Interference Nulling in Distributed Lossy Source Coding

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Abstract

We consider a problem of distributed lossy source coding with three jointly Gaussian sources (y_1, y_2, y_3), where y_1 and y_2 are positively correlated, $y_3 = y_1 - cy_2$, $c \geq 0$, and the decoder requires reconstructing y_3 with a target distortion. Inspired by results from the binary expansion model, the rate–distortion region of this problem is characterized within a bounded gap. Treating each source as a multi-layer input, it is shown that the layers required by the decoder are combined with some unneeded information, referred as interference. Therefore, the responsibility of a source coding algorithm is not only to eliminate the redundancy among the inputs, but also to manage the interference. It is shown that the interference can be effectively managed through some linear operations among the input layers. Binary expansion model demonstrates that some middle layers of y_1 and y_2 are not needed at the decoder, while some upper and lower layers are required. Structured (Lattice) quantizers enable the encoders to pre-eliminate unnecessary layers. While in the lossless distributed source coding cut-set outer bound is tight, it is shown that in the lossy ones, even for binary expansion models, this outer-bound has an unbounded gap with the rate distortion region. To prove the bounded-gap result, a new outer-bound is established.

I. Introduction

In this paper, we investigate a distributed lossy source coding problem with three jointly Gaussian sources (y_1, y_2, y_3), where y_1 and y_2 are positively correlated and $y_3 = y_1 - cy_2$, $c \geq 0$. Each source is observed with an isolated encoder, which sends sends an index message to a central decoder. The decoder requires reconstructing y_3 with a specific quadratic distortion.

In [1], Slepian and Wolf showed that in the distributed source coding problem, if the decoder requires all sources with no distortion, then the random-binning scheme achieves the optimal rate region. The Slepian-Wolf result suggests a natural encoding scheme for the lossy distributed source coding. In this scheme, first the sources are quantized and transformed to discrete sources and then Slepian-Wolf scheme

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efficiently exploits the correlation of the quantized sources to minimize the rates. In [2], it is shown that for two Gaussian sources, this scheme of quantization-binning (Q-B) achieves some parts of the rate-distortion boundaries. In addition, in [3], [4], it is proven that this scheme is optimal for a special form of distributed lossy source coding problems known as CEO problem. This result has been extended to the case, where the observation noises of the CEO problem are correlated with a specific tree structure [5]. In [6], it is proven that Q-B scheme achieves the entire rate-distortion region for two jointly Gaussian sources (y_1, y_2). Moreover, it is shown that if one wishes to reconstruct $y_1 + cy_2$, $c \geq 0$, with certain distortion, then Q-B is optimal. However, in [7], it is shown that if the objective is to reconstruct $y_1 - cy_2$, Q-B scheme is not optimal and a lattice-based scheme outperforms Q-B scheme in some scenarios. The scheme of [7] is motivated by an important result due to Korner and Marton, who showed that if the objective is to reproduce the modulo-two sum of two binary sources, observed by two isolated encoders, then linear coding achieves optimal performance, while random binning is strictly sub-optimal.

In [8], a linear finite-field deterministic model has been proposed to simplify understanding the key elements of the multi-user information theory. It is also suggested that the binary expansion model is used for source coding problems [8]. In [9], the connection of the deterministic (lossless) and the non-deterministic models (lossy) has been exploited to approximate the rate region of the Gaussian symmetric multiple description coding. In [10], the binary expansion model is used to show that Q-B is an appropriate scheme to approximate the rate-distortion region of a class of sources, with tree-structured three Gaussian sources.

In this paper, we start with developing a binary expansion model of the original Gaussian problem. This model by itself is a distributed lossy source coding problem, which demonstrates the connection among the sources in a simple and explicit way. Solving the associated source coding problem for the binary expansion model provides sensible intuition to develop schemes which achieve within a bounded gap of the rate-distortion region. Treating each source as a multi-layer input, we develop the achievable schemes based on the following observations: (i) Some middle layers of input are not needed at the decoder, while some upper and lower layers are required. Therefore, in an efficient achievable scheme, the encoders should avoid reporting these layers, otherwise the gap from the rate-distortion region becomes unbounded. (ii) The isolated encoders must employ certain network coding operations between the upper and lower layers of the inputs. These linear operations substantially reduce the load of reporting the required layers individually. In addition, in the underlying structure of the inputs, there are some unwanted information which is like interference, in the sense that it is combined with required information and each isolated encoder cannot avoid sending it by pre-elimination. The network coding operations will align these interference terms, such that at the decoder, these terms will be canceled out in the process reconstruction. (iii) The structured quantization enables the encoders to have access to the layers of inputs to pre-eliminate the unwanted layers.

II. Problem Formulation

Let $\{y_1(t), y_2(t), y_3(t)\}_{t=1}^n$ be a sequence of independent and identically distributed (i.i.d.) Gaussian random variables. It is assumed that $(y_1(t), y_2(t), y_3(t))$ has zero mean, the covariance of (y_1, y_2), denoted by $\mathbf{K}_{(y_1,y_2)}$ is equal to,

$$\mathbf{K}_y = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{1}$$

In addition, we assume that $y_3(t) = y_1(t) - cy_2(t)$. Here in this paper, we assume $\rho \geq 0$ and also $c \geq 0$.

In the problem of the distributed source coding, there are three non-cooperative encoders, where the j^{th} encoder observes $\{y_j(t)\}_{t=1}^n$, and sends a message, chosen from $\{1, 2, \ldots, M_j\}$, to the centralized decoder. The decoder receives the messages from the three encoders and estimates $\{\hat{y}_3(t)\}_{t=1}^n$.

We define Δ as $\Delta = \frac{1}{n}\sum_{t=1}^n E[(y_3^n(t) - \hat{y}_3^n(t))^2]$. The rate-distortion tuple (R_1, R_2, R_3, d) is admissible, if for every $\epsilon > 0$ and for a sufficiently large n, there exists a seven-tuple $(n, M_1, M_2, M_3, \Delta_3)$, such that $\frac{1}{n}\log(M_j) \leq R_j + \epsilon$, for $j = 1, 2, 3$, and $\Delta \leq d + \epsilon$. Without loss of generality, we assume that $0 \leq c \leq 1$. It is easy to see that any other cases can be transformed into this canonical form through some simple scaling of y_3 and distortion d. In [11], it is shown that for $R_3 = 0$ and $\rho \leq c$, the scheme of [7] is within a bounded gap from the rate-distortion region. In addition, in [11], it is proven that if $R_3 = 0$ and $c \leq \min\{\frac{1}{2\rho}, \rho\}$, Quantization-and-Binning scheme achieves within a bounded gap from an outer-bound in $R_1 + R_2$.

In this paper, we first consider the missing case, where $\frac{1}{2\rho} \leq c \leq \rho$. To clarify the interaction among the sources, we develop a binary expansion model. This model is simple to follow, but still rich enough to demonstrate the features of the problem. Using this model, we show that a specific interference management technique is need to achieve within a bounded gap of the rate-distortion region. Then, we focus on the cases, where $c \leq \min\{\frac{1}{2\rho}, \rho\}$ and $c \geq \rho$, and approximate the rate-distortion region within a bounded gap.

III. Distributed Coding for The Binary-Expansion Model for $\frac{1}{2\rho} \leq c \leq \rho$

Since the correlation between y_1 and y_2 is equal to ρ, we can write $y_1 = \rho y_2 + \sqrt{1-\rho^2}z$, where $z \sim \mathcal{N}(0, 1)$ and z is independent from y_2. We define r such that

$$\rho = 1 - 2^{-2r}. \tag{2}$$

In addition, we define x as $x = \rho y_2$. Noting that $c \leq \rho$, we define δ such that,

$$\frac{c}{\rho} = 1 - 2^{-\delta}. \tag{3}$$

In connection with the Gaussian problem, we introduce a simpler problem, which preserves many aspects of the original problem, but is easier to understand. In the new problem y_1, x and z are all uniformly distributed in $[0, 1]$. Therefore, x and z have the binary expansion representations as

$$x = 0.x^{[1]}x^{[2]}x^{[3]}\ldots, \tag{4}$$

$$z = 0.z^{[1]}z^{[2]}z^{[3]}\ldots. \tag{5}$$

In addition, we replace r and δ with the closest integers. The connection of y_1 and x is modeled as $y_1 = x \oplus 2^{-r}z$. More precisely,

$$y_1 = 0.y_1^{[1]}y_1^{[2]}y_1^{[3]}\ldots = 0.x^{[1]}x^{[2]}x^{[3]}\ldots x^{[r]}(x^{[r+1]} \oplus z^{[1]})(z^{[r+2]} \oplus z^{[2]})\ldots. \tag{6}$$

where $\oplus$ represents the bitwise modulo two addition. In the original problem, we have $y_3 = y_1 - cy_2 = (y_1 - x) + (1 - \frac{c}{\rho})x$. This equation is modeled as $y_3 = 2^{-\delta}x \oplus 2^{-r}z$ in the binary expansion model. This model is pictorially shown in Fig. 1.

We define b as $b = -\frac{1}{2}\log_2 d$. In the binary expansion model, b is replaced with the closest integer. Moreover, it is interpreted that the decoder needs the b most significant bits (MSBs) of y_3 after the radix point. In this model, the objective of distributed-source-coding problem is to find the the region for (R_1, R_2, R_3) such that the decoder can recover up to the b MSBs of y_3 after the radix point.

Here, we consider two cases, $\delta \leq r$ and $\delta \geq r$.

Case One – $\delta \leq r$: In this case, among the b MSBs of y_3, the first δ bits are always zero. Therefore, the decoder requires the remaining $(b - \delta)^+$ non-zero MSBs. These bits are shown by Block C_1 in the binary expansion of y_3, depicted in Fig. 1. In this figure, we assume $b \geq \delta$ and $b \leq r + \delta$.

Clearly, a simple achievable rate-vector is $(R_1, R_2, R_3) = (0, 0, (b - \delta)^+)$, denoted by P_1. In this achievable rate, the third encoder sends Block C_1 while the first and second encoders stay silent.

Now, let us consider a more exciting case, where R_3 is zero, and only the first and second encoders are allowed to pass messages to the decoder. The first question to be answered is which parts of y_1 and x are relevant to reconstruct Block C_1 at the decoder. In other words, the question is which bits of y_1 and x have a possible role in the reconstruction of Block C_1. By a simple comparison, it is easy to see that the first $(b-\delta)^+$ bits of y_1 and x, shown in Block A_1 and B_1 in Fig. 1, are relevant. In addition, we note that $z^{[1]}\ldots z^{[(b-r)^+]}$ are appeared in the construction of Block C_1, while only Block A_2 of y_1 has these bits in its structure, and therefore Block A_2 is also important. This makes Block B_2 relevant as well. The

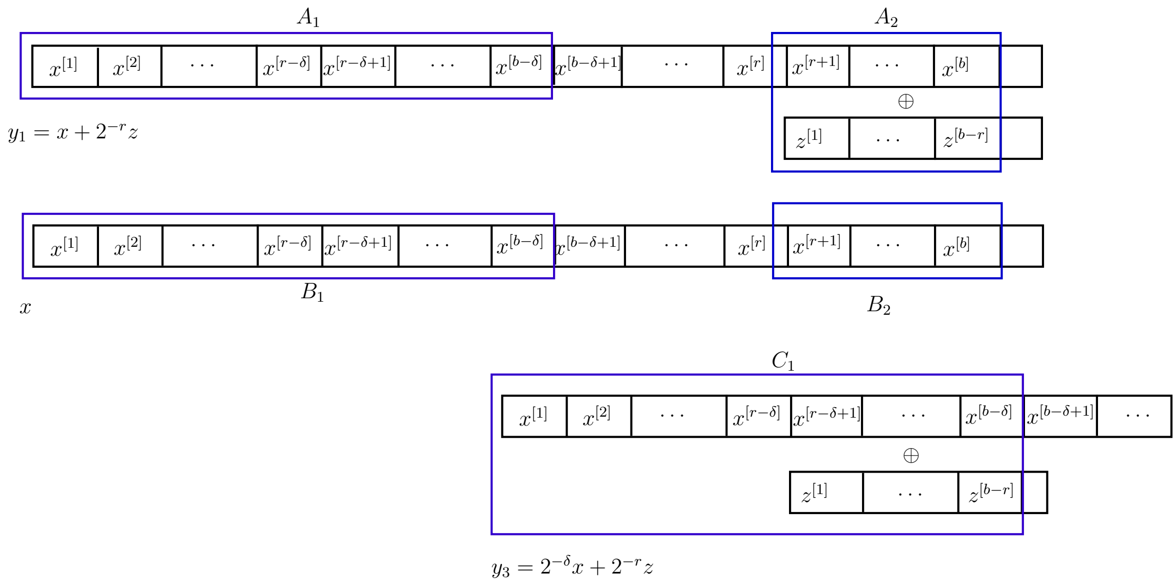

Fig. 1. Binary Expansion Model for Case I: $r \geq \delta$

reason is that Block A_2 is combined with the bits of Block B_2, and without having Block B_2, Block A_2 reveals no information about $z^{[1]} \dots z^{[(b-r)^+]}$.

