Convergence and Stability of a Distributed CSMA Algorithm for Maximal Network Throughput

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***Abstract*—Designing efficient scheduling algorithms is an important problem in a general class of networks with resource-sharing constraints, such as wireless networks and stochastic processing networks. In [4], we proposed a distributed scheduling algorithm that can achieve the maximal throughput in such networks under certain conditions. This algorithm was inspired by CSMA (Carrier Sense Multiple Access). In this paper, we prove the convergence and stability of the algorithm, with properly-chosen step sizes and adjustment periods. Convergence of the joint scheduling and rate-control algorithm for utility maximization in [4] can be proved similarly.**

***Index Terms*—Distributed scheduling, maximal throughput, stochastic approximation, Markov process, convex optimization**

I. Introduction

Efficient resource allocation is essential to achieve high utilization of a class of networks with resource-sharing constraints, such as wireless networks and stochastic processing networks (SPN [8]). In wireless networks, certain links can not transmit at the same time due to the interference constraints among them. In a task processing problem (further explained later), two tasks can not be processed simultaneously if they both require monopolizing a common resource. A scheduling algorithm determines which link to activate (or which task to process) at a given time without violating these constraints. Designing distributed scheduling algorithms to achieve high throughput is an important problem [1], [15].

This paper is devoted to a proof of the convergence and stability of a simple-to-implement distributed scheduling algorithm for such networks proposed in [4], [5]. For ease of reference, we review the algorithm below. The algorithm avoids the need to search for a maximum weighted independent set as required by Maximal-Weight Scheduling [15], an algorithm that is known to be throughput-optimal but is not easy to implement, especially in a distributed way. (In [11], a similar algorithm was independently proposed in the context of optical networks.) The paper [4] also describes an algorithm that maximizes the utility of flows by combining scheduling and flow control. The convergence of that algorithm can be proved using the same approach as in this paper.

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This paper is expanded from [6].

Consider a wireless networks where some links interfere. Packets arrive at the transmitters of the links with certain rates. Consider a "perfect CSMA" protocol [2], [3] that works as follows. The different transmitters choose independent exponentially-distributed backoff times. A transmitter decrements its backoff timer when it senses the channel idle and starts transmitting when its timer runs out. The packet transmission times are also exponentially distributed. (The process defines a "CSMA Markov chain".) The assumption in [2], [3] is that a transmitter hears any transmitter of a link that would interfere with it. That is, there are no hidden nodes. Moreover, the transmitters hear a conflicting transmission instantaneously. Accordingly, there are no collisions in perfect CSMA. In practice, other protocols such as RTS/CTS can be used to address the hidden node problems [2]. The optimality in the presence of collisions is analyzed in [16]. In the task processing problem, on the other hand, one can define a perfect CSMA protocol without considering collisions and hidden nodes.

The "adaptive CSMA" scheduling algorithm in [4] is as follows. Each link adjusts its transmission aggressiveness ("TA") based on its backlog. A link's TA is reflected in either its mean backoff time or mean transmission time. For example, the transmitter of a link sets its mean backoff time to be $\exp\{-\alpha \cdot Q\}$ where Q is the backlog of the link and $\alpha > 0$ is a small constant. That is, the link becomes more aggressive as its backlog increases. In [4], we have shown, under a time-scale-separation approximation, that such a simple algorithm is throughput-optimal (i.e., it stabilizes the queues if the arrival rates are strictly feasible). The approximation is that, as the links change their TA, the CSMA Markov chain instantaneously reaches its stationary distribution.

In this paper, we analyze the convergence and stability properties of the algorithm without the above approximation. In particular, we show that (i) For any strictly feasible arrival rates, using decreasing step sizes and increasing adjustment periods that satisfy certain conditions, the TA's of different links converge to the desired values. Although the intuition is to make the time-scale separation eventually hold, these conditions are quite intricate since the speed at which the CSMA Markov chain converges to its stationary distribution depends on the TA's. (ii) The maximal throughput can be arbitrarily approached by using constant step sizes and adjustment periods.

The rest of the paper is organized as follows. In section II, we describe the basic model and the throughput-optimality objective. Section III and IV present CSMA scheduling algorithms (adapted from [4], [5]), and give proofs of their convergence and/or stability under different sets of sufficient conditions. Section VI shows that very similar results apply to the joint algorithm in [4]. Section VII provides simulation studies that illustrate the main results. We conclude the paper and discuss future research in section VIII.

II. Basic Model and Problem statement

We first describe the basic model and objective as in [4].

A. Network Interference Model

There are K FIFO queues in the network. Not all queues can be served simultaneously, due to interference or resource-sharing constraints. These constraints are represented by a contention graph (or "CG") $G = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is the set of vertexes (each of them represents a queue) and $\mathcal{E}$ is the set of edges. Two queues cannot be served at the same time (i.e., "conflict") if and only if there is an edge between them.

In wireless networks, one can associate a queue with each *link*, which is an ordered transmitter-receiver pair. Two links cannot be activated at the same time if they interfere. Although this is a simplified model for wireless networks, it does provide a useful abstraction and has been used widely in literature (see, for example, [2], [1] and the references therein).

In the task processing problem, assume K different types of tasks and a finite set of resources $\mathcal{B}$. A queue is associated with each type of tasks. To perform a type-k task, one needs a subset $\mathcal{B}_k \subseteq \mathcal{B}$ of resources and these resources are then monopolized by the task while it is being performed. Note that two tasks cannot be performed simultaneously iff they require some common resources. Clearly, this can be modeled by a conflict graph G defined above.

In this paper we mostly use the terms in wireless networks since our algorithm is originally inspired by CSMA. But the algorithm and all results below can be applied to the task processing problem.

Assume that G has N different Independent Sets ("IS", not confined to "Maximal Independent Sets"), where each IS is a set of queues that can be served simultaneously. Denote the i'th IS as $x^i \in \{0,1\}^K$, a 0-1 vector that indicates which links are transmitting in this IS. That is, the k'th element of x^i, $x^i_k = 1$ if link k is transmitting, and $x^i_k = 0$ otherwise.

B. Throughput-optimality Objective

We first describe the *scheduling problem* which is the focus of the paper. Without loss of generality, assume that the capacity of each link is 1. Assume that traffic arrives at link k with an arrival rate $\lambda_k \in (0, 1)$. For simplicity, assume the following i.i.d. Bernoulli arrivals (although it can be readily generalized, see Appendix IX-G): Let $a_k(t) \in \{0, 1\}$ be the arrival process at link k. For $t \in [j, j + 1], j = 1, 2, \ldots$ (i.e., in a given "slot" with length 1), $a(t) = 1$ with probability λ_k and $a(t) = 0$ otherwise. Then, $A_k(t) := \int_0^t a_k(\tau)d\tau$, the cumulative amount of arrived traffic by time t, satisfies that $E(A_k(t))/t = \lambda_k$. Denote the vector of arrival rates as λ. We say that λ is *feasible* iff it can be written as $\lambda = \sum_i \bar{p}_i \cdot x^i$ where $\bar{p}_i \geq 0$ and $\sum_i \bar{p}_i = 1$. That is, there is a schedule of the independent sets (including the non-maximal ones) that can serve the arrivals. Denote the set of feasible λ by $\bar{\mathcal{C}}$. We say that λ is *strictly feasible* iff it can be written as $\lambda = \sum_i \bar{p}_i \cdot x^i$ where $\bar{p}_i > 0$ and $\sum_i \bar{p}_i = 1$.Denote the set of strictly feasible λ as $\mathcal{C}$. (The set $\mathcal{C}$ is the relative interior [12] of $\bar{\mathcal{C}}$.)

Our objective is to give a distributed scheduling algorithm such that any strictly feasible λ can be "supported". More formally, denote by $D_k(t)$ the cumulative traffic that has departed by t. The system is "rate stable" if $\lim_{t\to\infty}[A_k(t) - D_k(t)]/t = 0, \forall k$ almost surely. An algorithm is said to be "throughput-optimal" if for any $\lambda \in \mathcal{C}$, it makes the system rate stable. (There are also other definitions of throughput-optimality [15]. We use this definition here since in some of our algorithms, the underlying Markov chain is not time-homogeneous, in which case "positive recurrence" is not usually defined.)

An extension of the above *scheduling problem* is a *joint scheduling and congestion control problem*, where together with scheduling, the arrival rate λ is simultaneously adjusted by the sources in order to achieve certain fairness (or "maximal utility") objective among different links or multi-hop data flows. As shown in section VI, the convergence and stability results for our scheduling algorithms can be readily applied to the joint scheduling and congestion control algorithms proposed in [4].

III. A distributed CSMA Algorithm and its throughput-optimality

We first describe an idealized CSMA model proposed in [2], [3] which the algorithm in [4] in based on. Before transmitting, link k waits (or "backs-off") for a random period of time that is exponentially distributed with mean $1/R_k$. If it does not sense another transmission of a conflicting link during that time, then the link starts transmitting; otherwise, it suspends its backoff and resumes it after the conflicting transmission is over. (If more than one links share the same transmitter, the transmitter maintains independent backoff timers for these links.) The transmission time of link k is exponentially distributed with mean 1. Define $r_k = \log(R_k)$ as the "transmission aggressiveness" (TA) of link k. And let $\mathbf{r}$ be the vector of r_k's. Assuming that the sensing time is negligible, given the continuous distribution of the backoff times, the probability for two links to start transmission at the same time is zero, so collisions do not occur in the model of [2], [3].

Note that collision is not an issue in the task processing problem (cf. section II-A). In wireless networks, however, collisions occur since in practice the backoff time of each link is usually multiples of "minislots" due to the non-zero sensing time. Therefore the above idealize CSMA model provides an approximation. The approximation is more accurate when the transmission probability in each minislot is small which leads to small collision probability. In that case, the transmission time should be increased to compensate for the increases backoff time. In [16], we formulated a model which explicitly considers collisions among control packets such as RTS in 802.11, and it was shown that the algorithm in [4] can be naturally extended to that case. In this paper, we will focus on the case without collisions. The line of proof here is also useful for the collision case.

The transitions of the transmission states x^i form a Continuous Time Markov Chain, which is called the *CSMA Markov Chain*. Denote link k's neighboring set by $\mathcal{N}(k) := \{m : (k,m) \in \mathcal{E}\}$. If in state x^i, link k is not active ($x^i_k = 0$) and all of its conflicting links are not active (i.e., $x^i_m = 0, \forall m \in \mathcal{N}(k)$), then state x^i transits to state $x^i + \mathbf{e}_k$ with rate R_k, where $\mathbf{e}_k$ is the K-dimension vector whose k'th element is 1 and all other elements are 0's. Similarly, state $x^i + \mathbf{e}_k$ transits to state x^i with rate 1. However, if in state x^i, any link in its neighboring set $\mathcal{N}(k)$ is active, then state $x^i + \mathbf{e}_k$ does not exist. References [2], [3] showed that the Markov chain (with a given $\mathbf{r}$) is *time-reversible* [7], and in the stationary distribution, the probability of state x^i is

$$p(x^i;\mathbf{r}) = \frac{\exp(\sum_{k=1}^K x^i_k r_k)}{C(\mathbf{r})} \quad (1)$$

where

$$C(\mathbf{r}) = \sum_j \exp(\sum_{k=1}^K x^j_k r_k) \ . \quad (2)$$

(Note that an IS with larger $\sum_{k=1}^K x^i_k r_k$ has a higher probability.) Then, the probability that link k is active is

$$s_k(\mathbf{r}) := \sum_i [x^i_k \cdot p(x^i;\mathbf{r})]. \quad (3)$$

Since the link capacity is assumed to be 1, $s_k(\mathbf{r})$ is also the average service rate (or throughput) of link k given $\mathbf{r}$.

For simplicity, we assume that the arrival traffic can be viewed as "fluid". That is, upon transmission, the packet sizes may be different from the sizes of the arrived packets (by re-packetize the bits in the queue). This assumption, however, is not essential. More discussion is given in Appendix IX-H.

The key idea of the "adaptive CSMA algorithm" in [4] is that each link k should dynamically adjust r_k according to its empirical arrival rate and service rate. For example, if the empirical arrival rate is larger than the service rate (i.e., the queue length of link k increases), then r_k should be increased. Surprisingly, this simple algorithm can achieve the maximal throughput.

A. Review of the ideas behind the Algorithms

The algorithms in [4], [5] try to find or approximate, in a distributed way, the TA vector $\mathbf{r}$ in CSMA such that the induced service rates (3) at all links are not less than the arrival rates λ whenever λ is strictly feasible. In this section, we review some results in [4], [5] which state that the desired $\mathbf{r}$ can be obtained as the optimal dual variables in some convex optimization problems ((4) or (6)). (And our algorithms are distributed implementations to solve these optimization problems.)

Consider the convex optimization problem, where $\mathbf{u} \in \mathcal{R}^N_+$ is a distribution over the IS's (recall that N is the number of IS's)

$$\begin{array}{ll} \max_{\mathbf{u}} & -\sum_i u_i \log(u_i) \\ \text{s.t.} & \sum_i (u_i \cdot x^i_k) \geq \lambda_k, \forall k \\ & u_i \geq 0, \sum_i u_i = 1. \end{array} \quad (4)$$

where λ is strictly feasible, and $\sum_i$ is the summation over all IS's. The optimization problem chooses a distribution with the maximal entropy $H(\mathbf{u}) := -\sum_i u_i \log(u_i)$ among the set of distributions that satisfy the capacity constraints (i.e., such that for any link k, the service rate $\sum_i (u_i \cdot x^i_k)$ is not less than the arrival rate λ_k). It turns out that the solution $\mathbf{u}^*$ is the stationary distribution of the CSMA Markov chain with proper-chosen TA $\mathbf{r}^*$, as discussed below.

