begin{array}{rcl}h_{\mathbf{C}}(q)&=&0\quad\Rightarrow\quad\cos(\theta)=0\\A_{\mathbf{C}}(q)\dot{q}&=&0\quad\Rightarrow\quad\sin(\theta)\dot{\theta}=0.\end{array}$$

From the above two equations, we can compute precisely that $\theta=\mathrm{sign}(\theta)\pi/2$ and $\dot{\theta}=0$. The rest of the variables x and $\dot{x}$ have no extra constraints. Therefore, the complete reset map is:

$$R_{\mathbf{C}}^{\infty}(\theta,x,\dot{\theta},\dot{x})=\left(\left(\begin{array}{c}\mathrm{sign}(\theta)\pi/2\\x\end{array}\right),\left(\begin{array}{c}0\\\dot{x}\end{array}\right)\right).$$

Simulation verifies that the completed version of this hybrid system has the correct post-Zeno behavior; see Figure 5.15.

Example 5.9. As a final example, consider the hybrid system modeling a pendulum mounted on the floor, $\mathfrak{H}_{\mathbf{P}}$, as introduced in Example 3.7. Although this system was introduced as a Hamiltonian hybrid system, it can be converted without difficulty to a Lagrangian hybrid system through the Legendre transformation.

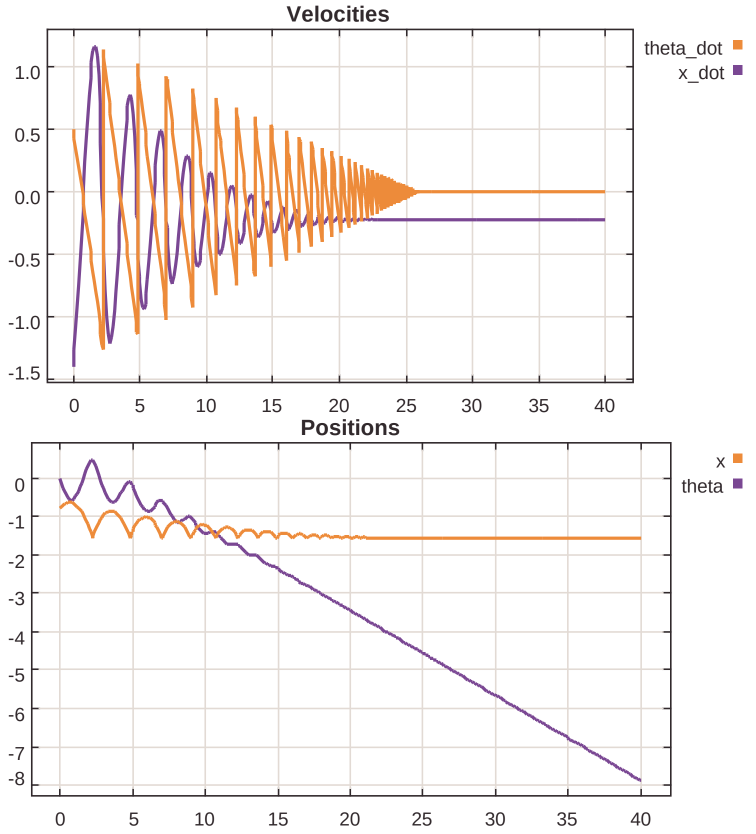


Figure 5.15: Simulation of the Cart that goes beyond the Zeno point. Velocities over the time (top) and displacements over the time (bottom).

In this example, the vector field for the holonomically constrained system $\mathfrak{D}_{\mathbf{P}}^{\infty}$ is given by

$$X_{\mathbf{P}}^{\infty}(q,\dot{q}) = \left(\dot{q}, \begin{pmatrix} -\cot(\theta)\dot{\theta}^2 \\ -2\cot(\theta)\dot{\theta}\dot{\varphi} \end{pmatrix}\right).$$

Note that

$$\begin{array}{rclcl} h_{\mathbf{P}}(q_{\infty}) & = & 0 & \Rightarrow & \cos(\theta) = 0 \\ A_{\mathbf{P}}(q_{\infty})\dot{q}_{\infty} & = & 0 & \Rightarrow & \sin(\theta)\dot{\theta} = 0. \end{array}$$

From the above two equations, we can calculate precisely that $\theta = \pi/2$ and $\dot{\theta} = 0$. Therefore, the complete reset map is:

$$R_{\mathbf{P}}^{\infty}(\theta,\varphi,\dot{\theta},\dot{\varphi}) = \left(\begin{pmatrix} \pi/2 \\ \varphi \end{pmatrix}, \begin{pmatrix} 0 \\ \dot{\varphi} \end{pmatrix}\right).$$

Simulation verifies that the completed version of this hybrid system has the correct post-Zeno behavior; See Figure 5.16.

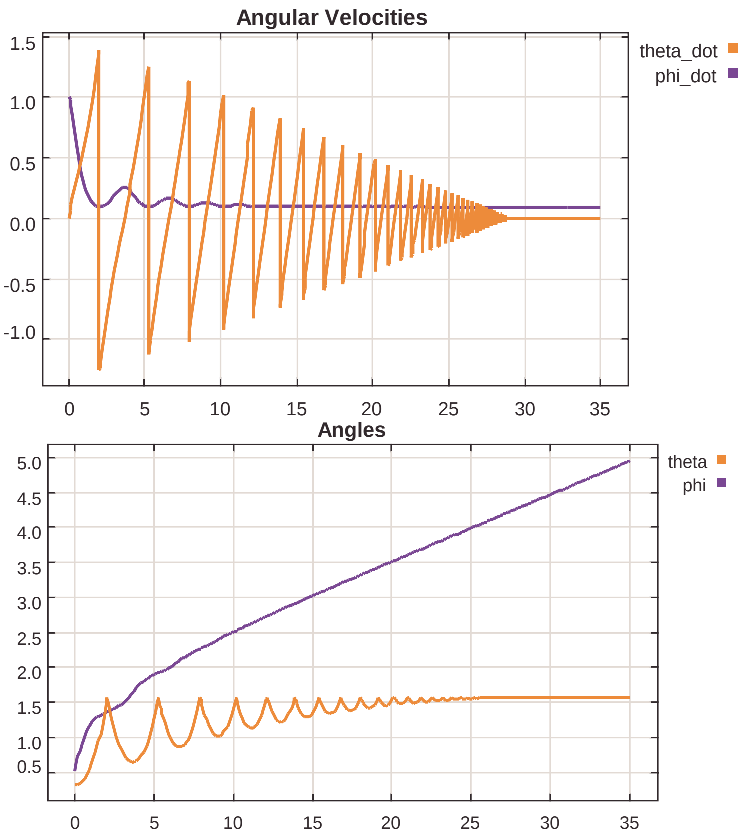


Figure 5.16: Simulation of the Pendulum goes beyond the Zeno point. Angular velocities over the time (top) and angles over the time (bottom).

Part III

Networked Systems

Chapter 6

Universally Composing Embedded Systems

In an embedded system, different components of the system evolve according to processes local to the specific components. Across the entire system, these typically heterogeneous processes may not be compatible, i.e., answering questions regarding the concurrency, timing and causality of the entire system—all of which are vital in the actual physical implementation of the system—can be challenging even if these questions can be answered for specific components. Denotational semantics provide a mathematical framework in which to study the behavior (signals, flows, executions, traces) of embedded systems or networks thereof. This framework is naturally applicable to the study of heterogeneous networks of embedded systems since signals always can be compared, regardless of the specific model of computation from which they were produced.

Tagged systems provide a denotational semantics for heterogeneous models of computation; they consist of a set of tags (a tag structure), variables and maps (behaviors) from the set of variables to the set of tags—hence, tagged systems are a specific case of the tagged signal model (cf. [76]). A heterogeneous network of embedded systems, e.g., a network consisting of both synchronous and asynchronous systems, can be modeled by a network of tagged systems with heterogeneous tag structures communicating through *mediator* tagged systems. Benveniste et al. [22], [23] and [24], introduced the notion of tagged systems and dealt with the issues we set forth in this chapter; this work extends and generalizes the ideas introduced in these papers. Of course, there is a wealth of literature on semantics preservation in heterogeneous networks, cf. [25], [40], [77], [97], and [117], the last of which approaches the problem from a categorical prospective.

A network of tagged systems can be implemented, or deployed, through heterogeneous parallel composition—obtained by taking the *conjunction* (intersection) of the behaviors that agree on the mediator tagged systems—which results in a single, homogeneous, tagged system. Thus, heterogeneous networks of tagged systems can be homogenized through the operation of composition. This chapter addresses the question:

When is semantics preserved by composition?

That is, when is the homogeneous tagged system obtained by composing a heterogeneous network of tagged systems semantically identical to the original network? Understanding this question is essential to understanding when networks of (possibly synchronous) embedded systems can be implemented asynchronously while preserving the semantics of the original system. Since implementing asynchronous systems is often more efficient (less overhead) when compared to the implementation of synchronous systems, deriving conditions on when this can be done while simultaneously preserving semantics would have many important implications.

In this chapter, taking a similar approach to Benveniste et al., we address the issue of semantics preservation. However, we use the formalism of *network objects* as introduced in Chapter 1, i.e., we introduce the category of tagged systems, TagSys, and demonstrate that a network of tagged systems corresponds to a network over the category TagSys. The result is necessary and sufficient conditions for semantics preservation.

Simple networks We begin by considering a network of two tagged systems $\mathscr{P}_1$ and $\mathscr{P}_2$ communicating through a mediator tagged system $\mathscr{M}$ as described by the diagram: $\mathscr{P}_1 \to \mathscr{M} \leftarrow \mathscr{P}_2$. The first contribution of this chapter is that we are able to show that the (classical notion of) heterogeneous composition of $\mathscr{P}_1$ and $\mathscr{P}_2$ over $\mathscr{M}$, $\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2$, is given by the *pullback* of this diagram:

$$\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2 = \mathscr{P}_1 \times_{\mathscr{M}} \mathscr{P}_2.$$

The importance of this result is that it implies that composition is endowed with a universal property; this universal property is fundamental in understanding when semantics is preserved. Consider the case when $\mathscr{P}_1$ and $\mathscr{P}_2$ have the same semantics, i.e., the same tag structure. Therefore, they always can communicate through the *identity mediator* tagged system, $\mathscr{I}$, and the homogeneous composition of $\mathscr{P}_1$ and $\mathscr{P}_2$, $\mathscr{P}_1 \| \mathscr{P}_2$, is given by the pullback $\mathscr{P}_1 \times_{\mathscr{I}} \mathscr{P}_2$ of the diagram: $\mathscr{P}_1 \to \mathscr{I} \leftarrow \mathscr{P}_2$. It is possible through this framework to give a precise statement of what it means to preserve semantics by composition over the mediator tagged system $\mathscr{M}$:

Semantics is preserved by composition if $\mathscr{P}_1 \| \mathscr{P}_2 \equiv \mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2$.

Through the universality of the pullback, we are able to give verifiable *necessary and sufficient conditions* on semantics preservation. A corollary of our result is the sufficient conditions on semantics preservation established by Benveniste et al..

General networks. A network of tagged systems is given by an oriented graph $\Gamma = (Q, E)$ together with a set of tagged systems $\mathscr{P} = \{\mathscr{P}_q\}_{q \in Q}$ communicating through a set of mediator tagged systems $\mathscr{M} = \{\mathscr{M}_e\}_{e \in E}$; that is, for every $e \in E$, there is a diagram in TagSys of the form:

$$\mathscr{P}_{\text{source}(e)} \xrightarrow{\alpha_e} \mathscr{M}_e \xleftarrow{\alpha'_e} \mathscr{P}_{\text{target}(e)}.$$

Equivalently, a network of tagged systems is given by a network over the category TagSys,

$$\mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)} : \mathfrak{N}_\Gamma \to \mathsf{TagSys},$$

where $\mathfrak{N}_\Gamma$ is a D^{op}-category obtained from Γ (see Section 1.5). As in the case of a network of two tagged systems, the heterogeneous composition of a network of tagged systems is given by the *limit* of this diagram,

$$\|_{\mathscr{M}} \mathscr{P} = \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)}),$$

and so composition again is defined by a universal property.

If all of the $\mathscr{P}_i'$*s* have the same semantics, then we can again consider the identity mediator $\mathscr{I}$ (which in this case is a set) for which there are associated diagrams in TagSys of the form:

$$\mathscr{P}_{\mathrm{source}(e)} \xrightarrow{\mathrm{Res}_e} \mathscr{M}_e \xleftarrow{\mathrm{Res}'_e} \mathscr{P}_{\mathrm{target}(e)}$$

with Res_e and Res'_e restriction maps. To this network of tagged systems there is a corresponding network object, $\mathbf{N}_{(\mathscr{P},\mathscr{I},\mathrm{Res})} : \mathfrak{N}_\Gamma \to \mathsf{TagSys}$, and the homogeneous composition of this network is given by

$$\| \mathscr{P} = \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{N}_{(\mathscr{P},\mathscr{I},\mathrm{Res})}).$$

The categorical formulation of networks of tagged systems allows us to give a precise statement of when semantics is preserved:

Semantics is preserved by composition if $\|_{\mathscr{M}} \mathscr{P} \equiv \| \mathscr{P}$.

The universality of composition allows us to derive concrete necessary and sufficient conditions on when semantics is preserved, indicating that this framework can produce results on semantics preservation that are both practical and verifiable.

Extensions. Although this work is centered around the formalism of tagged system, the results are easily extendable to arbitrary networks over categories and the composition thereof. That is, this work is the first step toward addressing the general question of how to compose systems in a general and systematic fashion in order to ensure that the composite system has the proper behavior.

Notation 6.1. In this section, we use calligraphic symbols to denote tag structures and tag systems; do not confuse these with D-categories (which will not be utilized in this section, although D^{op}-categories will be). The reason for this notation is historical precedence. Note that this is why we denote a D^{op}-category by the symbol $\mathfrak{N}$ (not to be confused with a hybrid system).

6.1 Universal Heterogeneous Composition

In this section, we begin by defining the category of tag structures. This definition is used to understand how to associate a common tag structure to a pair of tag structures which can communicate through a mediator tag structure. We then introduce the category of tagged systems, which will be instrumental in understanding how to form the heterogeneous composition of a network of tagged systems. Finally, we discuss how to take the composition of a “simple” network of embedded systems.

Given two tagged systems, $\mathscr{P}_1$ and $\mathscr{P}_2$, we would like to form their composition, i.e., a single tagged system obtainable from these two tagged systems. We begin by reviewing the “standard” definition of composition, followed by a categorical reformulation of composition. We demonstrate that the composition of two tagged systems corresponds to the pullback of a specific diagram in the category of tagged systems. This will allow us later to generalize the notion of composition.

6.1.1 Tag structures and the corresponding category. Fundamental to the notion of tagged systems is the notion of timing. This timing is encoded in a set of tags; these “tag" the occurrences of events, i.e., they index the events such that they are (partially) ordered. Hence, a *set of tags* or a *tag structure* is a partially ordered set $\mathscr{T}$, with the partial order denoted by $\leq$. The category of tags, Tag, can be defined as follows:

Objects: Partially ordered sets, i.e., tag structures.
Morphisms: Nondecreasing maps between sets $\rho : \mathscr{T} \to \mathscr{T}'$, i.e., if $t \leq t' \in \mathscr{T}$ then $\rho(t) \leq \rho(t') \in \mathscr{T}'$.
Composition: The standard composition of maps between sets.

Clearly, two objects in the category Tag are isomorphic if $\rho : \mathscr{T} \to \mathscr{T}'$ is a bijection: there exists a $\rho' : \mathscr{T}' \to \mathscr{T}$ such that $\rho \circ \rho' = \mathrm{id}_{\mathscr{T}'}$ and $\rho' \circ \rho = \mathrm{id}_{\mathscr{T}}$. Note also that the *terminal objects* in the category Tag are just one point sets $\mathscr{T}_{\mathrm{triv}} := \{*\}$ (called asynchronous tag structures), i.e., for all tag structures $\mathscr{T}$ there exists a unique morphism $\rho : \mathscr{T} \to \mathscr{T}_{\mathrm{triv}}$ defined by $\rho(t) \equiv *$ which desynchronizes the tag structure. The synchronous tag structure is given by $\mathscr{T}_{\mathrm{sync}} = \mathbb{N}$.