Regarding the above statements, we have the following interesting observation.

Observation One: To reconstruct y_3 with the required resolution, the decoder does not need some middle-layers of y_1 and x. More precisely, the decoder does not need $y_1^{[b-\delta+1]} \dots y_1^{[r]}$ and $x^{[b-\delta+1]} \dots x^{[r]}$, while it needs some upper and lower layers of y_1 and x.

The next question is how to efficiently send enough information to the decoder, such that the decoder can reproduce y_3 with the required resolution. One simple approach is as follows. Encoder one sends Block A_2 and encoder two sends Blocks B_1 and B_2. Then, the decoder forms $2^{-\delta}B_1 \oplus (A_2 \ominus B_2)$, to reconstruct Block C_1 as follows.

$$2^{-\delta}B_1 \oplus (A_2 \ominus B_2) \tag{7}$$

$$= 2^{-\delta}[0.x^{[1]} \dots x^{[(b-\delta)^+]}] \oplus \left(2^{-r}[0.y_1^{[r+1]} \dots y_1^{[b]}] \ominus 2^{-r}[0.x^{[r+1]} \dots x^{[b]}]\right) \tag{8}$$

$$= 2^{-\delta}[0.x^{[1]} \dots x^{[(b-\delta)^+]}] \oplus 2^{-r}\left([0.y_1^{[r+1]} \dots y_1^{[b]}] \ominus [0.x^{[r+1]} \dots x^{[b]}]\right) \tag{9}$$

$$= 2^{-\delta}[0.x^{[1]} \dots x^{[(b-\delta)^+]}] \oplus 2^{-r}\left(\left\{[0.x^{[r+1]} \dots x^{[b]}] \oplus [0.z^{[1]} \dots z^{[(b-r)^+]}]\right\}\right. \tag{10}$$

$$\left.\ominus [0.x^{[r+1]} \dots x^{[b]}]\right) \tag{11}$$

$$= 2^{-\delta}[0.x^{[1]} \dots x^{[(b-\delta)^+]}] \oplus 2^{-r}[0.z^{[1]} \dots z^{[(b-r)^+]}], \tag{12}$$

where $\oplus$ and $\ominus$ respectively represent bit-wise addition and subtraction, which are basically the same operations. However, we use different notations to resemble the corresponding operations for the original

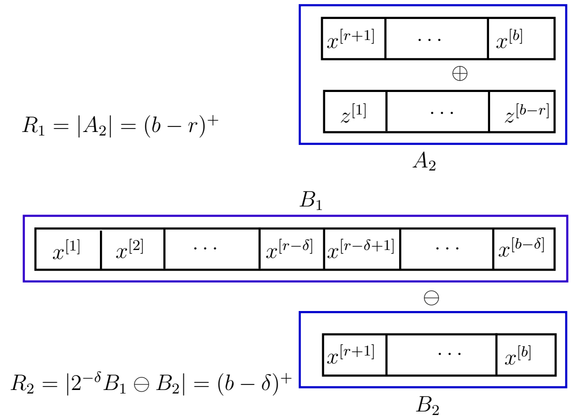

Fig. 2. Achievable Scheme for Case One ($r > \delta$), $P_2 = (R_1, R_2, R_3) = ((b-r)^+, (b-\delta)^+, 0)$

Gaussian problem. With this scheme, we achieve $(R_1, R_2, R_3) = (|A_2|, |B_1 \cup B_2|, 0) = ((b-r)^+, (b-\delta)^+ + (b-r)^+ - (r+\delta-b)^+, 0)$.

We notice that in the above equations, bits $0.x^{[r+1]} \dots x^{[b]}$ are like *interference* in the sense that these bits are basically unwanted at the decoder, and canceled out in the final result. The only reason that Block $B_2 = [x^{[r+1]} \dots x^{[b]}]$ is reported to the decoder is that the decoder can recover bits $[0.z^{[1]} \dots z^{[(b-r)^+]}]$ from Block A_2. On the other hand, the decoder does not need $[0.z^{[1]} \dots z^{[(b-r)^+]}]$ by itself, but it needs the addition of these bits with $0.x^{[r-\delta+1]} \dots x^{[b-\delta]}$. Now the question is if we can improve the performance of the achievable scheme regarding the above statements. The answer is yes. In the new scheme, the encoder two reports $2^{-\delta} B_1 \ominus B_2$, instead of sending B_1 and B_2 separately, while encoder one still sends A_2. In this case, the decoder forms $(2^{-\delta} B_1 \ominus B_2) \oplus A_2$, which is equal to $2^{-\delta} B_1 \oplus (A_2 \ominus B_2)$. It is important to note that the linear operation $2^{-\delta} B_1 \ominus B_2$ is formed such that the interference bits are aligned and canceled out in the final addition. In this case, we achieve the rate vector $(R_1, R_2, R_3) = (|A_2|, |B_1 \ominus 2^{\delta} B_2|, 0) = ((b-r)^+, (b-\delta)^+, 0)$, denoted by P_2. Note that the reduction in R_2 is unbounded. This encoding procedure has been shown in Fig. 2.

Therefore, we have the following observation:

Observation Two: Alignment and network coding among the layers of each source can improve the achievable scheme.

Since Blocks A_1 and B_1 are identical, then the load of transmitting part of B_1 can be shifted from encoder two to encoder one. Relying on this observation, we can develop the achievable scheme shown in Fig. 3. This scheme achieves the rate vector $(R_1, R_2, R_3) = (|2^{-\delta} A_1 \oplus A_2|, |B_2|, 0) = ((b-\delta)^+, (b-r)^+, 0)$, denoted by P_3, which has the same sum-rate as P_2.

Let us obtain an achievable rate vector in a cut of the region, where $R_1 = 0$. In this case, encoder three has to send bits of $y_3^{[r-\delta+1]} \dots y_3^{[(b-\delta)^+]}$. The reason is that these bits have $z^{[1]} \dots z^{[(b-r)^+]}$ in their

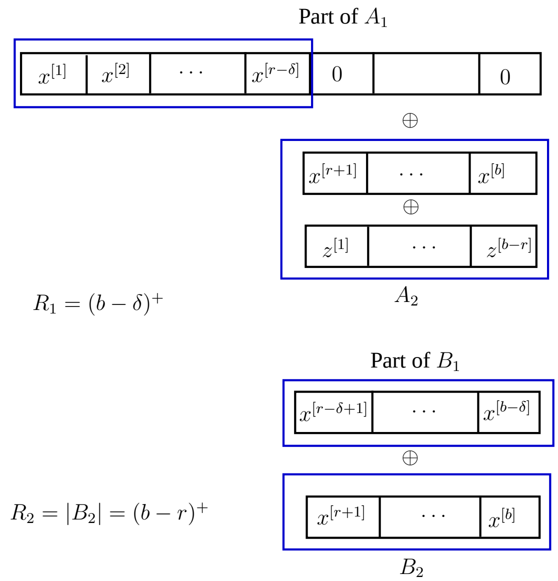

Fig. 3. Achievable Scheme for Case One ($r > \delta$), $P_3 = (R_1, R_2, R_3) = ((b-\delta)^+, (b-r)^+, 0)$

construction, and x has no information about them. Therefore, $R_3 \geq (b-r)^+$. About the bits $y_3^{[1]} \dots y_3^{[r-\delta]}$ one of encoders two or three can send it. If encoder two sends these bits, we achieve the rate vector $(R_1, R_2, R_3) = (0, r-\delta, (b-r)^+)$, denoted by P_4. If encoder three sends these bits, then we achieve $(R_1, R_2, R_3) = (0, 0, (b-\delta)^+)$ which is basically P_1.

Similarly, by setting $R_2 = 0$, the rate vector $(R_1, R_2, R_3) = (r-\delta, 0, (b-r)^+)$, denoted by P_5 is achievable. Using time-sharing among the achievable rate vectors, we will have the achievable region shown in Fig. 4.

This region is defined by the following six inequalities,

$$R_1 \geq 0, \tag{13}$$

$$R_2 \geq 0, \tag{14}$$

$$R_3 \geq 0, \tag{15}$$

$$R_1 + R_3 \geq (b-r)^+, \tag{16}$$

$$R_2 + R_3 \geq (b-r)^+, \tag{17}$$

$$R_1 + R_2 + R_3 \geq (b-\delta)^+, \tag{18}$$

$$R_1 + R_2 + 2R_3 \geq (b-r)^+ + (b-\delta)^+. \tag{19}$$

Case Two: $r \leq \delta$

For this case, the binary expansion of y_1, x, and y_3 are shown in Fig. 5. Block C_1 is what the decoder requires. Clearly, the rate vector $(R_1, R_2, R_3) = (0, 0, (b-r)^+)$, denoted by P_1', is achievable. Again, it is more exciting to consider the case where $R_3 = 0$. To reconstruct Block C_1, Blocks A_1, A_2 from

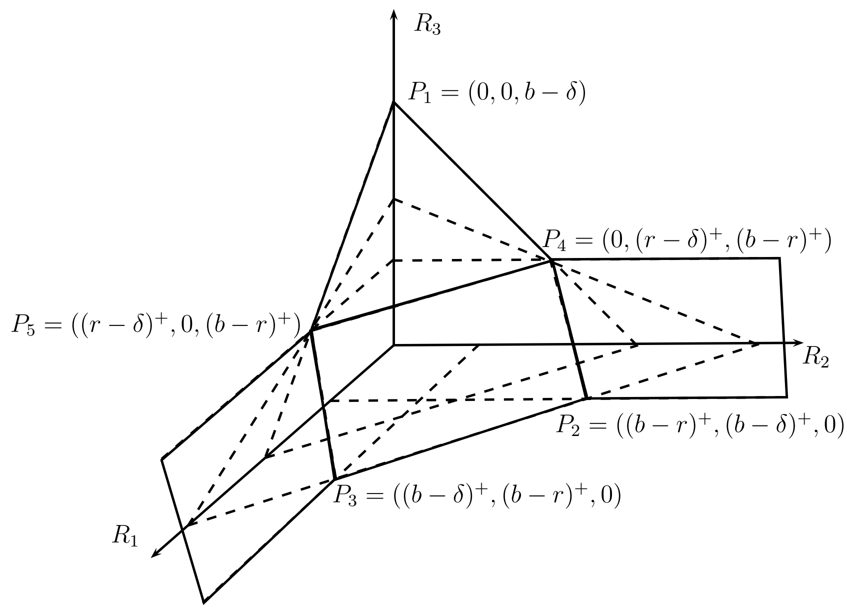

Fig. 4. Achievable Region for the Binary Expansion Model for Case One: $\delta \leq r$

y_1, and Blocks B_1 and B_2 from x are relevant. Therefore, here we have the same observation that some middle-layers of y_1 and x are not important. If decoder has access to A_2, B_1, and B_2, it can recover y_3 within the required resolution by forming $2^{-\delta} B_1 \oplus (A_2 \ominus B_2)$. Similar to Case 1, the second encoder can send $2^{-\delta} B_1 \ominus B_2$ to reduce its rate. Then as shown in Fig. 6, we achieve the rate vector $(R_1, R_2, R_3) = ((b - r)^+, (b - r)^+, 0)$, denoted by P_2'. Another approach is that encoder one sends $2^{-\delta} A_1 \oplus A_2$, while encoder two sends B_2, as shown in Fig. 7. Here, we achieve rate vector P_3', as $(R_1, R_2, R_3) = ((b-r)^+, (b-r)^+, 0)$, which is basically the same as P_2'. Therefore, in this case, the two of the corner points P_1 and P_2 we have in Case 1 collapse into one corner point $P_2' = P_3'$.

Let us consider the cut of the region, where $R_1 = 0$. We note that in this case, encoder three has to send all the bits in Block C_1. The reason is that Block C_1 has $z^{[1]} \ldots z^{[(b-r)^+]}$ in its construction, while x has no information about these bits. In other words, if encoder three does not send some bits of Block C_1, then it is impossible for the decoder to reconstruct those bits from information received from the second encoder. Therefore, the achievable rate vector is $(R_1, R_2, R_3) = (0, 0, (b - r)^+)$, which is basically the same as P_1'. We have the same situation where $R_2 = 0$. Therefore, three of the corner points P_1, P_2, and P_3 of Case 1 collapse to one corner point P_1' in Case 2.

Time sharing between the two rate vectors $(R_1, R_2, R_3) = ((b - r)^+, (b - r)^+, 0)$ and $(R_1, R_2, R_3) = (0, 0, (b - r)^+)$, we derive the achievable region shown in Fig 8.

