Impact of phase and amplitude errors on array performance

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Technical Report No. UCB/EECS-2009-1
http://www.eecs.berkeley.edu/Pubs/TechRpts/2009/EECS-2009-1.html

January 1, 2009

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Abstract

This paper analyzes the effect of errors in antenna weights on the performance of adaptive array systems, both in the case when an array is used to maximize the gain in a desired direction and in the case when an array is used to null interfering signals. We begin by deriving an explicit characterization of the loss in array gain due to phase errors in the optimal antenna weights. Then, we examine interference rejection in the presence of amplitude and phase errors in the antenna weights. We prove that the loss in interference rejection is independent of the number of antennas. For both cases, we give numerical simulations that validate our analysis.

I. INTRODUCTION

Many modern communication systems employ adaptive antennas in order to improve their capacity, coverage, and reliability. Unlike conventional fixed antenna systems, adaptive antenna arrays dynamically adjust their beam patterns in response to their environment. Adaptive arrays can extend the range by focusing most of the radio frequency (RF) power on a desired target. This is known as beamforming or beam-steering. Adaptive arrays can also reject unwanted interference signals by placing nulls in the direction of the interferers, which is known as beam-nulling or null-steering. Even if the directions of the interferers are unknown, adaptive arrays can still reduce signal propagation in undesired directions by using side lobe suppression.

Adaptive arrays are composed of multiple antenna elements that can be arranged in different geometries (antennas are usually spaced at least half a wavelength apart) [1]. Larger arrays provide more gain and degrees of freedom. The beam pattern (the locations of peaks and nulls, and the heights of the side lobes) is shaped by controlling the amplitudes and phases of the RF signals transmitted and received from each antenna element. For this reason, adaptive arrays are often referred to as phased arrays. Precise control over both amplitudes and phases is required to achieve good performance. However, various factors such as finite resolution, noise, mismatch in circuit elements, and channel uncertainty limit the precision that can be achieved in practice. Many of these error sources are random, and cannot be compensated for using pre-calibration or adaptive signal processing techniques. The limited precision will degrade the performance of the array (gain and interference rejection). In this paper, we examine the impact of phase and amplitude errors on the array gain (beamforming) in Section II and on interference rejection (beam-nulling) in Section III. We provide both mathematical proofs and simulation results that characterize the array performance as a function of phase and amplitude errors.

II. BEAMFORMING

Consider the array of N elements shown in Figure 1. A signal $s[n]$, sent by a remote transmitter, arrives at each antenna i in the array shifted in phase by ψ_i[1]. Antenna i will then apply a phase-shift ϕ_i to the incoming signal. Therefore, the overall complex (baseband) channel response H at the output of the receiver array is given by[2]:

$$H = \sum_{i=1}^{N} e^{j(\psi_i - \phi_i)}$$

[1]In general, signals arrive at different antenna elements with different delays. However, for narrowband signals, time delays can be approximated with phase shifts [2]. We define signals as narrowband when the fractional bandwidth (the ratio between the signal bandwidth and carrier frequency) is very small (e.g. less than 1%). In this paper, we shall assume that all signals are narrowband.

[2]The channel response is identical in the scenario with multiple transmitters and a single receive antenna.

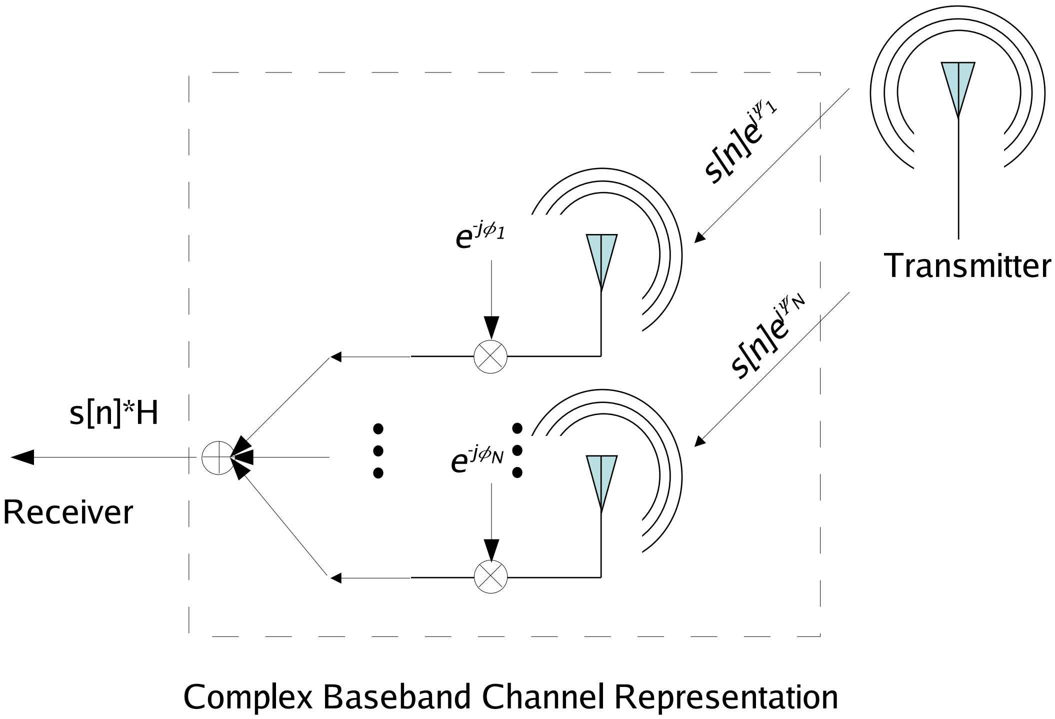

Fig. 1. A communication system with an adaptive array at the receiver. The narrowband signal $s[n]$ arrives at each antenna shifted in phase by ψ_i. The receiver applies a phase shift of ϕ_i at each antenna and sums the signals.