6.1.2 Common tag structures. Tags are fundamental in understanding tagged systems in that morphisms of tag structures will induce morphisms of tagged systems. To better understand this, we will discuss an important operation on tag structures: the pullback (the pullback of elements in a category will be used extensively in the chapter—see Appendix A for a formal definition). Consider two tag structures $\mathscr{T}_1$ and $\mathscr{T}_2$. We would like to find a tag structure that is more general than $\mathscr{T}_1$ and $\mathscr{T}_2$ and has morphisms to both of these tag structures, i.e., we would like to find a *common tag structure* for these two tag structures. To do this, first consider the diagram in Tag:

$$\mathscr{T}_1 \xrightarrow{\rho_1} \mathscr{T} \xleftarrow{\rho_2} \mathscr{T}_2, \tag{6.1}$$

where $\mathscr{T}$ is the *mediator tag structure*. We know that such a tag structure always exists since it always can be taken to be $\mathscr{T}_{\mathrm{triv}}$ (although this rarely is the wisest choice). We define the common tag structure to be the pullback (see Paragraph A.2.3) of this diagram:

$$\mathscr{T}_1 \times_{\mathscr{T}} \mathscr{T}_2 = \{(t_1, t_2) \in \mathscr{T}_1 \times \mathscr{T}_2 : \rho_1(t_1) = \rho(t_2)\}. \tag{6.2}$$

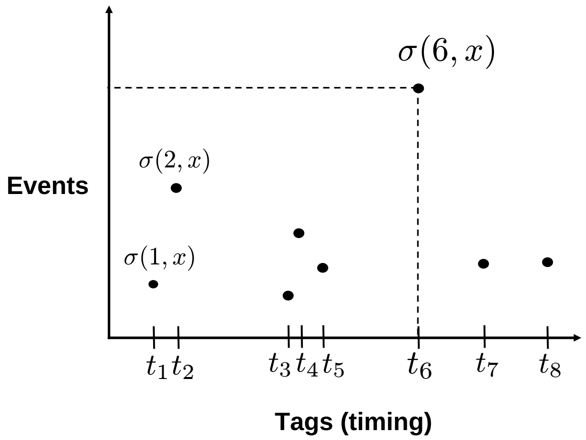
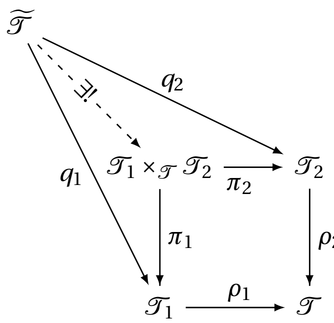


Figure 6.1: A graphical representation of a behavior of a tagged system.

The pullback is the desired common tag structure since it sits in a commutative diagram of the form:

$$\begin{array}{ccc} \mathscr{T}_1 \times_{\mathscr{T}} \mathscr{T}_2 & \xrightarrow{\pi_2} & \mathscr{T}_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \rho_2} \\ \mathscr{T}_1 & \xrightarrow{\rho_1} & \mathscr{T} \end{array} \tag{6.3}$$

Moreover, that fact that the common tag structure is the pullback implies that for any other tag structure that displays the properties of a common tag structure, there exists a unique morphism from this tag structure to *the* common tag structure. More precisely, for any tag structure $\widetilde{\mathscr{T}}$ such that the following diagram commutes:



(6.4)

there exists a unique morphism from $\widetilde{\mathscr{T}}$ to $\mathscr{T}_1 \times_{\mathscr{T}} \mathscr{T}_2$ also making the diagram commute. This construction on tag structures both motivates and mirrors constructions that will be performed throughout this chapter on tagged systems. To demonstrate this we must, as with tag structures, define tagged systems and the associated category.

6.1.3 Tagged systems. Following from [22], [23] and [24] (although our notation slightly deviates from theirs), we define a tagged system. We then proceed to introduce the category of tagged systems.

Let $\mathcal{V}$ be an underlying set of variables and D be the set of values that these variables can take, i.e., the domain of the variables. A tagged system is a tuple

$$\mathscr{P} = (V, \mathscr{T}, \Sigma),$$

where V is a finite subset of the underlying set of variables $\mathcal{V}$, $\mathscr{T}$ is a tag structure, i.e., an object of Tag, and Σ is a set of maps:

$$\sigma : \mathbb{N} \times V \to \mathscr{T} \times D.$$

Each of the elements of Σ, i.e., each of the maps σ, are referred to as V*-behaviors* (or just *behaviors* when the variable set is understood). It is required that for each $v \in V$, the map $\delta_v(n) := \pi_1(\sigma(n, v)) : \mathbb{N} \to \mathscr{T}$ is a morphism in Tag, that is, nondecreasing (and called a *clock* in [24]).

Remark 6.1. In defining the set of behaviors of a tagged system, we made an explicit choice for the domain of the behaviors: $\mathbb{N} \times V$. This choice is motivated by the fact that the behaviors of a tagged system are signals generated by a computer, and hence discrete in nature. It is possible to consider other domains for the behaviors, e.g., $\mathbb{R} \times V$, without any significant change to the theory introduced here. This indicates an interesting extension of this work to behavioral dynamical system theory (cf. [98, 111, 112]).

6.1.4 The category of tagged systems. We can use the formulation of tagged systems above in order to define the category of tagged systems, TagSys, as follows:

Objects: Tagged systems $\mathscr{P} = (V, \mathscr{T}, \Sigma)$.
Morphisms: A morphism of tagged systems $\alpha : \mathscr{P} = (V, \mathscr{T}, \Sigma) \to \mathscr{P}' = (V', \mathscr{T}', \Sigma')$ is a morphism (in the category of sets) of behaviors $\alpha : \Sigma \to \Sigma'$.
Composition: The standard composition of maps between sets. In other words, for $\alpha : \mathscr{P} \to \mathscr{P}'$ and $\alpha' : \mathscr{P}' \to \mathscr{P}''$ the composition of $\alpha : \Sigma \to \Sigma'$ and $\alpha' : \Sigma' \to \Sigma''$ is given by $\alpha' \circ \alpha : \Sigma \to \Sigma''$.

From the definition of morphisms in the category TagSys, it follows that two tagged systems, $\mathscr{P}$ and $\mathscr{P}'$, are isomorphic, $\mathscr{P} \cong \mathscr{P}'$, if and only if, to use the terminology from the literature, the two tagged systems are in bijective correspondence.

6.1.5 A "forgetful" functor. By the definition of the category TagSys, there is a *fully faithful* functor:

$$B : \mathsf{TagSys} \to \mathsf{Set},$$

where Set is the category of sets. This functor is defined on objects and morphisms in TagSys as follows: for every diagram in TagSys of the form:

$$\mathscr{P} = (V, \mathscr{T}, \Sigma) \xrightarrow{\alpha} \mathscr{P}' = (V', \mathscr{T}', \Sigma'),$$

the functor B is given by:

$$B\Big(\mathscr{P} = (V, \mathscr{T}, \Sigma) \xrightarrow{\alpha} \mathscr{P}' = (V', \mathscr{T}', \Sigma')\Big) = \Sigma \xrightarrow{\alpha} \Sigma'.$$

When discussing composition, we often will blur the distinction between the categories TagSys and Set, i.e., we often will define the composition of a diagram of tagged systems by the behaviors of the composite system, and hence implicitly view it as an object of Set. In this case, we always will construct an object in TagSys with behaviors isomorphic to the corresponding object in Set.

6.1.6 Induced morphisms of tagged systems. Suppose that there is a morphism of tag structures $\rho : \mathcal{T} \to \mathcal{T}'$. Then there exists a tagged system $\mathscr{P}^\rho$ together with an induced morphism of tagged systems $(\cdot)^\rho : \mathscr{P} \to \mathscr{P}^\rho$. First, if $\mathscr{P} = (V, \mathcal{T}, \Sigma)$ we define $\mathscr{P}^\rho = (V, \mathcal{T}', \Sigma^\rho)$ where

$$\Sigma^\rho \quad := \quad \{\sigma^\rho : \mathbb{N} \times V \to \mathcal{T}' \times D : \sigma^\rho(n, v) = (\rho(t), d) \text{ iff } (t, d) = \sigma(n, v) \text{ for some } \sigma \in \Sigma\}.$$

That is, Σ^ρ is defined by replacing t with $\rho(t)$ in the codomain of σ. With this definition of $\mathscr{P}^\rho$, we obtain a morphism $(\cdot)^\rho : \mathscr{P} \to \mathscr{P}^\rho$, called the *desynchronization morphism* and defined by, for each $\sigma \in \Sigma$,

$$\sigma^\rho(n, v) = (\rho(t), d) \quad \overset{\text{def}}{\Leftrightarrow} \quad \sigma(n, v) = (t, d).$$

Note that $(\cdot)^\rho$ is always surjective.

Example 6.1. Consider the following synchronous tagged systems, $\mathscr{P}_1$ and $\mathscr{P}_2$, defined as follows:

$$\begin{aligned} \mathscr{P}_1 &:= (V_1 = \{x\}, \mathcal{T}_{\text{sync}} = \mathbb{N}, \Sigma_1 = \{\sigma_1\}), \\ \mathscr{P}_2 &:= (V_2 = \{x, y\}, \mathcal{T}_{\text{sync}} = \mathbb{N}, \Sigma_2 = \{\sigma_2, \tilde{\sigma}_2\}), \end{aligned}$$

where

$$\begin{aligned} \sigma_1(n, x) &:= (m(n), \star), \\ \sigma_2(n, v) &:= \begin{cases} (m(n), \star) & \text{if} \quad v = x \in V_2 \\ (k(n), \star) & \text{if} \quad v = y \in V_2 \end{cases}, \\ \tilde{\sigma}_2(n, v) &:= \begin{cases} (m(n), \star) & \text{if} \quad v = x \in V_2 \\ (l(n), \star) & \text{if} \quad v = y \in V_2 \end{cases}. \end{aligned}$$

Here $m(n)$, $k(n)$ and $l(n)$ are any strictly increasing functions with $k(n) \neq l(n)$, and $\star$ is a (single) arbitrary value in D.

For $\rho : \mathcal{T}_{\text{sync}} \to \mathcal{T}_{\text{triv}}$ the desyncronization morphism, $\mathscr{P}_1 \cong \mathscr{P}_1^\rho$ because $\mathscr{P}_1$ consists of a single behavior. Since $\Sigma_2 = \{\sigma_2, \tilde{\sigma}_2\}$, $\Sigma_2^\rho = \{\sigma_2^\rho = \tilde{\sigma}_2^\rho\}$, i.e., Σ_2^ρ consists of a single behavior. Therefore, $\mathscr{P}_2$ is not in bijective correspondence with $\mathscr{P}_2^\rho$

6.1.7 Heterogeneous composition. Let $\mathscr{P}_1 = (V_1, \mathcal{T}_1, \Sigma_1)$ and $\mathscr{P}_2 = (V_2, \mathcal{T}_2, \Sigma_2)$ be two tagged systems. Consider a mediator tag structure $\mathcal{T}$ between the tag structures $\mathcal{T}_1$ and $\mathcal{T}_2$, i.e., there exists a diagram in Tag:

$$\mathcal{T}_1 \xrightarrow{\rho_1} \mathcal{T} \xleftarrow{\rho_2} \mathcal{T}_2.$$

Recall that the common tag structure to $\mathcal{T}_1$ and $\mathcal{T}_2$ (relative to $\mathcal{T}$) is given by the pullback of the above diagram in Tag (the fibered product as defined in (6.2)): $\mathcal{T}_1 \times_{\mathcal{T}} \mathcal{T}_2$.

We define the parallel composition of $\mathscr{P}_1$ and $\mathscr{P}_2$ over the mediator tag structure $\mathcal{T}$ by

$$\mathscr{P}_1 \parallel_{\mathcal{T}} \mathscr{P}_2 := (V_1 \cup V_2, \mathcal{T}_1 \times_{\mathcal{T}} \mathcal{T}_2, \Sigma_1 \wedge_{\mathcal{T}} \Sigma_2).$$

This notation for parallel composition is taken from [22]; the morphisms ρ_1 and ρ_2 are implicit in this notation. In the above definition, $\Sigma_1 \wedge_{\mathcal{T}} \Sigma_2$ is given by the set of behaviors

$$\sigma : \mathbb{N} \times (V_1 \cup V_2) \to (\mathcal{T}_1 \times_{\mathcal{T}} \mathcal{T}_2) \times D$$

such that the following condition holds: for all $(n, v) \in \mathbb{N} \times (V_1 \cup V_2)$, there exist unique $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ such that[1]

$$\sigma(n,v) = ((t_1,t_2),d) \quad \Leftrightarrow \quad \begin{array}{ll} \text{(i)} & \sigma_1(n,v) = (t_1,d) \text{ if } v \in V_1 \\ & \text{and} \\ \text{(ii)} & \sigma_2(n,v) = (t_2,d) \text{ if } v \in V_2 \\ & \text{and} \\ \text{(iii)} & \sigma_1^{\rho_1}(n,v) = \sigma_2^{\rho_2}(n,v) \text{ if } v \in V_1 \cap V_2. \end{array} \tag{6.5}$$

Since σ is uniquely determined by σ_1 and σ_2, and vise-versa, we write $\sigma = \sigma_1 \sqcup_{\mathcal{T}} \sigma_2$. We pick this notation so as to be consistent with the literature, cf. [22], where a pair $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$ is called *unifiable* when it satisfies condition (iii), and $\sigma_1 \sqcup_{\mathcal{T}} \sigma_2$ is called the *unification* of σ_1 and σ_2. We will always assume that such a pair exists; in this case composition is well-defined ($\Sigma_1 \wedge_{\mathcal{T}} \Sigma_2$ is not the empty set).

6.1.8 Universal heterogeneous composition. The common tag structure for the composition of two tagged systems is given by the pullback of a certain diagram. The natural question to ask is: can the composition of two tagged systems be realized as the pullback of a diagram of tagged systems of the form

$$\mathscr{P}_1 \xrightarrow{\alpha_1} \mathscr{M} \xleftarrow{\alpha_2} \mathscr{P}_2?$$

The importance of this question is that if the answer is yes, then the composition between two heterogeneous tagged systems is universal, i.e., defined by a universal property. We then can ask when the composition of two tagged systems is the same as the composition of these tagged systems with different tag structures, i.e., when semantics is preserved. It is possible to show that composition is in fact given by a universal property.

In order to define composition universally, we must define the tagged system $\mathscr{M}$ in the above diagram. In this vein, and using the same notation as the above paragraph, define

$$\mathscr{I}_{\mathcal{T}} := (V_1 \cap V_2, \mathcal{T}, \Sigma_1 \vee_{\mathcal{T}} \Sigma_2), \tag{6.6}$$

[1] Note that conditions (i) and (ii) imply condition (iii); this follows from the fact that $(t_1, t_2) \in \mathcal{T}_1 \times_{\mathcal{T}} \mathcal{T}_2$, so $\rho_1(t_1) = \rho_2(t_2)$ in condition (i) and (ii). Condition (iii) is stated for the sake of clarity.

where $\mathscr{T}$ is a mediator tag structure (between $\mathscr{T}_1$ and $\mathscr{T}_2$) and

$$\Sigma_1 \vee_{\mathscr{T}} \Sigma_2 \quad := \quad \{\sigma : \mathbb{N} \times (V_1 \cap V_2) \to \mathscr{T} \times D : \\ \sigma = \sigma_i^{\rho_i}|_{V_1 \cap V_2}, \text{ for } \sigma_i \in \Sigma_i, \ i = 1 \text{ or } 2\}. \tag{6.7}$$

Now for the tagged systems $\mathscr{P}_1$ and $\mathscr{P}_2$ there exist morphisms

$$\mathscr{P}_1 \xrightarrow{\mathrm{Res}_1^{\rho_1}} \mathscr{I}_{\mathscr{T}} \xleftarrow{\mathrm{Res}_2^{\rho_2}} \mathscr{P}_2 \tag{6.8}$$

defined as follows:

$$\mathrm{Res}_i^{\rho_i}(\sigma_i) = \sigma_i^{\rho_i}|_{V_1 \cap V_2} : \mathbb{N} \times (V_1 \cap V_2) \to \mathscr{T} \times D$$

for $\sigma_i \in \Sigma_i$, $i = 1, 2$. Clearly, such a morphism always exists.

Note that $\mathscr{I}_{\mathscr{T}}$ is a *mediator tagged system* or *channel* between the tagged systems $\mathscr{P}_1$ and $\mathscr{P}_2$; $\mathscr{I}_{\mathscr{T}}$ “communicates" between $\mathscr{P}_1$ and $\mathscr{P}_2$. In the case when $\mathscr{T} = \mathscr{T}_1 = \mathscr{T}_2$, $\mathscr{I} := \mathscr{I}_{\mathscr{T}}$ is exactly the *identity mediator tagged system* or *identity channel* introduced in [22].

Theorem 6.1. *Consider two tagged systems $\mathscr{P}_1 = (V_1, \mathscr{T}_1, \Sigma_1)$ and $\mathscr{P}_2 = (V_2, \mathscr{T}_2, \Sigma_2)$ with mediator tag structure $\mathscr{T}$, i.e., suppose that there is a diagram in* Tag*:*

$$\mathscr{T}_1 \xrightarrow{\rho_1} \mathscr{T} \xleftarrow{\rho_2} \mathscr{T}_2.$$

The parallel composition of $\mathscr{P}_1$ and $\mathscr{P}_2$ over this tag structure, $\mathscr{P}_1 \parallel_{\mathscr{T}} \mathscr{P}_2$, is the pullback of the diagram:

$$\mathscr{P}_1 \xrightarrow{\mathrm{Res}_1^{\rho_1}} \mathscr{I}_{\mathscr{T}} \xleftarrow{\mathrm{Res}_2^{\rho_2}} \mathscr{P}_2$$

in the category of tagged systems, TagSys.