Lemma 1: ([4]) For all k, let $r^*_k \geq 0$ be the (unique) optimal dual variable associated with the constraint $\sum_i (u_i \cdot x^i_k) \geq \lambda_k$ in (4). Then $\mathbf{r}^*$ satisfies that

$$s_k(\mathbf{r}^*) \geq \lambda_k, \forall k,$$

that is, with the TA vector $\mathbf{r}^*$, the service rate (3) at any link is high enough. (And the optimal $\mathbf{u}^*$ is the corresponding stationary distribution of the CSMA Markov chain.) Also, an iterative (subgradient dual) algorithm to find $\mathbf{r}^*$ (by solving the dual problem $\min_{\mathbf{r} \geq \mathbf{0}} L_1(\mathbf{r})$) is (for $i = 1, 2, \ldots$)

$$r_k(i) = [r_k(i-1) + \alpha(i)(\lambda_k - s_k(\mathbf{r}(i-1)))]_+, \forall k \quad (5)$$

where $\alpha(i)$ is some properly-chosen step size. *That is, link k increases r_k if the service rate is smaller than λ_k, and vice versa.*
The proof is given in Appendix IX-A.

The next optimization problem is an extension of (4) such that the optimal dual variables $\mathbf{r}^*$ satisfies $s_k(\mathbf{r}^*) > \lambda_k, \forall k$. The strict inequality can be used later to ensure that the queue lengths are stable and tend to be small.

$$\begin{array}{ll} \max_{\mathbf{u},\mathbf{w}} & -\sum_i u_i \log(u_i) + c\sum_k \log(w_k) \\ \text{s.t.} & \sum_i (u_i \cdot x^i_k) \geq \lambda_k + w_k, \forall k \\ & u_i \geq 0, \sum_i u_i = 1 \\ & 0 \leq w_k \leq \bar{w}, \forall k \end{array} \quad (6)$$

where λ is strictly feasible, and $c > 0, \bar{w} > 0$ are small constants.

Lemma 2: ([5]) For all k, let $r^*_k \geq 0$ be the (unique) optimal dual variable associated with the constraint $\sum_i (u_i \cdot x^i_k) \geq \lambda_k + w_k$ in problem (6). Then $\mathbf{r}^*$ satisfies that

$$s_k(\mathbf{r}^*) > \lambda_k, \forall k.$$

Also, a (subgradient dual) algorithm to find $\mathbf{r}^*$ (by solving the dual problem $\min_{\mathbf{r} \geq \mathbf{0}} L_2(\mathbf{r})$) is (for $i = 1, 2, \ldots$)

$$\begin{aligned} r_k(i) \quad = \quad & [r_k(i-1) + \alpha(i)(\lambda_k - s_k(\mathbf{r}(i-1)) + \\ & \min\{c/r_k(i-1), \bar{w}\})]_+, \forall k \end{aligned} \quad (7)$$

where $\alpha(i)$ is some properly-chosen step size. (The proof is similar to that of Lemma 1, and is given in Appendix IX-B.)

Remark: The key point of problem (6) is to generate a "gap" between $s_k(\mathbf{r}^*)$ and λ_k. The gap may also be obtained in the following way. If it is known that $\lambda + \mathbf{1}\cdot\epsilon$ is strictly feasible for some $\epsilon > 0$, then solving problem (4) with λ_k replaced by $\lambda_k + \epsilon$ also serves the purpose. However, it is difficult to choose ϵ such that $\lambda + \mathbf{1}\cdot\epsilon$ is strictly feasible, especially if λ is unknown. The advantage of solving problem (6) is that we can generate the gap without knowing λ or checking whether $\lambda + \mathbf{1}\cdot\epsilon$ is strictly feasible (see Algorithm 1 in the next section).

However, algorithms (5) or (7) require the knowledge of λ_k and $s_k(\mathbf{r}(i-1))$, which cannot be obtained directly in the network since both the traffic arrival and service processes are random. Therefore in the actual algorithm we need to properly average the randomness. The main complication here is that the time needed for the CSMA Markov chain to converge to its stationary distribution (i.e., the mixing time) depends on the varying $\mathbf{r}$. So the dynamics of the Markov chain and the dynamics of $\mathbf{r}$ are coupled in a complex way. The goal of this paper is to provide sufficient conditions to ensure the convergence of the algorithm with random arrivals and service.

B. TA adjustment Algorithm

Let $x_k(t) \in \{0,1\}$ be the instantaneous state of link k at (continuous) time t. For link k, define the cumulative "service" by time t as $S_k(t) = \int_{\tau=0}^{t} x_k(\tau)d\tau$, and the cumulative departure by time t as $D_k(t) = \int_{\tau=0}^{t} x_k(\tau)I(Q_k(\tau) > 0)d\tau$, where $I(\cdot)$ is the indicator function and $Q_k(\tau) := Q_k(0) + A_k(\tau) - D_k(\tau)$ is the queue length of link k at time τ. Note that there is no departure if the queue is empty but we allow $x_k(\tau) = 1$ even if $Q_k(\tau) = 0$ (in which case dummy packets are sent, further discussed below).

The adaptive CSMA algorithm which adjusts the TA is given below (Notice its similarity to (7)). We assume that there is a maximal instantaneous arrival rate $\bar{\lambda}$ for any link.

Algorithm 1: The vector $\mathbf{r}$ is updated at time t_i, $i = 1, 2, \ldots$. Let $t_0 = 0$ and $T_i := t_i - t_{i-1}, i = 1, 2, \ldots$. Define "period i" as the time between t_{i-1} and t_i, and $\mathbf{r}(i)$ be the value of $\mathbf{r}$ at the end of period i, i.e., at time t_i.

Initially, set $\mathbf{r}(0) = \mathbf{0}$.[1] Then at time t_i $(i = 1, 2, \ldots)$, update

$$r_k(i) = [r_k(i-1) + \alpha(i)(\lambda'_k(i) - s'_k(i) + \min\{c/r_k(i-1), \bar{w}\})]_+ \quad (8)$$

for all k, where $\lambda'_k(i)$ and $s'_k(i)$ are the empirical average arrival rate and service rate of link k in period i (i.e., $\lambda'_k(i) = [A_k(t_i) - A_k(t_{i-1})]/T_i \le \bar{\lambda}$, $s'_k(i) = [S_k(t_i) - S_k(t_{i-1})]/T_i$). $c > 0, \bar{w} > 0$ are small constants. We let link k transmit dummy packet with TA $r_k(i)$ even if the queue is empty. This ensures that the CSMA Markov chain (with parameter $\mathbf{r}(i)$) has the desired stationary distribution (1). (The transmitted dummy packets are included when computing $s'_k(i)$.)

[1]In fact, $\mathbf{r}(0)$ can be any finite value without affecting the result in Theorem 1. We assume $\mathbf{r}(0) = \mathbf{0}$ for simplicity.

Also, $\{\alpha(i)\}$ and $\{T_i\}$ are chosen such that $\{T_i\}$ is non-decreasing with i, and

$$\alpha(i) > 0, \textstyle\sum_i \alpha(i) = \infty, \sum_i \alpha(i)^2 < \infty \quad (9)$$

$$\textstyle\sum_{m=0}^{\infty}[\alpha(m+1)\sum_{i=1}^{m}\alpha(i)]^2 < \infty \quad (10)$$

$$\textstyle\sum_{m=0}^{\infty}[\alpha(m+1)\cdot\sum_{i=1}^{m}\alpha(i)\cdot f(m)/T_{m+1}] < \infty \quad (11)$$

where

$$f(m) = \exp\{(\tfrac{5}{2}K + 1)\cdot[\lambda_{max}\cdot\textstyle\sum_{i=1}^{m}\alpha(i) + \log(2)]\} \quad (12)$$

where K is the number of links, and $\lambda_{max} = \bar{\lambda} + \bar{w}$.

Remark 1: Since the update (8) only uses local information, Algorithm 1 is fully *distributed*. Also, it does not need to know λ_k explicitly.

Remark 2: In an alternative design, the mean backoff time of each link is 1, and the mean transmission time of link k is $\exp(r_k)$. For a given $\mathbf{r}$, the CSMA Markov chain has the same stationary distribution as in (1). In that case, the same Algorithm 1 can be used, with a minor difference in the definition of (12). See Appendix IX-I for more details.

Proposition 1: The setting $\alpha(i) = 1/[(i+1)\log(i+1)]$ and $T_i = i$ satisfies conditions (9), (10) and (11). Note that this setting does not depend on, or require the knowledge of K and λ_{max}, and thus can generally apply to any network.

The proof is given in Appendix IX-C.

Similarly, the same is true for the following settings. (i) $\alpha(i) = 1/[(i+1)\log(i+1)]$ and $T_i = i^{\gamma}$ for any $\gamma > 0$; (ii) $\alpha(i) = c_0/[(a\cdot i + b + 1)\log(a\cdot i + b + 1)]$ and $T_i = a\cdot i + b$ (with constants $a > 0, b > 0, c_0 > 0$).

The main result of the paper is the following:

Theorem 1: Assume that λ is strictly feasible (i.e., $\lambda \in \mathcal{C}$). Then with Algorithm 1, $\mathbf{r}$ converges to some $\mathbf{r}^*$ with probability 1. The vector $\mathbf{r}^*$ satisfies that $s_k(\mathbf{r}^*) > \lambda_k, \forall k$.

As a corollary, it can be further shown that the system is rate stable (Appendix IX-E).

The next two subsections (and Appendix IX-D) give the proof of Theorem 1. However, readers who are interested to have an overview of other algorithms and convergence/stability results could first skip to section III-E.

C. Some notation

Before proving Theorem 1, we need some further notation. Let $x^0(m-1)$ be the state of the CSMA Markov chain at the beginning of period m (i.e., at time t_{m-1}). Define the random vector $U(m-1) := (\mathbf{s}'(m-1), \lambda'(m-1), \mathbf{r}(m-1), x^0(m-1))$ for $m > 1$ and $U(0) = (\mathbf{r}(0) = \mathbf{0}, x^0(0))$. For $m \ge 1$, let $\mathcal{F}_{m-1}$ be the σ-field generated by $U(0), U(1), \ldots, U(m-1)$.

Given a vector of TA $\mathbf{r}(m-1)$ at the beginning of the period m of Algorithm 1, the vector $\mathbf{g}(m)$ whose k-th element $g_k(m) := s_k(\mathbf{r}(m-1)) - \lambda_k - (c/r_k(m-1)) \wedge \bar{w}$ is a subgradient of $L_2(\mathbf{r})$ (the dual problem of (6) is

$\min_{\mathbf{r}\geq\mathbf{0}} L_2(\mathbf{r})$). To find the desired $\mathbf{r}^*$ which solves the dual problem, the ideal algorithm (7) would follow the opposite direction of $\mathbf{g}(m)$. However, Algorithm 1 only has an estimation of $g_k(m)$, denoted by

$$g'_k(m) = s'_k(m) - \lambda'_k(m) - (c/r_k(m-1)) \wedge \bar{w}. \quad (13)$$

The "error bias" of $g'_k(m)$ is defined as

$$\begin{aligned} B_k(m) : &= E[g'_k(m)|\mathcal{F}_{m-1}] - g_k(m) \\ &= E[s'_k(m)|\mathcal{F}_{m-1}] - s_k(\mathbf{r}(m-1)) - \\ &\quad [E[\lambda'_k(m)|\mathcal{F}_{m-1}] - \lambda_k]. \end{aligned} \quad (14)$$

Define also the zero-mean "noise"

$$\begin{aligned} \eta_k(m) : &= (s'_k(m) - E[s'_k(m)|\mathcal{F}_{m-1}]) \\ &\quad -(\lambda'_k(m) - E[\lambda'_k(m)|\mathcal{F}_{m-1}]). \end{aligned}$$

Since both $s'_k(m)$ and $\lambda'_k(m)$ are bounded, the noise is also bounded: $|\eta_k(m)| \leq c_2$ for some $c_2 > 0$. Then, we have

$$g'_k(m) = g_k(m) + B_k(m) + \eta_k(m). \quad (15)$$

D. Proof of Theorem 1

The proof is composed of two parts. In the first part, we show that with Algorithm 1 and condition (11), the error bias (14) decreases "fast enough" with time. In the second part (Lemma 3), we use the result of part 1 and condition (10) to prove the convergence of $\mathbf{r}$ to $\mathbf{r}^*$.