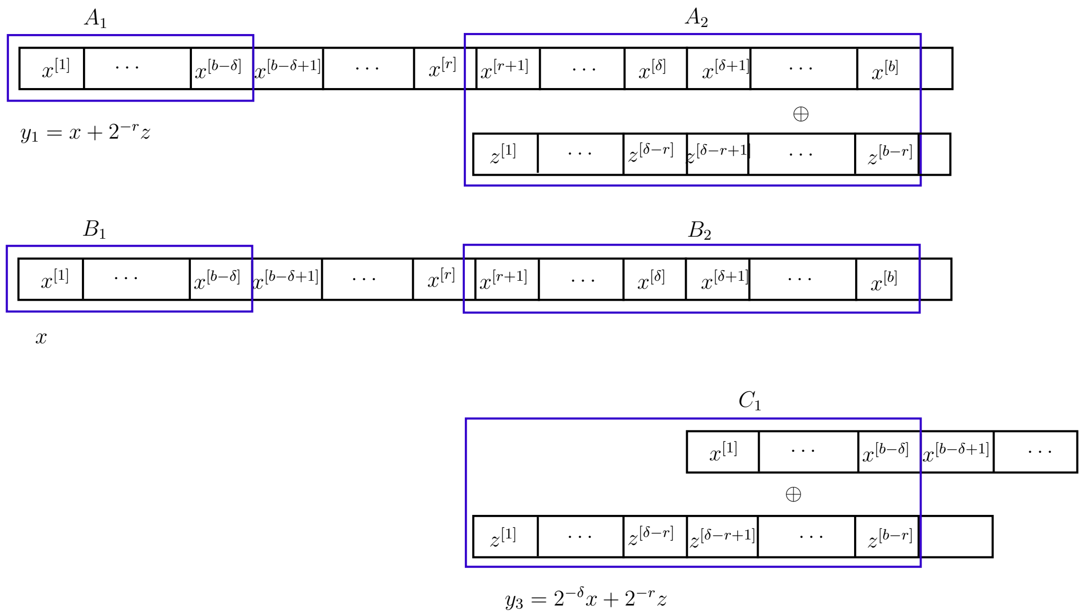

Fig. 5. Binary Expansion Model for Case Two: $r \leq \delta$

The achievable region for the case that $r \leq \delta$ is defined by the following inequalities,

$$R_1 \geq 0, \tag{20}$$

$$R_2 \geq 0, \tag{21}$$

$$R_3 \geq 0, \tag{22}$$

$$R_1 + R_3 \geq (b - r)^+, \tag{23}$$

$$R_2 + R_3 \geq (b - r)^+, \tag{24}$$

We note that the following inequality is dominated by the last two inequalities, however it touches the rate region at one point,

$$R_1 + R_2 + R_3 \geq (b - r)^+. \tag{25}$$

The two achievable regions that we derived for Case 1 and Case 2 can be unified by the region $\mathcal{R}_{in}$,

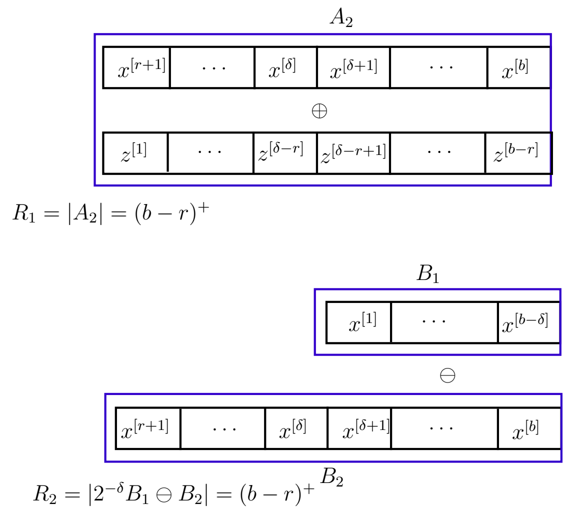

Fig. 6. Achievable Scheme for Case Two ($r \leq \delta$), $P_2' = (R_1, R_2, R_3) = ((b-r)^+, (b-r)^+, 0)$

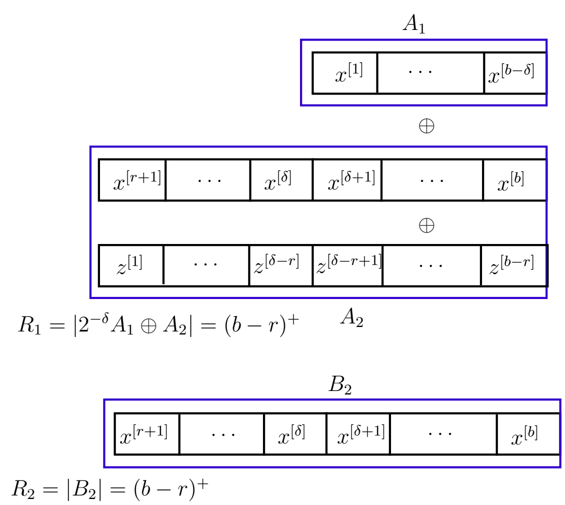

Fig. 7. Achievable Scheme for Case Two ($r \leq \delta$), $P_3' = (R_1, R_2, R_3) = ((b-r)^+, (b-r)^+, 0)$

defined as union of all $(R_1, R_2, R_3) \in \mathbb{R}^3$, satisfying:

$$R_1 \geq 0, \tag{26}$$

$$R_2 \geq 0, \tag{27}$$

$$R_3 \geq 0, \tag{28}$$

$$R_1 + R_3 \geq (b-r)^+, \tag{29}$$

$$R_2 + R_3 \geq (b-r)^+, \tag{30}$$

$$R_1 + R_2 + R_3 \geq \max\{(b-\delta)^+, (b-r)^+\}, \tag{31}$$

$$R_1 + R_2 + 2R_3 \geq (b-r)^+ + (b-\delta)^+. \tag{32}$$

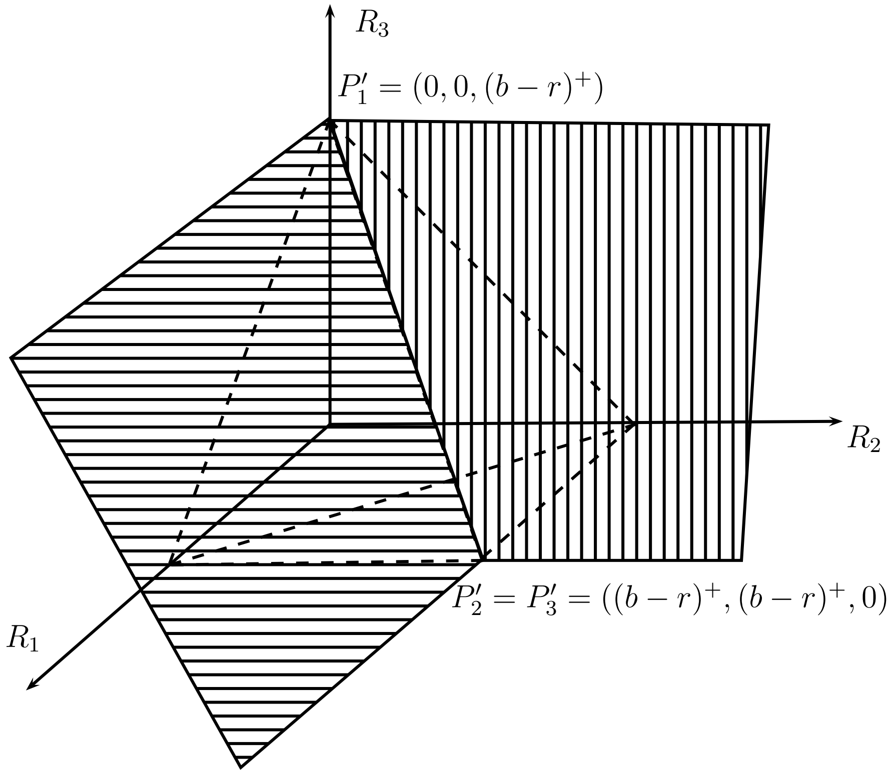

Fig. 8. Achievable Region for the Binary Expansion Model for Case Two: $r \leq \delta$

A. Outer-Bounds for Binary Expansion Model

Here we first try cut-set outer bounds as the most well-known outer-bounds. For any $S \subset \{1, 2, 3\}$, the cut-set lower-bound on $\sum_{j \in S} R_j$, is derived by assuming that a central encoder observes the sources in S, and sends an index to the decoder such that y_3 is able to be reconstructed with the target resolution, while all y_i, $i \in S^c$, are perfectly available at the decoder as the side information.

It is easy to see that inequalities (26) to (31) are derived by using cut-set outer bound. Therefore, to show the optimality of (26)-(32), we need to prove (32) is tight. In Theorem 1, we will prove a similar outer-bound for the Gaussian problem. Using similar arguments, we can show that (32) is tight as well. Therefore, the region $\mathcal{R}_{in}$ indeed characterizes the rate-distortion region of the binary-expansion model.

IV. Outer Bounds for the Gaussian Sources

The analysis of the binary expansion model supports the conjecture that the cut-set outer-bounds are useful to characterize some parts of the region within a bounded gap. It is easy to derive the cut-set outer-bounds as follows.

$$R_1 \geq \phi(1) = 0, \tag{33}$$

$$R_2 \geq \phi(2) = 0, \tag{34}$$

$$R_3 \geq \phi(3) = 0, \tag{35}$$

$$R_1 + R_2 \geq \phi(1,2) = 0, \tag{36}$$

$$R_1 + R_3 \geq \phi(1,3) = \frac{1}{2}\log_2^+ \frac{1-\rho^2}{d}, \tag{37}$$

$$R_2 + R_3 \geq \phi(2,3) = \frac{1}{2}\log_2^+ \frac{c^2(1-\rho^2)}{d}, \tag{38}$$

$$R_1 + R_2 + R_3 \geq \phi(1,2,3) = \frac{1}{2}\log_2^+ \frac{1+c^2-2c\rho}{d}. \tag{39}$$

However, from the binary expansion model, we learn that an extra lower-bound on $R_1 + R_2 + 2R_3$ is needed to approximate the region. In the following Theorem, we present the new bound for Gaussian sources.

Theorem 1

$$R_1 + R_2 + 2R_3 \geq \psi = \frac{1}{2}\log_2^+ \frac{1-\rho}{d} + \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} + \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}. \tag{40}$$

Proof: To have intuition the proof of this outer-bound, let us consider the case, where y_1 and y_2 are independent, i.e. $\rho = 0$. In this case, it is easy to see that the inequality (40) can be derived by adding two cut-set inequalities (37) and (38). This observation helps us to prove the inequality for the case that $\rho \neq 0$. For this case, we introduce a random variable $\tilde{x} \sim \mathcal{N}(0,1)$, such that

$$y_1 = \eta_1 \tilde{x} + \sqrt{1-\eta_1^2} z_1, \tag{41}$$

$$y_2 = \eta_2 \tilde{x} + \sqrt{1-\eta_2^2} z_2, \tag{42}$$

where for $0 \leq \eta_1 \leq 1$ and $0 \leq \eta_2 \leq 1$, $\eta_1\eta_2 = \rho$, $z_1 \sim \mathcal{N}(0,1)$, $z_2 \sim \mathcal{N}(0,1)$, and $\tilde{x}$, z_1, and z_2 are mutually independent. We note that y_1 and y_2, given $\tilde{x}$, are independent again. This observation suggests that we may be able to prove (40), by developing two inequalities for $R_1 + R_3$ and $R_2 + R_3$, extracting the contribution of $\tilde{x}$, and then add the two inequalities. In what follows, we elaborate the proof.