6.1.9 Implications of Theorem 6.1. Before proving Theorem 6.1, we discuss some of the implications of this theorem.

If we consider the following diagram in the category of sets, Set:

$$\Sigma_1 \xrightarrow{\mathrm{Res}_1^{\rho_1}} \Sigma_1 \vee_{\mathscr{T}} \Sigma_2 \xleftarrow{\mathrm{Res}_2^{\rho_2}} \Sigma_2 = B\left(\mathscr{P}_1 \xrightarrow{\mathrm{Res}_1^{\rho_1}} \mathscr{I}_{\mathscr{T}} \xleftarrow{\mathrm{Res}_2^{\rho_2}} \mathscr{P}_2\right),$$

the pullback of this diagram is given by:

$$\Sigma_1 \times_{\Sigma_1 \vee_{\mathscr{T}} \Sigma_2} \Sigma_2 = \{(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 : \mathrm{Res}_1^{\rho_1}(\sigma_1) = \mathrm{Res}_2^{\rho_2}(\sigma_2)\}.$$

It is important to note that the pullback of the above diagram (which is an object in Set) is related to—in fact, isomorphic to—the behavior of the tagged system $\mathscr{P}_1 \parallel_{\mathscr{T}} \mathscr{P}_2$. More precisely, for the functor B : TagSys $\to$ Set, we have:

$$\Sigma_1 \times_{\Sigma_1 \vee_{\mathscr{T}} \Sigma_2} \Sigma_2 \cong B(\mathscr{P}_1 \parallel_{\mathscr{T}} \mathscr{P}_2) = \Sigma_1 \vee_{\mathscr{T}} \Sigma_2. \tag{6.9}$$

This observation will allow us later to, justifiably, blur the distinction between pullbacks (and limits) in the categories TagSys and Set.

In order to produce the bijection given in (6.9), first note that there are projections defined by:

$$\pi_i : \Sigma_1 \wedge_{\mathscr{T}} \Sigma_2 \to \Sigma_i \tag{6.10}$$

where for each $\sigma_1 \sqcup_{\mathscr{T}} \sigma_2 \in \Sigma_1 \wedge_{\mathscr{T}} \Sigma_2$, $\pi_i(\sigma_1 \sqcup_{\mathscr{T}} \sigma_2) := \sigma_i$ for $i = 1,2$. Because of (6.5), it follows that any element $\sigma \in \Sigma_1 \wedge_{\mathscr{T}} \Sigma_2$ can be written as $\sigma = \pi_1(\sigma) \sqcup_{\mathscr{T}} \pi_2(\sigma)$.

Theorem 6.1 implies—by the universality of the pullback—that there is a bijection:

$$\begin{array}{rcl} (\pi_1,\pi_2) : \Sigma_1 \wedge_{\mathscr{T}} \Sigma_2 & \xrightarrow{\sim} & \Sigma_1 \times_{\Sigma_1 \vee_{\mathscr{T}} \Sigma_2} \Sigma_2 \\ \sigma = \sigma_1 \sqcup_{\mathscr{T}} \sigma_2 & \mapsto & (\sigma_1 = \pi_1(\sigma), \sigma_2 = \pi_2(\sigma)) \end{array} \tag{6.11}$$

where the inverse of this map is the unification operator:

$$\begin{array}{rcl} (\,\cdot\,) \sqcup_{\mathscr{T}} (\,\cdot\,) : \Sigma_1 \times_{\Sigma_1 \vee_{\mathscr{T}} \Sigma_2} \Sigma_2 & \xrightarrow{\sim} & \Sigma_1 \wedge_{\mathscr{T}} \Sigma_2 \\ (\sigma_1, \sigma_2) & \mapsto & \sigma_1 \sqcup_{\mathscr{T}} \sigma_2. \end{array} \tag{6.12}$$

This completes the description of the bijection given in (6.9).

Proof. (*of Theorem 6.1*) From the definition of $\Sigma_1 \wedge_{\mathscr{T}} \Sigma_2$ and $\Sigma_1 \vee_{\mathscr{T}} \Sigma_2$, it follows that the following diagram in Set:

$$\begin{array}{ccc} \Sigma_1 \wedge_{\mathscr{T}} \Sigma_2 & \xrightarrow{\pi_1} & \Sigma_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_1^{\rho_1}} \\ \Sigma_2 & \xrightarrow{\mathrm{Res}_2^{\rho_2}} & \Sigma_1 \vee_{\mathscr{T}} \Sigma_2 \end{array}$$

commutes, which implies, by the definition of morphisms of tagged systems, that the following diagram in TagSys:

$$\begin{array}{ccc} \mathscr{P}_1 \parallel_{\mathscr{T}} \mathscr{P}_2 & \xrightarrow{\pi_1} & \mathscr{P}_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_1^{\rho_1}} \\ \mathscr{P}_2 & \xrightarrow{\mathrm{Res}_2^{\rho_2}} & \mathscr{I}_{\mathscr{T}} \end{array}$$

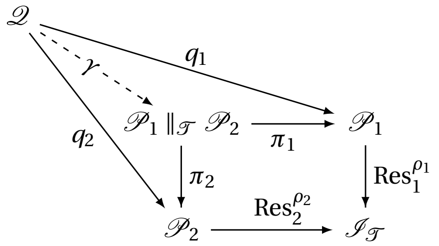
commutes. Consider a tagged system $\mathscr{Q}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{Q} & \xrightarrow{q_1} & \mathscr{P}_1 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_1^{\rho_1}} \\ \mathscr{P}_2 & \xrightarrow{\mathrm{Res}_2^{\rho_2}} & \mathscr{I}_{\mathscr{T}} \end{array}$$

We can define a morphism $\gamma : \mathscr{Q} \to \mathscr{P}_1 \parallel_{\mathscr{T}} \mathscr{P}_2$ by, for $\sigma \in \Sigma_{\mathscr{Q}}$,

$$\gamma(\sigma) = q_1(\sigma) \sqcup_{\mathscr{T}} q_1(\sigma).$$

It follows that there is a commuting diagram:



Moreover, by replacing γ with any other morphism making the diagram commute, say $\tilde{\gamma}$, it follows that for $\sigma \in \Sigma_{\mathscr{Q}}$

$$\tilde{\gamma}(\sigma) = \pi_1(\tilde{\gamma}(\sigma)) \sqcup_{\mathscr{T}} \pi_2(\tilde{\gamma}(\sigma)) = q_1(\sigma) \sqcup_{\mathscr{T}} q_2(\sigma) = \gamma(\sigma).$$

So $\gamma = \tilde{\gamma}$, i.e., γ is unique. $\square$

6.2 Equivalent Deployments of Tagged Systems

Standard composition is just the pullback of a specific diagram in TagSys; this observation naturally allows us to generalize composition. To perform this generalization, we introduce the notion of a general mediator tagged system, $\mathscr{M}$, and define composition to be the pullback of a diagram of the form:

$$\mathscr{P}_1 \xrightarrow{\alpha_1} \mathscr{M} \xleftarrow{\alpha_2} \mathscr{P}_2 \tag{6.13}$$

in TagSys. This process will be instrumental later in understanding how to take the composition of more general networks of embedded systems.

We conclude this section by reviewing the definition of semantics preservation and giving necessary and sufficient conditions on when semantics is preserved. We apply these results to the special case of semantics preservation through desyncronization.

6.2.1 Composition through mediation. Given the results of Theorem 6.1, we can develop a more intuitive notation for composition. Specifically, if $\mathscr{T}$ is the mediator tag structure, then we write $\mathscr{P}_1 \parallel_{\mathscr{T}} \mathscr{P}_2 = \mathscr{P}_1 \parallel_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2$ (in the case when $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{T}$ and $\rho_1 = \rho_2 = \mathrm{id}$ in (6.1), we just write $\mathscr{P}_1 \parallel \mathscr{P}_2 := \mathscr{P}_1 \parallel_{\mathscr{I}} \mathscr{P}_2$). The mathematical reason for this is that $\mathscr{P}_1 \parallel_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2$ is (isomorphic to) $\mathscr{P}_1 \times_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2$, i.e., the pullback of the diagram given in (6.8). The philosophical motivation for this notation is that the composition of $\mathscr{P}_1$ and $\mathscr{P}_2$ can be taken over general mediator tagged systems. In other words, the parallel composition of $\mathscr{P}_1$ and $\mathscr{P}_2$ over a general *mediator tagged system* $\mathscr{M}$, denoted by $\mathscr{P}_1 \parallel_{\mathscr{M}} \mathscr{P}_2$, is defined to be the pullback of the diagram given in (6.13):

$$\mathscr{P}_1 \parallel_{\mathscr{M}} \mathscr{P}_2 := \mathscr{P}_1 \times_{\mathscr{M}} \mathscr{P}_2.$$

This implies that if $\Sigma_{\mathscr{M}}$ is the set of behaviors of $\mathscr{M}$, then the set of behaviors for $\mathscr{P}_1 \parallel_{\mathscr{M}} \mathscr{P}_2$ is isomorphic to:

$$\Sigma_1 \times_{\Sigma_{\mathscr{M}}} \Sigma_2 = \{(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 : \alpha_1(\sigma_1) = \alpha_2(\sigma_2)\}. \tag{6.14}$$

The explicit construction of $\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2$ as a tagged system is not especially relevant, as we are interested only in its set of behaviors which must be isomorphic to the set of behaviors given in (6.14). That being said, this construction is a special case of a more general construction (given in Section 6.4) for which the construction of the tagged system is carried out. We only note that there are bijections

$$(\cdot) \sqcup_{\mathscr{M}} (\cdot) : \Sigma_1 \times_{\Sigma_{\mathscr{M}}} \Sigma_2 \xrightarrow{\sim} \Sigma_{\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2} \tag{6.15}$$

$$(\pi_1, \pi_2) : \Sigma_{\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2} \xrightarrow{\sim} \Sigma_1 \times_{\Sigma_{\mathscr{M}}} \Sigma_2 \tag{6.16}$$

which are generalizations of the unification and projection maps given in (6.12) and (6.11), respectively.

6.2.2 Specification vs. deployment. Consider two tagged systems $\mathscr{P}_1$ and $\mathscr{P}_2$ with a mediator tag structure $\mathscr{T}$. As in [23] (although with some generalization, since $\mathscr{P}_1$ and $\mathscr{P}_2$ are not assumed to have the same tag structure), we define the following semantics:

Specification Semantics: $\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2$
Deployment Semantics: $\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2$

for some mediator tagged system $\mathscr{M}$. The natural question to ask is when are the specification semantics and the deployment semantics "equivalent." Formally, and following from [23], we define a mediator $\mathscr{M}$ to be *semantics preserving with respect to* $\mathscr{I}_{\mathscr{T}}$, denoted by

$$\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2 \equiv \mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2 \tag{6.17}$$

if for all $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$,

$$\begin{array}{c} \exists\, \sigma' \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2} \text{ s.t. } \pi_1(\sigma') = \sigma_1 \text{ and } \pi_2(\sigma') = \sigma_2 \\ \Updownarrow \\ \exists\, \sigma \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2} \text{ s.t. } \pi_1(\sigma) = \sigma_1 \text{ and } \pi_2(\sigma) = \sigma_2. \end{array} \tag{6.18}$$

Utilizing Theorem 6.1, we have the following necessary and sufficient conditions on semantics preservation.

Theorem 6.2. *For two tagged systems $\mathscr{P}_1$ and $\mathscr{P}_2$,*

$$\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2 \equiv \mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2,$$

if and only if for all $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$:

$$\alpha_1(\sigma_1) = \alpha_2(\sigma_2) \quad \Leftrightarrow \quad \mathrm{Res}_1^{\rho_1}(\sigma_1) = \mathrm{Res}_2^{\rho_2}(\sigma_2).$$

Proof. **(Sufficiency:)** If

$$(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 \text{ s.t. } \alpha_1(\sigma_1) = \alpha_2(\sigma_2)$$

$$\begin{aligned}
&\overset{\text{by (6.14)}}{\Rightarrow} && (\sigma_1, \sigma_2) \in \Sigma_1 \times_{\Sigma_{\mathscr{M}}} \Sigma_2 \\
&\overset{\text{by (6.15)}}{\Rightarrow} && \sigma_1 \sqcup_{\mathscr{M}} \sigma_2 \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2} \text{ where} \\
& && \qquad \pi_1(\sigma_1 \sqcup_{\mathscr{M}} \sigma_2) = \sigma_1 \text{ and } \pi_2(\sigma_1 \sqcup_{\mathscr{M}} \sigma_2) = \sigma_2 \\
&\overset{\text{by (6.18)}}{\Rightarrow} && \exists\, \sigma \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2} \text{ s.t. } \pi_1(\sigma) = \sigma_1 \text{ and } \pi_2(\sigma) = \sigma_2 \\
&\overset{\text{by (6.16)}}{\Rightarrow} && (\sigma_1, \sigma_2) \in \Sigma_1 \times_{\Sigma_{\mathscr{I}_{\mathscr{T}}}} \Sigma_2 \\
&\overset{\text{by (6.14)}}{\Rightarrow} && \mathrm{Res}_1^{\rho_1}(\sigma_1) = \mathrm{Res}_2^{\rho_2}(\sigma_2).
\end{aligned}$$

The converse direction proceeds in the same manner: if

$$(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 \text{ s.t. } \mathrm{Res}_1^{\rho_1}(\sigma_1) = \mathrm{Res}_2^{\rho_2}(\sigma_2) \qquad \Rightarrow \qquad \alpha_1(\sigma_1) = \alpha_2(\sigma_2),$$

by (6.14), (6.15), (6.16), and (6.18).

(Necessity:) We have the following implications:

$$\exists\, \sigma' \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2} \text{ s.t. } \pi_1(\sigma') = \sigma_1 \text{ and } \pi_2(\sigma') = \sigma_2$$

$$\begin{aligned}
&\overset{\text{by (6.16)}}{\Rightarrow} && (\sigma_1, \sigma_2) \in \Sigma_1 \times_{\Sigma_{\mathscr{M}}} \Sigma_2 \\
&\overset{\text{by (6.14)}}{\Rightarrow} && \alpha_1(\sigma_1) = \alpha_2(\sigma_2) \\
&\Rightarrow && \mathrm{Res}_1^{\rho_1}(\sigma_1) = \mathrm{Res}_2^{\rho_2}(\sigma_2) \\
&\overset{\text{by (6.14)}}{\Rightarrow} && (\sigma_1, \sigma_2) \in \Sigma_1 \times_{\Sigma_{\mathscr{I}_{\mathscr{T}}}} \Sigma_2 \\
&\overset{\text{by (6.15)}}{\Rightarrow} && \sigma_1 \sqcup_{\mathscr{I}_{\mathscr{T}}} \sigma_2 \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2} \text{ and} \\
& && \quad \pi_1(\sigma_1 \sqcup_{\mathscr{I}_{\mathscr{T}}} \sigma_2) = \sigma_1 \text{ and } \pi_2(\sigma_1 \sqcup_{\mathscr{I}_{\mathscr{T}}} \sigma_2) = \sigma_2.
\end{aligned}$$

Therefore $\sigma_1 \sqcup_{\mathscr{I}_{\mathscr{T}}} \sigma_2$ is the element of $\Sigma_{\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2}$ such that $\pi_1(\sigma_1 \sqcup_{\mathscr{I}_{\mathscr{T}}} \sigma_2) = \sigma_1$ and $\pi_2(\sigma_1 \sqcup_{\mathscr{I}_{\mathscr{T}}} \sigma_2) = \sigma_2$, as desired.

The other direction follows in the same way:

$$\exists\, \sigma \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}}} \mathscr{P}_2} \text{ s.t. } \pi_1(\sigma) = \sigma_1 \text{ and } \pi_2(\sigma) = \sigma_2$$

$$\begin{aligned}
&\Rightarrow && \sigma_1 \sqcup_{\mathscr{M}} \sigma_2 \in \Sigma_{\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2} \text{ and} \\
& && \quad \pi_1(\sigma_1 \sqcup_{\mathscr{M}} \sigma_2) = \sigma_1 \text{ and } \pi_2(\sigma_1 \sqcup_{\mathscr{M}} \sigma_2) = \sigma_2,
\end{aligned}$$

by (6.14), (6.15) and (6.16). □

To demonstrate the power of Theorem 6.2, we prove the following theorem, which is a generalization of one of the two main theorems of [22]. Moveover, we show that the theorem in [22] is a corollary of this theorem; thus, our results are more general. First, we review the general set-up for this theorem.

6.2.3 Desynchronization. Consider the case when $\mathscr{P}_1$ and $\mathscr{P}_2$ have the same tag structure, i.e., $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{T}$. Consider a mediator tag structure $\mathscr{T}'$ of $\mathscr{T}$, i.e., suppose there exists a diagram in Tag:

$$\mathscr{T} \xrightarrow{\ \rho\ } \mathscr{T}' \xleftarrow{\ \rho\ } \mathscr{T}.$$

In this case, we ask when the mediator tagged system $\mathscr{I}_{\mathscr{T}'}$ is semantics preserving, i.e., when

$$\mathscr{P}_1 \| \mathscr{P}_2 \equiv \mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}'}} \mathscr{P}_2. \tag{6.19}$$

A very important example of when this framework is useful is in the desynchronization of tagged systems; in this case $\mathscr{T}' = \mathscr{T}_{\text{triv}} = \{*\}$, and $\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}'}} \mathscr{P}_2$ is the *desynchronization* of $\mathscr{P}_1$ and $\mathscr{P}_2$.