To maximize the magnitude of H, ϕ_i is chosen equal to ψ_i, in which case $|H|^2$ reaches its maximum value of $|H_{opt}|^2 = N^2$. In practice, however, factors such as quantization, clock jitter, and other sources of noise make it virtually impossible to realize the desired phase-shifts. Most of these errors are unpredictable and time varying, and are best modeled with random variables:

$$\hat{\psi}_i = \phi_i + \delta_i$$

We will assume that $\delta_i \sim U[-\delta_{max}, \delta_{max}]$, where $0 \leq \delta_{max} \leq 180°$ is an upper bound on the amplitude of phase deviation[3]. Furthermore, we assume that the errors are $i.i.d$ across different antennas. In this case the channel response becomes:

$$\hat{H}_{opt} = \sum_{i=1}^{N} e^{j\delta_i} = \sum_{i=1}^{N} \cos(\delta_i) + j\sum_{i=1}^{N} \sin(\delta_i)$$

We wish to characterize the effect of the phase errors on the square magnitude of the channel response. To simplify the analysis, we introduce two new random variables $X_i = \cos(\delta_i)$ and $Y_i = \sin(\delta_i)$ and compute the following expectations:

$$\mu_X = E[X_i] = E[\cos(\delta_i)] = \frac{1}{2\delta_{max}} \int_{-\delta_{max}}^{\delta_{max}} \cos(x)dx$$

$$= \frac{1}{\delta_{max}} \int_{0}^{\delta_{max}} \cos(x)dx = \frac{\sin(\delta_{max})}{\delta_{max}}$$

$$\mu_{X^2} = E[X_i^2] = \frac{1}{2\delta_{max}} \int_{-\delta_{max}}^{\delta_{max}} \cos^2(x)dx = \frac{1}{\delta_{max}} \int_{0}^{\delta_{max}} \cos^2(x)dx$$

[3]We assume a uniform distribution to simplify the calculations. Note that no assumptions were made regarding the geometry of the array or the direction of arrival, so the result holds for an arbitrary array.

$$= \frac{1}{2\delta_{max}} \int_0^{\delta_{max}} (1 + \cos(2x))dx = \frac{1}{2} + \frac{\sin(2\delta_{max})}{4\delta_{max}}$$

$$\mu_Y = E[Y_i] = E[\sin(\delta_i)] = 0 \quad \text{(by symmetry)}$$

$$\mu_{Y^2} = E[Y_i^2] = \frac{1}{2\delta_{max}} \int_{-\delta_{max}}^{\delta_{max}} \sin^2(x)dx = \frac{1}{\delta_{max}} \int_0^{\delta_{max}} \sin^2(x)dx$$

$$= \frac{1}{2\delta_{max}} \int_0^{\delta_{max}} (1 - \cos(2x))dx = \frac{1}{2} - \frac{\sin(2\delta_{max})}{4\delta_{max}}$$

Now, we can rewrite the expression for the channel response as:

$$\hat{H}_{opt} = \sum_{i=1}^{N} X_i + j \sum_{i=1}^{N} Y_i$$

$$\Rightarrow |\hat{H}_{opt}|^2 = \left(\sum_{i=1}^{N} X_i \right)^2 + \left(\sum_{i=1}^{N} Y_i \right)^2 = \sum_{k=1}^{N} \sum_{l=1}^{N} (X_k X_l + Y_k Y_l)$$

$$\Rightarrow E[|\hat{H}_{opt}|^2|] = \sum_{k=1}^{N} \sum_{l=1}^{N} (E[X_k X_l] + E[Y_k Y_l])$$

$$E[X_k X_l] = \begin{cases} E[X_k]E[X_l] = \mu_X^2 & \text{when } k \neq l \text{ (using independence)} \\ E[X_k^2] = \mu_{X^2} & \text{when } k = l \end{cases}$$

$$E[Y_k Y_l] = \begin{cases} E[Y_k]E[Y_l] = \mu_Y^2 & \text{when } k \neq l \text{ (using independence)} \\ E[Y_k^2] = \mu_{Y^2} & \text{when } k = l \end{cases}$$

$$\Rightarrow E[|\hat{H}_{opt}|^2|] = (N^2 - N)(\mu_X^2 + \mu_Y^2) + N(\mu_{X^2} + \mu_{Y^2})$$

$$= (N^2 - N) \left(\frac{\sin^2(\delta_{max})}{\delta_{max}^2} \right) + N = (N^2) \left(\frac{\sin^2(\delta_{max})}{\delta_{max}^2} \right) + N \left(1 - \frac{\sin^2(\delta_{max})}{\delta_{max}^2} \right)$$