In the following consider period $m+1$ (i.e., from t_m to t_{m+1}). At time t_m with the TA vector $\mathbf{r}(m)$, denote the corresponding CSMA Markov chain by $X(t)$ (for convenience we drop the index $m+1$). $X(t)$ is a continuous time Markov chain (CTMC). By (1), the probability of state $x \in \{0,1\}^K$ in the stationary distribution of $X(t)$ is

$$\pi_x(\mathbf{r}(m)) = p(x; \mathbf{r}(m)) = \frac{1}{C(\mathbf{r}(m))} \exp(\sum_k x_k r_k(m)).$$

Since $\mathbf{r}(m) \geq \mathbf{0}$, using (2),

$$C(\mathbf{r}(m)) \leq \sum_{x'} \exp(\mathbf{1}^T \mathbf{r}(m)) \leq 2^K \exp(\mathbf{1}^T \mathbf{r}(m))$$

since there are at most 2^K states. Also, $\exp(\sum_k x_k r_k(m)) \geq 1$ (since $r_k(m) \geq 0$ in Algorithm 1). So, the minimal probability in the stationary distribution

$$\pi_{min}(\mathbf{r}(m)) := \min_x \pi_x(\mathbf{r}(m)) \geq \exp(-\mathbf{1}^T \mathbf{r}(m) - K \cdot \log(2)).$$

Since $\lambda'_k(i) + \min\{c/r_k(i), \bar{w}\} \leq \lambda_{max}$ and $s'_k(\mathbf{r}(i)) \geq 0$, we have $r_k(i+1) \leq r_k(i) + \alpha(i)\lambda_{max}, \forall i, k$. Recall that $r_k(0) = 0, \forall k$. So $r_k(m) \leq \lambda_{max} \sum_{i=1}^m \alpha(i), \forall k$. Thus,

$$\pi_{min}(\mathbf{r}(m)) \geq \exp\{-K \cdot [\lambda_{max} \sum_{i=1}^m \alpha(i) + \log(2)]\}. \quad (16)$$

To proceed with the proof, we first "uniformize" $X(t)$. Recall that for the Markov chain $X(t)$, if each element of its transition rate matrix Q has an absolute value less than a constant A_{m+1}, then we can write $X(t) = Z(N(t))$ where $Z(n)$ is a discrete time Markov chain with probability transition matrix $P = I + Q/A_{m+1}$, where I is the identity matrix, and $N(t)$ is an independent Poisson process with rate A_{m+1}. We claim that $A_{m+1} = K \cdot \exp(\lambda_{max} \sum_{i=1}^m \alpha(i))$ suffices for the need. (Proof: $\because$ $r_k(m) \leq \lambda_{max} \sum_{i=1}^m \alpha(i)$, we have $Q(x, x') \leq \exp(\lambda_{max} \sum_{i=1}^m \alpha(i)), \forall x, x'$. Also, for any state x, $Q(x, x') > 0$ for at most K different x', i.e., state x can at most transit to K other states by changing the state of any one of the K links, so $\sum_{x' \neq x} Q(x, x') \leq A_{m+1}$.)

Now we estimate how far $E[s'_k(m+1)|\mathcal{F}_m]$ is from the desired value $s_k(\mathbf{r}(m))$. Let the vector $\mu_m(t) = \{\mu_m(t, x)\}$ be the probabilities of all states at time $t_m + t$ (where $0 \leq t \leq T_{m+1}$), given that the initial state at time t_m is $x^0(m)$ and that the TA's during $[t_m, t_{m+1})$ are $\mathbf{r}(m)$. Let $x(t_m + t) = \{x_k(t_m + t)\}$ be the state at time $t_m + t$. Then

$$\begin{aligned} &E[s'_k(m+1)|\mathcal{F}_m] \\ &\quad = E[\int_0^{T_{m+1}} I(x_k(t_m + t) = 1) dt / T_{m+1}] \\ &\quad = \int_0^{T_{m+1}} P(x_k(t_m + t) = 1) dt / T_{m+1} \\ &\quad = \sum_{x': x'_k = 1} [\int_0^{T_{m+1}} \mu_m(t, x') dt / T_{m+1}] \\ &\quad = \sum_{x': x'_k = 1} \bar{\mu}_m(x') \end{aligned}$$

where

$$\bar{\mu}_m(x') = \int_0^{T_{m+1}} \mu_m(t, x') dt / T_{m+1}$$

is the time-averaged probability of state x' in the interval. Let $\bar{\mu}_m := \{\bar{\mu}_m(x)\}$ be the vector of such probabilities of all states.

Let $\pi_{x^0}(\mathbf{r}(m))$ be the probability of $x^0(m)$, simply written as x^0, in the stationary distribution of $X(t)$. Use $||\cdot||_{var}$ to denote the variation distance between two distributions (expressed below). Let β_1 be the second largest eigenvalue of P, and the vector $\pi(\mathbf{r}(m)) := \{\pi_x(\mathbf{r}(m))\}$. The following inequality [9] has used the fact that $X(t)$ is equivalent to $Z(N(t))$,

$$\begin{aligned} &||\mu_m(t) - \pi(\mathbf{r}(m))||_{var} := \sum_x [\mu_m(t, x) - \pi_x(\mathbf{r}(m))]/2 \\ &\quad \leq \frac{1}{2} \sqrt{\frac{1 - \pi_{x^0}(\mathbf{r}(m))}{\pi_{x^0}(\mathbf{r}(m))}} \exp(-A_{m+1}(1 - \beta_1)t) \\ &\quad \leq \frac{1}{2} \sqrt{\frac{1}{\pi_{min}(\mathbf{r}(m))}} \exp(-A_{m+1}(1 - \beta_1)t). \end{aligned}$$

So,

$$\begin{aligned} &||\bar{\mu}_m - \pi(\mathbf{r}(m))||_{var} \\ &\quad = ||\int_0^{T_{m+1}} [\mu_m(t) - \pi(\mathbf{r}(m))] dt / T_{m+1}||_{var} \\ &\quad \leq \int_0^{T_{m+1}} ||\mu_m(t) - \pi(\mathbf{r}(m))||_{var} dt / T_{m+1} \\ &\quad \leq \frac{1}{2} \sqrt{\frac{1}{\pi_{min}(\mathbf{r}(m))}} \frac{1}{A_{m+1}(1 - \beta_1) T_{m+1}} \end{aligned} \quad (17)$$

where the first inequality has used the fact that $||\cdot||_{var}$ is a convex function.

Also, β_1 can be bounded by Cheeger's inequality [9]

$$\beta_1 \leq 1 - \phi^2/2 \tag{18}$$

where ϕ is the "conductance" of P, defined as

$$\phi := \min_{S\subset\Omega, \pi(S)\in(0,1/2]} \frac{F(S,S^c)}{\pi_S(\mathbf{r}(m))}$$

where $\pi_S(\mathbf{r}(m)) = \sum_{x\in S}\pi_x(\mathbf{r}(m))$, and $F(S,S^c)$ is the "ergodic flow" from S to S^c: $F(S,S^c) = \sum_{x\in S, x'\in S^c} F(x,x') = \sum_{x\in S,x'\in S^c}\pi_x(\mathbf{r}(m))P(x,x')$. Then similar to [11], we have

$$\begin{aligned}\phi &\geq \min_{S\subset\Omega,\pi(S)\in(0,1/2]} F(S,S^c)\\ &\geq \min_{x\neq x', P(x,x')>0} F(x,x')\\ &= \min_{x\neq x', P(x,x')>0}\{\pi_x(\mathbf{r}(m))\cdot P(x,x')\}.\end{aligned}$$

For any $x \neq x'$ such that $P(x,x') > 0$, it must be that $Q(x,x') > 0$. Note that $Q(x,x') = 1$ or $Q(x,x') = \exp(r_k(m))$ for some k, so $Q(x,x') \geq 1$. Hence, $P(x,x') = Q(x,x')/A_{m+1} \geq 1/A_{m+1}$. Combined with the last inequality, we find

$$\phi \geq \min_x \pi_x(\mathbf{r}(m))/A_{m+1} = \pi_{min}(\mathbf{r}(m))/A_{m+1}.$$

Using (18), $\beta_1 \leq 1 - [\pi_{min}(\mathbf{r}(m))/A_{m+1}]^2/2$. Thus $1/(1-\beta_1) \leq 2\cdot A_{m+1}^2[\pi_{min}(\mathbf{r}(m))]^{-2}$. Plugging this into (17) and use (16), we have

$$\begin{aligned}&||\bar{\mu}_m - \pi(\mathbf{r}(m))||_{var}\\ &\leq A_{m+1}[\pi_{min}(\mathbf{r}(m))]^{-5/2}/T_{m+1}.\\ &\leq K\cdot f(m)/T_{m+1}\end{aligned}$$

where $f(m)$ is defined in (12). So,

$$\begin{aligned}&|E[s_k'(m+1)|\mathcal{F}_m] - s_k(\mathbf{r}(m))|\\ &= |\sum_{x':x_k'=1}\bar{\mu}_m(x') - s_k(\mathbf{r}(m))|\\ &\leq 2||\bar{\mu}_m - \pi(\mathbf{r}(m))||_{var}\\ &\leq 2\cdot K\cdot f(m)/T_{m+1}, \forall k.\end{aligned} \tag{19}$$

Also, with the Bernoulli arrival process $a_k(t)$ assumed in section II-B, it is easy to show that

$$|E[\lambda_k'(m+1)|\mathcal{F}_m] - \lambda_k| \leq 1/T_{m+1}. \tag{20}$$

Therefore, the error bias $B_k(m+1)$, defined in (14), satisfies $|B_k(m+1)| \leq 2K\cdot f(m)/T_{m+1} + 1/T_{m+1} \leq 3K\cdot f(m)/T_{m+1}$. Denote by $\mathbf{B}(m)$ the vector of $B_k(m)$'s. Since $|r_k(m) - r_k^*| \leq \bar{r} + \lambda_{max}\sum_{i=1}^m \alpha(i)$, where $\bar{r} = \max_k r_k^*$, we show that the term $(\mathbf{r}(m)-\mathbf{r}^*)^T\cdot\mathbf{B}(m+1)$ is diminishing:

$$\begin{aligned}&\sum_{m=0}^{\infty}\alpha(m+1)|(\mathbf{r}(m)-\mathbf{r}^*)^T\cdot\mathbf{B}(m+1)|\\ \leq\ & 3K^2\sum_{m=0}^{\infty}\{\alpha(m+1)[\bar{r}+\lambda_{max}\sum_{i=1}^m\alpha(i)]\cdot f(m)/T_{m+1}]\}\\ <\ & \infty\end{aligned} \tag{21}$$

where the last step is obtained using condition (11).

Lemma 3: If (21) and (10) hold, then with Algorithm 1, $\mathbf{r}$ converges to $\mathbf{r}^*$ with probability 1.
The line of proof is similar to that of Theorem 3.1 in [10], but with more intricacies. The complete proof is given in Appendix IX-D.

To conclude, with Algorithm 1, $\mathbf{r}$ converges to the optimal value $\mathbf{r}^*$, such that $s_k(\mathbf{r}^*) > \lambda_k, \forall k$.

In a related work, reference [14] used a differential-equation method to analyze the convergence of the utility maximization algorithm in [4]. In [14], it is required that an upper bound of $\mathbf{r}^*$ is known to the algorithm. Therefore, it is not obvious whether the proof there can be directly applied to the scheduling problem above without a priori upper bound of $\mathbf{r}^*$. In the case of [14], however, the required step size conditions are weaker than those in Algorithm 1.

E. An Algorithm with bounded TA (reduced capacity)

We have shown above that Algorithm 1 is throughput-optimal in that it can support any $\lambda \in \mathcal{C}$, the set of strictly feasible arrival rates. No upper bound of TA is imposed in Algorithm 1. In this section, we give a similar algorithm which simply upper-bound the TA by a constant $r_{max} > 0$. The algorithm's capacity region is smaller than $\mathcal{C}$. But it allows weaker conditions on the step sizes and adjustment periods. Also, one can choose the parameters of the algorithm to make its capacity region arbitrarily close to $\mathcal{C}$.

Algorithm 2: The vector $\mathbf{r}$ is updated at time t_i, $i = 1, 2, \ldots$.

$$r_k(i) = \min\{r_{max}, [r_k(i-1) + \alpha(i)(\lambda_k'(i) + \epsilon - s_k'(i))]_+\}, \forall k. \tag{22}$$

where $\epsilon > 0$. Algorithm 2 tries to solve problem (4) (notice its similarity to (5)), except that it "pretends" to serve the arrival rates $\lambda + \epsilon\cdot\mathbf{1}$ which are higher than the actual arrival rates λ, in order to ensure that the average service rate is strictly higher than the arrival rate after convergence.

Also, $\{\alpha(i)\}$ and non-decreasing $\{T_i\}$ are required to satisfy (9) and

$$\sum_{m=0}^{\infty}[\alpha(m+1)/T_{m+1}] < \infty \tag{23}$$

For example, $\alpha(i) = 1/i$ and $T_i = i^\gamma$ for any $\gamma > 0$ satisfy (9) and (23).

The following theorem states that the capacity region of Algorithm 2 is (at least)

$$\begin{aligned}\mathcal{C}_{II}(r_{max},\epsilon) = \{&\lambda|\lambda+\epsilon\cdot\mathbf{1}\in\mathcal{C} \text{ and}\\ &\mathbf{r}_{II}^*(\lambda+\epsilon\cdot\mathbf{1})\in[0,r_{max}]^K\}\end{aligned}$$

where $\mathbf{r}_{II}^*(\lambda+\epsilon\cdot\mathbf{1})$ is the optimal vector of dual variables $\mathbf{r}^*$ of problem (4) with arrival rates $\lambda+\epsilon\cdot\mathbf{1}$.

Theorem 2: With Algorithm 2, if $\lambda \in \mathcal{C}_{II}(r_{max},\epsilon)$, then $\mathbf{r}(i)$ converges to $\mathbf{r}_{II}^*(\lambda+\epsilon\cdot\mathbf{1})$ with probability 1. (And $s_k(r_{II}^*(\lambda+\epsilon\cdot\mathbf{1})) \geq \lambda_k + \epsilon > \lambda_k, \forall k$.)

Remark: Similar to the case of Algorithm 1, it can be further shown that the system is rate stable (Appendix IX-E).

Proof: The proof is very similar to that of Theorem 1. So here we only present the differences. Following the proof of Theorem 1, in the first step, we bound the error bias. Since $r_k(m) \in [0, r_{max}], \forall k, m$, we have (different from inequality (16)),

$$\pi_{min}(\mathbf{r}(m)) \geq C_1 := \exp\{-K \cdot [r_{max} + \log(2)]\}.$$

Also, the definition of A_{m+1} is changed to $A_{m+1} = K \cdot \exp(r_{max})$. Then, (19) is changed into

$$|E[s'_k(m+1)|\mathcal{F}_m] - s_k(\mathbf{r}(m))| \leq C_2/T_{m+1}$$

for a constant $C_2 > 0$. Therefore condition (23) is sufficient to ensure that $\sum_{m=0}^{\infty} \alpha(m+1)|(\mathbf{r}(m)-\mathbf{r}^*)^T \cdot \mathbf{B}(m+1)| < \infty$.