Using the notation of this paragraph, we have the following theorem and its corollary.

Theorem 6.3. *$\mathscr{I}_{\mathscr{T}'}$ is semantics preserving w.r.t. $\mathscr{I}$, $\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}'}} \mathscr{P}_2 \equiv \mathscr{P}_1 \| \mathscr{P}_2$, if and only if for all* $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$*:*

$$\sigma_1^\rho|_{V_1 \cap V_2} = \sigma_2^\rho|_{V_1 \cap V_2} \qquad \Rightarrow \qquad \sigma_1|_{V_1 \cap V_2} = \sigma_2|_{V_1 \cap V_2}.$$

Proof. Note that by the definition of the desynchronization morphism $(\cdot)^\rho$ (and the fact that it is always surjective), it follows that

$$\sigma_1|_{V_1 \cap V_2} = \sigma_2|_{V_1 \cap V_2} \qquad \Rightarrow \qquad \sigma_1^\rho|_{V_1 \cap V_2} = \sigma_2^\rho|_{V_1 \cap V_2}.$$

Therefore, this result is a corollary of Theorem 6.2. □

Corollary 6.1. *If $\mathscr{P}_i^\rho$ is in bijection with $\mathscr{P}_i$ for $i = 1, 2$ and $(\mathscr{P}_1 \| \mathscr{P}_2)^\rho = \mathscr{P}_1^\rho \| \mathscr{P}_2^\rho$, then $\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}'}} \mathscr{P}_2 \equiv \mathscr{P}_1 \| \mathscr{P}_2$ ($\mathscr{I}_{\mathscr{T}'}$ is semantics preserving w.r.t. $\mathscr{I}$).*

Proof. We need only show that the suppositions of the theorem imply for all $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$

$$\sigma_1^\rho|_{V_1 \cap V_2} = \sigma_2^\rho|_{V_1 \cap V_2} \qquad \Rightarrow \qquad \sigma_1|_{V_1 \cap V_2} = \sigma_2|_{V_1 \cap V_2}.$$

The result then follows from Theorem 6.3.

To see that the desired implication holds, note that we have the following chain of implications:

$$\sigma_1^\rho|_{V_1 \cap V_2} = \sigma_2^\rho|_{V_1 \cap V_2}$$

$$\begin{array}{lrcll} \Rightarrow & (\sigma_1, \sigma_2) & \in & \Sigma_1 \times_{\Sigma_{\mathscr{I}_{\mathscr{T}'}}} \Sigma_2 \cong \Sigma_{\mathscr{P}_1 \|_{\mathscr{I}_{\mathscr{T}'}} \mathscr{P}_2} & \\ \Rightarrow & (\sigma_1^\rho, \sigma_2^\rho) & \in & \Sigma_1^\rho \times_{\Sigma_{\mathscr{I}_{\mathscr{T}'}}} \Sigma_2^\rho \cong \Sigma_{\mathscr{P}_1^\rho \| \mathscr{P}_2^\rho} & \\ \Rightarrow & \sigma_1^\rho \sqcup_{\mathscr{I}_{\mathscr{T}'}} \sigma_2^\rho & \in & \Sigma^\rho_{\mathscr{P}_1 \| \mathscr{P}_2} & \text{(since } (\mathscr{P}_1 \| \mathscr{P}_2)^\rho = \mathscr{P}_1^\rho \| \mathscr{P}_2^\rho) \\ \Rightarrow & \exists\, \tilde{\sigma} & \in & \Sigma_{\mathscr{P}_1 \| \mathscr{P}_2} & \text{s.t.} \quad \tilde{\sigma}^\rho = \sigma_1^\rho \sqcup_{\mathscr{I}_{\mathscr{T}'}} \sigma_2^\rho. \end{array}$$

Setting $\tilde{\sigma}_i = \pi_i(\tilde{\sigma})$, the last of these implications implies that $(\tilde{\sigma}_1^\rho, \tilde{\sigma}_2^\rho) = (\sigma_1^\rho, \sigma_2^\rho) \in \Sigma_1^\rho \times_{\Sigma_{\mathscr{I}_{\mathscr{T}'}}} \Sigma_2^\rho$. Now, the fact that $\mathscr{P}_i^\rho$ is in bijection with $\mathscr{P}_i$ for $i = 1, 2$ implies that $\tilde{\sigma}_i = \sigma_i$, or:

$$(\sigma_1, \sigma_2) = (\tilde{\sigma}_1, \tilde{\sigma}_2) \in \Sigma_1 \times_{\Sigma_{\mathscr{I}_{\mathscr{T}}}} \Sigma_2 \qquad \Rightarrow \qquad \sigma_1|_{V_1 \cap V_2} = \sigma_2|_{V_1 \cap V_2}.$$

□

Example 6.2. We would like to know semantics is preserved by desynchronization for the tagged systems given in Example 6.1, i.e., for $\rho : \mathcal{T}_{\text{sync}} \to \mathcal{T}_{\text{triv}}$ the desyncronization morphism, is $\mathscr{P}_1 \| \mathscr{P}_2 \equiv \mathscr{P}_1 \|_{\mathscr{I}_{\mathcal{T}_{\text{triv}}}} \mathscr{P}_2$?

First we apply the necessary and sufficient conditions given in Theorem 6.3. Since $V_1 \cap V_2 = \{x\}$,

$$\begin{aligned} \sigma_1^\rho(n,x) = \sigma_2^\rho(n,x) \qquad &\Rightarrow \qquad \sigma_1(n,x) = \sigma_2(n,x) \\ \sigma_1^\rho(n,x) = \tilde{\sigma}_2^\rho(n,x) \qquad &\Rightarrow \qquad \sigma_1(n,x) = \tilde{\sigma}_2(n,x) \end{aligned}$$

because $\sigma_1(n,x) = \sigma_2(n,x)$ and $\sigma_1(n,x) = \tilde{\sigma}_2(n,x)$. Therefore, semantics is preserved.

Note that Corollary 6.1 would not tell us whether semantics is preserved, because $\mathscr{P}_2$ is not in bijective correspondence with $\mathscr{P}_2^\rho$, and so the conditions of the corollary do not hold. This demonstrates that Theorem 6.3 is a stronger result than Corollary 6.1.

6.3 Networks of Tagged Systems

In this section, we introduce the notion of a network of tag structures, tagged systems and behaviors. Moreover, we are able to show that these objects correspond to networks over the categories Tag, TagSys and Set, respectively (see Section 1.5). This observation will be fundamental in defining composition for these networks.

6.3.1 Networks of tag structures. We begin by defining a network of tag structures as in [23] (although we state the definition in a slightly different manner). A *network of tag structures* is defined to be a tuple

$$(\Gamma, \mathcal{T}, \mathscr{S}, \rho),$$

where

- $\Gamma = (Q, E)$ is a graph.
- $\mathcal{T} = \{\mathcal{T}_q\}_{q \in Q}$ is a set of tag structures.
- $\mathscr{S} = \{\mathscr{S}_e\}_{e \in E}$ is a set of mediator tag structures, mediating between $\mathcal{T}_{\mathsf{sor}(e)}$ and $\mathcal{T}_{\mathsf{tar}(e)}$.
- $\rho = \{(\rho_e, \rho'_e)\}_{e \in E}$ is a set of pairs of morphisms in Tag, such that for every $e \in E$, there is the following diagram in Tag:

$$\mathcal{T}_{\mathsf{sor}(e)} \xrightarrow{\rho_e} \mathscr{S}_e \xleftarrow{\rho'_e} \mathcal{T}_{\mathsf{tar}(e)}.$$

Networks of tagged systems are defined in an analogous manner.

6.3.2 Networks of tagged systems. A *network of tagged systems* is defined to be a tuple

$$(\Gamma, \mathscr{P}, \mathscr{M}, \alpha),$$

where

- ⋄ $\Gamma = (Q, E)$ is a graph.
- ⋄ $\mathscr{P} = \{\mathscr{P}_q\}_{q\in Q}$ is a set of tag structures.
- ⋄ $\mathscr{M} = \{\mathscr{M}_e\}_{e\in E}$ is a set of mediator tagged systems, mediating between $\mathscr{P}_{\mathsf{sor}(e)}$ and $\mathscr{P}_{\mathsf{tar}(e)}$.
- ⋄ $\alpha = \{(\alpha_e, \alpha'_e)\}_{e\in E}$ is a set of pairs of morphisms in TagSys, such that for every $e \in E$, there is the following diagram in TagSys:

$$\mathscr{P}_{\mathsf{sor}(e)} \xrightarrow{\alpha_e} \mathscr{M}_e \xleftarrow{\alpha'_e} \mathscr{P}_{\mathsf{tar}(e)}.$$

Suppose we have a network of tag structures $(\Gamma, \mathscr{T}, \mathscr{S}, \rho)$ and a collection of tagged systems $\mathscr{P} = \{\mathscr{P}_q\}_{q\in Q}$ such that $\mathscr{P}_q$ has tag structure $\mathscr{T}_q$. Then we can associate to this set of tagged systems a network of tagged systems, $(\Gamma, \mathscr{P}, \mathscr{I}_{\mathscr{S}}, \mathrm{Res}^{\rho})$ with:

$$\mathscr{I}_{\mathscr{S}} = \{\mathscr{I}_{\mathscr{S}_e}\}_{e\in E}, \qquad \mathrm{Res}^{\rho} = \{(\mathrm{Res}^{\rho_e}_{\mathsf{sor}(e)}, \mathrm{Res}^{\rho'_e}_{\mathsf{tar}(e)})\}_{e\in E}$$

where $\mathscr{I}_{\mathscr{S}_e}$ is defined as in (6.6), and $\mathrm{Res}^{\rho_e}_{\mathsf{sor}(e)}$ and $\mathrm{Res}^{\rho'_e}_{\mathsf{tar}(e)}$ are defined as in (6.8), i.e., there is a diagram in TagSys of the form:

$$\mathscr{P}_{\mathsf{sor}(e)} \xrightarrow{\mathrm{Res}^{\rho_e}_{\mathsf{sor}(e)}} \mathscr{I}_{\mathscr{S}_e} \xleftarrow{\mathrm{Res}^{\rho'_e}_{\mathsf{tar}(e)}} \mathscr{P}_{\mathsf{tar}(e)}$$

for every $e \in E$.

6.3.3 Networks of behaviors. We can define a *network of behaviors* from the network of tagged systems, $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$, as a tuple:

$$(\Gamma, \Sigma_{\mathscr{P}}, \Sigma_{\mathscr{M}}, \alpha),$$

where

- ⋄ $\Gamma = (Q, E)$ is a graph.
- ⋄ $\Sigma_{\mathscr{P}} = \{\Sigma_{\mathscr{P}_q}\}_{q\in Q}$, where $\Sigma_{\mathscr{P}_q}$ is the set of behaviors for $\mathscr{P}_q$.
- ⋄ $\Sigma_{\mathscr{M}} = \{\Sigma_{\mathscr{M}_e}\}_{e\in E}$, where $\Sigma_{\mathscr{M}_e}$ is the set of behaviors for $\mathscr{M}_e$.
- ⋄ $\alpha = \{(\alpha_e, \alpha'_e)\}_{e\in E}$ is a set of pairs of morphisms in Set, such that for every $e \in E$, there is the following diagram in Set:

$$\Sigma_{\mathscr{P}_{\mathsf{sor}(e)}} \xrightarrow{\alpha_e} \Sigma_{\mathscr{M}_e} \xleftarrow{\alpha'_e} \Sigma_{\mathscr{P}_{\mathsf{tar}(e)}}.$$

The association of a network of behaviors from a network of tagged systems can be viewed categorically. For the network of tagged systems $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$, the functor $B : \mathsf{TagSys} \to \mathsf{Set}$ yields the corresponding network of behaviors $(\Gamma, \Sigma_{\mathscr{P}}, \Sigma_{\mathscr{M}}, \alpha)$ because

$$\begin{aligned} \Sigma_{\mathscr{P}} &= \{\Sigma_{\mathscr{P}_q}\}_{q\in Q} = \{B(\mathscr{P}_q)\}_{q\in Q} \\ \Sigma_{\mathscr{M}} &= \{\Sigma_{\mathscr{M}_e}\}_{e\in E} = \{B(\mathscr{M}_e)\}_{e\in E}. \end{aligned}$$

More generally,

$$\Sigma_{\mathscr{P}_{\mathsf{sor}(e)}} \xrightarrow{\alpha_e} \Sigma_{\mathscr{M}_e} \xleftarrow{\alpha'_e} \Sigma_{\mathscr{P}_{\mathsf{tar}(e)}} = B\left(\mathscr{P}_{\mathsf{sor}(e)} \xrightarrow{\alpha_e} \mathscr{M}_e \xleftarrow{\alpha'_e} \mathscr{P}_{\mathsf{tar}(e)}\right)$$

for every $e \in E$.

6.3.4 Networks as networks over a category. We define a network of tag structures and a network of tagged systems to be networks over the categories Tag and TagSys:

$$\mathbf{T} : \mathfrak{N} \to \mathsf{Tag}, \qquad \mathbf{N} : \mathfrak{N} \to \mathsf{TagSys},$$

where $\mathfrak{N}$ is an D^{op}-category. For example, if $\mathfrak{N}$ is given by the diagram: $\bullet \longrightarrow \bullet \longleftarrow \bullet$, then the network of tag structures given in (6.1) and the network of tagged systems given in (6.13) are defined, respectively, by the functors:

$$\mathbf{T}(\bullet \longrightarrow \bullet \longleftarrow \bullet) = \left(\mathscr{T}_1 \xrightarrow{\rho_1} \mathscr{T} \xleftarrow{\rho_2} \mathscr{T}_2\right), \tag{6.20}$$

$$\mathbf{N}(\bullet \longrightarrow \bullet \longleftarrow \bullet) = \left(\mathscr{P}_1 \xrightarrow{\alpha_1} \mathscr{M} \xleftarrow{\alpha_2} \mathscr{P}_2\right). \tag{6.21}$$

More generally, we can associate to a network of tag structures $(\Gamma, \mathscr{T}, \mathscr{S}, \rho)$ and a network of tagged systems $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$ functors:

$$\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)} : \mathfrak{N}_\Gamma \to \mathsf{Tag}, \qquad \mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)} : \mathfrak{N}_\Gamma \to \mathsf{TagSys},$$

where $\mathfrak{N}_\Gamma$ is the D^{op}-category associated with Γ, and $\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)}$ and $\mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)}$ are defined by:

$$\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)}(\mathsf{sor}(e) \longrightarrow e \longleftarrow \mathsf{tar}(e)) := \left(\mathscr{T}_{\mathsf{sor}(e)} \xrightarrow{\rho_e} \mathscr{S}_e \xleftarrow{\rho'_e} \mathscr{T}_{\mathsf{tar}(e)}\right), \tag{6.22}$$

$$\mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)}(\mathsf{sor}(e) \longrightarrow e \longleftarrow \mathsf{tar}(e)) := \left(\mathscr{P}_{\mathsf{sor}(e)} \xrightarrow{\alpha_e} \mathscr{M}_e \xleftarrow{\alpha'_e} \mathscr{P}_{\mathsf{tar}(e)}\right), \tag{6.23}$$

for every $e \in E$.

If $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$ is a network of tagged systems, and $(\Gamma, \Sigma_{\mathscr{P}}, \Sigma_{\mathscr{M}}, \alpha)$ is the associated network of behaviors, then there is a functor

$$\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)} : \mathfrak{N}_\Gamma \to \mathsf{Set},$$

where $\mathfrak{N}_\Gamma$ is the D^{op}-category associated with Γ, and $\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}$ is defined to be the composite:

$$\mathfrak{N}_\Gamma \xrightarrow{\mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)}} \mathsf{TagSys} \xrightarrow{B} \mathsf{Set}.$$

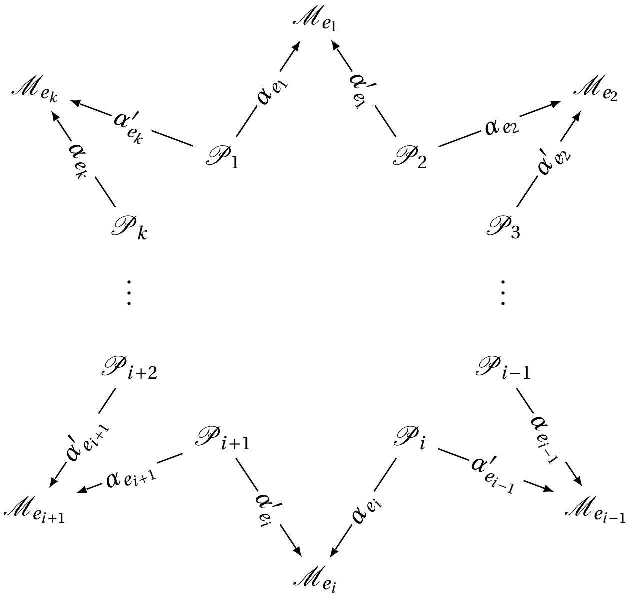
In other words, $\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}$ is defined by:

$$\begin{aligned}\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}(\mathsf{sor}(e) \longrightarrow e \longleftarrow \mathsf{tar}(e)) &:= \left(\Sigma_{\mathscr{P}_{\mathsf{sor}(e)}} \xrightarrow{\alpha_e} \Sigma_{\mathscr{M}_e} \xleftarrow{\alpha'_e} \Sigma_{\mathscr{P}_{\mathsf{tar}(e)}}\right),\\ &= B\left(\mathscr{P}_{\mathsf{sor}(e)} \xrightarrow{\alpha_e} \mathscr{M}_e \xleftarrow{\alpha'_e} \mathscr{P}_{\mathsf{tar}(e)}\right)\end{aligned}$$

for every $e \in E$.