If we normalize $E[|\hat{H}_{opt}|^2|]$ by dividing by the maximum value $|H_{opt}|^2 = N^2$, we obtain:

$$\Phi_N(\delta_{max}) = \frac{E[|\hat{H}_{opt}|^2|]}{N^2} = \frac{\sin^2(\delta_{max})}{\delta_{max}^2} + \frac{1}{N} \left(1 - \frac{\sin^2(\delta_{max})}{\delta_{max}^2} \right)$$

$$\Rightarrow \Phi(\delta_{max}) = \lim_{N \to \infty} \Phi_N(\delta_{max}) = \frac{\sin^2(\delta_{max})}{\delta_{max}^2}$$

Figure 2(a) shows a plot of $\Phi(\delta_{max})$ for $0 \leq \delta_{max} \leq 180°$. Figure 2(b) shows the same function in dB scale. Figure 2(b) also shows that the calculated array gain closely matches simulation results. Figure 2(c) shows that the actual distribution of the phase errors has little impact on the loss in array gain. Notice that using a single bit of phase resolution corresponds to $\delta_{max} = 90° = \frac{\pi}{2}$, and $\Phi(\frac{\pi}{2}) = (\frac{2}{\pi})^2 \approx .4 \approx -3.9dB$! We expect the bound to become tighter as N increases, due to the law of large numbers. The graphs in Figure 3 show the loss in array gain as a result of quantizing the phase to one and two bits. Figures 3(g,h) show that quantization does not increase the width of the main lobe. Also, notice that when $\delta_{max} = 180°$, which corresponds to completely randomizing the phase of each antenna, the normalized array response $\Phi_N(\pi) = \frac{1}{N}$, which reduces the array gain to that of an omni-directional antenna. So a simple way of creating an omni-like beampattern without reducing the radiated power is to choose the phases randomly[4].

[4]With omni-directional antennas, the absolute, non-normalized power of the signals adds.

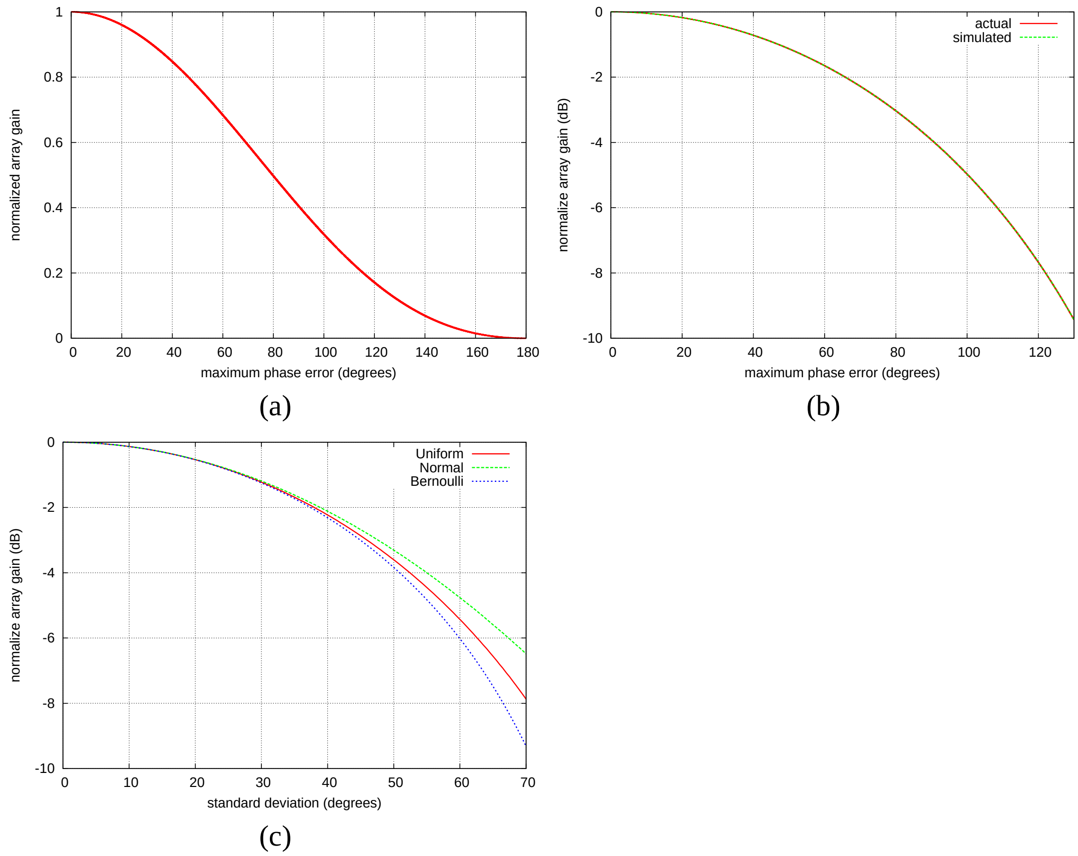

Fig. 2. (a) The normalized array gain $\Phi(\delta_{max})$ as a function of the maximum phase error δ_{max}. (b) The normalized array gain in dB scale, $10\log\Phi(\delta_{max})$. The plot shows both the calculated gain and the simulated gain for a 10000 element array. (c) The simulated gain (dB) for a 10000 element array with different phase error distributions. For a uniform distribution, the standard deviation is $\sigma_\delta = \delta_{max}/\sqrt{3}$.