In the second step, we show $\mathbf{r}(m)$ converges to $\mathbf{r}^*_{II}(\lambda + \epsilon \cdot \mathbf{1})$. First notice that $\mathbf{r}^*_{II}(\lambda + \epsilon \cdot \mathbf{1}) \in [0, r_{max}]^K$ by the definition of $\mathcal{C}_{II}(r_{max}, \epsilon)$. Second, in (34), $\lambda_{max} \sum_{i=1}^{m-1} \alpha(i)$ can be replaced by r_{max}. Therefore, (9) suffices to ensure the convergence. ■

Clearly, $\mathcal{C}_{II}(r_{max}, \epsilon) \rightarrow \mathcal{C}$ as $r_{max} \rightarrow +\infty$ and $\epsilon \rightarrow 0$. So we can choose r_{max}, ϵ to achieve arbitrarily close approximations of the maximal capacity region $\mathcal{C}$.

IV. Constant step size and adjustment period

Now we consider Algorithm 2 with a constant step size $\alpha(i) = \alpha, \forall i$ and a constant adjustment period $T_i = T, \forall i$.

Theorem 3: If $\lambda \in \mathcal{C}_{II}(r_{max}, \epsilon)$, then there exists $\alpha > 0, T > 0$ such that the queues are stable using Algorithm 2 with $\alpha(i) = \alpha, T_i = T, \forall i$.

The proof is given in Appendix IX-F.

V. Algorithms without the "gap"

The following Algorithm 3 is similar to Algorithm 1.

Algorithm 3:

$$r_k(i) = [r_k(i-1) + \alpha(i)(\lambda'_k(i) - s'_k(i))]_+ \qquad (24)$$

where $\alpha(i)$ and T_i satisfy the conditions specified in Algorithm 1.

Theorem 4: Assume that λ is strictly feasible (i.e., $\lambda \in \mathcal{C}$). Then with Algorithm 3, $\mathbf{r}$ converges to $\mathbf{r}^*$, the optimal vector of dual variables of problem (1) with probability 1, where$\mathbf{r}^*$ satisfies that $s_k(\mathbf{r}^*) \geq \lambda_k, \forall k$. Also, the system is rate stable.

The proof is virtually the same as that of Theorem 1.

The following Algorithm 4 is similar to Algorithm 2.

Algorithm 4:

$$r_k(i) = \min\{r_{max}, [r_k(i-1) + \alpha(i)(\lambda'_k(i) - s'_k(i))]_+\}, \forall k.$$

Theorem 5: Assume that $\lambda \in \mathcal{C}_{II}(r_{max}, 0)$.

(i) If $\alpha(i)$ and T_i satisfy (9) and (23), and $\{T_i\}$ is non-decreasing, then $\mathbf{r}$ converges to $\mathbf{r}^*$, the optimal vector of dual variables of problem (1) with probability 1, where$\mathbf{r}^*$ satisfies that $s_k(\mathbf{r}^*) \geq \lambda_k, \forall k$; and the system is rate stable.

(ii) For any $\delta > 0$, there exist $\alpha, T > 0$ such that if we use $\alpha(i) = \alpha, T_i = T, \forall i$, the long-term average service rate of link k is larger than $\lambda_k - \delta$ for all k.

The proof of (i) is the same as before. Part (ii) can be proved similarly to Theorem 3, by showing that in the long term, $\mathbf{r}(i)$ is near $\mathbf{r}^*$ with arbitrarily high probability by choosing small enough α and large enough T. Thus, the average service rate of link k can be arbitrarily close to λ_k.

VI. Joint scheduling and congestion control

In [4], [5], we also designed a joint CSMA scheduling and congestion control algorithm to approach the maximal utility in the network. For simplicity, here we assume that each link k generates a one-hop data flow with adjustable rate $f_k \in [0, 1]$. Write $\mathbf{f} \in [0, 1]^K$ as the vector of f_k's. Flow k has an increasing, strictly concave utility function $v_k(f_k)$.

Algorithm 5: Joint scheduling and congestion control algorithm

The vectors $\mathbf{r}$ and $\mathbf{f}$ are updated at time t_i, $i = 1, 2, \ldots$. Let $t_0 = 0$ and $T_i := t_i - t_{i-1}, i = 1, 2, \ldots$. Define "period i" as the time between t_{i-1} and t_i, and $\mathbf{r}(i)$ be the value of $\mathbf{r}$ at the end of period i, i.e., at time t_i. Initially, link k set $r_k(0) = 0, f_k(0) = 1, \forall k$. Then at time t_i, $i = 1, 2, \ldots$, do the following:

- Congestion control: Link k sets $f_k(i) = \rho \cdot \hat{f}_k(i)$ where ρ is slightly smaller than 1 and $\hat{f}_k(i) = \arg\max_{f' \in [0,1]}[\beta \cdot v_k(f') - r_k(i) f']$, where $\beta > 0$ is constant "weighting factor".
- r_k is updated as follows: $r_k(i) = [r_k(i-1) + \alpha(i)(\hat{f}_k(i-1) - s'_k(i))]_+$.

Theorem 6: If $\{\alpha(i)\}$ and $\{T_i\}$ satisfy the conditions specified in Algorithm 1, then the algorithm converges, i.e., $\mathbf{r}(i) \rightarrow \mathbf{r}^*$, $\hat{f}_k(i) \rightarrow \hat{f}^*_k = s_k(\mathbf{r}^*)$ as $i \rightarrow \infty$ with probability 1, and all queues are stable. Also,

$$\sum_k v_k(\hat{f}^*_k) \geq \bar{W} - \log(N)/\beta \qquad (25)$$

where $\bar{W}$ is the maximal achievable total utility, and N is the number of IS's. We see that when β is large, the algorithm achieves close-to-optimal total utility.

Proof: The proof of convergence is almost the same as Theorem 1. The performance bound (25) has been shown in [4]. ■

Theorem 7: Assume that $v'_k(0) < G < \infty$ for all k. If $\alpha(i)$ is non-increasing with i, $\sum_i \alpha(i) = \infty$, $\sum_i \alpha(i)^2 < \infty$, and condition (23) is satisfied (for example, $\alpha(i) = 1/i$ and $T_i = i^\gamma$ where $\gamma > 0$), then the algorithm converges, i.e., $\mathbf{r}(i) \rightarrow \mathbf{r}^*$, $\hat{f}_k(i) \rightarrow \hat{f}^*_k = s_k(\mathbf{r}^*)$ as $i \rightarrow \infty$ with probability 1, and all queues are stable. Also, (25) holds.

Proof: First notice that r_k is always upper-bounded in the algorithm: $r_k(i) \leq \beta \cdot G + \alpha(1), \forall i$. Proof by induction: (a) $r_k(0) = 0 \leq \beta \cdot G + \alpha(1)$; (b) Assume that $r_k(i-1) \leq \beta \cdot G + \alpha(1)$. Consider two cases. If $\beta \cdot G \leq r_k(i-1) \leq \beta \cdot G + \alpha(1)$, then $\hat{f}_k(i-1) = 0$. So $r_k(i) \leq [r_k(i-1)]_+ = r_k(i-1) \leq \beta \cdot G + \alpha(1)$. If $r_k(i-1) < \beta \cdot G$, since $\hat{f}_k(i), s'_k(i) \in [0, 1]$, we have $r_k(i) \leq [r_k(i-1) + \alpha(i)]_+ \leq \beta \cdot G + \alpha(1)$. Combining (a) and (b) gives the desired result. Similarly, it can be also shown that r_k is always lower-bounded.

Then, like the proof of Theorem 2, the algorithm converges: $\mathbf{r}(i) \to \mathbf{r}^*$ and $\hat{f}_k(i) \to \hat{f}_k^* = s_k(\mathbf{r}^*)$. Since the actual input rate $f_k < \hat{f}_k$, all queues are stable and the total utility is close to $\sum_k v_k(\hat{f}_k^*)$ if ρ is close to 1. The bound (25) has been shown in [4]. ■

Remark: In fact, similar to the proof of Theorem 1 but with more arguments, one can show that if $T_i = T$, a constant, and $\sum_i \alpha(i) = \infty$, $\sum_i \alpha(i)^2 < \infty$, the same conclusion in Theorem 7 holds. A different approach using ODE is provided in [14].

VII. Numerical Examples

A. CSMA scheduling: i.i.d. input traffic with fixed average rates

In our C++ simulations, the transmission time of all links is exponentially distributed with mean 1ms, and the backoff time of link k is exponentially distributed with mean $1/\exp(r_k)$ ms. Assume that the capacity of each link is 1(data unit)/ms. The initial TA $r_k(0) = 0$ for all link k. To show the negative drift of queues, assume that initially, all queue lengths are 300 data units. Then r_k is adjusted using Algorithm 1, with step sizes $\alpha(i) = 0.46/[(2+i/1000)\log(2+i/1000)]$ and adjustment periods $T_i = (2+i/1000)$ ms. The constants $c = 0.01$, and $\bar{w} = 0.02$.

There are 6 links in the network, whose conflict graph is shown in Fig. 1 (a). (Each link only needs to know the set of links that conflict with itself.) Define $0 \le \rho < 1$ as the "load factor", and let $\rho = 0.98$ in this simulation. The arrival rate vector is set to λ=ρ*[0.2*(1,0,1,0,0,0) + 0.3*(1,0,0,1,0,1) + 0.2*(0,1,0,0,1,0) + 0.3*(0,0,1,0,1,0)] = ρ*(0.5,0.2,0.5,0.3,0.5,0.3) (data units/ms). We have multiplied by ρ a convex combination of some maximal IS's to ensure that λ is in the interior of the capacity region.

As expected, the TA vector $\mathbf{r}$ tends to converge (Fig. 1 (b)). Also, the queues tend to decrease and are stable (Fig. 1 (c)).

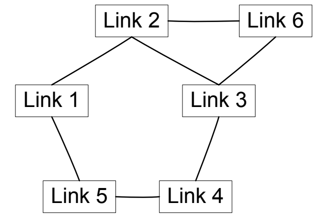
Fig. 2 shows the simulation results of the same network using Algorithm 2 with constant step size $\alpha(i) = \alpha = 0.23$, $T_i = T = 5ms$ and $r_{max} = 8$. (All other parameters remain unchanged.) We observe that the queues are also stable and tend to decrease. (Note that the conditions on α and T in the constructive proof of Theorem 3 are sufficient, but not necessary in general.)

Comparing Fig. 1 and Fig. 2, we see that with decreasing $\alpha(i)$ and increasing T_i, although $\mathbf{r}$ indeed converges, there are more oscillations in the queue lengths. This is because when $\alpha(i)$ becomes smaller when i is large, $\mathbf{r}(i)$ becomes less responsive to the variations of queue lengths.

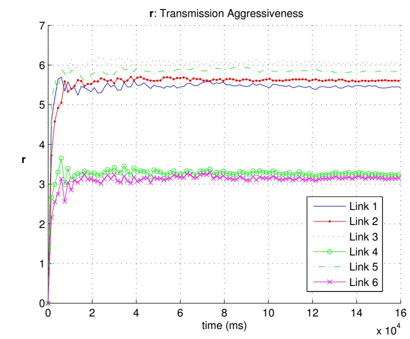
B. Joint CSMA scheduling and congestion control

The link contention graph is shown in Fig. 3. The simulation results are shown in Fig. 4. The utility function is $v_k(f_k) = \log(f_k + 0.1)$. The weighting factor $\beta = 1.5$. The adjustment period is $T_i = T = 5ms$, and $\alpha(i) = 0.14/(2 + i/100)$.[2] The initial queue lengths are 300 data units.

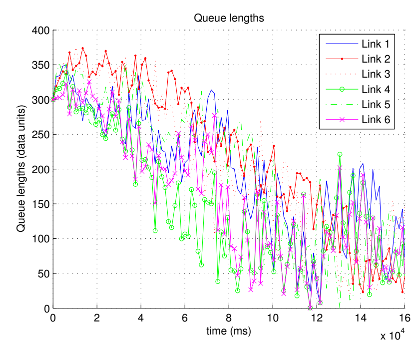
[2]Here we mainly simulate decreasing step sizes since constant step size has been evaluated in [4].



(a) Link Contention Graph



(b) $\mathbf{r}$: the vector of TA



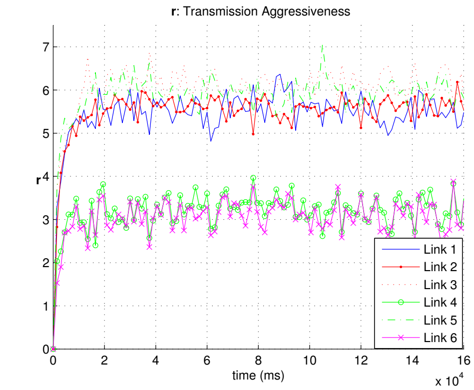
(c) Queue lengths

Fig. 1: CSMA Scheduling with varying step sizes and adjustment periods

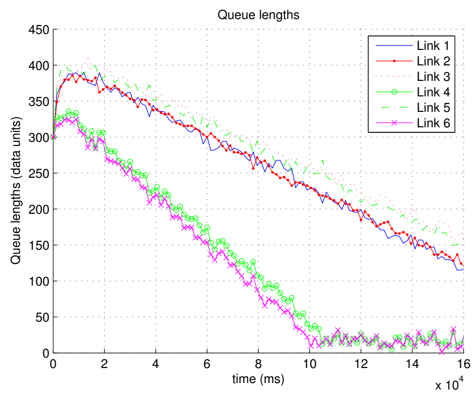
VIII. Conclusion

In this paper, we have proved the convergence and/or stability property of the distributed CSMA scheduling algorithm proposed in [4], [5] with properly chosen step sizes and adjustment periods. Similar results also apply to the cross-layer algorithm (joint CSMA scheduling and congestion control) in [4], [5].