Example 6.3. An example of a network of tagged systems associated to the D^{op}-category $\mathfrak{N}_{C_k}$ (as introduced in Example 1.24) is given in the following diagram:

$\mathbf{N}(\mathfrak{N}_{C_k}) =$



6.4 Universally Composing Networks of Tagged Systems

In this section, we give a categorical formulation for composition. Since a network is just a diagram, the composition of a network is the limit of this diagram. To illustrate this concept, we first consider a network of tag structures and demonstrate how taking the composition of this network is consistent with the notion of a common tag structure as first introduced in Section 6.1. We then discuss how these ideas can be generalized to networks of tagged systems. Finally, we explicitly relate the composite of a network of tagged systems with the composite of a network of behaviors. This relationship will be important when attempting to prove results relating to semantics preservation.

6.4.1 Composing networks of tag structures. Recall that for two tag structures, $\mathscr{T}_1$ and $\mathscr{T}_2$, communicating through a mediator tag structure, $\mathscr{T}$, we obtained a single, common, tag structure that was unique up to isomorphism; the common tag structure, $\mathscr{T}_1 \times_{\mathscr{T}} \mathscr{T}_2$, was the pullback of the diagram given in (6.1). But the pullback is just a special case of the limit of a functor (see Appendix A), i.e.,

$$\mathscr{T}_1 \times_{\mathscr{T}} \mathscr{T}_2 = \mathrm{Lim}_{(\bullet \longrightarrow \bullet \longleftarrow \bullet)}(\mathbf{T}) = \mathrm{Lim}_{(\bullet \longrightarrow \bullet \longleftarrow \bullet)}\left(\mathscr{T}_1 \xrightarrow{\ \rho_1\ } \mathscr{T} \xleftarrow{\ \rho_2\ } \mathscr{T}_2\right),$$

where $\mathbf{T} : (\bullet \longrightarrow \bullet \longleftarrow \bullet) \to \mathsf{Tag}$ is defined as in (6.20).

Therefore, we can define a tag structure common to an entire network of tag structures by taking the limit (defined in Appendix A) of the corresponding diagram in Tag describing this network. If $(\Gamma, \mathscr{T}, \mathscr{S}, \rho)$ is a network of tag structures and

$$\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)} : \mathfrak{N}_\Gamma \to \mathsf{Tag}$$

is the corresponding functor and D^{op}-category, we define the common tag structure to be $\mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)})$, which because of the special structure of a D^{op}-category is given by:

$$\mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)}) = \left\{ (t_q)_{q\in Q} \in \prod_{q\in Q} \mathscr{T}_q : \rho_e(t_{\mathsf{sor}(e)}) = \rho'_e(t_{\mathsf{tar}(e)}), \quad \forall\ e \in E \right\},$$

which corresponds to the common tag structure defined in [23]. By the properties of the limit, we know that this is in fact the desired common tag structure since for every $e \in E$, we have a diagram of the form

$$\begin{array}{ccc} \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)}) & \xrightarrow{\pi_{\mathsf{tar}(e)}} & \mathscr{T}_{\mathsf{tar}(e)} \\ {\scriptstyle \pi_{\mathsf{sor}(e)}}\downarrow & & \downarrow{\scriptstyle \rho'_e} \\ \mathscr{T}_{\mathsf{sor}(e)} & \xrightarrow{\rho_e} & \mathscr{S}_e \end{array}$$

which is a direct generalization of (6.3). Moreover, the limit is universal in the same sense as (6.4) (see Appendix A for a complete discussion).

6.4.2 Composing networks of tagged systems. As with networks of tag structures, we can consider the limit of a network of tagged systems (when viewed as a diagram)—this is the heterogeneous composition of the network. This is justified by the discussion in Section 6.2, where the composition of a network of two tagged systems, $\mathscr{P}_1$ and $\mathscr{P}_2$, was defined to be the pullback of these systems over the mediator tagged system, $\mathscr{M}$:

$$\mathscr{P}_1 \|_{\mathscr{M}} \mathscr{P}_2 = \mathrm{Lim}_{(\bullet \longrightarrow \bullet \longleftarrow \bullet)}(\mathbf{N}) = \mathrm{Lim}_{(\bullet \longrightarrow \bullet \longleftarrow \bullet)}\left(\mathscr{P}_1 \xrightarrow{\ \alpha_1\ } \mathscr{M} \xleftarrow{\ \alpha_2\ } \mathscr{P}_2 \right),$$

where $\mathbf{N} : (\bullet \longrightarrow \bullet \longleftarrow \bullet) \to \mathsf{Tag}$ is defined as in (6.20).

This indicates a general, and universal, way of taking the composition of a network of tagged systems: through the limit. Consider a network of tagged systems $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$, with the tagged systems $\mathscr{P}_q$ and $\mathscr{M}_e$ given by

$$\begin{array}{rcll} \mathscr{P}_q & = & (V_q, \mathscr{T}_q, \Sigma_q), & q \in Q, \\ \mathscr{M}_e & = & (V_e, \mathscr{S}_e, \Sigma_e), & e \in E. \end{array}$$

For the corresponding functor and D^{op}-category:

$$\mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)} : \mathfrak{N}_\Gamma \to \mathsf{TagSys}$$

denote the heterogeneous composition of $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$ by $\|_{\mathscr{M}} \mathscr{P}$ (to be consistent with the notation of [23]) and define it by (unlike [23])

$$\|_{\mathscr{M}} \mathscr{P} := \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)}) = \left(\bigcup_{q\in Q} V_q, \prod_{q\in Q} \mathscr{T}_q, \Sigma_{\|_{\mathscr{M}} \mathscr{P}} \right), \tag{6.24}$$

where $\Sigma_{\|_{\mathscr{M}}\mathscr{P}}$ is the set of behaviors

$$\begin{aligned} \sigma : \mathbb{N} \times \bigcup_{q\in Q} V_q &\to \prod_{q\in Q} \mathscr{T}_q \times D \\ \sigma(n,v) &\mapsto \big((t_q)_{q\in Q}, d\big) \end{aligned}$$

such that the following conditions hold: for all $(n,v) \in \mathbb{N} \times \bigcup_{q\in Q} V_q$, there exists unique $(\sigma_q)_{q\in Q} \in \prod_{q\in Q}\Sigma_q$ such that[2] for all $e \in E$

$$\sigma(n,v) = \big((t_q)_{q\in Q}, d\big) \Leftrightarrow \begin{array}{ll} (\mathrm{i}') & \sigma_{\mathsf{sor}(e)}(n,v) = (t_{\mathsf{sor}(e)}, d) \text{ if } v \in V_{\mathsf{sor}(e)} \\ & \text{and} \\ (\mathrm{ii}') & \sigma_{\mathsf{tar}(e)}(n,v) = (t_{\mathsf{tar}(e)}, d) \text{ if } v \in V_{\mathsf{tar}(e)} \\ & \text{and} \\ (\mathrm{iii}') & \alpha_e(\sigma_{\mathsf{sor}(e)})(n,w) = \alpha'_e(\sigma_{\mathsf{tar}(e)})(n,w) \\ & \forall\ w \in V_e. \end{array} \tag{6.25}$$

Because $\sigma \in \Sigma_{\|_{\mathscr{M}}\mathscr{P}}$ is uniquely determined by $(\sigma_q)_{q\in Q} \in \prod_{q\in Q}\Sigma_q$ satisfying the right-hand side of (6.25), we write

$$\sigma = \bigsqcup_{\mathscr{M}} (\sigma_q)_{q\in Q} \in \Sigma_{\|_{\mathscr{M}}\mathscr{P}}$$

for the corresponding element on the left-hand side of (6.25), and call it the *unification* of $(\sigma_q)_{q\in Q} \in \prod_{q\in Q}\Sigma_q$. Conversely, every element of $\Sigma_{\|_{\mathscr{M}}\mathscr{P}}$ can be written as the unification of an element of $\prod_{q\in Q}\Sigma_q$, so there are projection maps π_q, $q \in Q$, given by

$$\begin{aligned} \pi_q : \Sigma_{\|_{\mathscr{M}}\mathscr{P}} &\to \Sigma_q \\ \sigma = \bigsqcup_{\mathscr{M}} (\sigma_q)_{q\in Q} &\mapsto \sigma_q = \pi_q(\sigma). \end{aligned}$$

We can obtain a better understanding of the behaviors of the tagged system $\|_{\mathscr{M}}\mathscr{P}$ by considering the associated network of behaviors.

6.4.3 Composing networks of behaviors. Let $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$ be a network of tagged systems, $(\Gamma, \Sigma_{\mathscr{P}}, \Sigma_{\mathscr{M}}, \alpha)$ the associated network of behaviors, and

$$\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)} = B \circ \mathbf{N}_{(\mathscr{P},\mathscr{M},\alpha)} : \mathfrak{N}_\Gamma \to \mathsf{Set},$$

the associated functor and D^{op}-category. Because of the special structure of an D^{op}-category, we can explicitly compute the limit of the functor $\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}$; it is given by

$$\begin{aligned} &\mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}) = \\ &\quad \left\{ (\sigma_q)_{q\in Q} \in \prod_{q\in Q}\Sigma_q : \alpha_e(\sigma_{\mathsf{sor}(e)}) = \alpha'_e(\sigma_{\mathsf{tar}(e)}), \quad \forall\ e \in E \right\}. \end{aligned} \tag{6.26}$$

[2]Unlike (6.5), the third condition stated here is no longer redundant.

Now, there is a bijection:

$$\Sigma_{\|_{\mathscr{M}}\mathscr{P}} \cong \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}). \tag{6.27}$$

The map from $\Sigma_{\|_{\mathscr{M}}\mathscr{P}}$ to $\mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)})$ is given by

$$\begin{aligned}(\pi_q)_{q\in Q} : \Sigma_{\|_{\mathscr{M}}\mathscr{P}} &\xrightarrow{\sim} \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}) \\ \sigma &\mapsto (\pi_q(\sigma))_{q\in Q}.\end{aligned} \tag{6.28}$$

The inverse of this map is given by the unification operator generalized to the network case. That is

$$\begin{aligned}\bigsqcup_{\mathscr{M}}(\cdot) : \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}) &\xrightarrow{\sim} \Sigma_{\|_{\mathscr{M}}\mathscr{P}} \\ (\sigma_q)_{q\in Q} &\mapsto \sigma = \bigsqcup_{\mathscr{M}}(\sigma_q)_{q\in Q},\end{aligned} \tag{6.29}$$

where σ is given as in the left-hand side of equation (6.25), which is well defined because of the definition of $\mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)})$, i.e., because an element $(\sigma_q)_{q\in Q} \in \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)})$ automatically satisfies the right-hand side of (6.25) by (6.26).

6.4.4 Composition over identity mediator tagged systems. An especially interesting case is when the network of tagged systems is obtained from a network of tag structures $(\Gamma, \mathscr{T}, \mathscr{S}, \rho)$, i.e., the network of tagged systems is given by $(\Gamma, \mathscr{P}, \mathscr{I}_{\mathscr{S}}, \mathrm{Res}^{\rho})$. Here we explicitly carry out the construction of $\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}$, and demonstrate how this yields the correct definition of $\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}$ so as to be consistent with [23].

If $\mathscr{P}_q = (V_q, \mathscr{T}_q, \Sigma_q)$ for all $q \in Q$, then

$$\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P} = \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{N}_{(\mathscr{P},\mathscr{I}_{\mathscr{S}},\mathrm{Res}^{\rho})}) = \left(\bigcup_{q\in Q} V_q, \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)}), \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}}\right),$$

where $\Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}}$ is defined in the same way as $\Sigma_{\|_{\mathscr{M}}\mathscr{P}}$ with the appropriate modifications, i.e., $\Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}}$ is the set of behaviors

$$\begin{aligned}\sigma : \mathbb{N} \times \bigcup_{q\in Q} V_q &\to \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)}) \times D \subset \prod_{q\in Q}\mathscr{T}_q \times D \\ \sigma(n,v) &\mapsto \big((t_q)_{q\in Q}, d\big),\end{aligned}$$

such that the following conditions hold: for all $(n,v) \in \mathbb{N} \times \bigcup_{q\in Q} V_q$, there exists unique $(\sigma_q)_{q\in Q} \in \prod_{q\in Q}\Sigma_q$ such that[3] for all $e \in E$

$$\sigma(n,v) = \big((t_q)_{q\in Q}, d\big) \Leftrightarrow \begin{array}{ll} \text{(i}''\text{)} & \sigma_{\mathsf{sor}(e)}(n,v) = (t_{\mathsf{sor}(e)}, d) \text{ if } v \in V_{\mathsf{sor}(e)} \\ & \text{and} \\ \text{(ii}''\text{)} & \sigma_{\mathsf{tar}(e)}(n,v) = (t_{\mathsf{tar}(e)}, d) \text{ if } v \in V_{\mathsf{tar}(e)} \\ & \text{and} \\ \text{(iii}''\text{)} & \sigma^{\rho_e}_{\mathsf{sor}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}} = \sigma^{\rho'_e}_{\mathsf{tar}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}}. \end{array} \tag{6.30}$$

[3]Like (6.5) and unlike (6.25), the third condition stated here is again redundant because we are taking $\mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)})$ as our tag structure.

Note that the fact that $\sigma \in \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}} \mathscr{P}}$ takes values in $\mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{T}_{(\mathscr{T},\mathscr{S},\rho)})$, rather than $\prod_{q\in Q}\mathscr{T}_q$, is exactly because of condition (iii$''$).

The conditions given in (6.30) demonstrate that our definition of $\Sigma_{\|_{\mathscr{I}_{\mathscr{S}}} \mathscr{P}}$ is consistent with the one given in [23] (although our definition of $\Sigma_{\|_{\mathscr{M}} \mathscr{P}}$ is more general that anything defined in that chapter). Moreover, (6.26), (6.27) and (6.30) imply that we have the following bijection

$$\begin{aligned} \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}} \mathscr{P}} &\cong \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{|_{\mathscr{S}}},\mathrm{Res}^{\rho})}) \\ &= \left\{ (\sigma_q)_{q\in Q} \in \prod_{q\in Q} \Sigma_q : \sigma^{\rho_e}_{\mathsf{sor}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}} = \sigma^{\rho'_e}_{\mathsf{tar}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}},\ \forall\ e\in E \right\} \end{aligned} \tag{6.31}$$

as defined in (6.28) and (6.29). Here

$$\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{|_{\mathscr{S}}},\mathrm{Res}^{\rho})} = B \circ \mathbf{N}_{(\mathscr{P},\mathscr{I}_{\mathscr{S}},\mathrm{Res}^{\rho})} : \mathfrak{N}_\Gamma \to \mathsf{Set}$$

is the functor corresponding to the network of behaviors $(\Gamma, \Sigma_{\mathscr{P}}, \Sigma_{\mathscr{I}_{\mathscr{S}}}, \mathrm{Res}^{\rho})$ obtained from the network of tagged systems $(\Gamma, \mathscr{P}, |_{\mathscr{S}}, \mathrm{Res}^{\rho})$.

6.5 Semantics Preserving Deployments of Networks

Using the framework established in this chapter, we are able to introduce a general notion of semantics preservation. After this concept is introduced, we state the main result of this work: necessary and sufficient conditions for semantics preservation. We conclude this section by applying this result to the specific case of network desyncronization.

6.5.1 Network specification vs. network deployment. Generalizing the notion of specification vs. deployment given in Section 6.2, we define the following semantics (using the notation of the previous paragraph):

Network Specification Semantics: $\|_{\mathscr{I}_{\mathscr{S}}} \mathscr{P}$
Network Deployment Semantics: $\|_{\mathscr{M}} \mathscr{P}$

The set of mediator tagged systems $\mathscr{M}$ is said to be *semantics preserving with respect to $\mathscr{I}_{\mathscr{S}}$*, denoted by

$$\|_{\mathscr{M}} \mathscr{P} \equiv \|_{\mathscr{I}_{\mathscr{S}}} \mathscr{P}$$

if for all $(\sigma_q)_{q\in Q} \in \prod_{q\in Q}\Sigma_q$

$$\begin{gathered} \exists\ \sigma' \in \Sigma_{\|_{\mathscr{M}} \mathscr{P}} \text{ s.t. } \pi_q(\sigma') = \sigma_q \quad \forall\ q\in Q \\ \Updownarrow \\ \exists\ \sigma \in \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}} \mathscr{P}} \text{ s.t. } \pi_q(\sigma) = \sigma_q \quad \forall\ q\in Q. \end{gathered} \tag{6.32}$$

We now are able to generalize the results given in Theorem 6.2 on semantics preservation to the networks of tagged systems case.