A second method of proving a lower bound on the array gain is by using the mean of the random variable, which is often easier to compute, instead of the mean of the square of the random variable. By Jensen's inequality, the square of the mean of a random variable is less than or equal to the mean of its square:

$$E[X]^2 \leq E[X^2]$$

for any random variable X. More generally:

$$f(E[X]) \leq E[f(X)] \quad \text{when } f(\cdot) \text{ is a convex function.}$$

Using this fact, and the expected value of the channel response, we see that:

$$E[\hat{H}_{opt}] = E\left[\sum_{i=1}^{N} X_i + j\sum_{i=1}^{N} Y_i\right] = N\mu_X = N\frac{\sin(\delta_{max})}{\delta_{max}}$$

$$\Rightarrow E[|\hat{H}_{opt}|^2] \geq E[\hat{H}_{opt}]^2 = N^2\left(\frac{\sin(\delta_{max})}{\delta_{max}}\right)^2$$

$$\Phi_N(\delta_{max}) = \frac{E[|\hat{H}_{opt}|^2|]}{N^2} \geq \left(\frac{\sin(\delta_{max})}{\delta_{max}}\right)^2 = \Phi(\delta_{max})$$

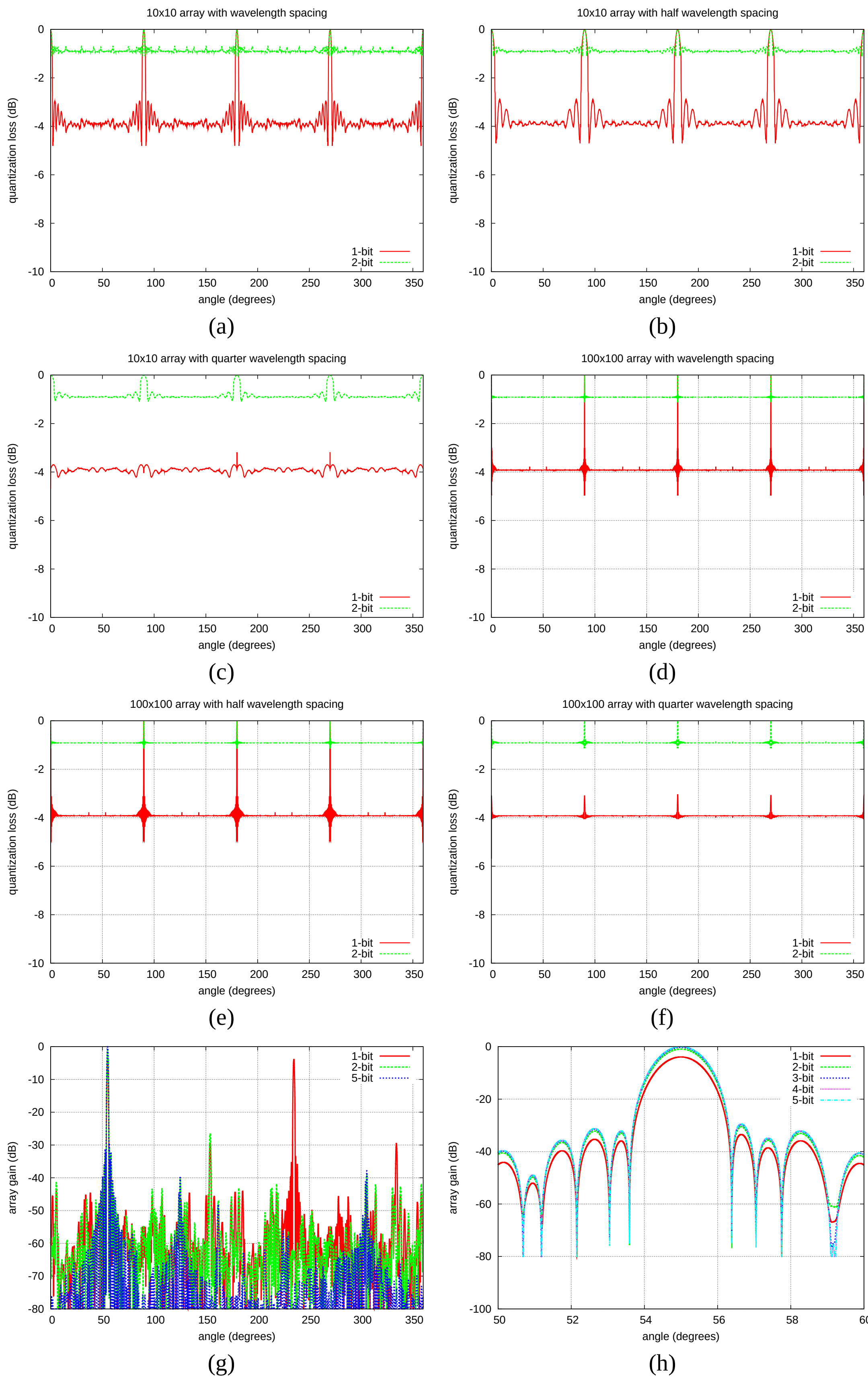

Fig. 3. (a)-(f) Array loss as a result of phase error for different size arrays. (g)-(h) 2-dimensional horizontal beampattern of a 100x100 array with $\lambda/2$ spacing (steered towards 55 degrees) for different phase resolutions.