The conditions on the step sizes and adjustment periods given here are sufficient for the convergence/stability of the algorithms. However, since certain bounds in the proof may not be tight, it is possible that these conditions are not necessary. Also, we have assumed general conflict graphs. In many networks of practical interest, however, the conflict graphs may have particular structures. For example, if the conflict graph is a full graph (corresponding to a network where all links conflict to each other), then it can be shown that the mixing time is much smaller than the worst-case bound used in this paper. In the future, we would like to

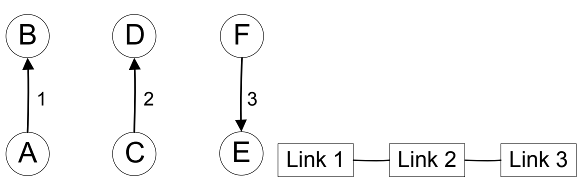


(a) $\mathbf{r}$: the vector of TA



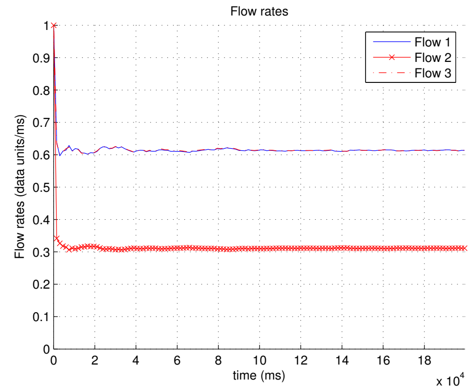
(b) Queue lengths

Fig. 2: CSMA Scheduling with constant step size and adjustment period (Network 1)

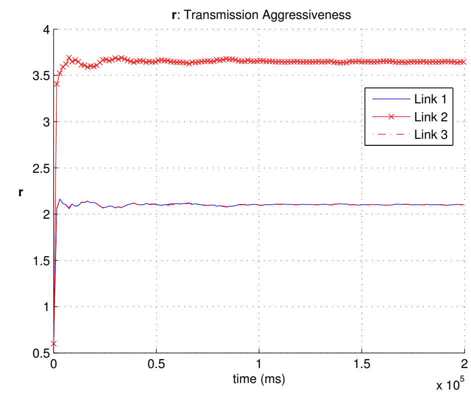


(a) Network topology (b) Link contention graph

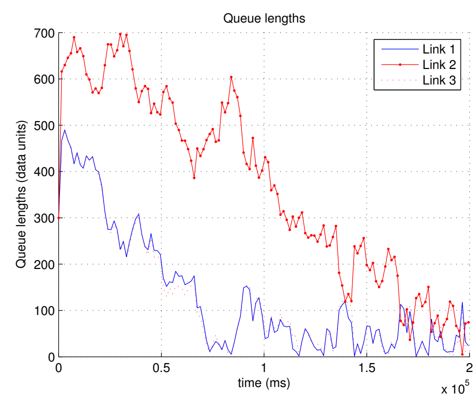
Fig. 3: Network 2



(a) Flow rates



(b) $\mathbf{r}$: the vector of TA



(c) Queue lengths

Fig. 4: Joint CSMA Scheduling and congestion control, without collisions (Network 2)

study whether some of the conditions can be relaxed, either generally or in networks with certain structure.

IX. Appendices

A. Proof of Lemma 1

Proof: Since λ is strictly feasible, there exists $\mathbf{u} \succ \mathbf{0}$ (i.e., $\mathbf{u} = \bar{\mathbf{p}}$) satisfying the constraints of problem (4). Therefore (4) is strictly feasible (satisfying the Slater condition [12]). So there exist (finite) optimal dual variables r_k^*'s.

Given some finite dual variables $\mathbf{r}$, a partial Lagrangian of problem (4) is

$$\mathcal{L}(\mathbf{u};\mathbf{r}) = -\sum_i u_i \log(u_i) + \sum_k r_k (\sum_i u_i \cdot x_k^i - \lambda_k). \quad (26)$$

Denote $\mathbf{u}(\mathbf{r}) = \arg\max_{\mathbf{u}} \mathcal{L}(\mathbf{u};\mathbf{r})$, subject to that $\mathbf{u}$ is a distribution. Since $\sum_i u_i = 1$, if we can find some v, and $\mathbf{u}(\mathbf{r}) > 0$ such that

$$\frac{\partial \mathcal{L}(\mathbf{u}(\mathbf{r});\mathbf{r})}{\partial u_i} = -\log(u_i(\mathbf{r})) - 1 + \sum_k r_k x_k^i = v, \forall i,$$

then $\mathbf{u}(\mathbf{r})$ is the desired distribution. Solving the above equation yields $u_i(\mathbf{r}) = p(x^i;\mathbf{r}), \forall i$ (cf. (1)), that is, $u_i(\mathbf{r})$ is exactly the stationary probability of state i in the CSMA Markov chain given the TA vector $\mathbf{r}$. Since the optimal solution of $\mathbf{u}$, $u_i(\mathbf{r}^*) = p(x^i;\mathbf{r}^*), \forall i$ must be feasible to problem (4), we have $\sum_i (u_i(\mathbf{r}^*) \cdot x_k^i) = \sum_i (x_k^i \cdot p(x^i;\mathbf{r}^*)) =$

$s_k(\mathbf{r}^*) \geq \lambda_k, \forall k$.

Let $L_1(\mathbf{r}) := \max_{\mathbf{u}} \mathcal{L}(\mathbf{u};\mathbf{r})$ subject to the constraints that $u_i \geq 0, \sum_i u_i = 1$. It follows from the optimization theory [12] that the dual problem of (4) is $\min_{\mathbf{r}\geq\mathbf{0}} L_1(\mathbf{r})$, and the vector $\mathbf{g}(\mathbf{r}) \in \mathcal{R}^K$ whose k'th element $g_k(\mathbf{r}) := \sum_i u_i(\mathbf{r}) \cdot x_k^i - \lambda_k = s_k(\mathbf{r}) - \lambda_k$ is a subgradient of $L_1(\cdot)$ at $\mathbf{r}$. Therefore the subgradient dual algorithm (5) follows.

On the uniqueness of $\mathbf{r}^*$: Note that the objective function of (1) is strictly concave. Therefore $\mathbf{u}^*$, the optimal solution of (1) is unique. Consider two state $\mathbf{e}_k$ and $\mathbf{0}$, where $\mathbf{e}_k$ is the K-dimensional vector whose k'th element is 1 and all other elements are 0's. We have $u^*_{\mathbf{e}_k} = p(\mathbf{e}_k;\mathbf{r}^*)$ and $u^*_{\mathbf{0}} = p(\mathbf{0};\mathbf{r}^*)$. Then

$$u^*_{\mathbf{e}_k}/u^*_{\mathbf{0}} = \exp(r_k^*). \tag{27}$$

Suppose that r^* is not unique, that is, there exist $\mathbf{r}^*_I \neq \mathbf{r}^*_{II}$ but both are optimal $\mathbf{r}$. Then, $r^*_{I,k} \neq r^*_{II,k}$ for some k. This contradict to (27) and the uniqueness of $\mathbf{u}^*$. Therefore $\mathbf{r}^*$ is unique. Note that $\mathbf{r}^*$ is also the unique solution of the dual problem $\min_{\mathbf{r}\geq\mathbf{0}} L_1(\mathbf{r})$. ■

B. Proof of Lemma 2

Proof: Since λ is strictly feasible, by definition, it can be written as $\lambda = \sum_i \bar{p}_i \cdot x^i$ where $\bar{p}_i > 0$ and $\sum_i \bar{p}_i = 1$. For convenience, in the vector $\bar{\mathbf{p}}$, let $\bar{p}_0$ be the probability of the all-0 IS, and $\bar{p}_k, k = 1,2,\ldots K$ be the probability of the IS $\mathbf{e}_k$ (i.e., the IS where only link k is active). Define another distribution $\bar{\mathbf{p}}'$ (over the IS's) as follows. Let $\bar{p}'_0 = \bar{p}_0/(K+1) > 0$ and $\bar{p}'_k = \bar{p}_k + \bar{p}_0/(K+1) > 0, k = 1,2,\ldots,K$, and all other IS's have the same probabilities. Then we have $\forall k, \sum_i(\bar{p}'_i \cdot x_k^i) = \sum_i(\bar{p}_i \cdot x_k^i) + \bar{p}_0/(K+1) = \lambda_k + \bar{p}_0/(K+1)$. That is, there exist $\mathbf{u} \succ \mathbf{0}, \mathbf{w} \succ \mathbf{0}$ (i.e., $\mathbf{u} = \bar{\mathbf{p}}', \mathbf{w} = \bar{p}_0/(K+1) \cdot \mathbf{1}$) satisfying the constraints of problem (4). Therefore problem (6) is strictly feasible and satisfies the Slater condition. Hence, there exists (finite) optimal dual variables r_k^*'s. With r_k's as dual variables, a partial Lagrangian of (6) is

$$\begin{aligned}\mathcal{L}(\mathbf{u},\mathbf{w};\mathbf{r}) &= -\textstyle\sum_i u_i \log(u_i) + c\sum_k \log(w_k) + \\ &\quad \textstyle\sum_k [r_k(\sum_i u_i \cdot x_k^i - \lambda_k - w_k)] \\ &= [-\textstyle\sum_i u_i \log(u_i) + \sum_k (r_k \sum_i u_i \cdot x_k^i)] + \\ &\quad \textstyle\sum_k [c\cdot\log(w_k) - r_k w_k] - \sum_k (r_k\lambda_k).\end{aligned} \tag{28}$$

Let $L_2(\mathbf{r}) := \max_{\mathbf{u},\mathbf{w}} \mathcal{L}(\mathbf{u},\mathbf{w};\mathbf{r})$ subject to $u_i \geq 0, \sum_i u_i = 1$ and $0 \leq w_k \leq \bar{w}, \forall k$. Denote by $\mathbf{u}(\mathbf{r})$ and $\mathbf{w}(\mathbf{r})$ the maximizers. Similar to the last lemma, $\sum_i u_i(\mathbf{r}) \cdot x_k^i = s_k(\mathbf{r})$. And it is easy to find $w_k(\mathbf{r}) = (c/r_k) \wedge \bar{w}$. Since the optimal solutions of $\mathbf{u}, \mathbf{w}$, i.e., $u_i(\mathbf{r}^*) = p(x^i;\mathbf{r}^*), \forall i$ and $w_k(\mathbf{r}^*)$ are feasible to problem (6), we have $\sum_i(u_i(\mathbf{r}^*) \cdot x_k^i) = s_k(\mathbf{r}^*) \geq \lambda_k + w_k(\mathbf{r}^*) > \lambda_k, \forall k$. Also, the vector $\mathbf{g}(\mathbf{r}) \in \mathcal{R}^K$ whose k'th element $g_k(\mathbf{r}) := \sum_i u_i(\mathbf{r}) \cdot x_k^i - \lambda_k - w_k(\mathbf{r}) = s_k(\mathbf{r}) - \lambda_k - w_k(\mathbf{r})$ is a subgradient of $L_2(\cdot)$ at $\mathbf{r}$. So the subgradient dual algorithm (7) follows.

The uniqueness of $\mathbf{r}^*$ follows from the same argument in the proof of Lemma 1. ■

C. Proof of Proposition 1

Proof: Condition (9) is easy to check: $\sum_i \alpha(i) \geq \int_2^\infty [1/(y\log y)]dy = \log(\log(y))|_2^\infty = \infty$, and $\sum_i \alpha(i)^2 \leq [\frac{1}{\log(2)}]^2 \sum_i \frac{1}{(i+1)^2} < \infty$.

For $m \geq 1$, $0 \leq \sum_{i=1}^m \alpha(i) \leq \alpha(1) + \int_1^m 1/[(x+1)\log(x+1)]dx \leq c_1 + \log\log(m+1)$ where $c_1 = \alpha(1) - \log\log 2 > 0$. So

$$\begin{aligned} f(m) \quad &\leq \quad \exp\{(\frac{5}{2}K+1)[\lambda_{max}\log\log(m+1) + \\ &\qquad \lambda_{max}c_1 + \log(2)]\}\end{aligned}$$

for $m \geq 1$. When $m = 0$, $\alpha(m+1) \cdot \sum_{i=1}^m \alpha(i) \cdot f(m)/T_{m+1} = 0$, so the L.H.S. of (11) is

$$\begin{aligned}&\sum_{m=1}^\infty [\alpha(m+1) \cdot \sum_{i=1}^m \alpha(i) \cdot f(m)/T_{m+1}] \\ &\leq \exp\{(\frac{5}{2}K+1)\cdot[\lambda_{max}c_1 + \log(2)]\} \cdot \\ &\quad \sum_{m=1}^\infty \frac{[\log(m+1)]^{(\frac{5}{2}K+1)\cdot\lambda_{max}}[\log\log(m+1)+c_1]}{(m+1)(m+2)\log(m+2)}.\end{aligned}$$

When $m \geq M$ for a large enough M, $[\log(m+1)]^{(\frac{5}{2}K+1)\cdot\lambda_{max}}[\log\log(m+1)+c_1] \leq m^{1/2}$. Thus

$$\begin{aligned}&\sum_{m=M}^\infty \frac{[\log(m+1)]^{(\frac{5}{2}K+1)\cdot\lambda_{max}}[\log\log(m+1)+c_1]}{(m+1)(m+2)\log(m+2)} \\ &\leq \sum_{m=M}^\infty \frac{m^{1/2}}{m^2\log(M+2)} \\ &= \frac{1}{\log(M+2)} \sum_{m=M}^\infty m^{-3/2} < \infty.\end{aligned}$$