Theorem 6.4. *For the networks,* $(\Gamma, \mathscr{P}, \mathscr{M}, \alpha)$ *and* $(\Gamma, \mathscr{P}, \mathscr{I}_{\mathscr{S}}, \mathrm{Res}^{\rho})$*,*

$$\|_{\mathscr{M}} \mathscr{P} \equiv \|_{\mathscr{I}_{\mathscr{S}}} \mathscr{P}$$

if and only if for all $(\sigma_q)_{q\in Q} \in \prod_{q\in Q} \Sigma_q$ *and all* $e \in E$*:*

$$\alpha_e(\sigma_{\mathsf{sor}(e)}) = \alpha'_e(\sigma_{\mathsf{tar}(e)}) \quad \Leftrightarrow \quad \mathrm{Res}^{\rho_e}_{\mathsf{sor}(e)}(\sigma_{\mathsf{sor}(e)}) = \mathrm{Res}^{\rho'_e}_{\mathsf{tar}(e)}(\sigma_{\mathsf{tar}(e)}).$$

Proof. **(Sufficiency:)** If

$$(\sigma_q)_{q\in Q} \in \prod_{q\in Q} \Sigma_q \text{ s.t. } \alpha_e(\sigma_{\mathsf{sor}(e)}) = \alpha'_e(\sigma_{\mathsf{tar}(e)}) \quad \forall\ e \in E$$

$$\begin{aligned}
&\overset{\text{by (6.26)}}{\Rightarrow} && (\sigma_q)_{q\in Q} \in \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}) \\
&\overset{\text{by (6.29)}}{\Rightarrow} && \textstyle\bigsqcup_{\mathscr{M}}(\sigma_q)_{q\in Q} \in \Sigma_{\|_{\mathscr{M}}\mathscr{P}} \text{ where} \\
& && \qquad \pi_q\left(\textstyle\bigsqcup_{\mathscr{M}}(\sigma_q)_{q\in Q}\right) = \sigma_q \quad \forall\ q \in Q \\
&\overset{\text{by (6.32)}}{\Rightarrow} && \exists\ \sigma \in \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}} \text{ s.t. } \pi_q(\sigma) = \sigma_q \quad \forall\ q \in Q \\
&\overset{\text{by (6.28)}}{\Rightarrow} && (\sigma_q)_{q\in Q} \in \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathsf{I}_{\mathscr{S}}},\mathrm{Res}^{\rho})}) \\
&\overset{\text{by (6.31)}}{\Rightarrow} && \mathrm{Res}^{\rho_e}_{\mathsf{sor}(e)}(\sigma_{\mathsf{sor}(e)}) = \mathrm{Res}^{\rho'_e}_{\mathsf{tar}(e)}(\sigma_{\mathsf{tar}(e)}) \quad \forall\ e \in E.
\end{aligned}$$

The converse direction proceeds in the same manner: if

$$(\sigma_q)_{q\in Q} \in \prod_{q\in Q} \Sigma_q \text{ s.t. } \mathrm{Res}^{\rho_e}_{\mathsf{sor}(e)}(\sigma_{\mathsf{sor}(e)}) = \mathrm{Res}^{\rho'_e}_{\mathsf{tar}(e)}(\sigma_{\mathsf{tar}(e)}) \quad \forall\ e \in E$$

$$\begin{aligned}
&\overset{\text{by (6.29)}}{\Rightarrow} && \textstyle\bigsqcup_{\mathscr{I}_{\mathscr{S}}}(\sigma_q)_{q\in Q} \in \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}} \text{ where} \\
& && \qquad \pi_q\left(\textstyle\bigsqcup_{\mathscr{I}_{\mathscr{S}}}(\sigma_q)_{q\in Q}\right) = \sigma_q \quad \forall\ q \in Q \\
&\overset{\text{by (6.32)}}{\Rightarrow} && \exists\ \sigma' \in \Sigma_{\|_{\mathscr{M}}\mathscr{P}} \text{ s.t. } \pi_q(\sigma') = \sigma_q \quad \forall\ q \in Q \\
&\overset{\text{by (6.28)}}{\Rightarrow} && (\sigma_q)_{q\in Q} \in \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}) \\
&\overset{\text{by (6.26)}}{\Rightarrow} && \alpha_e(\sigma_{\mathsf{sor}(e)}) = \alpha'_e(\sigma_{\mathsf{tar}(e)}) \quad \forall\ e \in E.
\end{aligned}$$

(Necessity:) We have the following implications:

$$\exists\ \sigma' \in \Sigma_{\|_{\mathscr{M}}\mathscr{P}} \text{ s.t. } \pi_q(\sigma') = \sigma_q \quad \forall\ q \in Q$$

$$\begin{aligned}
&\overset{\text{by (6.28)}}{\Rightarrow} && (\sigma_q)_{q\in Q} \in \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{S}_{(\Sigma_{\mathscr{P}},\Sigma_{\mathscr{M}},\alpha)}) \\
&\overset{\text{by (6.26)}}{\Rightarrow} && \alpha_e(\sigma_{\mathsf{sor}(e)}) = \alpha'_e(\sigma_{\mathsf{tar}(e)}) \quad \forall\ e \in E \\
&\Rightarrow && \mathrm{Res}^{\rho_e}_{\mathsf{sor}(e)}(\sigma_{\mathsf{sor}(e)}) = \mathrm{Res}^{\rho'_e}_{\mathsf{tar}(e)}(\sigma_{\mathsf{tar}(e)}) \quad \forall\ e \in E \\
&\overset{\text{by (6.29)}}{\Rightarrow} && \textstyle\bigsqcup_{\mathscr{I}_{\mathscr{S}}}(\sigma_q)_{q\in Q} \in \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}} \text{ and} \\
& && \qquad \pi_q\left(\textstyle\bigsqcup_{\mathscr{I}_{\mathscr{S}}}(\sigma_q)_{q\in Q}\right) = \sigma_q \quad \forall\ q \in Q.
\end{aligned}$$

Therefore $\bigsqcup_{\mathscr{I}_{\mathscr{S}}}(\sigma_q)_{q\in Q}$ is the element of $\Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}}$ such that $\pi_q\left(\bigsqcup_{\mathscr{I}_{\mathscr{S}}}(\sigma_q)_{q\in Q}\right) = \sigma_q$ for all $q \in Q$.

The other direction follows in the same way:

$$\exists\, \sigma \in \Sigma_{\|_{\mathscr{I}_{\mathscr{S}}}\mathscr{P}} \text{ s.t. } \pi_q(\sigma) = \sigma_q \quad \forall\, q \in Q$$

$$\Rightarrow \qquad \textstyle\bigsqcup_{\mathscr{M}} (\sigma_q)_{q\in Q} \in \Sigma_{\|_{\mathscr{M}}\mathscr{P}} \text{ and } \pi_q\left(\bigsqcup_{\mathscr{M}} (\sigma_q)_{q\in Q}\right) = \sigma_q \quad \forall\, q \in Q,$$

by (6.26), (6.28) and (6.29), so $\bigsqcup_{\mathscr{M}} (\sigma_q)_{q\in Q}$ is the element of $\Sigma_{\|_{\mathscr{M}}\mathscr{P}}$ such that

$$\pi_q\left(\bigsqcup_{\mathscr{M}} (\sigma_q)_{q\in Q}\right) = \sigma_q$$

for all $q \in Q$. □

6.5.2 Network desyncronization. Let $\mathscr{T}$ and $\mathscr{T}'$ be two tag structures and $\rho : \mathscr{T} \to \mathscr{T}'$ be a morphism between these tag structures. By slight abuse of notation, let $(\Gamma, \mathscr{T}, \mathscr{T}', \rho)$ denote the network of tag structures such that $\mathscr{T}_q = \mathscr{T}$ for all $q \in Q$ and $\mathscr{T}'_e = \mathscr{T}'$, $\rho_e = \rho'_e = \rho$ for all $e \in E$; denote the corresponding network of tagged systems by $(\Gamma, \mathscr{P}, \mathscr{I}_{\mathscr{T}'}, \mathrm{Res}^{\rho})$. Similarly, let $(\Gamma, \mathscr{T}, \mathscr{T}, \mathrm{id})$ denote the network of tag structures with $\mathscr{T}_{\mathsf{sor}(e)} = \mathscr{T}_{\mathsf{tar}(e)} = \mathscr{T}_e = \mathscr{T}$ for all $e \in E$, and with all morphisms of tag structures being the identity; denote the corresponding network of tagged systems by $(\Gamma, \mathscr{P}, \mathscr{I}, \mathrm{Res}^{\mathrm{id}})$. Therefore, this network consists of a set of tagged systems, all with the same tag structure, communicating through the identity tagged system. A special case in which this framework in interesting is when $\mathscr{T}' = \mathscr{T}_{\mathrm{triv}} = \{*\}$; in this case $(\Gamma, \mathscr{P}, \mathscr{I}_{\mathscr{T}'}, \mathrm{Res}^{\rho})$ is the *desynchronization* of $(\Gamma, \mathscr{P}, \mathscr{I}, \mathrm{Res}^{\mathrm{id}})$.

Utilizing the notation of Section 6.4, and generalizing the discussion on desynchronization given in this Section 6.2, we are interested in when

$$\|\mathscr{P} := \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{N}_{(\mathscr{T},\mathscr{T},\mathrm{id})}) \equiv \mathrm{Lim}_{\mathfrak{N}_\Gamma}(\mathbf{N}_{(\mathscr{T},\mathscr{T}',\rho)}) =: \|_{\mathscr{I}_{\mathscr{T}'}}\mathscr{P}.$$

In other words, we would like to know when $\mathscr{I}_{\mathscr{T}'}$ is semantics preserving. The following corollary (of Theorem 6.4) says that this happens exactly when every element of $\mathscr{I}_{\mathscr{T}'}$ is semantics preserving.

Corollary 6.2. *$\mathscr{I}_{\mathscr{T}'}$ is semantics preserving, $\|\mathscr{P} \equiv \|_{\mathscr{I}_{\mathscr{T}'}}\mathscr{P}$, if and only if for all $(\sigma_q)_{q\in Q} \in \prod_{q\in Q}\Sigma_q$ and all $e \in E$:*

$$\sigma^{\rho}_{\mathsf{sor}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}} = \sigma^{\rho}_{\mathsf{tar}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}} \quad \Rightarrow \quad \sigma_{\mathsf{sor}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}} = \sigma_{\mathsf{tar}(e)}|_{V_{\mathsf{sor}(e)}\cap V_{\mathsf{tar}(e)}}.$$

Appendix A

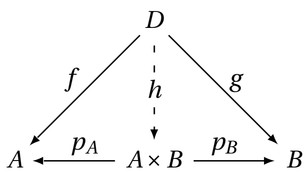
Limits

Central to category theory, and hence to all of mathematics, is the notion of a *universal property*, which characterizes objects that share a certain property, i.e., objects displaying such a property are unique up to isomorphism. Examples abound in category theory (e.g., products and limits) but also appear in engineering, although almost never recognized (e.g., stability is a universal property).

This appendix reviews some of the most fundamental universal constructions in a category: products, equalizers, pullbacks and limits. This is done in order to provide the necessary preliminaries for Chapter 6.

A.1 Products

A.1.1 Binary products. Let C be a category and A and B two objects in C. The *product* of these objects, if it exists, is an object $C = A \times B$ of C together with a pair of morphisms $p_A : A \times B \to A$ and $p_B : A \times B \to B$ (called *projections*). It must satisfy the *universal property* that for any other object D of C together with a pair of morphisms $f : D \to A$ and $g : D \to B$, there exists a unique morphism $h : D \to A \times B$ making the following diagram:



commute. The morphism h is typically denoted by (f, g).

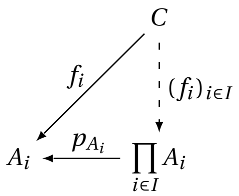
Example A.1. For two sets A and B, the binary product of these sets is the usual cartesian product:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

A.1.2 Products. Let C be a category, I be a set and $\{A_i\}_{i\in I}$ a set of objects of C. The product of these objects is an object $\prod_{i\in I} A_i$ of C together with *projections* $p_{A_i} : \prod_{i\in I} A_i \to A_i$ satisfying the universal property that for any other object C of C with morphisms $f_i : C \to A_i$, there exists a unique morphism

$$(f_i)_{i\in I} : C \to \prod_{i\in I} A_i$$

making the following diagram

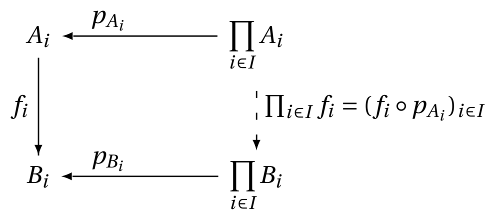


commute for all $i \in I$.

Given two sets of objects $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ together with morphisms $f_i : A_i \to B_i$, there is a unique induced morphism

$$\prod_{i\in I} f_i = (f_i \circ p_{A_i})_{i\in I} : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

making the following diagram



commute.

Definition A.1. A category C is said to **have products**, or products **exist** in C, if for any set of objects $\{A_i\}_{i\in I}$ in C, the product $\prod_{i\in I} A_i$ exists.

Remark A.1. Products of the form given in Definition A.1 are often referred to as *small products.* Sometimes finite products—I is a finite set—are often of interest.

Example A.2. The category of sets, Set, has products. For a set of sets $\{X_i\}_{i\in I}$, the product is the usual cartesian product:

$$\prod_{i\in I} X_i = \{(x_i)_{i\in I} : x_i \in X_i\}.$$

The projections are defined as

$$\begin{aligned} p_i : \prod_{i\in I} X_i &\to X_i \\ (x_i)_{i\in I} &\mapsto x_i. \end{aligned}$$

To verify the universal property of the product, consider a set D and functions $f_i : D \to X_i$. From these we obtain a function $f : D \to \prod_{i\in I} X_i$ given by:

$$f(y) = (f_i(y))_{i\in I}$$

for all $y \in D$.

A.1.3 Products of graphs. The category of graphs, Grph, has products. For a set of graphs $\{\Gamma_i = (Q_i, E_i)\}_{i\in I}$, the product is induced from the product on sets as follows:

$$\prod_{i\in I} \Gamma_i = (\prod_{i\in I} Q_i, \prod_{i\in I} E_i).$$

The source and target maps for the product graph $\prod_{i\in I}\Gamma_i$ are defined to be the unique maps making the following diagrams commute:

$$\begin{array}{ccc} E_i & \xleftarrow{p_{E_i}} & \prod_{i\in I} E_i \\ {\scriptstyle \mathsf{sor}_i}\downarrow & & \downarrow{\scriptstyle \prod_{i\in I}\mathsf{sor}_i} \\ Q_i & \xleftarrow{p_{Q_i}} & \prod_{i\in I} Q_i \end{array} \qquad \begin{array}{ccc} E_i & \xleftarrow{p_{E_i}} & \prod_{i\in I} E_i \\ {\scriptstyle \mathsf{tar}_i}\downarrow & & \downarrow{\scriptstyle \prod_{i\in I}\mathsf{tar}_i} \\ Q_i & \xleftarrow{p_{Q_i}} & \prod_{i\in I} Q_i \end{array}$$

Specifically, for $(e_i)_{i\in I} \in \prod_{i\in I} E_i$,

$$\prod_{i\in I} \mathsf{sor}_i((e_i)_{i\in I}) = (\mathsf{sor}_i(e_i))_{i\in I}, \qquad \prod_{i\in I} \mathsf{tar}_i((e_i)_{i\in I}) = (\mathsf{tar}_i(e_i))_{i\in I}.$$

Note that the projections are defined by:

$$p_{\Gamma_i} := (p_{E_i}, p_{Q_i}) : \coprod_{i\in I} \Gamma_i \to \Gamma_i.$$

Finally, we must verify the universal property of the product. Consider a graph $\Gamma = (Q, E)$ together with a collection of morphisms $F_i = ((F_Q)_i, (F_E)_i) : \Gamma \to \Gamma_i$. It follows from the universality of products in the category of sets that the diagrams given in Table A.1 are commutative. So $F = (F_Q, F_E) : \Gamma \to \prod_{i\in I}\Gamma_i$ is the desired unique morphism.

A.1.4 Products in categories of hybrid objects. The existence of products in C relates to the existence of products in $\mathsf{Hy(C)}$. In order to establish this relationship, we need to show that products exist in Dcat and that if products exist for C then they exist for C^{J} for any small category J. These two results are then "glued" together to yield products in $\mathsf{Hy(C)}$.