III. Beam-nulling

In addition to maximizing SNR by steering the direction of the beam towards desired locations, many communication systems are faced with unwanted interference. In most of these systems, simply steering the direction of the peak is not sufficient to suppress large interferers and signal jammers. Other techniques, such as null-steering and side lobe suppression, are required to provide the necessary rejection of interfering signals.

Adaptive systems that require precise control of the locations of the nulls and side lobe levels need to adjust both the phase and amplitude response of each antenna element. In this case, we need to account for both phase and amplitude errors. Analyzing the combined effect of phase and amplitude errors is easier when we consider the problem in the spatial domain where the optimal complex beamforming weights and channel responses can be represented as complex vectors in the N-dimensional Euclidean space, where N is the number of antennas in the array. Let us assume that we have $K+1$ vectors: a desired vector $\mathbf{h}_d$ corresponding to the direction of the desired signal[5], and K interferering vectors $\mathbf{h}_i \;\; \forall_{1\leq i \leq K}$ corresponding to the directions of K interfering signals.

$$\mathbf{h}_d = [\alpha_{1d}e^{j\beta_{1d}}, \ldots, \alpha_{Nd}e^{j\beta_{Nd}}]^\top$$
$$\mathbf{h}_i = [\alpha_{1i}e^{j\beta_{1i}}, \ldots, \alpha_{Ni}e^{j\beta_{Ni}}]^\top \quad \forall_{1\leq i \leq K}$$

The incoming signal at the input of the array $\mathbf{y}[n]$ is the sum of the desired signal and interference and noise:

$$\mathbf{y}[n] = \mathbf{h}_d d[n] + \sum_{i=1}^{K} \mathbf{h}_i d_i[n] + \mathbf{v}[n]$$

where $d[n]$ is the desired signal, $d_i[n]$ is interfering signal i, and $\mathbf{v}[n]$ is the white noise vector at the receiver (the variance of each component of $\mathbf{v}[n]$ is σ_v^2). For simplicity, we shall assume that the desired and interfering signals have the same power. Using beamforming weights $\mathbf{w}$ (without loss of generality, we can restrict $|\mathbf{w}| = 1$), the signal at the output of the array will be $\mathbf{w}^H\mathbf{y}[n]$. The output signal to noise plus interference ratio (SINR) is given by:

$$\text{SINR}_{out} = \frac{|\mathbf{w}^H\mathbf{h}_d|^2}{|\sum_{i=1}^{K}\mathbf{w}^H\mathbf{h}_i|^2 + \sigma_v^2}$$

where $(\cdot)^H$ denotes the complex conjugate transpose. Let $\mathbf{H}_I = [\mathbf{h}_1, \ldots, \mathbf{h}_K]$ be the matrix whose columns are the interference vectors. Complete interference rejection can be achieved by choosing a beamforming weight vector $\mathbf{w}$ that is the projection of the desired vector $\mathbf{h}_d$ onto the subspace orthogonal to the column space of $\mathbf{H}_I$ (or the null-space of $\mathbf{H}_I^\top$, which is also known as the left nullspace of $\mathbf{H}_I$), as described in [3]:

$$\mathbf{w}_{opt} = \mathbf{w}_{projection} = \mathbf{h}_d - \mathbf{H}_I(\mathbf{H}_I^H\mathbf{H}_I)^{-1}\mathbf{H}_I^H\mathbf{h}_d$$

We can see that rejecting all the interfering signals is only possible when the left nullspace is non-empty. This is guaranteed when $K < N$. The projection-based beamformer does not take noise into account. In general, maximizing the output SINR does not necessarily require complete interference rejection; reducing the interference to the noise level may be sufficient. Optimizing the output SINR leads to the Minimum Variance Distortionless Response (MVDR) beamformer [4]. If we define the noise+interference correlation matrix $\mathbf{R}_{N+I}$ as:

$$\mathbf{R}_{N+I} = \sum_{i=1}^{K}\mathbf{h}_i^H\mathbf{h}_i + \sigma_\nu^2 I_N$$

where I_N is the $N \times N$ identity matrix, then the output SINR can be maximized by choosing $\mathbf{w}_{opt}$:

$$\mathbf{w}_{opt} = \mathbf{w}_{\text{MVDR}} = \frac{\mathbf{R}_{N+I}^{-1}\mathbf{h}_d}{\mathbf{h}_d^H\mathbf{R}_{N+I}^{-1}\mathbf{h}_d}$$

The denominator is a normalizing factor. When the interference power is much larger than the noise power, both projection and MVDR yield virtually identical results. In practice, however, phase and amplitude errors degrade the performance of both beamformers[6]. We will assume that an optimum beamformer $\mathbf{w}_{opt}$ is computed using

[5]We will denote scalars in lower case, vectors in bold lower case, and matrices in bold upper case.