Assume that encoder j sends message M_j to the decoder, then we have,

$$\begin{aligned} n(R_1 + R_3) &\geq H(M_1) + H(M_3) \\ &\geq H(M_1, M_3 | M_2) \\ &= H(M_1, M_3 | M_2, \tilde{x}(1:n)) + I(M_1, M_3; \tilde{x}(1:n) | M_2), \end{aligned}$$

and

$$\begin{aligned}
n(R_2+R_3) &\geq H(M_2)+H(M_3)\\
&= H(M_2|M_1)+H(M_3)+I(M_1;M_2)\\
&\geq H(M_2|M_1)+H(M_3|M_1)+I(M_1;M_2)\\
&\geq H(M_2,M_3|M_1)+I(M_1;M_2)\\
&= H(M_2,M_3|M_1,\tilde{x}(1:n))+I(M_2,M_3;\tilde{x}(1:n)|M_1)+I(M_1;M_2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&n(R_1+R_2+2R_3)\geq\\
&\qquad I(M_1,M_3;\tilde{x}(1:n)|M_2)+I(M_2,M_3;\tilde{x}(1:n)|M_1)+I(M_1;M_2)\\
&\qquad + H(M_1,M_3|M_2,\tilde{x}(1:n))+H(M_2,M_3|M_1,\tilde{x}(1:n)).
\end{aligned}$$

On the other hand, we have,

$$\begin{aligned}
I(M_1,M_3;\tilde{x}(1:n)|M_2)+&I(M_2,M_3;\tilde{x}(1:n)|M_1)+I(M_1;M_2)\\
&= I(M_1,M_2,M_3;\tilde{x}(1:n))-I(M_2;\tilde{x}(1:n))+I(M_2;\tilde{x}(1:n)|M_1)\\
&+I(M_3;\tilde{x}(1:n)|M_1,M_2)+I(M_1;M_2)\\
&\overset{(a)}{\geq} I(M_1,M_2,M_3;\tilde{x}(1:n))-H(M_2)+H(M_2|\tilde{x}(1:n))\\
&+H(M_2|M_1)-H(M_2|M_1,\tilde{x}(1:n))+I(M_1;M_2)\\
&= I(M_1,M_2,M_3;\tilde{x}(1:n))-H(M_2|M_1,\tilde{x}(1:n))+H(M_2|\tilde{x}(1:n))\\
&\overset{(b)}{\geq} I(M_1,M_2,M_3;\tilde{x}(1:n)),
\end{aligned}$$

where (a) follows from $I(M_3;\tilde{x}(1:n)|M_1,M_2)\geq 0$, and (b) relies on $I(M_2;M_1|\tilde{x}(1:n))\geq 0$

In addition, we have,

$$\begin{aligned}
H(M_1,M_3|M_2,\tilde{x}(1:n)) &\geq H(M_1,M_3|M_2,\tilde{x}(1:n),y_2(1:n))\\
&\geq I(M_1,M_3;y_1(1:n)|M_2,\tilde{x}(1:n),y_2(1:n)).
\end{aligned}$$

Similarly,

$$H(M_2,M_3|M_1,\tilde{x}(1:n))\geq I(M_2,M_3;y_2(1:n)|M_1,\tilde{x}(1:n),y_1(1:n)).$$

Having M_j, $j=1,2,3$, the decoder reconstructs $\hat{y}_3(1:n)$, such that,

$$\mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}(\hat{y}_3(i)-y_3(i))^2]\leq d+\epsilon \tag{43}$$

for some small ϵ.

Then, we have the following three observations:

Observation 1: The decoder can reconstruct $(\eta_1 - c\eta_2)\tilde{x}$ with distortion $(\sqrt{d}+\sqrt{1-\eta_1^2+c^2(1-\eta_2^2)})^2$.

We note that,

$$y_3 = (\eta_1 - c\eta_2)\tilde{x} + \sqrt{1-\eta_1^2}z_1 - c\sqrt{1-\eta_2^2}z_2. \tag{44}$$

We use $\hat{y}_3$ as an estimation for $(\eta_1 - c\eta_2)\tilde{x}$. We have,

$$\begin{aligned}
&\frac{1}{n}E[\sum_{i=1}^{n}[\hat{y}_3(i) - (\eta_1 - c\eta_2)\tilde{x}(i)]^2] \\
&=\frac{1}{n}\sum_{i=1}^{n}E[\hat{y}_3(i) - y_3(i) + \sqrt{1-\eta_1^2}z_1(i) - c\sqrt{1-\eta_2^2}z_2(i)]^2 \\
&\leq\frac{1}{n}\sum_{i=1}^{n}\left\{\left(E[(\hat{y}_3(i)-y_3(i))^2]\right)^{\frac{1}{2}} + \left(E[(\sqrt{1-\eta_1^2}z_1(i) - c\sqrt{1-\eta_2^2}z_2(i))^2]\right)^{\frac{1}{2}}\right\}^2 \\
&\leq\left\{\left(\frac{1}{n}\sum_{i=1}^{n}E\left[(\hat{y}_3(i)-y_3(i))^2\right]\right)^{\frac{1}{2}} + \left(\frac{1}{n}\sum_{i=1}^{n}E\left[(\sqrt{1-\eta_1^2}z_1(i) - c\sqrt{1-\eta_2^2}z_2(i))^2\right]\right)^{\frac{1}{2}}\right\}^2 \\
&\leq(\sqrt{d+\epsilon}+\sqrt{1-\eta_1^2+c^2(1-\eta_2^2)})^2 \\
&\leq(\sqrt{d}+\sqrt{1-\eta_1^2+c^2(1-\eta_2^2)})^2 + \hat{\epsilon},
\end{aligned}$$

for a small $\hat{\epsilon} \geq 0$.

As a result,

$$I(M_1, M_2, M_3; \tilde{x}(1:n)) \geq \frac{n}{2}\log_2^+ \frac{(\eta_1 - c\eta_2)^2}{(\sqrt{d}+\sqrt{1-\eta_1^2+c^2(1-\eta_2^2)})^2}. \tag{45}$$

Observation 2: If $\tilde{x}(1:n)$ and $y_2(1:n)$ are available at the decoder in addition to M_j, $j = 1, 2, 3$, then $\sqrt{1-\eta_1^2}z_1$ can be reconstructed with distortion d.

We note that

$$y_3 = y_1 - cy_2 = \eta_1\tilde{x} + \sqrt{1-\eta_1^2}z_1 - cy_2. \tag{46}$$

We form $\sqrt{1-\eta_1^2}\hat{z}_1(i) = \hat{y}_3(i) - \eta_1\tilde{x}(i) + cy_2(i)$ as an estimation for $\sqrt{1-\eta_1^2}z_1(i)$. Then, it is easy to confirm the Observation 2. Therefore,

$$I(M_1, M_3; y_1(1:n)|M_2, \tilde{x}(1:n), y_2(1:n)) = I(M_1, M_2, M_3; \sqrt{1-\eta_1^2}z_1|\tilde{x}(1:n), y_2(1:n)) \tag{47}$$

$$\geq \frac{n}{2}\log_2^+ \frac{1-\eta_1^2}{d}. \tag{48}$$

Similarly we have the following observation.

Observation 3: If $\tilde{x}(1:n)$ and $y_1(1:n)$ are available at the decoder in addition to M_j, $j = 1, 2, 3$, then $c\sqrt{1-\eta_2^2}z_2$ can be reconstructed with distortion d.

Therefore,

$$I(M_2, M_3; y_2(1:n)|M_1, \tilde{x}(1:n), y_1(1:n)) = I(M_1, M_2, M_3; \sqrt{1-\eta_2^2} z_2 | \tilde{x}(1:n), y_1(1:n)) \tag{49}$$

$$\geq \frac{n}{2} \log_2^+ \frac{c^2(1-\eta_2^2)}{d}. \tag{50}$$

Therefore, we have

$$\begin{aligned} n(R_1 + R_2 + 2R_3) \geq & I(M_1, M_2, M_3; \tilde{x}(1:n)) + I(M_1, M_3; y_1(1:n)|M_2, \tilde{x}(1:n), y_2(1:n)) \\ & + I(M_2, M_3; y_2(1:n)|M_2, \tilde{x}(1:n), y_1(1:n)) \\ \geq & \log_2^+ \frac{(\eta_1 - c\eta_2)^2}{(\sqrt{d} + \sqrt{1-\eta_1^2 + c^2(1-\eta_2^2)})^2} + \frac{n}{2} \log_2^+ \frac{1-\eta_1^2}{d} + \frac{n}{2} \log_2^+ \frac{c^2(1-\eta_2^2)}{d}. \end{aligned}$$

By choosing $\eta_1 = \eta_2 = \sqrt{\rho}$, the result follows. ■

Note that this outer-bound is general, and does not depend on the condition $\frac{1}{2\rho} \leq c \leq \rho$.

V. Achievable Scheme for $\frac{1}{2\rho} \leq c \leq \rho$

In this section, efficient achievable schemes are developed inspired by the result from the binary expansion model. Here, corresponding to any corner point of the regions, depicted Fig. 4 and Fig. 8, we suggest an achievable rate vector for the original Gaussian problem.

A. Achievable Scheme Corresponding to P_2 in Fig. 4 and P_2' in Fig. 8

Let us focus on the achievable scheme which is inspired by the schemes showed in Fig. 2 and Fig. 6 for the binary expansion model. This achievable scheme corresponds to point P_2 in Fig. 4 and point P_2' in Fig. 8. In Fig. 9, the block diagram of this achievable scheme is shown. To clarify this achievable scheme, the role of each lattice has been shown in Fig. 10. We note that in Fig. 9, the quantizer Λ_M and subtraction after it form the operation $\mod \Lambda_M$. In this figure, we have lattices $3.\Lambda_M$ which is not justified by binary expansion model. The role of these lattices will be explained later.

All the front quantizes Λ_{F1}, Λ_{F_2}, and Λ_{F_3} are fine lattice-quantizers with the second moments *around* d. The precise choices of the second moments will be given later. The quantizer Λ_M has the second moment $\sigma^2_{\Lambda_M}$ which is around the covariance of $y_1 - \rho y_2$ i.e. $1-\rho^2$. The coarse quantizer Λ_C has the second moment $\sigma^2_{\Lambda_C}$ around the covariance of $y_1 - cy_2$ which is $1 + c^2 - 2c\rho$. Roughly speaking, if the second moment of a quantizer is σ^2, it quantizes the input with the resolution of $-\frac{1}{2}\log_2 \sigma^2$ bits after the radix point. It is insightful to let $d = 2^{-2b}$, $\rho = 1 - 2^{-2r}$, and $c = 1 - 2^{-\delta}$ and check that the binary expansion model shown in Fig. 10 matches with the achievable scheme of Fig. 9.

In what follows, we elaborate the achievable scheme. In Fig. 9, we have some chains of nested lattices $\Lambda_{F_1} \subset \Lambda_M \subset \Lambda_C$. In addition, we have either $\Lambda_{F_2} \subset \Lambda_{F_3} \subset \Lambda_M \subset \Lambda_C$ or $\Lambda_{F_3} \subset \Lambda_{F_2} \subset \Lambda_M \subset \Lambda_C$. In this paper, we assume that all lattices are simultaneously good channel and source lattice codes. To see definition of the good lattices, refer to [12] and to find the proof of the existence of such chains of nested lattices, see [7, Appendix B].

Let u_i^n be mutually independent random vectors uniformly distributed in the Voronoi regions of Λ_{F_i}, for $i = 1, 2, 3$. The second moment of Λ_{F_i} is denoted by $\sigma^2_{\Lambda_{F_i}}$.

We define $e^n_{\Lambda_{F_i}}$ as the error of the quantizer Λ_{F_i}, i.e.

$$e^n_{\Lambda_{F_1}} = Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - (y_1^n + u_1^n), \tag{51}$$

$$e^n_{\Lambda_{F_2}} = Q_{\Lambda_{F_1}}(\rho y_2^n + u_2^n) - (\rho y_2^n + u_2^n), \tag{52}$$

$$e^n_{\Lambda_{F_3}} = Q_{\Lambda_{F_3}}((c-\rho) y_3^n + u_3^n) - ((c-\rho) y_3^n + u_3^n). \tag{53}$$

We have the following facts about $e^n_{\Lambda_{F_i}}$, $i = 1, 2, 3$.

Fact 1: Remember u_i^n is uniformly distributed in the Voronoi region of Λ_{F_i}. Then it is easy to show that $e^n_{\Lambda_{F_i}}$ has the same distribution as u_i^n and is independent of the input of the lattice Λ_{F_i} [13].

Fact 2: Since Λ_{F_i} is an optimal lattice, then as $n \to \infty$, then $e^n_{\Lambda_{F_1}}$ converges to a white Gaussian noise in Kullback-Leibler distance sense [13].

Fact 3: In addition, $e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}}$ and $e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}} - e^n_{\Lambda_{F_3}}$ tend to a white Gaussian noise in Kullback-Leibler divergence sense [7].

We define z^n as

$$z^n = Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) = y_1^n - \rho y_2^n + e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}}, \tag{54}$$

Then, then we have,

$$\frac{1}{n}\mathbb{E}\|z^n\| = 2(1-\rho) + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}}, \tag{55}$$

where we use the fact that the second moment of the lattice Λ_{F_i} is $\sigma^2_{\Lambda_{F_i}}$, and also $y_1^n - \rho y_2^n$, $e^n_{\Lambda_{F_1}}$, $e^n_{\Lambda_{F_2}}$ are mutually independent (Fact 1). We choose Λ_M a good channel-code lattice with the second moment of,

$$\sigma^2_{\Lambda_M} = 2(1-\rho) + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}}. \tag{56}$$

Fact 4: Since $e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}}$ tends to an i.i.d. Gaussian random process, then z^n converges to an i.i.d. Gaussian random process as well. Since lattice Λ_M is a good channel code, then the probability that z^n is not in the Voronoi region of Λ_M goes to zero exponentially fast, as $n \to \infty$ and the effect of deviation of $e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}}$ from Gaussian on that probability is sub-exponential [7].

As we will see later, at the decoder requires $Q_{\Lambda_M}[Q_{\Lambda_{F_1}}(y_1^n + u_1^n)] - Q_{\Lambda_M}[Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)]$ to reconstruct $\hat{y}_3$. In Fact 4, we stated that $Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)$ is in the Voronoi region of the lattice Λ_M. Then, it is easy to show the following fact.