So (11) holds. To check condition (10), we have

$$\begin{aligned}&\sum_{m=0}^\infty [\alpha(m+1)\sum_{i=1}^m \alpha(i)]^2 \\ \leq &\sum_{m=1}^\infty [\frac{c_1 + \log\log(m+1)}{(m+2)\log(m+2)}]^2 \\ \leq &\sum_{m=1}^\infty [\frac{c_1 + \log\log(m+1)}{(m+2)}]^2 \\ < &\ \infty.\end{aligned}$$

■

D. Proof of Lemma 3

Let $\mathbf{r}^*$ be the optimal dual variables of problem (6). Use $||\cdot||$ to denote the L2 norm. Since $r_k(m) = [r_k(m-1) - \alpha(m)\cdot g'_k(m)]_+$ by Algorithm 1, we have

$$\begin{aligned}&||\mathbf{r}(m) - \mathbf{r}^*||^2 \\ &\quad \leq ||\mathbf{r}(m-1) - \alpha(m)\cdot g'_k(m) - \mathbf{r}^*||^2 \\ &\quad = ||\mathbf{r}(m-1) - \mathbf{r}^*||^2 - \alpha(m)\cdot[\mathbf{r}(m-1) - \mathbf{r}^*]^T\mathbf{g}'(m) \\ &\qquad + \alpha^2(m)||\mathbf{g}'(m)||^2\end{aligned}$$

where the first inequality follows from the fact that the projection $[\cdot]_+$ is non-expansive [12]. Denote $d(m) =$

$||\mathbf{r}(m)-\mathbf{r}^*||^2$. Since $||\mathbf{g}'(m)||^2$ is bounded (cf. (13)), write $||\mathbf{g}'(m)||^2 \le C$. Using this and (15),

$$\begin{aligned} d(m) \;\le\; & d(m-1)+\alpha(m)\cdot[\mathbf{r}^*-\mathbf{r}(m-1)]^T\mathbf{g}(m) \\ & +\alpha(m)\cdot[\mathbf{r}^*-\mathbf{r}(m-1)]^T[\mathbf{B}(m)+\eta(m)] \\ & +\alpha^2(m)\cdot C. \end{aligned} \tag{29}$$

Assume that $\mathbf{r}(m-1)\notin H_\mu := \{\mathbf{r}|L_2(\mathbf{r})\le \mu + L_2(\mathbf{r}^*)\}$ (recall that the dual problem of (6) is $\min_{\mathbf{r}\ge\mathbf{0}} L_2(\mathbf{r})$). Since $\mathbf{g}(m)$ is a subgradient of $L_2(.)$ at $\mathbf{r}(m-1)$, we have $[\mathbf{r}^*-\mathbf{r}(m-1)]^T\mathbf{g}(m)\le L_2(\mathbf{r}^*)-L_2(\mathbf{r})\le -\mu$. So

$$\begin{aligned} & E(d(m)|\mathcal{F}_{m-1})\le d(m-1)-\alpha(m)\mu \\ & +\alpha(m)\cdot[\mathbf{r}^*-\mathbf{r}(m-1)]^T\mathbf{B}(m)+\alpha^2(m)\cdot C. \end{aligned} \tag{30}$$

By inequality (21), $|\sum_m\{\alpha(m)\cdot[\mathbf{r}^*-\mathbf{r}(m-1)]^T\mathbf{B}(m)\}|<\infty$ in any realization and $\sum_m \alpha^2(m)\cdot C<\infty$. Then we use the same super-martingale lemma (Lemma A.1) in [10] to conclude that the set H_μ is recurrent for $\{\mathbf{r}(m)\}$.

Next, by (29) we have for $n\ge m$,

$$\begin{aligned} d(n) \le\; & d(m-1) \\ & + \textstyle\sum_{i=m}^n\{\alpha(i)\cdot[\mathbf{r}^*-\mathbf{r}(i-1)]^T\mathbf{g}(i)\} \\ & + \textstyle\sum_{i=m}^n\{\alpha(i)\cdot[\mathbf{r}^*-\mathbf{r}(i-1)]^T[\mathbf{B}(i)+\eta(i)]\} \\ & + C\textstyle\sum_{i=m}^n\alpha^2(i). \end{aligned} \tag{31}$$

Since $C\sum_{i=1}^\infty\alpha^2(i)<\infty$, we have

$$\lim_{m\to\infty} C\sum_{i=m}^\infty \alpha^2(i)=0. \tag{32}$$

Also, $\sum_{i=1}^\infty|\alpha(i)\cdot[\mathbf{r}^*-\mathbf{r}(i-1)]^T\mathbf{B}(i)|<\infty$ by (21). So

$$\lim_{m\to\infty}\sum_{i=m}^\infty|\alpha(i)\cdot[\mathbf{r}^*-\mathbf{r}(i-1)]^T\mathbf{B}(i)|=0. \tag{33}$$

Finally, $W(n):=\sum_{i=1}^n\{\alpha(i)\cdot[\mathbf{r}^*-\mathbf{r}(i-1)]^T\eta(i)\}$ is a martingale [13]. To see this, note that (a) $W(n)\in\mathcal{F}_n$; (b) $E|W(n)|<\infty,\forall n$; and (c) $E(W(n)|\mathcal{F}_{n-1})-W(n-1)=\alpha(n)\cdot[\mathbf{r}^*-\mathbf{r}(n-1)]^TE[\eta(n)|\mathcal{F}_{n-1}]=0$. Also, since

$$|(\mathbf{r}^*-\mathbf{r}(m-1))^T\eta(m)|\le K\cdot c_2[\bar{r}+\lambda_{max}\sum_{i=1}^{m-1}\alpha(i)]$$

(recall that $|\eta_k(m)|\le c_2$), we have

$$\begin{aligned} & E\{[\alpha(m)\cdot(\mathbf{r}^*-\mathbf{r}(m-1))^T\eta(m)]^2\} \\ & \quad =\alpha(m)^2E[|(\mathbf{r}^*-\mathbf{r}(m-1))^T\eta(m)|^2] \\ & \quad \le\alpha(m)^2K^2c_2^2[\bar{r}+\lambda_{max}\textstyle\sum_{i=1}^{m-1}\alpha(i)]^2. \end{aligned}$$

Therefore

$$\begin{aligned} & \sup_n E(W(n)^2) \\ = \; & \sup\textstyle\sum_{m=1}^n E\{[\alpha(m)\cdot(\mathbf{r}^*-\mathbf{r}(m-1))^T\eta(m)]^2\} \\ \le \; & \textstyle\sum_{m=1}^\infty E\{[\alpha(m)\cdot(\mathbf{r}^*-\mathbf{r}(m-1))^T\eta(m)]^2\} \\ \le \; & \textstyle\sum_{m=1}^\infty\{\alpha(m)^2K^2c_2^2[\bar{r}+\lambda_{max}\sum_{i=1}^{m-1}\alpha(i)]^2\}<\infty \end{aligned} \tag{34}$$

where the last step follows from condition (10). By the L2 Martingale Convergence Theorem [13], $W(n)$ converges with probability 1. So

$$\begin{aligned} & \sup_{n\ge m\ge N_0}|\sum_{i=m}^n\{\alpha(i)\cdot[\mathbf{r}^*-\mathbf{r}(i-1)]^T\eta(i)\}| \\ & \quad =\sup_{n\ge m\ge N_0}|W(n)-W(m-1)|\to 0 \end{aligned} \tag{35}$$

as $N_0\to\infty$ with probability 1.

Combining (32), (33) and (35), we know that with probability 1, for any $\epsilon>0$, after $\mathbf{r}(m-1)$ returns to H_μ for some large enough m (due to recurrence of H_μ),

$$\sum_{i=m}^n\{\alpha(i)\cdot[\mathbf{r}^*-\mathbf{r}(i-1)]^T[\mathbf{B}(i)+\eta(i)]\}+C\sum_{i=m}^n\alpha^2(i)\le\epsilon$$

for any $n\ge m$. In (31), since $[\mathbf{r}^*-\mathbf{r}(i-1)]^T\mathbf{g}(i)\le 0$, we have $d(n)\le d(m-1)+\epsilon,\forall n\ge m$. In other words, $\mathbf{r}$ cannot move far away from H_μ after iteration $m-1$. Since the above argument hold for H_μ with arbitrarily small μ and any $\epsilon>0$, $\mathbf{r}$ converge to $\mathbf{r}^*$ with probability 1.

E. Rate stability of Algorithm 1 and Algorithm 2

The rate-stability of Algorithm 1 and 2 follows from Theorem 8 below. To prove it we need a lemma which is intuitively clear.

Recall that for link k, the cumulative arrival process is $A_k(t)$, the cumulative service process is $S_k(t)$, and the cumulative departure process by $D_k(t)$. When the queue is empty, there is no departure but there can be service, so $S_k(t)\ge D_k(t)$. Assume that the initial queue lengths are zero, then it is clear that $D_k(t)\le A_k(t)$, and link k's queue length is $Q_k(t)=A_k(t)-D_k(t)$.

By assumption, $\lim_{t\to\infty}A_k(t)/t=\lambda_k$ a.s.. Also, we have the following lemma.

Lemma 4: $\lim_{t\to\infty}S_k(t)/t=s_k(\mathbf{r}^*),\forall k$, a.s..

Proof: This is a quite intuitive result since $\mathbf{r}\to\mathbf{r}^*$ a.s.. In the following we give a proof. Recall that $\mathbf{r}$ is adjusted at time $t_i, i=1,2,\dots$. And $T_i=t_i-t_{i-1}\to\infty$ as $i\to\infty$ (This can be derived from the conditions on T_i in Algorithm 1 and 2 and the fact that T_i is non-decreasing). Fix a $T>0$, we construct a sequence of time $\{\tau_j\}$ as follows. Let $\tau_0=t_0=0$. Denote $t_{(j)}:=\min\{t_i|t_i>\tau_j\}$, i.e., $t_{(j)}$ is the nearest time in the sequence $\{t_i,i=1,2,\dots\}$ that is larger than τ_j. The following defines $\tau_j, j=1,2,\dots$ recursively. If $t_{(j)}-\tau_j<2T$, then let $\tau_{j+1}=t_{(j)}$. If $t_{(j)}-\tau_j\ge 2T$, then let $\tau_{j+1}=\tau_j+T$. Also, define $U_j:=\tau_j-\tau_{j-1}, j=1,2,\dots$.

Denote $i^*(T)=\min\{i|T_{i+1}\ge T\}$, and $j^*(T)=\min\{j|\tau_j=t_{i^*(T)}\}$. From the above construction, we have

$$T\le U_j\le 2T,\forall j>j^*(T) \tag{36}$$

Define $\hat{s}_j:=[S_k(\tau_{j+1})-S_k(\tau_j)]/U_{j+1}$. Write and $\hat{s}_j=s_k(\mathbf{r}(\tau_j))+b_j+m_j$, where the "error bias" $b_j=E_j(\hat{s}_j)-s_k(\mathbf{r}(\tau_j))$ ($E_j(\cdot)$ is the expectation conditions on the state at time τ_j), and the martingale noise $m_j=\hat{s}_j-E_j(\hat{s}_j)$ (note that $E_j(m_j)=0$). For convenience, we have dropped the subscript k in $\hat{s}_j, b_j, m_j$. But all discussion below is for link k.

First show that $\lim_{N\to\infty}[\sum_{j=0}^{N}(m_j \cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}] = 0$ a.s.. Since m_j is bounded, $E(m_j^2) \le c_1$ for some $c_1 > 0$. Clearly, $M_N := \sum_{j=0}^{N}(m_j \cdot U_{j+1}), N = 0,1,\ldots$ is a martingale (define $M_{-1} = 0$). We have $E(M_N^2) = \sum_{j=0}^{N}(E(m_j^2) \cdot U_{j+1}^2) \le c_1\sum_{j=0}^{N}U_{j+1}^2$. Therefore,

$$\begin{aligned}\sum_{N=0}^{\infty}\frac{E(M_N^2)-E(M_{N-1}^2)}{(\sum_{j=0}^{N}U_{j+1})^2} &= \sum_{N=0}^{\infty}\frac{E(m_N^2)\cdot U_{N+1}^2}{(\sum_{j=0}^{N}U_{j+1})^2}\\ &\le c_1\sum_{N=0}^{\infty}\frac{U_{N+1}^2}{(\sum_{j=0}^{N}U_{j+1})^2}\\ &= c_1\sum_{N=0}^{j^*(T)-1}\frac{U_{N+1}^2}{(\sum_{j=0}^{N}U_{j+1})^2}+\\ &\quad c_1\sum_{N=j^*(T)}^{\infty}\frac{U_{N+1}^2}{(\sum_{j=0}^{N}U_{j+1})^2}.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{N=j^*(T)}^{\infty}\frac{U_{N+1}^2}{(\sum_{j=0}^{N}U_{j+1})^2}\\ \le\ &\sum_{N=j^*(T)}^{\infty}\frac{4T^2}{(\sum_{j=0}^{N}U_{j+1})^2}\\ \le\ &\sum_{N=j^*(T)}^{\infty}\frac{4T^2}{(\sum_{j=j^*(T)}^{N}U_{j+1})^2}\\ \le\ &\sum_{N=j^*(T)}^{\infty}\frac{4T^2}{(N-j^*(T)+1)^2T^2}\\ =\ &\sum_{N=j^*(T)}^{\infty}\frac{4}{(N-j^*(T)+1)^2}<\infty,\end{aligned}$$

we have $\sum_{N=0}^{\infty}\frac{E(M_N^2)-E(M_{N-1}^2)}{(\sum_{j=0}^{N}U_{j+1})^2}<\infty$. Using Theorem 2.1 in [17], we conclude that

$$\lim_{N\to\infty}[\sum_{j=0}^{N}(m_j\cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}]=0,\ \text{a.s.} \tag{37}$$

We know that with probability 1, $\mathbf{r}\to\mathbf{r}^*$. Consider a realization where $\mathbf{r}\to\mathbf{r}^*$ and (37) holds. Choose $t_0>\tau_{j^*(T)}$ large enough such that $\forall t\ge t_0$, $||\mathbf{r}(t)-\mathbf{r}^*||<\epsilon$. That is, after t_0, $\mathbf{r}(t)$ is near $\mathbf{r}^*$ and is thus bounded. Using an argument similar to before [todo], we have $|b_j|\le c_2(\epsilon)/U_{j+1}$ for some constant $c_2(\epsilon)$, for any j satisfying $\tau_j>t_0$. Then, for any large-enough N,

$$\begin{aligned}&|\sum_{j:\tau_j>t_0}^{N}(b_j\cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}|\\ \le\ &(\sum_{j:\tau_j>t_0}^{N}c_2(\epsilon))/(\sum_{j:\tau_j>t_0}^{N}U_{j+1})\\ \le\ &c_2(\epsilon)/T.\end{aligned}$$

Therefore, $\limsup_{N\to\infty}\sum_{j=0}^{N}(b_j\cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}\le c_2(\epsilon)/T$ and similarly $\limsup_{N\to\infty}\sum_{j=0}^{N}(b_j\cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}\ge -c_2(\epsilon)/T$.