[6]The errors can also result from uncertainties about the channel responses for both desired and interfering signals. Up to this point, we have assumed perfect knowledge of the channels.

projection, and $\hat{\mathbf{w}}_{opt}$ takes into account both phase and amplitude errors:

$$\mathbf{w}_{opt} = [\alpha_{1w}e^{j\beta_{1w}}, \ldots, \alpha_{Nw}e^{j\beta_{Nw}}]^\top$$

$$\hat{\mathbf{w}}_{opt} = [\alpha_{1w}(1+\epsilon_1)e^{j(\beta_{1w}+\delta_1)}, \ldots, \alpha_{Nw}(1+\epsilon_N)e^{j(\beta_{Nw}+\delta_N)}]^\top$$

where $\epsilon_i \ \forall_{1\leq i\leq N}$ are $i.i.d$ zero mean real random variables with variance $E[\epsilon_i^2] = \sigma_\epsilon^2$, and $\delta_i \ \forall_{1\leq i\leq N}$ are $i.i.d$ zero mean real random variables with variance $E[\delta_i^2] = \sigma_\delta^2$. We also assume that the phase and amplitude errors are independent of each other. Furthermore, we scale the weights so that $\mathbf{w}_{opt}$ has unit norm (i.e. $\sum_{i=1}^{N} \alpha_{iw}^2 = 1$).

The phase and amplitude errors result in $\hat{\mathbf{w}}_{opt}$ deviating from $\mathbf{w}_{opt}$ by an angle θ. Note that θ does not necessarily correspond to a physical angle or direction. This deviation will result in a reduction in the signal strength in the desired direction as well as an increase in the interference power, since $\hat{\mathbf{w}}_{opt}$ will no longer be orthogonal the interference subspace. The desired power is proportional to $\cos(\theta)$, and the increase in interference (leakage) is proportional to $\sin(\theta)$ (see Figure 4). For small angles θ, we can use the standard approximations[7] $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$. Thus, we can characterize the effect of phase and amplitude errors on beam nulls by considering how the mean square angle $\sigma_\theta^2(\sigma_\delta, \sigma_\epsilon, N) = E[\theta^2]$ behaves as a function of σ_δ, σ_ϵ, and N.

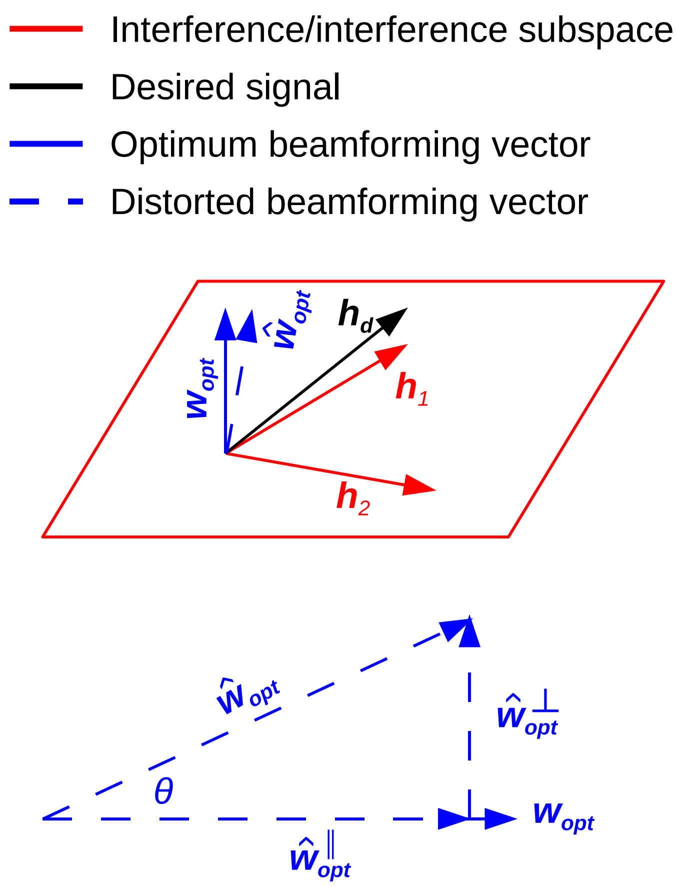

Fig. 4. The optimum beamforming vector $\mathbf{w}_{opt}$ can be viewed as a projection of the desired signal onto the subspace orthogonal to the interference subspace. The distorted beamforming vector $\hat{\mathbf{w}}_{opt}$ can decomposed into two orthogonal components: $\hat{\mathbf{w}}_{opt} = \hat{\mathbf{w}}_{opt}^{\perp} + \hat{\mathbf{w}}_{opt}^{\parallel}$. $\hat{\mathbf{w}}_{opt}^{\parallel}$, which is parallel to $\mathbf{w}_{opt}$, represents the potential loss in beamforming gain, and is proportional to $\cos(\theta)$. $\hat{\mathbf{w}}_{opt}^{\perp}$, which is orthogonal to $\mathbf{w}_{opt}$, represents the potential leakage into the interference subspace, and is proportional to $\sin(\theta)$.

[7]This explains why nulls are more sensitive than peaks to phase and amplitude errors, since $\sin(\theta)$ changes more rapidly than $\cos(\theta)$ when θ is small.