Fact 5: $Q_{\Lambda_M}[Q_{\Lambda_{F_1}}(y_1^n + u_1^n)] - Q_{\Lambda_M}[Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)]$ is in Voronoi region of $3.\Lambda_M$. We define $3.\Lambda_M$ as the lattice with generating matrix which is equal to 3 times of generating matrix of Λ_M.

Therefore, Θ_1, shown in Fig. 9, is equal to,

$$\Theta_1 = R_{3.\Lambda_M}\Big(R_{3.\Lambda_M}Q_{\Lambda_M}[Q_{\Lambda_{F_1}}(y_1^n + u_1^n)] - R_{3.\Lambda_M}Q_{\Lambda_M}[Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)]\Big) \tag{57}$$

$$\overset{(a)}{=} R_{3.\Lambda_M}\Big(Q_{\Lambda_M}[Q_{\Lambda_{F_1}}(y_1^n + u_1^n)] - Q_{\Lambda_M}[Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)]\Big) \tag{58}$$

$$\overset{(b)}{\doteq} Q_{\Lambda_M}[Q_{\Lambda_{F_1}}(y_1^n + u_1^n)] - Q_{\Lambda_M}[Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)], \tag{59}$$

where (a) relies on the properties of lattices, and (b) is based on Fact 5. In fact the probability that (b) is not valid goes to zero exponentially fast as $n \to \infty$. Here, we use the short-hand notation $R_{\Lambda}(x)$ for $x \mod \Lambda$.

Let us define

$$\Pi^n = z^n - Q_{\Lambda_{F_3}}((c-\rho)y_2^n + u_3^n) \tag{60}$$

$$= z^n - (c-\rho)y_2^n - e^n_{\Lambda_{F_3}} \tag{61}$$

$$= y_3^n + e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}} - e^n_{\Lambda_{F_3}}, \tag{62}$$

where z^n is defined in (54). Then from Facts 1, 2, and 3, Π^n converges to an i.i.d. Gaussian random sequence, with covariance

$$\frac{1}{n}\mathbb{E}\|\Pi^n\| = 1 + c^2 - 2c\rho + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}} + \sigma^2_{\Lambda_{F_3}}. \tag{63}$$

We choose $\sigma^2_{\Lambda_C}$ as

$$\sigma^2_{\Lambda_C} = \frac{1}{n}\mathbb{E}\|\Pi^n\| = 1 + c^2 - 2c\rho + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}} + \sigma^2_{\Lambda_{F_3}}. \tag{64}$$

Fact 6: Since Λ_C is a good channel code and Π^n converges to an i.i.d. Gaussian, then the probability that Π^n is not in the Voronoi region of Λ_C goes to zero exponentially fast as $n \to \infty$.

Now we are ready to evaluate the output of the encoder.

$$\begin{aligned}
\hat{y}_3^n &= \eta R_{\Lambda_C}\Big[R_{\Lambda_M}\left(Q_{\Lambda_{F_1}}(y_1^n + u_1^n)\right) \\
&- R_{\Lambda_C}\left\{R_{\Lambda_M}(Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)) + Q_{\Lambda_{F_2}}((c-\rho)y_2^n + u_3^n)\right\} + u_1^n - u_2^n - u_3^n + \Theta_1\Big] \\
&\overset{(a)}{=} \eta R_{\Lambda_C}\Big[R_{\Lambda_M}(Q_{\Lambda_{F_1}}(y_1^n + u_1^n)) - R_{\Lambda_M}(Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)) - Q_{\Lambda_{F_3}}((c-\rho)y_2^n + u_3^n) + u_1^n - u_2^n - u_3^n + \Theta_1\Big] \\
&\overset{(b)}{\doteq} \eta R_{\Lambda_C}\Big[Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) - Q_{\Lambda_{F_3}}((c-\rho)y_2^n + u_3^n) + u_1^n - u_2^n - u_3^n\Big] \\
&\overset{(c)}{=} \eta R_{\Lambda_C}\Big[y_3^n + e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}} - e^n_{\Lambda_{F_3}}\Big] \\
&\overset{(d)}{\doteq} \eta(y_3^n + e^n_{\Lambda_{F_1}} - e^n_{\Lambda_{F_2}} - e^n_{\Lambda_{F_3}}),
\end{aligned}$$

where (a) follows from properties of the operation R and (b) follows from (57). In fact, the probability that (b) is not valid goes to zero exponentially fast as $n \to \infty$. (c) follows from the definition of $e^n_{\Lambda_{F_i}}$ and (d) relies on Fact 6.

We choose η as

$$\eta = \frac{\sigma^2_{y_3}}{\sigma^2_{\Lambda_{y_3}} + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}} + \sigma^2_{\Lambda_{F_3}}}. \tag{65}$$

Then, the distortion of the achievable scheme is $\frac{1}{n}\mathbb{E}\|y_3^n - \hat{y}_3^n\|$. We choose $\sigma^2_{\Lambda_{F_i}}$, $i = 1, 2, 3$, such that $\frac{1}{n}\mathbb{E}\|y_3^n - \hat{y}_3^n\|$ is equal to the target distortion d, i.e.

$$d = \frac{1}{n}\mathbb{E}\|y_3^n - \hat{y}_3^n\| = \frac{\sigma^2_{y_3}(\sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}} + \sigma^2_{\Lambda_{F_3}})}{\sigma^2_{\Lambda_{y_3}} + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}} + \sigma^2_{\Lambda_{F_3}}}. \tag{66}$$

Then, we can rewire η as

$$\eta = \frac{\sigma^2_{y_3} - d}{\sigma^2_{y_3}}. \tag{67}$$

Refereing to Fig. 9, R_1 and R_2 have two components:

$$R_1 = R_{11} + R_{12}, \tag{68}$$

$$R_2 = R_{21} + R_{22}. \tag{69}$$

We have

$$R_{21} = R_{12} = \frac{\sigma^2_{3.\Lambda_M}}{\sigma^2_{\Lambda_M}} = \log_2 3. \tag{70}$$

In addition, we have

$$R_{11} = \frac{1}{2}\log_2 \frac{\sigma^2_{\Lambda_M}}{\sigma^2_{\Lambda_{F_1}}} = \frac{1}{2}\log_2 \frac{2(1-\rho) + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}}}{\sigma^2_{\Lambda_{F_1}}}. \tag{71}$$

and,

$$R_{22} = \frac{1}{2}\log_2 \frac{\sigma^2_{\Lambda_C}}{\min\{\sigma^2_{\Lambda_{F_2}}, \sigma^2_{\Lambda_{F_3}}\}} = \frac{1}{2}\log_2 \frac{1 + c^2 - 2c\rho + \sigma^2_{\Lambda_{F_1}} + \sigma^2_{\Lambda_{F_2}} + \sigma^2_{\Lambda_{F_3}}}{\min\{\sigma^2_{\Lambda_{F_2}}, \sigma^2_{\Lambda_{F_3}}\}}. \tag{72}$$

Then, we have the following result.

Theorem 2 *Any rate vector* $(R_1, R_2, 0)$ *that satisfies* (68)-(72) *is achievable.*

Let us choose a particular achievable rate vector $P_2^G = (R_1^{(2)}, R_2^{(2)}, 0)$, by choosing $\sigma^2_{\Lambda_{F_1}} = \sigma^2_{\Lambda_{F_2}} = \sigma^2_{\Lambda_{F_3}} = q$. Then from (66), we have

$$q = \frac{\sigma^2_{y_3} d}{3(\sigma^2_{y_3} - d)}. \tag{73}$$

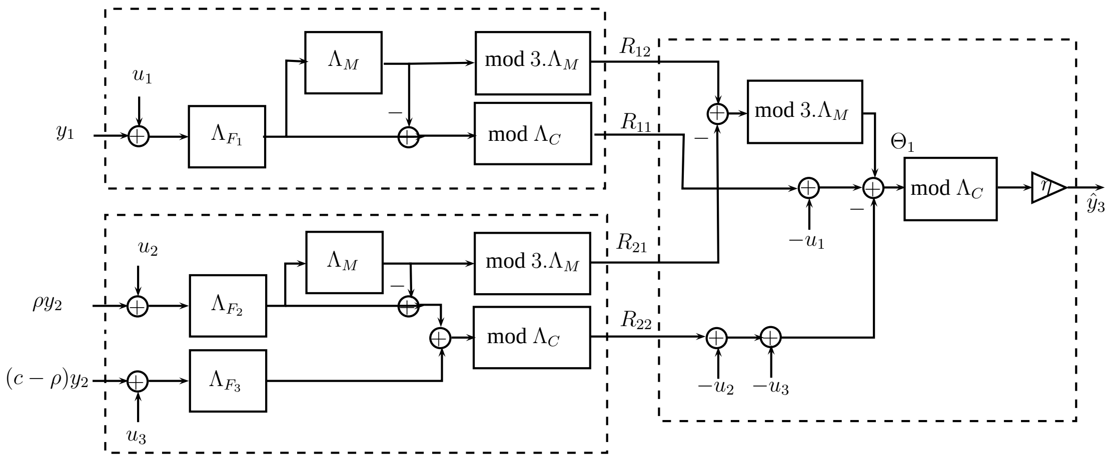

Fig. 9. Achievable Scheme for P_2^G

Therefore, we have,

$$R_{11}^{(2)} = \frac{1}{2}\log_2^+(2 + \frac{6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d}), \tag{74}$$

$$R_{22}^{(2)} = \frac{1}{2}\log_2^+ \frac{3\sigma_{y_3}^2}{d}. \tag{75}$$

Lemma 3 *For the achievable point* $P_2^G = (R_1^{(2)}, R_2^{(2)}, 0)$*, we have*

(i) $R_1^{(2)} - \phi(1,3) \leq \frac{1}{2}\log_2 72 = 3.08$ *bits per sample.*

(ii) $R_2^{(2)} - \phi(2,3) \leq \frac{1}{2}\log_2 108 = 3.38$ *bits per sample, if* $1 - c \leq \sqrt{1-\rho}$*.*

(iii) $R_1^{(2)} + R_2^{(2)} - \psi \leq 3\log_2 3 + \frac{1}{2}\log_2 12(1+\sqrt{5})^2 = 8.44$ *bits per sample, if* $1 - c \geq \sqrt{1-\rho}$*.*

Proof: Refer to Appendix A. ∎

B. Achievable Scheme Corresponding to P_3 *in Fig. 4 and* P_3' *in Fig. 8*

Here we focus on another achievable scheme which corresponds to the achievable scheme shown in Fig. 3, or the corner point P_3 in Fig. 4 for the binary expansion model when $r \geq \delta$. Refereing to Fig. 10, we note that in the first achievable scheme, bits $x^{[1]} \ldots x^{[r-\delta]}$ is sent by the second encoder. However, the first encoder has access to these bits and can take care of it. Shifting the responsibility of sending these bits from encoder one to encoder two, we obtain another corner point of the region, referred as P_3. Obviously, this statement is valid for case $r \geq \delta$. In the Gaussian case, we modify the achievable scheme of Fig. 9 to form the achievable scheme of Fig. 11, for the case where $1 - c \geq \sqrt{1-\rho}$. The role of each lattice has been shown in the binary expansion model in Fig. 12.

All the facts, equations and definitions from (51) to (57) are valid for the second achievable scheme. We note that the input of $\mod 2.\Lambda_M$ is the summation of outputs of two $\mod \Lambda_M$ blocks. Therefore, this summation is evidently in the Voronoi region of the lattice $\mod 2.\Lambda_M$. Therefore, the operation

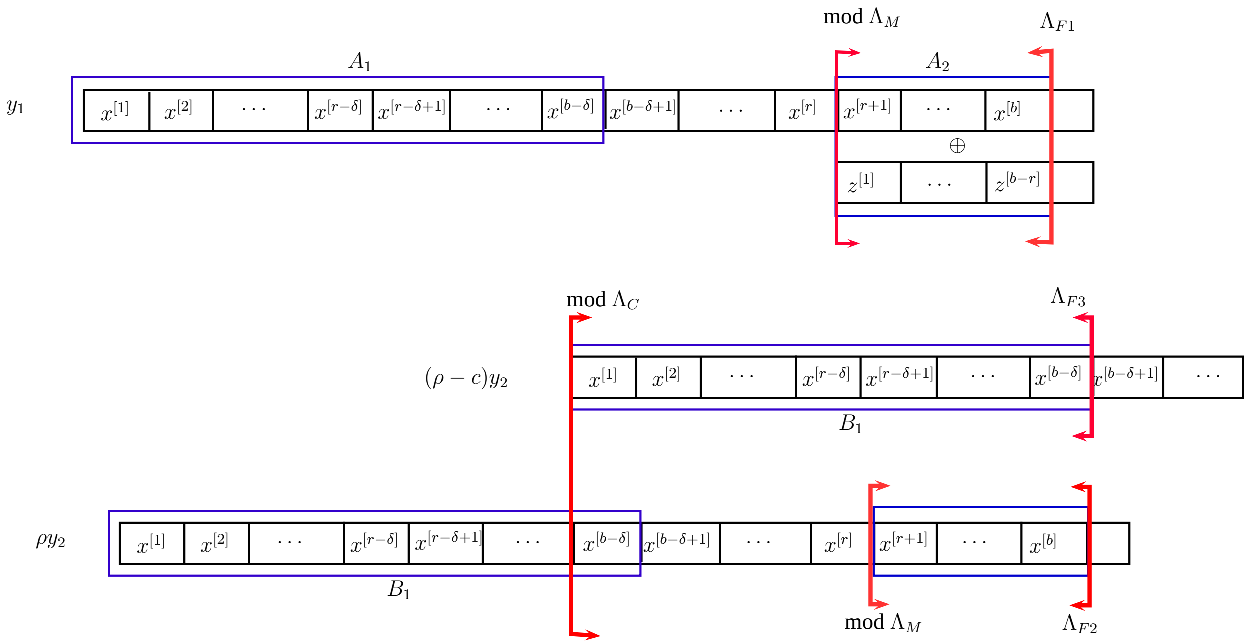

Fig. 10. The Role of Each Lattice of Fig. 9 on Binary Expansion Model

$\mod\ 2.\Lambda_M$ does not do any thing and can be eliminated. We just use it here to show how the rates are calculated.