Proposition A.1. *Products exist in* Dcat. *Specifically, for* $\{\mathscr{A}_i\}_{i\in I}$ *a set of D-categories, the product* $\prod_{i\in I}\mathscr{A}_i$ *exists and is given by:*

$$\prod_{i\in I} \mathscr{A}_i = \mathrm{dcat}(\prod_{i\in I} \mathrm{grph}(\mathscr{A}_i)),$$

where $\prod_{i\in I}\mathrm{grph}(\mathscr{A}_i)$ *is the product of graphs.*

Proof. This follows from the fact that dcat and grph are isomorphisms between categories (Theorem 1.1). Specifically, the projections:

$$P_i : \prod_{i\in I} \mathrm{grph}(\mathscr{A}_i) \to \mathrm{grph}(\mathscr{A}_i)$$

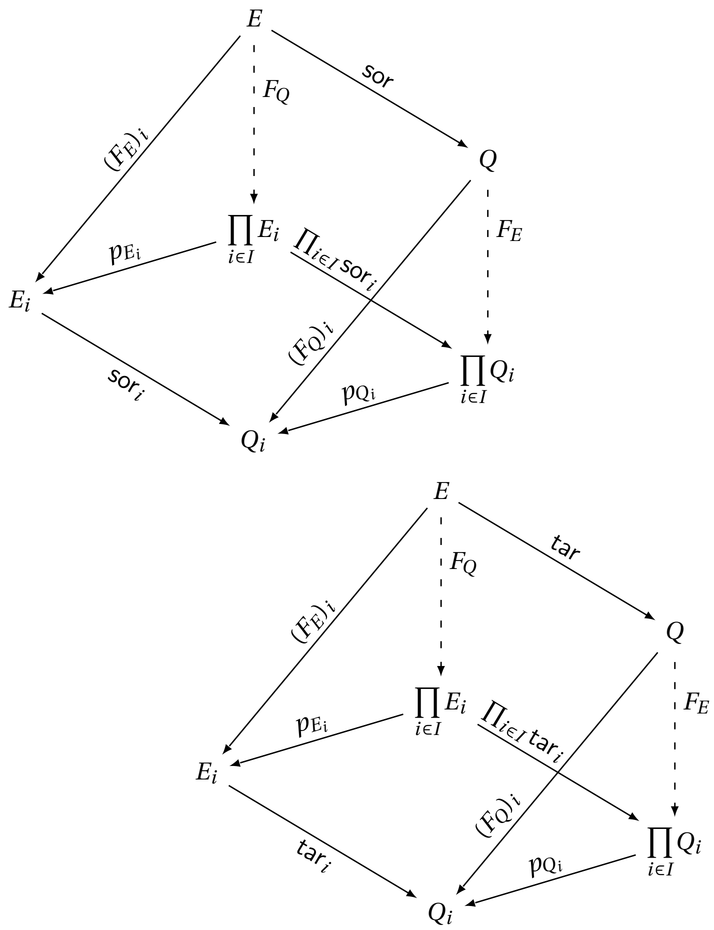
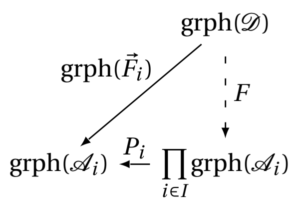


Table A.1: Commuting diagrams verifying the universality of the product in Grph.

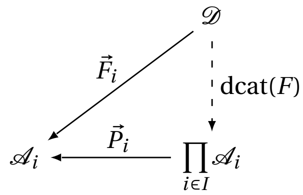
yield projections of D-categories:

$$\vec{P}_i := \mathrm{dcat}(P_i) : \prod_{i\in I} \mathscr{A}_i = \mathrm{dcat}(\prod_{i\in I} \mathrm{grph}(\mathscr{A}_i)) \to \mathscr{A}_i = \mathrm{dcat}(\mathrm{grph}(\mathscr{A}_i)).$$

Now, to verify universality, for any other D-category $\mathscr{D}$ with morphisms $\vec{F}_i : \mathscr{D} \to \mathscr{A}_i$ there is a graph $\mathrm{grph}(\mathscr{D})$ and morphisms $\mathrm{grph}(\vec{F}_i) : \mathrm{grph}(\mathscr{D}) \to \mathrm{grph}(\mathscr{A}_i)$. By the universality of the product in Grph, there exists a unique morphism F making the following diagram



commute. Applying the functor dcat yields a commuting diagram



where dcat(F) must be unique; if there were another morphisms making the diagram commute, it would also make the corresponding diagram of graphs commute, thus violating the uniqueness of F. □

Lemma A.1. *If products exist in* C*, then products exists in* C^{J} *for any small category* J*. Specifically, for a set of functors* $F_i : \mathsf{J} \to \mathsf{C}$*,* $i \in I$*, the product is given on objects and morphisms by:*

$$(\prod_{i\in I} F_i)(a) = \prod_{i\in I} F_i(a), \qquad (\prod_{i\in I} F_i)(\alpha) = \prod_{i\in I} F_i(\alpha).$$

Proof. See [74], Theorem 1, page 115. □

Proposition A.2. *If products exist in* C*, then products exist in* $\mathsf{Hy}(\mathsf{C})$*. Specifically, for a set of hybrid objects* $\{(\mathscr{A}_i, \mathbf{A}_i)\}_{i\in I}$*, the product is given by:*

$$\prod_{i\in I}(\mathscr{A}_i, \mathbf{A}_i) = (\prod_{i\in I}\mathscr{A}_i, \prod_{i\in I}\vec{P}_i^*(\mathbf{A}_i))$$

where $\prod_{i\in I}\mathscr{A}_i$ *is the product of D-categories,* $\prod_{i\in I}\vec{P}_i^*(\mathbf{A}_i)$ *is the product in* $\mathsf{C}^{\prod_{i\in I}\mathscr{A}_i}$ *with* $\vec{P}_i : \prod_{i\in I}\mathscr{A}_i \to \mathscr{A}_i$ *the projection morphisms in* Dcat*.*

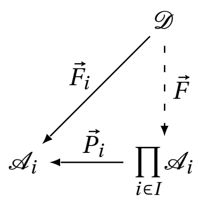
Proof. The projection morphisms are given by:

$$(\vec{P}_i, \vec{p}_i) : \prod_{i\in I}(\mathscr{A}_i, \mathbf{A}_i) \to (\mathscr{A}_i, \mathbf{A}_i),$$

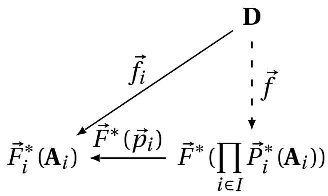
where $\vec{P}_i : \prod_{i\in I}\mathscr{A}_i \to \mathscr{A}_i$ and

$$\vec{p}_i : \prod_{i\in I}\vec{P}_i^*(\mathbf{A}_i) \stackrel{\cdot}{\to} \vec{P}_i^*(\mathbf{A}_i)$$

is objectwise the projection in C. We must verify the universality of the product. Consider a hybrid object $(\mathscr{D}, \mathbf{D})$ together with morphisms $(\vec{F}_i, \vec{f}_i) : (\mathscr{D}, \mathbf{D}) \to (\mathscr{A}_i, \mathbf{A}_i)$. By the universality of the product in Dcat, there exists a unique morphism $\vec{F} : \mathscr{D} \to \prod_{i\in I}\mathscr{A}_i$ yielding a commuting diagram



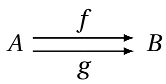
Therefore, we need only find a unique natural transformation $\vec{f} : \mathbf{D} \dot{\to} \vec{F}^*(\prod_{i\in I} \vec{P}_i^*(\mathbf{A}_i))$ in $\mathsf{C}^{\mathscr{D}}$. Since $\vec{f}_i : \mathbf{D} \dot{\to} \vec{F}_i^*(\mathbf{A}_i)$ there is a commuting diagram



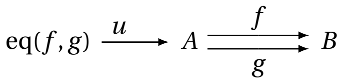
in $\mathsf{C}^{\mathscr{D}}$ where the existence and uniqueness of $\vec{f}$ follows from the universal property of the product in $\mathsf{C}^{\mathscr{D}}$. □

A.2 Equalizers and Pullbacks

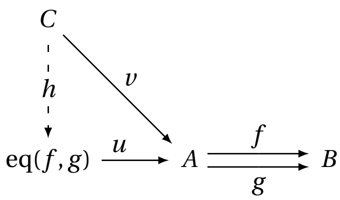
A.2.1 Equalizers For a category C and a pair of morphisms:



between two object A and B of C, the equalizer of this pair is an object $\mathrm{eq}(f,g)$ of C together with a morphism $u : \mathrm{eq}(f,g) \to B$ making diagram



commute, i.e., $f \circ u = g \circ u$. In addition, it must satisfying the universal property that for any other object C with a morphism $v : C \to A$ such that $f \circ v = g \circ v$, there exists a unique morphism $h : C \to \mathrm{eq}(f,g)$ such that the following diagram commutes:



Definition A.2. A category C is said to **have equalizers** if for any pair of morphism $f, g : A \to B$ between any pair of objects in C, the equalizer exists.

Example A.3. In the category of sets, Set, equalizers exist. For two sets X and Y and two functions $f, g : X \to Y$, the equalizer is given by:

$$\mathrm{eq}(f,g) = \{x \in X : f(x) = g(x)\},$$

with $u : \mathrm{eq}(f,g) \to X$ the inclusion. For a set Z and a morphism $h : Z \to X$ such that $f \circ h = g \circ h$, then $h : Z \to \mathrm{eq}(f,g)$ by the definition of $\mathrm{eq}(f,g)$, and hence is unique.

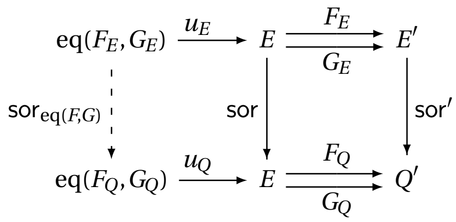
A.2.2 Equalizers in Grph**.** For a diagram in Grph of the form:

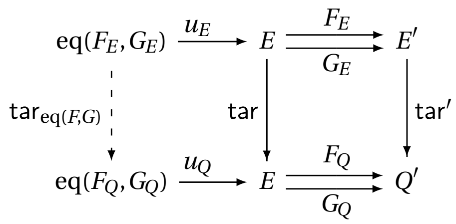
$$\Gamma = (Q,E) \underset{G=(G_Q,G_E)}{\overset{F=(F_Q,F_E)}{\rightrightarrows}} \Gamma' = (Q',E'),$$

the equalizer of this pair of morphisms exists. It is given by:

$$\mathrm{eq}(F,Q) = (\mathrm{eq}(F_Q,G_Q), \mathrm{eq}(F_E,G_E)),$$

where the equalizers on the right are in the category of sets. The source and target functions for $\mathrm{eq}(F,G)$ are given uniquely by requiring that the following diagrams commute:





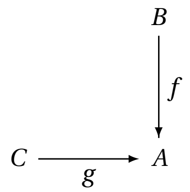
Note that the uniqueness of the source and target functions are due to the universality of equalizers in Set. It also follows from the definition of equalizers in Set that

$$\mathsf{sor}_{\mathrm{eq}(F,G)} = \mathsf{sor}|_{\mathrm{eq}(F_E,G_E)}, \qquad \mathsf{tar}_{\mathrm{eq}(F,G)} = \mathsf{tar}|_{\mathrm{eq}(F_E,G_E)},$$

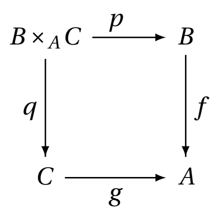
since u_E and u_Q are inclusions.

The universality of the equalizer in Grph is easy to verify (it is a simple exercise in diagram chasing).

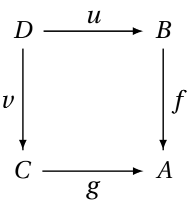
A.2.3 Pullbacks. Consider a category C and a diagram of the form:



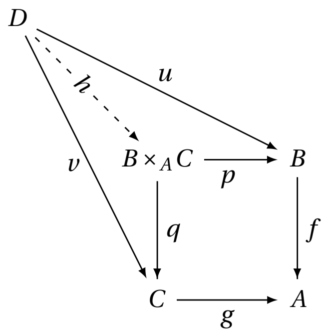
The pullback of this diagram is an object $B \times_A C$ of C together with two morphisms p and q such that the following diagram



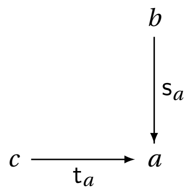
commutes. It is universal in the following sense: for any other object D of C with morphisms u and v making the following diagram commute



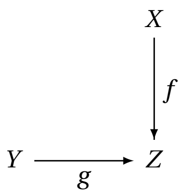
there exists a unique morphism $h : D \to B \times_A C$ such that the following diagram commutes:



Pullbacks are very useful when dealing with networks over a category since the canonical D^{op}-category is of the form:



Example A.4. In the category of sets, pullbacks exists. Specifically, for a diagram of sets of the form:



The pullback is given by:

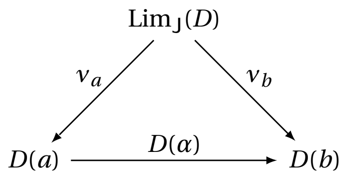
$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

A.3 Limits

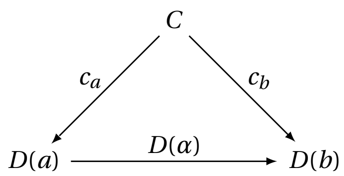
A.3.1 Limits For a category C and a functor $D : \mathsf{J} \to \mathsf{C}$ the limit, if it exists, is an object of C, denoted by $\mathrm{Lim}_{\mathsf{J}}(D)$, together with morphisms:

$$v_a : \mathrm{Lim}_{\mathsf{J}}(D) \to D(a), \qquad a \in \mathsf{Ob}(\mathsf{J}),$$

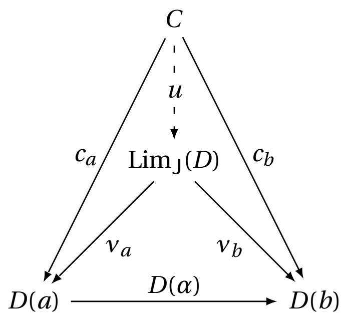
such that for every $\alpha : a \to b$ in J, the following diagram



commutes. In addition it is required to satisfy the universal property that for any object C of C with morphisms $c_a : C \to D(a)$, $a \in \mathsf{Ob}(\mathsf{J})$, such that there is a commuting diagram:



there exists a unique morphism $u : C \to \mathrm{Lim}_{\mathsf{J}}(D)$ making the following diagram

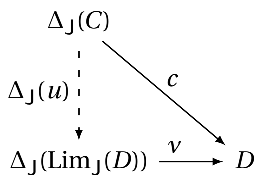


commute.

The notion of a limit perhaps can be better understood utilizing the language of natural transformations. For the constant functor $\Delta_{\mathsf{J}} : \mathsf{C} \to \mathsf{C}^{\mathsf{J}}$, the limit of D is an object $\mathrm{Lim}_{\mathsf{J}}(D)$ of C together with a *universal* natural transformation:

$$\nu : \Delta_{\mathsf{J}}(\mathrm{Lim}_{\mathsf{J}}(D)) \xrightarrow{\bullet} D.$$

It must be *universal* in the following sense: for any other object C of C and natural transformation $c : \Delta_{\mathsf{J}}(C) \xrightarrow{\bullet} D$, there exists a unique morphism $u : C \to \mathrm{Lim}_{\mathsf{J}}(D)$ such that the following diagram commutes



in C^{J}.

Definition A.3. A category C is **complete** if for every small category J and every functor $D : \mathsf{J} \to \mathsf{C}$, the limit exists.

Example A.5. The category of sets, Set, is the canonical example of a complete category.

The category of small categories Cat is complete; this completeness is directly a result of the completeness of Set. In fact, one might be tempted to say that the category of D-categories is complete since the category of small categories is complete. The problem with this logic is that there is no guarantee that the limit of diagram in Dcat is again in Dcat.

Another example of a complete category is the category of graphs, Grph, which is again complete because the category of sets is complete. It turns out that the completeness of this category does imply the completeness of Dcat, which is not surprising in light of the isomorphism Dcat $\cong$ Grph.

A.3.2 Special cases of the limit. The limit includes as a special case all of the previous universal constructions we have introduced. Specifically,

Products. The limit of a functor

$$D : \mathsf{I} \to \mathsf{C},$$

where I is the discrete category obtained from an indexing set I.

Equalizers. The limit of a functor

$$D : (\bullet \rightrightarrows \bullet) \to \mathsf{C}.$$

Pullback. The limit of a functor

$$D : (\bullet \to \bullet \leftarrow \bullet) \to \mathsf{C}.$$

Interestingly enough, the existence of limits in a category is related to the existence of equalizers and products.

Proposition A.3. *A category* C *is complete iff it has equalizers and products.*

Proof. See Corollary 2, page 113, [74]. □

Corollary A.1. *The category of graphs,* Grph*, is complete.*

A corollary of this is that the category of D-categories is complete. Before stating this result, we introduce some notation.

Notation A.1. To differentiate, when necessary, between limits in different categories, we sometimes write $\mathrm{Lim}_{\mathsf{J}}^{\mathsf{C}}$ for the limit of a functor $D : \mathsf{J} \to \mathsf{C}$. Similarly, we sometimes write $\Delta_{\mathsf{J}}^{\mathsf{C}}$.