If we assume that the phase and amplitude variations are small, and given that $\mathbf{w}_{opt}$ is unit norm, then we can approximate the error angle θ with the error vector:

$$\mathbf{\Delta w} = \mathbf{w}_{opt} - \hat{\mathbf{w}}_{opt} = [\alpha_{1w} e^{j\beta_{1w}}(1-(1+\epsilon_1)e^{j\delta_1}), \ldots, \alpha_{Nw} e^{j\beta_{Nw}}(1-(1+\epsilon_N)e^{j\delta_N})]^\top$$

We can further simplify the above expression using the approximations $\cos(\delta_i) \approx 1$, $\sin(\delta_i) \approx \delta_i$, and $\delta_i \epsilon_i \approx 0$.

$$\begin{aligned}\mathbf{\Delta w} &= [\alpha_{1w} e^{j\beta_{1w}}(1-1-\epsilon_1 - j\delta_1), \ldots, \alpha_{Nw} e^{j\beta_{Nw}}(1-1-\epsilon_N - j\delta_N)]^\top \\ &= [-\alpha_{1w} e^{j\beta_{1w}}(\epsilon_1 + j\delta_1), \ldots, -\alpha_{Nw} e^{j\beta_{Nw}}(\epsilon_N + j\delta_N)]^\top \\ &\Rightarrow |\mathbf{\Delta w}|^2 = (\mathbf{\Delta w})^H (\mathbf{\Delta w}) = \sum_{i=1}^{N} \alpha_{iw}^2(\epsilon_i^2 + \delta_i^2)\end{aligned}$$

By taking the expectation of this expression:

$$\begin{aligned}\sigma_\theta^2 \approx E[|\mathbf{\Delta w}|^2] &= E\left[\sum_{i=1}^{N} \alpha_{iw}^2(\epsilon_i^2 + \delta_i^2)\right] = \sum_{i=1}^{N} \alpha_{iw}^2(E[\epsilon_i^2] + E[\delta_i^2]) \\ &= \sum_{i=1}^{N} \alpha_{iw}^2(\sigma_\epsilon^2 + \sigma_\delta^2) = (\sigma_\epsilon^2 + \sigma_\delta^2)\sum_{i=1}^{N} \alpha_{iw}^2 = \sigma_\epsilon^2 + \sigma_\delta^2\end{aligned}$$

As we can see, the mean square error angle σ_θ^2 is equal to the sum of the mean square phase error σ_δ^2 and the mean square amplitude error σ_ϵ^2. The key conclusion that we draw from this result is that the angle error is independent of N, the number of antennas[8].

The simulation results shown in Figure 5 verify this result. Figure 5(a) shows a linear relationship between $10\log(\sigma_\epsilon^2 + \sigma_\delta^2)$ (x-axis) and $10\log(\sigma_\theta^2)$ (y-axis), with slope equal to 1. Figure 5(b) shows that the relationship between $10\log(\sigma_\delta^2)$ and $10\log(\sigma_\theta^2)$, when the amplitude errors ϵ_i are set to 0, is also linear with slope equal to 1, which demonstrates that the phase and amplitude errors contribute equally to the overall error angle θ. In both Figures 5(a) and 5(b), we simulated a 100 element array. We repeated the same experiment for a 1000 element array and the results are identical, as shown in Figures 5(c) and 5(d). Figure 6 also shows identical results, where we plot the interference rejection as a function of phase and amplitude errors for different array sizes. This shows that the number of antenna elements has no effect on the mean square error angle σ_θ^2 or the interference rejection. This means that the depth of beam nulls is limited by gain and phase accuracy, and is independent of the size of the array N and the number of interferers K, as long as $N > K$.

IV. CONCLUSION

Adaptive antenna arrays are a key component in many modern communication systems. They are used to both increase the gain in the direction of a desired signal as well as to reject interfering signals. However, when these adaptive arrays are implemented, a variety of practical considerations will cause the actual antenna weights to differ from the optimal weights, which in turn degrades the performance of the array.

In this paper, we analyzed the performance loss due to phase and amplitude errors in the weights. We began by considering a beamforming system, which maximizes the gain in a desired direction. We derived an expression for the loss in gain due to uniform phase errors, and provided simulations that validate this result. Then, we considered a beam-nulling system, which rejects interfering signals. We analyzed the effect of uniform amplitude and phase errors, and again provide numerical simulations. We showed that the interference rejection is a function of the errors in the weights, and is independent of the number of antennas, assuming that there are more antennas than interferers.

[8]The power leakage into the interference subspace is independent of the number of antennas. However, increasing the number of antennas can still increase the output SINR (peak to null ratio) by increasing the power gain of the desired signal. Increasing the number of antennas also increases the degrees freedom necessary to null more interferers.