In what follows, we use the short-hand notation $\mu = c - \rho$. We know that μy_1^n is almost surely is in the Voronoi region of a good lattice code for channel with second moment μ^2. On the other hand, we have,

$$Q_{\Lambda_M}(\mu y_1^n) = \mu y_1^n - R_{\Lambda_M}(\mu y_1^n). \tag{76}$$

Therefore, it is easy to see that $Q_{\Lambda_M}(\mu y_1^n)$ is almost surely in the Voronoi region of a good lattice $\Lambda_{\hat{C}}$ with the second moment $\sigma_{\hat{C}}^2$, if

$$\sigma_{\hat{C}}^2 \geq (|\mu| + \sigma_{\Lambda_M})^2. \tag{77}$$

Then, the probability that the equation

$$R_{\Lambda_{\hat{C}}} Q_{\Lambda_M}(\mu y_1^n) \doteq Q_{\Lambda_M}(\mu y_1^n) \tag{78}$$

is not valid goes to zero exponentially fast as $n \to 0$. We choose $\sigma_{\hat{C}}^2$ as

$$\sigma_{\Lambda_{\hat{C}}}^2 = \max\{(|\mu| + \sigma_{\Lambda_M})^2, \sigma_{\Lambda_C}^2\}. \tag{79}$$

Therefore, Θ_2, shown in Fig. 11, is equal to

$$\begin{aligned}\Theta_2 :=& R_{\Lambda_D}\Big[R_{\Lambda_{\hat{C}}}Q_{\Lambda_M}(\mu y_1^n) - R_{\Lambda_D}Q_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n)\Big] \\ \overset{(a)}{\doteq}& R_{\Lambda_D}\Big[Q_{\Lambda_M}(\mu y_1^n) - R_{\Lambda_D}Q_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n)\Big] \\ \overset{(b)}{=}& R_{\Lambda_D}\Big[Q_{\Lambda_M}(\mu y_1^n) - Q_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n)\Big] \\ \overset{(c)}{=}& R_{\Lambda_D}\Big[\mu y_1^n - R_{\Lambda_M}(\mu y_1^n) - Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n) + R_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n)\Big] \\ \overset{(d)}{=}& R_{\Lambda_D}\Big[\mu y_1^n - R_{\Lambda_M}(\mu y_1^n) - \mu y_2^n - u_3^n - e_{\Lambda_{F_3}}^n + R_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n)\Big] \\ =& R_{\Lambda_D}\Big[\mu(y_1^n - y_2^n) - R_{\Lambda_M}(\mu y_1^n) - u_3^n - e_{\Lambda_{F_3}}^n + R_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n)\Big] \\ \overset{(e)}{\doteq}& \mu(y_1^n - y_2^n) - R_{\Lambda_M}(\mu y_1^n) - u_3^n - e_{\Lambda_{F_3}}^n + R_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n),\end{aligned} \tag{80}$$

where (a) is based on (78), (b) is based on the basic property of the operation R_{Λ_D}, (c) follows from $\mu y_1^n = R_{\Lambda_M}(\mu y_1^n) + Q_{\Lambda_M}(\mu y_1^n)$, and $Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n) = R_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n) + Q_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n)$, and (d) is based on (53). In addition, the probability that (e) is not correct goes to zero exponentially fast, as $n \to \infty$. The reason is that we choose Λ_D as a good lattice for channel code with the second moment,

$$\sigma_{\Lambda_D}^2 = \left(2\sigma_{\Lambda_M} + |\mu|\sqrt{2(1-\rho)} + 2\sigma_{\Lambda_{F_3}}\right)^2. \tag{81}$$

Then, from Fig. 11, we have

$$\begin{aligned}\hat{y}_3^n =& \eta R_{\Lambda_{\hat{C}}}\Big[R_{\Lambda_M}Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - R_{\Lambda_C}Q_{\Lambda_M}(\mu y_1^n) \\ &- R_{\Lambda_M}Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) - R_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n) - u_1^n + u_2^n + u_3^n + \Theta_1 + \Theta_2\Big] \\ \overset{(a)}{\doteq}& \eta R_{\Lambda_{\hat{C}}}\Big[R_{\Lambda_M}Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - Q_{\Lambda_M}(\mu y_1^n) \\ &- R_{\Lambda_M}Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) - R_{\Lambda_M}Q_{\Lambda_{F_3}}(\mu y_2^n + u_3^n) - u_1^n + u_2^n + u_3^n + \Theta_1 + \Theta_2\Big] \\ \overset{(b)}{\doteq}& \eta R_{\Lambda_{\hat{C}}}\Big[R_{\Lambda_M}Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - R_{\Lambda_M}Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) - u_1^n + u_2^n + \Theta_1 - \mu y_2^n - e_{\Lambda_{F_3}}^n\Big] \\ \overset{(c)}{\doteq}& \eta R_{\Lambda_{\hat{C}}}\Big[Q_{\Lambda_{F_1}}(y_1^n + u_1^n) - Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) - u_1^n + u_2^n - \mu y_2^n - e_{\Lambda_{F_3}}^n\Big] \\ \overset{(d)}{=}& \eta(y_3^n + e_{\Lambda_{F_1}}^n - e_{\Lambda_{F_2}}^n - e_{\Lambda_{F_3}}^n),\end{aligned}$$

where (a) is based on (78), (b) follows from (80), and $\mu y_1^n = R_{\Lambda_M}(\mu y_1^n) + Q_{\Lambda_M}(\mu y_1^n)$, (c) relies on (57) and $Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) = R_{\Lambda_M}Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n) + Q_{\Lambda_M}Q_{\Lambda_{F_2}}(\rho y_2^n + u_2^n)$, and $Q_{\Lambda_{F_1}}(y_1^n + u_1^n) = R_{\Lambda_M}Q_{\Lambda_{F_1}}(y_1^n + u_1^n) + Q_{\Lambda_M}Q_{\Lambda_{F_1}}(y_1^n + u_1^n)$, finally (d) is based in (51)-(53).

Again here we choose η as (65). In addition, the distortion of the scheme is derived by (66).

Refereing to Fig. 11, R_1 and R_2 are equal to:

$$R_1 = R_{11} + R_{12} + R_{13}, \tag{82}$$

$$R_2 = R_{21} + R_{22} + R_{23}, \tag{83}$$

where

$$R_{11} = \frac{1}{2}\log_2 \frac{\sigma^2_{\Lambda_M}}{\sigma^2_{\Lambda_{F_1}}}, \tag{84}$$

$$R_{12} = R_{21} = \frac{1}{2}\log_2 \frac{\sigma^2_{3.\Lambda_M}}{\sigma^2_{\Lambda_M}} = \log_2 3, \tag{85}$$

$$R_{13} = \frac{1}{2}\log_2 \frac{\sigma^2_{\Lambda_{\hat{C}}}}{\sigma^2_{\Lambda_M}}, \tag{86}$$

$$R_{22} = \frac{1}{2}\log_2 \frac{\sigma^2_{2.\Lambda_M}}{\min\{\sigma^2_{\Lambda_{F_2}}, \sigma^2_{\Lambda_{F_3}}\}} = 1 + \frac{1}{2}\log_2 \frac{\sigma^2_{\Lambda_M}}{\min\{\sigma^2_{\Lambda_{F_2}}, \sigma^2_{\Lambda_{F_3}}\}}, \tag{87}$$

$$R_{23} = \frac{1}{2}\log_2 \frac{\sigma^2_{\Lambda_D}}{\sigma^2_{\Lambda_M}}. \tag{88}$$

Theorem 4 *The rate vectors $(R_1, R_2, 0)$ which satisfy* (82)–(88)*, are achievable.*

Let focus on a specific rate vector $P_3^G = (R_1^{(3)}, R_2^{(3)}, 0)$, achieved by choosing $\sigma^2_{\Lambda_{F_1}} = \sigma^2_{\Lambda_{F_2}} = \sigma^2_{\Lambda_{F_3}} = q$. Then from (66), we have

$$q = \frac{\sigma^2_{y_3} d}{3(\sigma^2_{y_3} - d)}. \tag{89}$$

Therefore, we have,

$$R_{11}^{(3)} = \frac{1}{2}\log_2(2 + \frac{6(1-\rho)(\sigma^2_{y_3} - d)}{\sigma^2_{y_3} d}), \tag{90}$$

$$R_{22}^{(3)} = \frac{1}{2}\log_2(2 + \frac{6(1-\rho)(\sigma^2_{y_3} - d)}{\sigma^2_{y_3} d}) + 1. \tag{91}$$

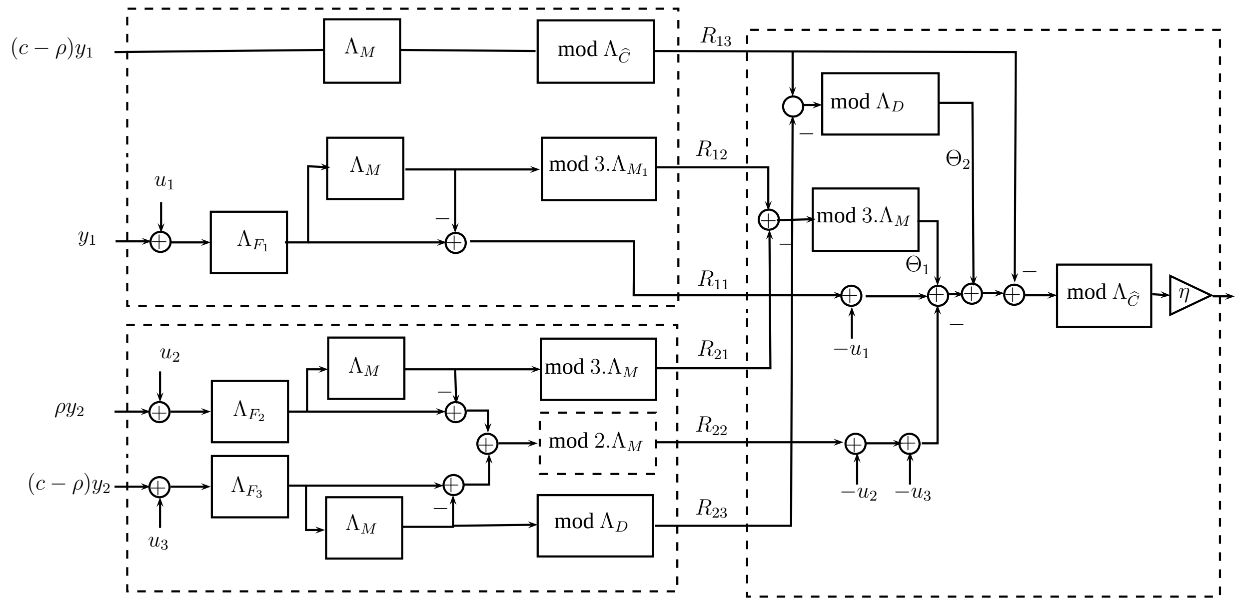

Fig. 11. Achievable Scheme for P_3^G

In this case, we have

$$R_{23}^{(3)} = \frac{1}{2}\log_2 \frac{\sigma^2_{\Lambda_D}}{\sigma^2_{\Lambda_M}} \tag{92}$$

$$\overset{(a)}{=} \log_2 \frac{2\sigma_{\Lambda_M} + (\rho-c)\sqrt{2(1-\rho)} + 2\sqrt{q}}{\sigma_{\Lambda_M}} \tag{93}$$

$$\overset{(b)}{=} \log_2 \left(2 + \frac{(\rho-c)\sqrt{2(1-\rho)} + 2\sqrt{q}}{\sqrt{2(1-\rho)+2q}}\right) \tag{94}$$

$$= \log_2 \left(2 + \frac{(\rho-c)\sqrt{2(1-\rho)}}{\sqrt{2(1-\rho)+2q}} + \frac{2\sqrt{q}}{\sqrt{2(1-\rho)+2q}}\right) \tag{95}$$

$$\leq \log_2 \left(2 + \frac{(\rho-c)\sqrt{2(1-\rho)}}{\sqrt{2(1-\rho)}} + \frac{2\sqrt{q}}{\sqrt{2q}}\right) \tag{96}$$

$$\overset{(c)}{\leq} \log_2 \left(2 + 0.5 + \sqrt{2}\right) = \log_2 \left(2.5 + \sqrt{2}\right) < 2, \tag{97}$$

where (a) and (b) follow from (56) and (81), respectively, and (c) is based on $\rho - c \leq \frac{1}{2}$, for $\frac{1}{2\rho} \leq c \leq \rho$.