Theorem A.1. *The category of D-categories,* Dcat*, is complete. Specifically, for a functor* $D : \mathsf{J} \to \mathsf{Dcat}$*,* J *small, the limit is given by:*

$$\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(D) = \mathrm{dcat}\left(\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Grph}}(\mathrm{grph}_*(D))\right).$$

Proof. Follows from the fact that dcat and grph are isomorphisms of categories; the proof is analogous to the proof of Proposition A.1. □

A.3.3 The limit as a functor. If C is a complete category, then the limit exists for every diagram over a small category J, i.e., for every functor $D : \mathsf{J} \to \mathsf{C}$. In fact, the universality of the limit implies that it defines a functor

$$\mathrm{Lim}_{\mathsf{J}} : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}.$$

Specifically, consider two functors $D, D' : \mathsf{J} \to \mathsf{C}$ together with the corresponding universal natural transformations:

$$\begin{aligned} \nu : \Delta_{\mathsf{J}}(\mathrm{Lim}_{\mathsf{J}}(D)) &\overset{\cdot}{\to} D, \\ \nu' : \Delta_{\mathsf{J}}(\mathrm{Lim}_{\mathsf{J}}(D')) &\overset{\cdot}{\to} D'. \end{aligned}$$

The object function of the limit (as a functor) associates to these functors their limit. For a morphism $f : D \overset{\cdot}{\to} D'$, the limit of this morphism is the unique morphism $\mathrm{Lim}_{\mathsf{J}}(f) : \mathrm{Lim}_{\mathsf{J}}(D) \to \mathrm{Lim}_{\mathsf{J}}(D')$ making the following diagram:

$$\begin{array}{ccc} \Delta_{\mathsf{J}}(\mathrm{Lim}_{\mathsf{J}}(D)) & \xrightarrow{\nu} & D \\ {\scriptstyle \Delta_{\mathsf{J}}(\mathrm{Lim}_{\mathsf{J}}(f))}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta_{\mathsf{J}}(\mathrm{Lim}_{\mathsf{J}}(D')) & \xrightarrow{\nu'} & D' \end{array}$$

commute.

An especially useful result is outlined in the following proposition (see [74], Theorem 1, page 115).

Proposition A.4. *If C is complete, then C^{K} is complete for every small category K. Specifically, for $D : \mathsf{J} \to \mathsf{C}^{\mathsf{K}}$, the limit is given on objects and morphisms of K by:*

$$\mathrm{Lim}_{\mathsf{J}}^{\mathsf{C}^{\mathsf{K}}}(D)(a) = \mathrm{Lim}_{\mathsf{K}}^{\mathsf{C}}(D(a)), \qquad \mathrm{Lim}_{\mathsf{J}}^{\mathsf{C}^{\mathsf{K}}}(D)(\alpha) = \mathrm{Lim}_{\mathsf{K}}^{\mathsf{C}}(D(\alpha)).$$

A.4 Limits in Categories of Hybrid Objects

As an application of these ideas, we will prove that if C is complete, then $\mathsf{Hy}(\mathsf{C})$ is complete. This result was established in [6]; in addition, it was proven that if C is cocomplete (the dual notion to completeness) then $\mathsf{Hy}(\mathsf{C})$ is complete.

A.4.1 Diagrams in categories of hybrid objects. By slight abuse of notation, we denote a diagram in $\mathsf{Hy}(\mathsf{C})$ by

$$(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C}).$$

That is, for every $\alpha : a \to b$ in J, there are corresponding hybrid objects and morphisms:

$$(\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a)) \xrightarrow{(\mathscr{D}^{\mathsf{J}}(\alpha), \mathbf{D}^{\mathsf{J}}(\alpha))} (\mathscr{D}^{\mathsf{J}}(b), \mathbf{D}^{\mathsf{J}}(b)).$$

In particular, $\mathscr{D}^{\mathsf{J}}(\alpha) : \mathscr{D}^{\mathsf{J}}(a) \to \mathscr{D}^{\mathsf{J}}(b)$ is a morphism of D-categories and

$$\mathbf{D}^{\mathsf{J}}(\alpha) : \mathbf{D}^{\mathsf{J}}(a) \stackrel{\centerdot}{\to} (\mathscr{D}^{\mathsf{J}}(\alpha))^{*}(\mathbf{D}^{\mathsf{J}}(b))$$

is a morphism in $\mathsf{C}^{\mathscr{D}^{\mathsf{J}}(a)}$.

Note that by the definition of a hybrid object, we can without ambiguity write $\mathscr{D}^{\mathsf{J}} : \mathsf{J} \to \mathsf{Dcat}$; note that $\mathscr{D}^{\mathsf{J}}$ is *not* a D-category, but a diagram of such categories. Since the category of D-categories is complete, there exists a D-category $\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})$ together with a universal natural transformation:

$$\vec{V} : \Delta_{\mathsf{J}}^{\mathsf{Dcat}}(\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})) \stackrel{\centerdot}{\to} \mathscr{D}^{\mathsf{J}}.$$

The motivation for denoting this natural transformation by $\vec{V}$ is that for every diagram of the form $\alpha : a \to b$ in J, there is a diagram of D-categories:

$$\begin{array}{ccc} & \mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}}) & \\ {}^{\vec{V}_a}\swarrow & & \searrow^{\vec{V}_b} \\ \mathscr{D}^{\mathsf{J}}(a) & \xrightarrow{\mathscr{D}^{\mathsf{J}}(\alpha)} & \mathscr{D}^{\mathsf{J}}(b) \end{array} \tag{A.1}$$

For the diagram $(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C})$ in $\mathsf{Hy}(\mathsf{C})$, the limit of $\mathscr{D}^{\mathsf{J}} : \mathsf{J} \to \mathsf{Dcat}$ yields a functor:

$$\vec{V}^{*}(\mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{C}^{\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}$$

defined on objects and morphisms of J by:

$$\vec{V}^{*}(\mathbf{D}^{\mathsf{J}})(a) := \vec{V}_a^{*}(\mathbf{D}^{\mathsf{J}}(a)), \qquad \vec{V}^{*}(\mathbf{D}^{\mathsf{J}})(\alpha) := \vec{V}_{\mathsf{dom}(\alpha)}^{*}(\mathbf{D}^{\mathsf{J}}(\alpha)).$$

Note that $\vec{V}^{*}(\mathbf{D}^{\mathsf{J}})$ is well-defined because of the commutativity of (A.1).

Using this notation, we can now prove that categories of hybrid objects are complete and give an explicit formula for the limit of a diagram.

Theorem A.2. *If* C *is complete, then* $\mathsf{Hy}(\mathsf{C})$ *is complete. Specifically, for* $(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C})$, *the limit is given by:*

$$\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Hy}(\mathsf{C})}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) = \left(\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}}), \mathrm{Lim}_{\mathsf{J}}^{\mathsf{C}^{\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{V}^{*}(\mathbf{D}^{\mathsf{J}}))\right).$$

Proof. The first step is to find the universal natural transformation in $\mathsf{Hy}(\mathsf{C})^{\mathsf{J}}$

$$\nu : \Delta_{\mathsf{J}}^{\mathsf{Hy}(\mathsf{C})}\left(\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Hy}(\mathsf{C})}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}})\right) \stackrel{\centerdot}{\to} (\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}).$$

There are universal natural transformations

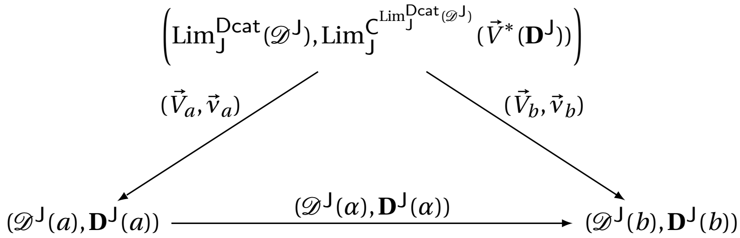
$$\begin{aligned} \vec{V} : \Delta_{\mathsf{J}}^{\mathsf{Dcat}}\left(\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})\right) &\stackrel{\centerdot}{\to} \mathscr{D}^{\mathsf{J}} \\ \vec{v} : \Delta_{\mathsf{J}}^{\mathsf{C}^{\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}\left(\mathrm{Lim}_{\mathsf{J}}^{\mathsf{C}^{\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{V}^{*}(\mathbf{D}^{\mathsf{J}}))\right) &\stackrel{\centerdot}{\to} \vec{V}^{*}(\mathbf{D}^{\mathsf{J}}) \end{aligned}$$

in $\mathsf{Dcat}^{\mathsf{J}}$ and $(\mathsf{C}^{\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})})^{\mathsf{J}}$.

The claim is that the universal transformation ν is given by $\nu = (\vec{V}, \vec{v})$. To verify this, first note that for an object a of J,

$$\nu_a = (\vec{V}_a, \vec{v}_a) : \mathrm{Lim}_{\mathsf{J}}^{\mathsf{Hy(C)}}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) = \left(\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}}), \mathrm{Lim}_{\mathsf{J}}^{\mathsf{C}^{\mathrm{Lim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{V}^*(\mathbf{D}^{\mathsf{J}}))\right) \stackrel{\bullet}{\to} (\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a))$$

since $\vec{V}_a^*(\mathbf{D}^{\mathsf{J}}(a)) = \vec{V}^*(\mathbf{D}^{\mathsf{J}})(a)$. Now, we need to verify that defining $\nu = (\vec{V}, \vec{v})$ in fact yields a natural transformation. That is, for $\alpha : a \to b$ in J, we need to show that there is a commuting diagram:

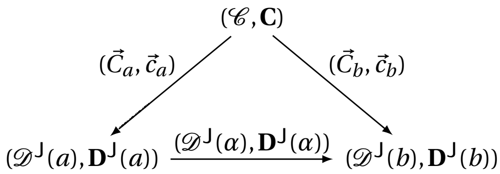


By the commutativity of (A.1), this follows from the fact that

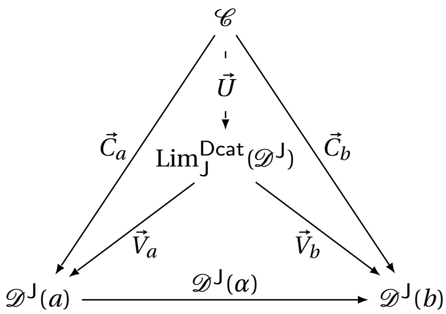
$$\vec{V}_a^*(\mathbf{D}^{\mathsf{J}}(\alpha)) \bullet \vec{v}_a = \vec{V}^*(\mathbf{D}^{\mathsf{J}})(\alpha) \bullet \vec{v}_a = \vec{v}_b,$$

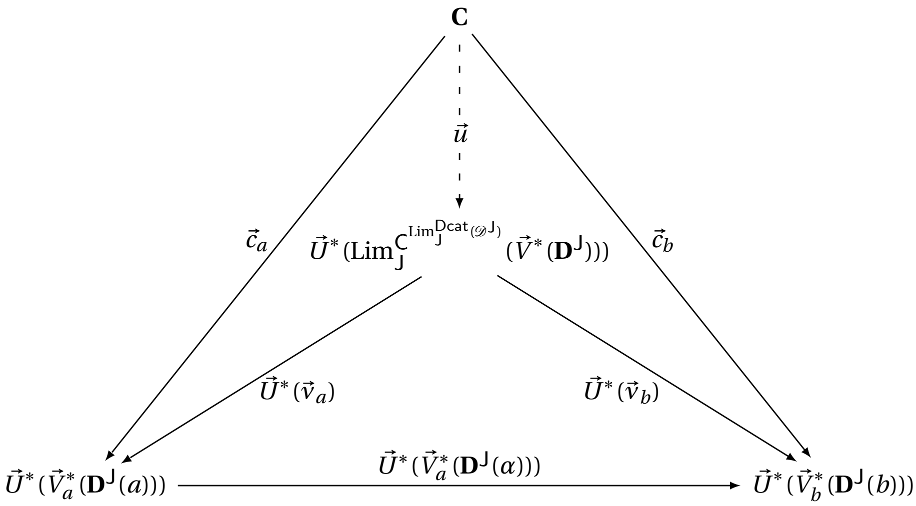
which is implied by the naturality of $\vec{v}$ and the definition of $\vec{V}^*(\mathbf{D}^{\mathsf{J}})$.

To conclude, we need only show the universality of $\nu = (\vec{V}, \vec{v})$. Suppose that there is a hybrid object $(\mathscr{C}, \mathbf{C})$ together with a collection of morphisms $(\vec{C}_a, \vec{c}_a) : (\mathscr{C}, \mathbf{C}) \to (\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a))$ of hybrid objects making the following following diagram



commute. This yields commuting diagrams and unique morphisms:

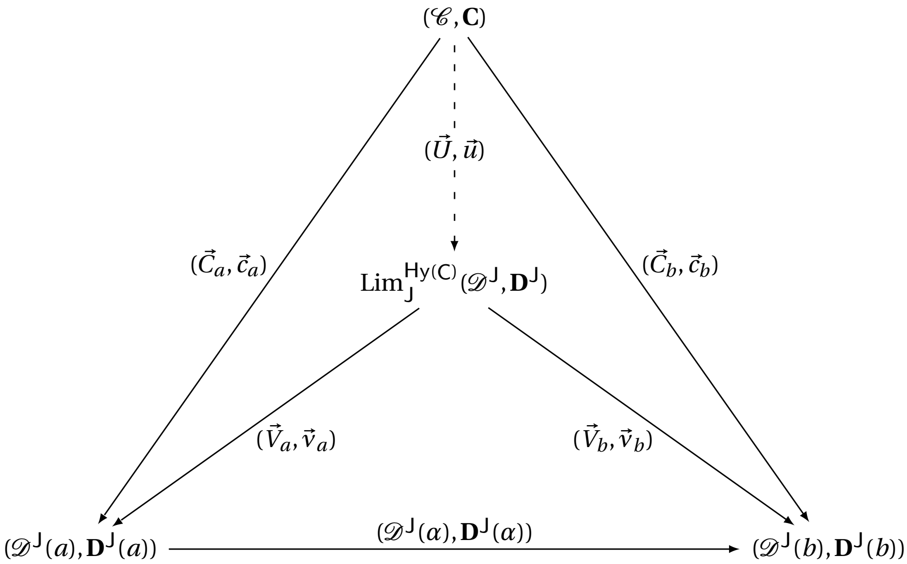




That is, we obtain a unique morphism of hybrid objects:

$$(\vec{U}, \vec{u}) : (\mathscr{C}, \mathbf{C}) \to \mathrm{Lim}_{\mathsf{J}}^{\mathsf{Hy(C)}}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}})$$

that makes the following diagram



commute as desired. □

Future Directions

This dissertation developed a unifying mathematical theory of hybrid systems, i.e., a theory satisfying Properties I-IV discussed in the introduction. Yet there is still much to be done. We, therefore, briefly discuss some future research directions.

Mathematical Foundations. The connections between modern mathematics and our categorical framework for hybrid systems have only begun to be established. The theory of hybrid objects makes the possibility of further connections not only likely, but certain. For example, hybrid objects allow us to utilize the theory of model categories, which provides a method for "doing homotopy theory" on general categories satisfying certain axioms. Understanding hybrid systems in the context of model categories allows one to understand the homotopy-theoretic properties of these systems, laying the ground work for *hybrid homotopy theory*. This promises to play a fundamental role in understanding the topological properties of hybrid systems. We refer the reader to the author's master's thesis [6] for more details.

Hybrid Systems. This dissertation discussed applications of the theory of hybrid objects to hybrid systems, but there is still much to be done. Understanding the implications of the results presented on a practical level provide important research directions. For example, we gave "Lyapunov-like" conditions on the existence of Zeno behavior. Can explicit "hybrid Lyapunov" functions be constructed in the case of linear hybrid systems? Can the conditions on the stability of Zeno equilibria be used to give analogous conditions on the stability of other types of "hybrid" equilibria? Applications of hybrid reduction to bipedal robotic walking also were discussed, although we restricted our attention to walkers without hips. Can these ideas be extended to the hipped walker case? Answers to these questions promises to further the general understanding of hybrid systems.

In addition to extensions of the ideas presented in this dissertation, categories of hybrid objects can provide a framework in which to address questions related to the relationship between different hybrid systems—this is one of the general strengths of category theory. For example, bisimulation relations have been well-studied in the hybrid systems community. Because there is a categorical formulation of bisimulation relations, it seems likely that it is possible to completely characterize bisimulation relations for hybrid objects, given a characterization for their non-hybrid counterparts.

Networked Systems. The first steps toward understanding how to compose networked systems in a "behavior preserving manner" were taken in the final chapter of this dissertation. Yet these first steps are tentative. We considered a very specific behavioral representation of embedded systems: the tagged system model. Can these ideas be extended to general behavioral representations of systems? We briefly discuss what such an extension may entail.

In networked systems, one typically considers a collection of simple systems whose behavior is understood. These systems are then interconnected. In general, there is no guarantee that properties of the simple systems constituting the network are preserved through interconnection. That is, one of the fundamental questions in networked systems is: how does one compose a network while preserving the behavior of the components of the network? A theory of behavior preserving composition, or *deep compositionality*, is needed. This is related to the notion of "semantics preservation" discussed in this dissertation, but semantics preservation does not appear to address this issue in its full generality. Is there a more general concept capturing the notion of property preserving composition? To answer this question, the notion of a property needs to be formulated mathematically and conditions need to be given on when taking the composition of a network (the limit) preserves a given property. The ability to do so could greatly increase the general understanding of networked systems.

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