Also, since $\mathbf{r}\to\mathbf{r}^*$ in the realization, it is easy to show that

$$\lim_{N\to\infty}[\sum_{j=0}^{N}(s_k(\mathbf{r}(\tau_j))\cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}]=s_k(\mathbf{r}^*).$$

Combining the above facts, we know that with probability 1, $\limsup_{t\to\infty}S_k(t)/t = \limsup_{N\to\infty}[\sum_{j=0}^{N}(\hat{s}_j\cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}] \le s_k(\mathbf{r}^*)+c_2(\epsilon)/T$ and $\liminf_{t\to\infty}S_k(t)/t = \liminf_{N\to\infty}[\sum_{j=0}^{N}(\hat{s}_j\cdot U_{j+1})/\sum_{j=0}^{N}U_{j+1}]\ge s_k(\mathbf{r}^*)-c_2(\epsilon)/T$.

Since the above argument holds for any $T>0$. Letting $T\to\infty$, we have $\lim_{t\to\infty}S_k(t)/t=s_k(\mathbf{r}^*)$ with probability 1. ■

We are now ready to prove the rate stability.

Theorem 8: If $s_k(\mathbf{r}^*)\ge\lambda_k,\forall k$, and Lemma 4 holds, then $\lim_{t\to\infty}D_k(t)/t=\lambda_k,\forall k$, a.s.. That is, the system is "rate stable".

Proof: (i) We first show that $\liminf_{t\to\infty}[A_k(t)-D_k(t)]/t=0$ a.s.. For this purpose, we show that $\forall\epsilon>0$, $P(\liminf_{t\to\infty}[A_k(t)-D_k(t)]/t>\epsilon)=0$. If in a realization,

$$\liminf_{t\to\infty}[A_k(t)-D_k(t)]/t>\epsilon \tag{38}$$

, then $\exists T_0>1/\epsilon$, s.t. $\forall t\ge T_0$, $[A_k(t)-D_k(t)]/t\ge\epsilon$, i.e., $Q_k(t)\ge\epsilon\cdot t$. Since $T_0>1/\epsilon$, we have $Q_k(t)>1,\forall t\ge T_0$, i.e., the queue is not empty after T_0. Therefore, if $t\ge T_0$, then $S_k(t)=S_k(T_0)+[S_k(t)-S_k(T_0)]=S_k(T_0)+[D_k(t)-D_k(T_0)]\le T_0+D_k(t)$. So

$$\begin{aligned}\limsup_{t\to\infty}S_k(t)/t &\le \limsup_{t\to\infty}\frac{T_0+D_k(t)}{t}\\ &= \limsup_{t\to\infty}D_k(t)/t.\end{aligned}$$

By the assumption (38), $\limsup_{t\to\infty}D_k(t)/t<\liminf_{t\to\infty}A_k(t)/t-\epsilon$. So $\limsup_{t\to\infty}S_k(t)/t<\liminf_{t\to\infty}A_k(t)/t-\epsilon$. Therefore the intersection of events

$$\begin{aligned}&\{\lim_{t\to\infty}S_k(t)/t\ge\lim_{t\to\infty}A_k(t)/t\}\cap\\ &\{\liminf_{t\to\infty}[A_k(t)-D_k(t)]/t>\epsilon\} = \emptyset\end{aligned} \tag{39}$$

.

On the other hand, with probability 1, $\lim_{t\to\infty}A_k(t)/t=\lambda_k$ and $\lim_{t\to\infty}S_k(t)/t=s_k(\mathbf{r}^*)$. Since $s_k(\mathbf{r}^*)\ge\lambda_k$, $P(\lim_{t\to\infty}S_k(t)/t\ge\lim_{t\to\infty}A_k(t)/t)=1$. In view of (39), we have $P(\liminf_{t\to\infty}[A_k(t)-D_k(t)]/t>\epsilon)=0$. Since this holds for any $\epsilon>0$, we conclude that $\liminf_{t\to\infty}[A_k(t)-D_k(t)]/t=0$ a.s.

(ii) Second, we show that $\limsup_{t\to\infty}[A_k(t)-D_k(t)]/t=0$ a.s..

From (i), we know that for an arbitrary $a>0$, with probability 1 $[A_k(t)-D_k(t)]/t\le a$ infinitely often ("i.o."), and $\lim_{t\to\infty}[A_k(t)-S_k(t)]/t\le 0$. Consider a realization in which the above two conditions hold, and

$\limsup_{t\to\infty}[A_k(t) - D_k(t)]/t > 2a$. Then, $[A_k(t) - D_k(t)]/t \geq 2a$ i.o..

By the above assumptions, $Q_k(t) = A_k(t) - D_k(t) \leq a \cdot t$ and $Q_k(t) = A_k(t) - D_k(t) \geq 2a \cdot t$ i.o.. Also note that in a time interval of 1, $Q_k(t)$ at most change by 1, i.e., $|Q_k(t) - Q_k(t+1)| \leq 1$. So, for any T_1 (satisfying $a \cdot T_1 \geq 2$), there exist $t_2 > t_1 \geq T_1$ such that $Q_k(t_1) \leq a \cdot t_1$, $Q_k(t_2) \geq 2a \cdot t_2$, and $Q_k(t) \geq 2$ for any $t_1 < t < t_2$. Since the queue is not empty from time t_1 to t_2, we have

$$S_k(t_2) - S_k(t_1) = D_k(t_2) - D_k(t_1).$$

Denote $B_k(t) := A_k(t) - S_k(t)$, then

$$\begin{aligned} & B_k(t_2) \\ = \quad & B_k(t_1) + [B_k(t_2) - B_k(t_1)] \\ = \quad & B_k(t_1) + \{[A_k(t_2) - A_k(t_1)] - [S_k(t_2) - S_k(t_1)]\} \\ = \quad & B_k(t_1) + \{[A_k(t_2) - A_k(t_1)] - [D_k(t_2) - D_k(t_1)]\} \\ = \quad & B_k(t_1) + Q_k(t_2) - Q_k(t_1) \\ \geq \quad & B_k(t_1) + 2a \cdot t_2 - a \cdot t_1 \end{aligned}$$

Therefore

$$B_k(t_2)/t_2 \geq B_k(t_1)/t_2 + 2a - a \cdot t_1/t_2$$

Then,

$$B_k(t_2)/t_2 - B_k(t_1)/t_1 \geq \frac{B_k(t_1)}{t_1}(\frac{t_1}{t_2} - 1) + 2a - a\frac{t_1}{t_2}.$$

Since $\lim_{t\to\infty} B_k(t)/t := b \leq 0$, we choose T_1 large enough such that $\forall t \geq T_1$, $|B_k(t)/t - b| \leq a/3$. Then,

$$|B_k(t_1)/t_1 - B_k(t_2)/t_2| \leq (2/3) \cdot a \tag{40}$$

. Also, $B_k(t_1)/t_1 \leq b + a/3 \leq a$. Since $\frac{t_1}{t_2} - 1 < 0$, we have

$$\begin{aligned} B_k(t_2)/t_2 - B_k(t_1)/t_1 \quad & \geq \quad a \cdot (\frac{t_1}{t_2} - 1) + 2a - a\frac{t_1}{t_2} \\ & = \quad a \end{aligned}$$

which contradict to (40). Therefore, $P(\limsup_{t\to\infty}[A_k(t) - D_k(t)]/t > 2a) = 0$. Since this holds for any $a > 0$, we conclude that $\limsup_{t\to\infty}[A_k(t) - D_k(t)]/t = 0$ a.s..

Combining (i) and (ii) gives $\lim_{t\to\infty}[A_k(t) - D_k(t)]/t = 0$ a.s.. So $\lim_{t\to\infty} D_k(t)/t = \lambda_k, \forall k$, a.s.. That is, the system is "rate stable". ■

F. Proof of Theorem 3

Proof: The proof is constructive. That is, we find an α and T that ensures that the queues are stable. The basic idea is to show that when T is large enough and α is small enough, then $g'_k(m)$ approximate $g_k(m)$ well, and over time, $\mathbf{r}(m)$ is near $\mathbf{r}^*$ with high probability. Therefore the queues can be made "stable" since $s_k(\mathbf{r}^*) > \lambda_k, \forall k$. The following is the detailed proof.

Similar to (19), we have that for any link k, any time step m

$$|B_k(m)| \leq 2 \cdot K \cdot \bar{f}/T \tag{41}$$

where the constant $\bar{f} = \exp\{(\frac{5}{2}K + 1) \cdot [\lambda_{max} \cdot r_{max} + \log(2)]\} + 1$.

Define the set $H_\mu := \{\mathbf{r}|L(\mathbf{r}) \leq \mu + L(\mathbf{r}^*)\}$. Recall that $s_k(\mathbf{r}^*) - \lambda_k \geq \delta(\lambda), \forall k$ for some $\delta(\lambda) > 0$. Since $s_k(\mathbf{r})$ is continuous in $\mathbf{r}$, we can choose $\mu > 0$ small enough such that for any $\mathbf{r} \in H_\mu$, $s_k(\mathbf{r}) - \lambda_k \geq \delta(\lambda)/2, \forall k$. Similar to (42), if $\mathbf{r}(m-1) \notin H_\mu := \{\mathbf{r}|L(\mathbf{r}) \leq \mu + L(\mathbf{r}^*)\}$, then

$$\begin{aligned} & E(d(m)|\mathcal{F}_{m-1}) \\ & \quad \leq d(m-1) - \alpha \cdot \mu \\ & \quad\quad + \alpha \cdot [\mathbf{r}^* - \mathbf{r}(m-1)]^T \mathbf{B}(m) \\ & \quad\quad + \alpha^2 \cdot C. \end{aligned} \tag{42}$$

Since $0 \leq r_k^* \leq r_{max}$ and $0 \leq r_k(m-1) \leq r_{max}$ for any k, combined with (41), we have

$$\begin{aligned} & E(d(m)|\mathcal{F}_{m-1}) \\ & \quad \leq d(m-1) - \alpha \cdot \mu \\ & \quad\quad + C_\mu. \end{aligned} \tag{43}$$

where $C_\mu := 2\alpha \cdot K^2 \cdot r_{max} \cdot \bar{f}/T + \alpha^2 \cdot C$.

If $\mathbf{r}(m-1) \in H_\mu$, then similar to (43),

$$E(d(m)|\mathcal{F}_{m-1}) - d(m-1) \leq C_\mu. \tag{44}$$

For convenience, we always choose T to be integer, so that with the assumption of Bernoulli arrivals, $U(m) := (\mathbf{s}'(m), \lambda'(m), \mathbf{r}(m), x_0(m))$ (when $m = 0$, let $\mathbf{s}'(m) = \lambda'(m) = 0$ by default) is a Markov process. Define another Markov process $V(m) := (\mathbf{r}(m), x_0(m), \mathbf{Q}(m))$ where $\mathbf{Q}(m)$ is the vector of queue lengths of all links at time t_m. Note that the first two components of $V(m)$ are bounded. We will show that for some properly chosen $\bar{Q} > 0$ and $F \in \mathcal{Z}_{++}$, if $Q_k(m_0) \geq \bar{Q}$ (with other components of $V(m_0)$ being arbitrary), then $E(Q_k(m_0 + F)|V(m_0)) - Q_k(m_0) \leq -\delta$ where $\delta > 0$. (That is, the queue length has negative drift.)