Lemma 5 *If* $1 - c \geq \sqrt{1-\rho}$*, then we have,*

(i) $R_2^{(3)} - \phi(2,3) \leq \log_2 3 + 3 + \frac{1}{2}\log_2 28 = 6.98$ *bits per sample*.

(ii) $R_1^{(3)} + R_2^{(3)} - \psi \leq 12.42$ *bits per sample*.

Proof: Refer to Appendix B. ■

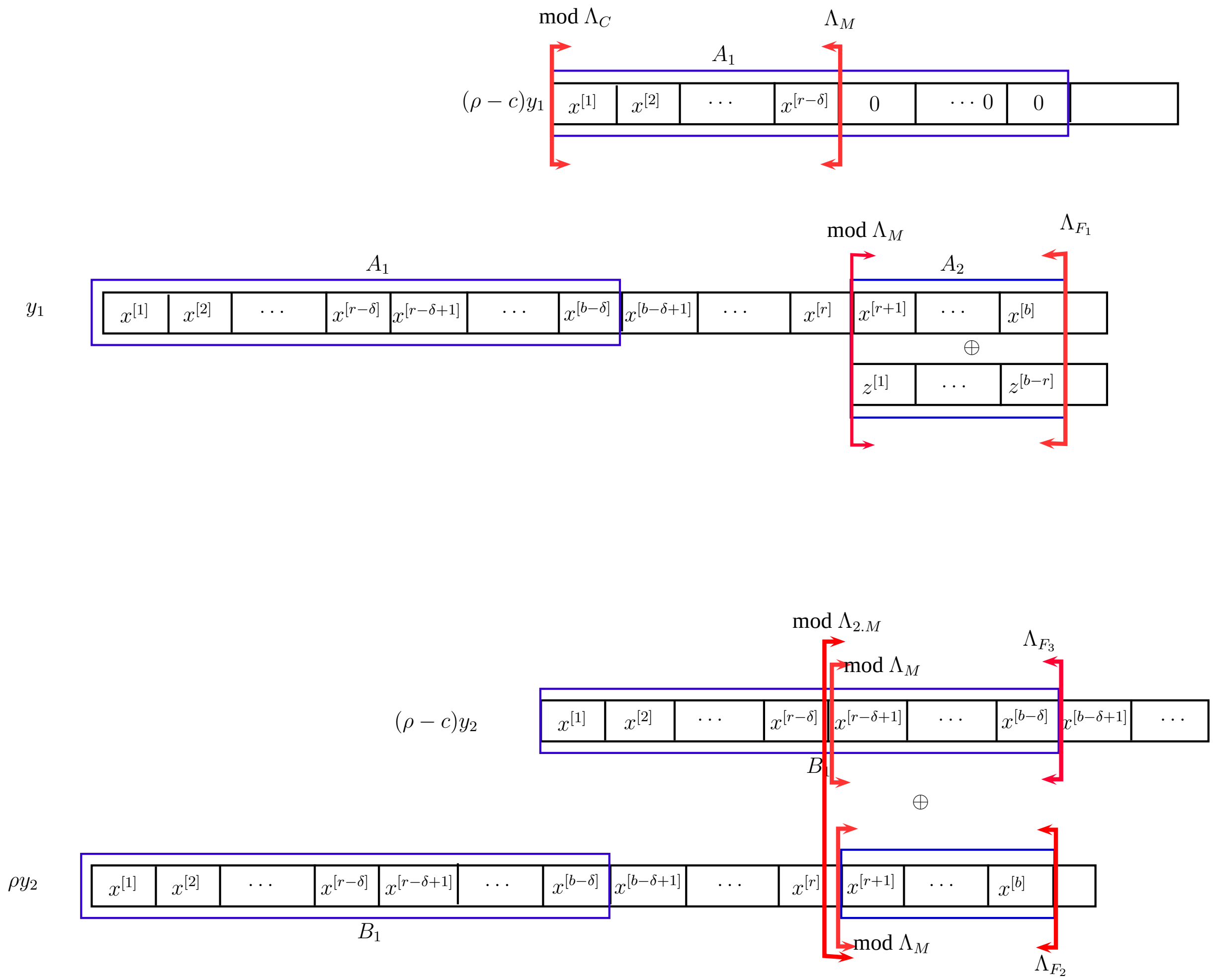

Fig. 12. The Role of Each Lattice of Fig. 11 on Binary Expansion Model

C. Achievable Scheme Corresponding to P_1 in Fig. 4 and P_1' in Fig. 8

It is obvious that the rate vector $P_1^G = (0, 0, R_3^{(1)})$, where

$$R_3^{(1)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d}, \tag{98}$$

is achievable.

Lemma 6 *If $1 - c \leq \sqrt{1-\rho}$, then*

(i) $R_3^{(1)} - \phi(1,3) \leq 0.30$ *bits per sample.*

(ii) $R_3^{(1)} - \phi(2,3) \leq 1.21$ *bits per sample.*

Proof: Refer to Appendix C. ■

D. Achievable Scheme Corresponding to P_5 in Fig. 4 and P_5' in Fig. 8

Let us find achievable rate vectors, for which $R_2 = 0$. From [2], we know the quantization-and-binning scheme achieves all rate vectors $(R_1, 0, R_3)$, satisfying

$$R_3 = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2\left(1-\rho_{13}^2+\rho_{13}^2 e^{-2R_1}\right)}{d}. \tag{99}$$

Let us choose a particular rate vector $P_5^G = (R_1^{(5)}, 0, R_3^{(5)})$ from the above sets of achievable rates, as

$$R_1^{(5)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2\left(1-\rho_{13}^2\right)}{d} = \min\left\{\frac{1}{2}\log_2\frac{\sigma_{y_3}^2}{d}, \frac{1}{2}\log_2\frac{1}{1-\rho_{13}^2}\right\}, \tag{100}$$

and

$$R_3^{(5)} = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2\left(1-\rho_{13}^2+\rho_{13}^2 e^{-2R_1^{(5)}}\right)}{d}. \tag{101}$$

Lemma 7 *If $1-c \geq \sqrt{1-\rho}$, then we have*

(i) $R_3^{(5)} - \phi(2,3) \leq 1$ *bits per samples.*

(ii) $R_1^{(5)} + R_3^{(5)} - \phi(1,2,3) \leq 1$ *bits per samples.*

(ii) $R_1^{(5)} + 2R_3^{(5)} - \psi \leq 6.2$ *bits per samples.*

Proof: Refer to Appendix D. ■

E. Achievable Scheme Corresponding to P_4 in Fig. 4 and P_4' in Fig. 8

Similarly, we can find achievable rate vectors for which $R_1 = 0$. Again from [2], we know that the quantization-and-binning scheme achieves all rate vectors $(0, R_2, R_3)$, satisfying

$$R_3 = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2\left(1-\rho_{23}^2+\rho_{23}^2 e^{-2R_2}\right)}{d}. \tag{102}$$

A particular rate vector $P_4^G = (0, R_2^{(4)}, R_3^{(4)})$ from the above sets of achievable rates is obtained by

$$R_2^{(4)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2\left(1-\rho_{23}^2\right)}{d}, \tag{103}$$

and

$$R_3^{(4)} = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2\left(1-\rho_{23}^2+\rho_{43}^2 e^{-2R_1^{(4)}}\right)}{d}. \tag{104}$$

Lemma 8 *If $1-c \geq \sqrt{1-\rho}$, then we have*

(i) $R_3^{(4)} - \phi(1,3) \leq 1$ *bits per samples.*

(ii) $R_2^{(4)} + R_3^{(4)} - \phi(1,2,3) \leq 1$ *bits per samples.*

(ii) $R_1^{(4)} + 2R_3^{(4)} - \psi \leq 5.84$ *bits per samples.*

Proof: Refer to Appendix E. ■

VI. Bounded Gap Result for $\frac{1}{2\rho} \leq c \leq \rho$

Therefore, we have the following results.

Theorem 9 *If $\frac{1}{2\rho} \leq c \leq \rho$, and $1-c \geq \sqrt{1-\rho}$, then,*

- $P_1^G = (0, 0, R_3^{(1)})$ *has a zero gap from the outer-bounds* $R_1 = 0$, $R_2 = 0$, *and* $R_1+R_2+R_3 = \phi(1,2,3)$.
- $P_2^G = (R_1^{(2)}, R_2^{(2)}, 0)$ *has a bounded gap from the outer-bounds* $R_1+R_3 = \phi(1,3)$, $R_1+R_2+2R_3 = \psi$, *and* $R_3 = 0$.
- $P_3^G = (R_1^{(3)}, R_2^{(3)}, 0)$ *has a bounded gap from the outer-bounds* $R_2+R_3 = \phi(2,3)$, $R_1+R_2+2R_3 = \psi$, *and* $R_3 = 0$.
- $P_4^G = (0, R_2^{(4)}, R_3^{(4)})$ *has a bounded gap from the outer-bounds* $R_1+R_3 = \phi(1,3)$, $R_1+R_2+R_3 = \phi(1,2,3)$, $R_1+R_2+2R_3 = \psi$, *and* $R_1 = 0$.
- $P_5^G = (R_1^{(5)}, 0, R_3^{(5)})$ *has a bounded gap from the outer-bounds* $R_2+R_3 = \phi(2,3)$, $R_1+R_2+R_3 = \phi(1,2,3)$, $R_1+R_2+2R_3 = \psi$, *and* $R_2 = 0$.

Therefore, the convex hull of the achievable rate vectors P_i^G, $i = 1,2,3,4,5$, is within a bounded gap from the outer-bound formed by the cut-set outer-bounds (33)-(39), *and also the outer-bound* (40).

Theorem 10 *If $\frac{1}{2\rho} \leq c \leq \rho$, and $1-c \leq \sqrt{1-\rho}$, then,*

- $P_1^G = (0, 0, R_3^{(1)})$ *has a bounded gap from the outer-bounds* $R_1+R_3 = \phi(1,3)$, $R_2+R_3 = \phi(2,3)$, $R_1 = 0$, *and* $R_2 = 0$.
- $P_2^G = (R_1^{(2)}, R_2^{(2)}, 0)$ *has a bounded gap from the outer-bounds* $R_1+R_3 = \phi(1,3)$, $R_2+R_3 = \phi(2,3)$, *and* $R_3 = 0$.

Therefore, the convex hull of the achievable rate vectors P_i^G, $i = 1,2$, is within a bounded gap from the outer-bound formed by the of cut-set outer-bounds (33)-(38).

VII. Rate Distortion Region for $c \leq \{\frac{1}{2\rho}, \rho\}$.

Here in this section, we assume that $c \leq \min\{\frac{1}{2\rho}, \rho\}$. Therefore,

$$c \leq \min\{\rho, \frac{1}{2\rho}\} \leq \max\min\{\rho, \frac{1}{2\rho}\} = \frac{\sqrt{2}}{2}. \tag{105}$$

Here again, we first characterize the rate region for the corresponding binary expansion model.