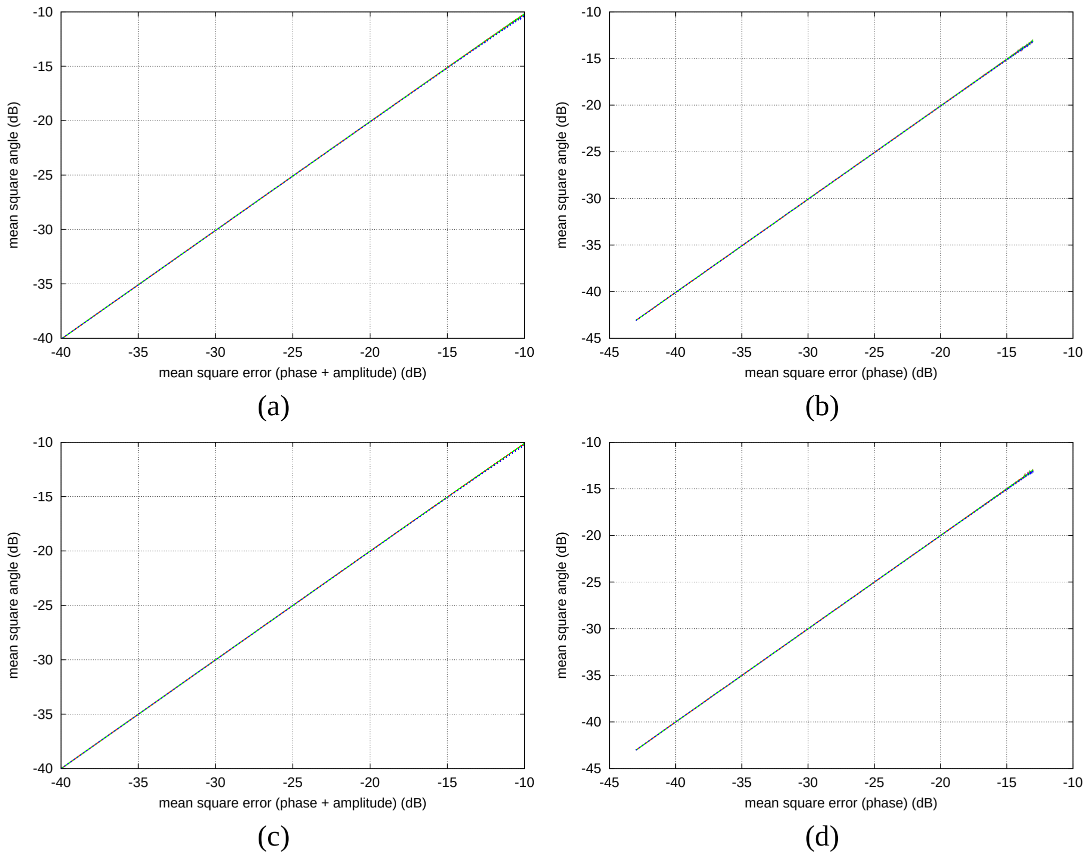

Fig. 5. Simulated relationship between phase and amplitude errors and the mean square error angle: (a),(c) $10\log(\sigma_\epsilon^2 + \sigma_\delta^2)$ on the x-axis, $10\log(\sigma_\theta^2)$ on the y-axis. (b),(d) $10\log(\sigma_\delta^2)$ on the x-axis, $10\log(\sigma_\theta^2)$ on the y-axis. In (a) and (b), the simulated array had 100 elements. In (c) and (d), the simulated array had 1000 elements.

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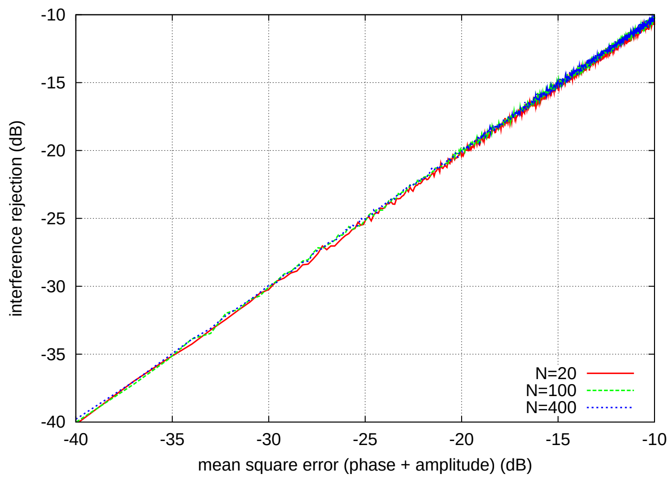

Fig. 6. Simulated power leakage (interference rejection) as a function of phase and amplitude errors: $10\log(\sigma_\epsilon^2 + \sigma_\delta^2)$ on the x-axis versus the interference rejection in dB on y-axis. Interference rejection (IR) is defined as the ratio of the interferer power after beam-nulling to the interferer power before beam-nulling. Before beam-nulling, we choose the beamforming vector $\mathbf{w}_{before}$ along the direction of the desired signal $\mathbf{h}_d$ (i.e. $\mathbf{w}_{before} = \frac{\mathbf{h}_d}{|\mathbf{h}_d|}$). In this case, the input power at the receiver from interferer $\mathbf{h}_i$ will be $|\mathbf{w}_{before}^H \mathbf{h}_i|^2$. After beam-nulling, we choose the beamforming vector $\mathbf{w}_{after}$ as the projection of the desired vector onto the subspace orthogonal to the interference subspace. In this case, the input power at the receiver from interferer $\mathbf{h}_i$ after beam-nulling (and phase and amplitude distortion) will be $|\hat{\mathbf{w}}_{after}^H \mathbf{h}_i|^2$. Thus, on the y-axis, the interference rejection IR $= 20\log\frac{|\hat{\mathbf{w}}_{after}^H \mathbf{h}_i|}{|\mathbf{w}_{before}^H \mathbf{h}_i|}$. The relationship is plotted for several array sizes. Both $\mathbf{h}_d$ and $\mathbf{h}_i$ are complex random vectors whose components (both real and imaginary parts) are sampled independently from a standard Gaussian distribution.