In the following, all expectations and probabilities are conditioned on $V(m_0)$. For convenience we drop the notation. Denote $\Delta(m) = d(m) - d(m-1)$. And define $M := m_0 + F$. We have for any $M > m_0$,

$$\sum_{m=m_0+1}^{M} \Delta(m) = d(M) - d(m_0) \in [-D_1, D_1] \tag{45}$$

for some $D_1 > 0$, since both $d(M)$ and $d(m_0)$ are bounded. So

$$E[\sum_{m=m_0+1}^{M} \Delta(m)] \in [-D_1, D_1].$$

Note that

$$
\begin{aligned}
& E[\sum_{m=m_0+1}^{M} \Delta(m)] = \sum_{m=m_0+1}^{M} E[\Delta(m)] \\
= & \sum_{m=m_0+1}^{M} E\{E[\Delta(m)|\mathcal{F}_{m-1}]\} \\
\leq & \sum_{m=m_0+1}^{M} E\{I(\mathbf{r}(m-1) \notin H_\mu)(-\alpha \cdot \mu + C_\mu) + \\
& I(\mathbf{r}(m-1) \in H_\mu)C_\mu\} \\
= & \sum_{m=m_0+1}^{M} [(1 - P_\mu(m-1))(-\alpha \cdot \mu + C_\mu) + P_\mu(m-1)C_\mu]
\end{aligned}
$$

where $P_\mu(m-1) := Pr\{\mathbf{r}(m-1) \in H_\mu\}$. Denote $\bar{P}_\mu(F) := \frac{1}{F}\sum_{m=m_0+1}^{M} P_\mu(m-1)$ (recall that $F = M - m_0$), then

$$
\begin{aligned}
-\frac{D_1}{F} & \leq \frac{1}{F}E[\sum_{m=m_0+1}^{M} \Delta(m)] \\
& \leq (1 - \bar{P}_\mu(F))(-\alpha \cdot \mu + C_\mu) + \bar{P}_\mu(F)C_\mu.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\bar{P}_\mu(F) & \geq \frac{\alpha \cdot \mu - C_\mu - D_1/F}{\alpha \cdot \mu} \\
& = 1 - \frac{2K^2 \cdot r_{max} \cdot \bar{f}/T + \alpha \cdot C + D_1/[\alpha \cdot F]}{\mu} \quad (46)
\end{aligned}
$$

If $T \geq 8K \cdot \bar{f}/\delta(\lambda)$, then $1/[\frac{1}{2}\delta(\lambda) - 2 \cdot K \cdot \bar{f}/T + 1] \leq 1/[\frac{1}{4}\delta(\lambda) + 1]$. By properly choosing $T \in \mathcal{Z}_+$, $T \geq 8K \cdot \bar{f}/\delta(\lambda)$, α, F, the RHS of (46) can be made close to 1, such that $\bar{P}_\mu(F) \geq (1 + \epsilon')/[\frac{1}{4}\delta(\lambda) + 1]$ for some $\epsilon' \in (0, 1)$.

If $Q_k(m_0) \geq \bar{Q} := T \cdot F + 1$, then the queue is non-empty before time t_M (since the maximal decreasing rate is 1). So, the expected decrease of queue k from time t_{m_0} to t_M is

$$
\begin{aligned}
& \sum_{m=m_0+1}^{M} T \cdot [E(s'_k(m)) - \lambda_k] \\
= & T \sum_{m=m_0+1}^{M} \{E[E(s'_k(m)|\mathcal{F}_{m-1})] - \lambda_k\}
\end{aligned}
$$

Recall that if $\mathbf{r} \in H_\mu$, then $s_k(\mathbf{r}) - \lambda_k \geq \delta(\lambda)/2$. By (41), we have $E(s'_k(m)) \geq \lambda_k + \delta(\lambda)/2 - 2 \cdot K \cdot \bar{f}/T \geq \lambda_k + \delta(\lambda)/4$ if $\mathbf{r}(m-1) \in H_\mu$. Then,

$$
\begin{aligned}
& \sum_{m=m_0+1}^{M} T \cdot [E(s'_k(m)) - \lambda_k] \\
\geq & T \cdot \sum_{m=m_0+1}^{M} [P_\mu(m-1)(\lambda_k + \delta(\lambda)/4) - \lambda_k] \\
= & T \cdot F[\bar{P}_\mu(F)(\lambda_k + \delta(\lambda)/4) - \lambda_k] \\
\geq & T \cdot F[\bar{P}_\mu(F)(1 + \delta(\lambda)/4) - 1] \\
\geq & T \cdot F \cdot \epsilon'
\end{aligned}
$$

The above inequality implies that if $Q_k(m_0) \geq \bar{Q}$,

$$E(Q_k(m_0 + F)|V(m_0)) - Q_k(m_0) \leq -T \cdot F \cdot \epsilon' \quad (47)$$

as desired. Also, clearly

$$E(Q_k(m_0 + F)|V(m_0)) - Q_k(m_0) \leq \bar{Q}, \text{ if } Q_k(m_0) < \bar{Q}. \quad (48)$$

Now consider the Markov process $\{V(m' \cdot F)\}, m' = 0, 1, 2, \ldots$. We claim that

$$
\begin{aligned}
& \Delta L_k(m') \\
:= & E\{[Q_k((m'+1) \cdot F)]^2 - [Q_k(m' \cdot F)]^2 | V(m' \cdot F)\} \\
\leq & -2T \cdot F \cdot \epsilon' \cdot Q_k(m' \cdot F) + 6 \cdot \bar{Q}^2. \quad (49)
\end{aligned}
$$

To see (49), consider two cases: (i) If $Q_k(m' \cdot F) \geq \bar{Q}$, then with $\Delta Q_k(m' \cdot F) := Q_k((m'+1) \cdot F) - Q_k(m' \cdot F)$, we have $\Delta L_k(m') = 2Q_k(m' \cdot F) \cdot E\{\Delta Q_k(m' \cdot F)|V(m' \cdot F)\} + E\{[\Delta Q_k(m' \cdot F)]^2 | V(m' \cdot F)\} \leq -2T \cdot F \cdot \epsilon' \cdot Q_k(m' \cdot F) + \bar{Q}^2$ where (47) has been used. (ii) If $Q_k(m' \cdot F) < \bar{Q}$, then $Q_k((m'+1) \cdot F) \leq 2\bar{Q}$. So $\Delta L_k(m') \leq 4\bar{Q}^2 \leq -2T \cdot F \cdot \epsilon' \cdot Q_k(m' \cdot F) + 6 \cdot \bar{Q}^2$ since $Q_k(m' \cdot F) < \bar{Q}$ and $T \cdot F \cdot \epsilon' < \bar{Q}$.

Define the Lyapunov function $L(m') := \sum_k [Q_k(m' \cdot F)]^2$. By (49),

$$
\begin{aligned}
& E\{L(m'+1) - L(m')|V(m' \cdot F)\} \\
\leq & -2T \cdot F \cdot \epsilon' \cdot \sum_k Q_k(m' \cdot F) + 6K \cdot \bar{Q}^2. \quad (50)
\end{aligned}
$$

Then we consider two notions of queue stability.

Proof of "strong stability"

Definition: Assume that $Q_k(0) < \infty, \forall k$. The queues are "strongly stable" [18] iff

$$\limsup_{n \to \infty} \sum_{m'=0}^{n-1} E[Q_k(m')]/n < \infty, \forall k.$$

Taking expectations on both sides of (50),

$$
\begin{aligned}
& E\{L(m'+1) - L(m')\} \\
\leq & -2T \cdot F \cdot \epsilon' \cdot \sum_k E[Q_k(m' \cdot F)] + 6K \cdot \bar{Q}^2 \quad (51)
\end{aligned}
$$

Similar to [18], summing (51) over $m' = 0, 1, 2, \cdot, n-1$, we have $E\{L(n) - L(0)\}/n \leq -2T \cdot F \cdot \epsilon' \sum_{m'=0}^{n-1} \sum_k E[Q_k(m' \cdot F)]/n + 6K \cdot \bar{Q}^2$. Then, since $Q_k(0) < \infty, \forall k$,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{m'=0}^{n-1} \sum_k E[Q_k(m' \cdot F)] \leq \frac{3K \cdot \bar{Q}^2}{T \cdot F \cdot \epsilon'} < \infty. \quad (52)$$

This implies $\limsup_{n \to \infty} \sum_{m'=0}^{n-1} E[Q_k(m' \cdot F)]/n < \infty, \forall k$. Due to the bounded increment of Q_k in each interval of T, we also have $\limsup_{n \to \infty} \sum_{m'=0}^{n-1} E[Q_k(m')]/n < \infty, \forall k$.

Positive Harris recurrence

Using (50) and following the line of proof in [19], it can be shown that the Markov process $\{V(m)\}$ is postive Harris recurrent. ■

G. Comments on the arrival processes

More complicated arrival processes than Bernoulli arrivals can be assumed. As long as the following two conditions hold

$$\lambda'_k(i) = [A_k(t_i) - A_k(t_{i-1})]/T_i \le \bar{\lambda}, \forall k, i \quad (53)$$

$$E[\lambda'_k(m+1)|\mathcal{F}_m] - \lambda_k| \le c_4/T_{m+1}, \forall k, m \quad (54)$$

for some constant $\bar{\lambda} > 0, c_4 > 0$, then the whole proof in the paper still applies.

For example, assume that $a_k(t)$ is a continuous time Markov chain which has two states 0 and 1. The transition rate from 0 to 1 is $\lambda_k/(1-\lambda_k)$, and the transition rate from 1 to 0 is 1. Then, in the stationary distribution, $a_k(t) = 1$ with probability λ_k. Therefore $a_k(t)$ has a long-term average of λ_k. Also, $\lambda'_k(i) = A_k(t_i) - A_k(t_{i-1})/T_i \le 1$ is upper-bounded. Thirdly, following a similar analysis in section III-D, it can be shown that (54) holds for some constant c_4 due to the "mixing" of the arrival Markov chain.

H. Comments on the fluid assumption

For simplicity, in the paper we have assumed that the traffic is fluid: upon transmission, the bits in the queues form packets which can have different sizes from the arrived packets. Consider the following non-fluid model. The sizes of the packets are independent and exponentially distributed with mean 1. In the CSMA protocol, these packets are transmitted in their entirety. If there is no packet in the queue then a dummy packet with the same distribution is transmitted. This leads to the CSMA Markov chain defined before. For link k, packets with the above distribution arrive at time instances $0, 1, 2, \dots$ according to Bernoulli distribution with parameter λ_k. That is, at time $j \in \mathcal{Z}_+$, a packet arrives with probability λ_k. Let $N_k(t)$ be the number of packets arrived at link k by time t. And define the empirical arrival rate a little differently in the algorithms: $\lambda'_k(i) = N_k(t_i) - N_k(t_{i-1})/T_i$. Then, since $T_i \ge \min_j T_j > 0$ and the assumption on $N_k(t)$, $\lambda'_k(i)$ is upper-bounded. Also, (20) is satisfied. Then the previous proof still applies to this case. Therefore the system is "rate stable" in the sense that $\lim_{t\to\infty}(N_k(t) - D_k(t))/t = 0$. Since the mean of packet sizes is 1, by the law of large numbers, $\lim_{t\to\infty} A_k(t)/N_k(t) = 1$. So $\lim_{t\to\infty} A_k(t)/t = \lim_{t\to\infty}[A_k(t)/N_k(t)] \cdot [N_k(t)/t] = \lambda_k$. Therefore $\lim_{t\to\infty}(A_k(t) - D_k(t))/t = 0$, i.e., the system is also rate stable in the original sense.

One can also relax the assumption that packets arrive at time instances $0, 1, 2, \dots$. For example, consider the following arrival process to link k similar to Poisson arrivals but with a fixed "waiting time" W_k. When a packet arrives, it takes a fixed time W_k for the queue to "process" it and store it at the end of the queue. During this time no other packet can be accepted. After W, the next packet arrives after an exponentially-distributed duration with mean $\bar{m}_k$. Assume that the average arrival rate induced by W_k and $\bar{m}_k$ is λ_k. The lengths of all packet are independent and exponentially distributed with mean 1. With these assumptions, and with $\lambda'_k(i)$ interpreted as $N_k(t_i) - N_k(t_{i-1})/T_i$, conditions (53) and (54) are satisfied. Then the analysis in the paper applies.

I. Comments on adjusting transmission times instead of backoff times

In an alternative design, the random backoff time of each link is exponentially distributed with mean 1, and the transmission duration of link k is exponentially distributed with mean $\exp(r_k)$. Clearly, when $\mathbf{r}$ is fixed, the CSMA Markov chain has the same stationary distribution as before.

In the new design, r_k is still adjusted according to equation (8).

Theorem 9: The following conditions ensure the convergence of the algorithm: $\sum_i \alpha(i) = \infty$, $\sum_i \alpha(i)^2 < \infty$, conditions (10) and (11), but with $f(m)$ redefined as

$$f(m) = \exp\{(\frac{5}{2}K + 2) \cdot [\lambda_{max} \cdot \sum_{i=1}^{m} \alpha(i) + \log(2)]\}. \quad (55)$$

Remark: The setting $\alpha(i) = 1/[(i+1)\log(i+1)]$ and $T_i = i$ satisfies these conditions. So are the other examples mentioned in section III-B.

Proof: The proof is similar to that of Theorem 1, but with minor changes in the calculation of the mixing time. Specifically, since in this design $Q(x, x') \le 1 \forall x, x'$ (recall that $\mathbf{r}(m) \ge \mathbf{0}$), we let $A_{m+1} = K$ in the uniformization. Also, for any x, x' such that $Q(x, x') > 0$, we have $Q(x, x') \ge 1/\exp(\lambda_{max} \sum_{i=1}^{m} \alpha(i))$. So

$$\begin{aligned} P(x, x') &= Q(x, x')/A_{m+1} \\ &\ge 1/[K \cdot \exp(\lambda_{max} \sum_{i=1}^{m} \alpha(i))]. \end{aligned}$$

Then the conductance

$$\phi \ge \pi_{min}(\mathbf{r}(m))/[K \cdot \exp(\lambda_{max} \sum_{i=1}^{m} \alpha(i))].$$

Combined with (18), we have

$$1/(1-\beta_1) \le 2 \cdot K^2 \exp(2\lambda_{max} \sum_{i=1}^{m} \alpha(i))[\pi_{min}(\mathbf{r}(m))]^{-2}.$$

Plugging this into (17) and use (16), we have

$$\begin{aligned} &||\bar{\mu}_{\mathbf{x}_0}(\mathbf{r}(m); T_{m+1}) - \pi(\mathbf{r}(m))||_{var} \\ &\le K \cdot \exp(2\lambda_{max} \sum_{i=1}^{m} \alpha(i))[\pi_{min}(\mathbf{r}(m))]^{-5/2}/T_{m+1}. \\ &\le K \cdot f(m)/T_{m+1} \end{aligned}$$

where $f(m)$ is newly defined in (55). Other parts of the proof are the same as Theorem 1. ■

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