A. Rate-Distortion Region for The Binary-Expansion Model

1) The Binary-Expansion Model: Since the correlation between y_1 and y_2 is equal to ρ, we can write $y_1 = \rho y_2 + \sqrt{1-\rho^2} z$, where $z \sim \mathcal{N}(0,1)$ and z is independent from y_2. We define r such that

$$\rho = 1 - 2^{-2r}. \tag{106}$$

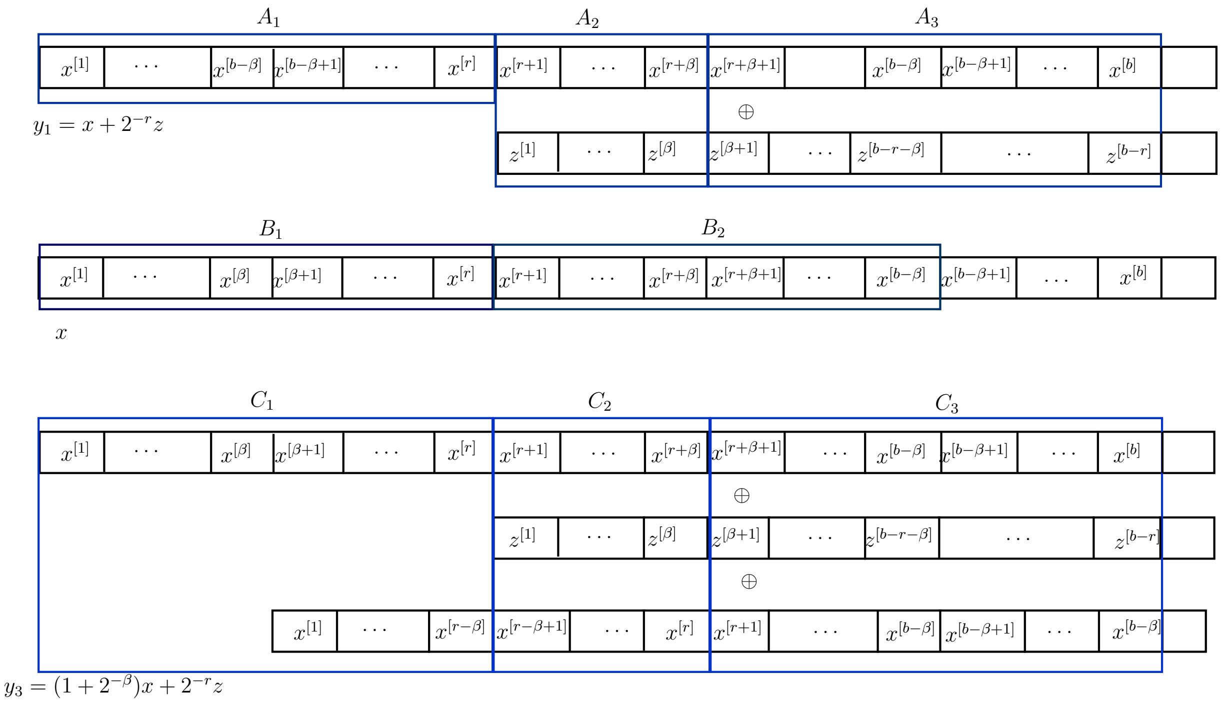

Fig. 13. Binary Expansion Model for $c \leq \{\frac{1}{2\rho}, \rho\}$

In addition, we define x as $x = \rho y_2$. Noting that $c \leq \rho$, we define β such that,

$$\frac{c}{\rho} = 2^{-\beta}. \tag{107}$$

In connection with the Gaussian problem, we introduce a binary expansion model. In this model, y_1, x and z are all uniformly distributed in $[0, 1]$, with binary expansion representations, as detailed in Section III. In addition, we replace r and β with the closest integers. The connection of y_1 and x is modeled as $y_1 = x \oplus 2^{-r}z$. In the Gaussian problem, we have

$$y_3 = y_1 - cy_2 = (y_1 - x) + (1 - \frac{c}{\rho})x. \tag{108}$$

This equation is modeled as $y_3 = 2^{-r}z \oplus x \ominus 2^{-\beta}x$ in the binary expansion model, as shown in Fig. 13.

2) Achievable Region for the Binary Expansion Model: The achievable region for the binary expansion model is shown Fig. 14, where we have

$$\bar{P}_1 = (0,\ 0,\ b)\,, \tag{109}$$

$$\bar{P}_2 = \left((b-r)^+,\ b + (b-\beta-r)^+ - (b-r)^+,\ 0\right), \tag{110}$$

$$\bar{P}_3 = \left(b,\ (b-\beta-r)^+,\ 0\right), \tag{111}$$

$$\bar{P}_4 = \left(0,\ b-(b-r)^+,\ (b-r)^+\right), \tag{112}$$

$$\bar{P}_5 = \left(b-(b-r-\beta)^+,\ 0,\ (b-r-\beta)^+\right), \tag{113}$$

$$\bar{P}_6 = \left((b-r)^+ - (b-r-\beta)^+,\ b-(b-r)^+,\ (b-r-\beta)^+\right). \tag{114}$$

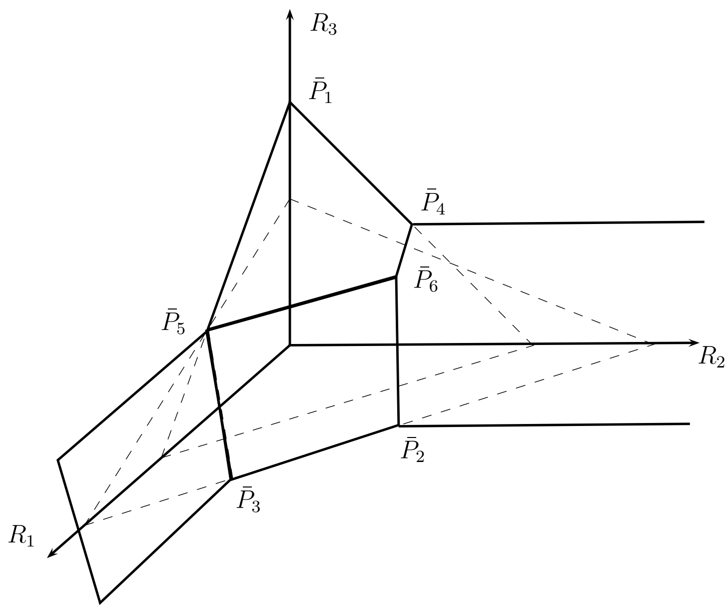

Fig. 14. Achievable Region of the Binary Expansion Model, Corresponding to the Case $c \leq \{\frac{1}{2\rho}, \rho\}$

Referring to Fig. 13, we list the schemes to achieve the corner points of Fig. 14, as follows.

- To achieve $\bar{P}_1$: encoder 3 sends C_1, C_2, and C_3.
- To achieve $\bar{P}_2$: encoder 1 sends A_2, and A_3, and encoder 2 sends B_1.
- To achieve $\bar{P}_3$: encoder 1 sends A_1, A_2, and A_3, and encoder 2 sends B_2.
- To achieve $\bar{P}_4$: encoder 2 sends B_1, encoder 3 sends C_2, and C_3 (Note that having B_1, the decoder can reconstruct C_1).
- To achieve $\bar{P}_5$: encoder 1 sends A_1 and A_2, and encoder 3 sends C_3.
- To achieve $\bar{P}_6$: encoder 1 sends A_2, encoder 2 sends B_1, and encoder 3 sends C_3.

One can observe that the concatenation of quantization and binning can achieve all the corner points as follows.

To achieve $\bar{P}_1$, $\bar{P}_4$, $\bar{P}_5$, and $\bar{P}_6$,

- form v_1 as the quantized version of y_1 with resolution $\min\{r + \beta, b\}$ bits after the radix point.
- form v_2 as the quantized version of $x = \rho y_2$ with resolution $\min\{r, b\}$ bits after the radix point.
- form v_3 as the quantized version of y_1 with resolution b bits after the radix point.
- use the distributed lossless source coding scheme (Slepian-Wolf) to report v_1, v_2, and v_3 to the decoder.

It is easy to see that $\bar{P}_1$, $\bar{P}_4$, $\bar{P}_5$, and $\bar{P}_6$ are the corner points of the above scheme.

To achieve $\bar{P}_2$, $\bar{P}_3$,

- form v_1 as the quantized version of y_1 with resolution b bits after the radix point.
- form v_2 as the quantized version of $x = \rho y_2$ with resolution $(b - \beta)^+$ bits after the radix point.
- use the distributed lossless source coding scheme (Slepian-Wolf) to report u_1 and u_2 to the decoder.

It is easy to see that $\bar{P}_2$ and $\bar{P}_3$ are the corner points of this scheme.

The rate region $\mathcal{R}_{in}$ can be characterized as union of all $(R_1, R_2, R_3) \in \mathbb{R}^3$, satisfying:

$$R_1 \geq 0, \tag{115}$$

$$R_2 \geq 0, \tag{116}$$

$$R_3 \geq 0, \tag{117}$$

$$R_1 + R_3 \geq (b - r)^+, \tag{118}$$

$$R_2 + R_3 \geq (b - r - \beta)^+, \tag{119}$$

$$R_1 + R_2 + R_3 \geq b, \tag{120}$$

$$R_1 + R_2 + 2R_3 \geq b + (b - r - \beta)^+. \tag{121}$$

3) Outer Bounds: The inequalities (115)-(120) match with the cut-set outer bounds. In addition, following the arguments of the proof of Theorem 1, we can show that (121) is tight. Therefore, $\mathcal{R}_{in}$ is indeed the rate-distortion region of the developed binary-expansion-model.

B. Achievable Scheme for the Jointly Gaussian Sources

Motivated by the results from the binary expansion model, we conjecture that using Quantization-and-Binning scheme, we can achieve within a bounded gap of the rate-distortion region. In this sub-section, corresponding to each corner point of the region, shown in Fig. 14, we introduce an achievable rate vector for the Gaussian problem, using Quantization-and-Binning scheme.

1) Achievable Scheme Corresponding to $\bar{P}_1$ in Fig. 14: We choose $\bar{P}_1^G = (0, 0, \bar{R}_3^{(1)})$ equal to P_1^G in Sub-Section V-C, i.e.

$$\bar{R}_3^{(1)} = \frac{1}{2} \log_2 \frac{\sigma_{y_3}^2}{d}. \tag{122}$$

2) Achievable Scheme Corresponding to $\bar{P}_5$ in Fig. 14: We choose $\bar{P}_5^G = (\bar{R}_1^{(5)}, 0, \bar{R}_3^{(5)})$ equal to P_5^G in Sub-Section V-D.

Lemma 11 *We have*

(i) $\bar{R}_3^{(5)} - \phi(2, 3) \leq 1$ *bits per samples.*

(ii) $\bar{R}_1^{(5)} + \bar{R}_3^{(5)} - \phi(1, 2, 3) \leq 1$ *bits per samples.*

(iii) *If* $\frac{c^2(1-\rho^2)}{d} \geq 1$, $\bar{R}_1^{(5)} + 2\bar{R}_3^{(5)} - \psi \leq 5$ *bits per samples.*

(iv) *If* $\frac{c^2(1-\rho^2)}{d} \leq 1$, $\bar{R}_3^{(5)} \leq 0.5$ *bits per samples.*

Proof: Refer to Appendix G. ■

3) Achievable Scheme Corresponding to $\bar{P}_4$ in Fig. 14: We choose $\bar{P}_4^G = (0, \bar{R}_2^{(4)}, \bar{R}_3^{(4)})$ equal to P_4^G in Sub-Section V-E.

Lemma 12 *We have*

(i) $\bar{R}_3^{(4)} - \phi(1,3) \leq 1$ *bits per samples.*

(ii) $\bar{R}_2^{(4)} + \bar{R}_3^{(4)} - \phi(1,2,3) \leq 1$ *bits per samples.*

Proof: Refer to Appendix H. ■

4) Achievable Scheme Corresponding to $\bar{P}_6$ in Fig. 14: Corresponding to $\bar{P}_6$, we introduce the achievable rate vector $\bar{P}_6^G = (\bar{R}_1^{(6)}, \bar{R}_2^{(6)}, \bar{R}_3^{(6)})$ as follows.

In this scheme, we quantize source y_j to v_j, with the quadratic distortion d_j, $j = 1, 2, 3$, where,

$$d_1 = \max\{c^2(1-\rho), d\}, \tag{123}$$

$$d_2 = \max\{1-\rho, d\}, \tag{124}$$

$$d_3 = d. \tag{125}$$

Then we use Slepian-Wolf scheme to report the quantized version of the sources to the decoder. Since v_3 will be available at the decoder, with vanishing probability of error, therefore, the decoder has y_3 with the quadratic distortion $d_3 = d$. The test channels for the quantization part are as follows:

$$v_j = \eta_j y_j + w_j, \tag{126}$$

where $w_j \sim \mathcal{N}(0, 1-\eta_j^2)$, and z_j is independent of y_j, and

$$\eta_1 = \sqrt{1-d_1}, \tag{127}$$

$$\eta_2 = \sqrt{1-d_2}, \tag{128}$$

$$\eta_3 = \sqrt{1-\frac{d_3}{\sigma_{y_3}^2}}. \tag{129}$$

Then, from Burger-Tung theorem, we can show that the following rate vector is achievable,

$$\bar{R}_3^{(6)} = I\left(y_1, y_2, y_3; u_1, u_2, u_3 | u_1, u_2\right) = I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) - I\left(y_1, y_2; u_1, u_2\right), \tag{130}$$

$$\bar{R}_1^{(6)} = I\left(y_1, y_2; u_1, u_2 | u_2\right) = I\left(y_1, y_2; u_1, u_2\right) - I\left(y_2; u_2\right), \tag{131}$$

$$\bar{R}_2^{(6)} = I\left(y_2; u_2\right), \tag{132}$$

where

$$I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) \tag{133}$$

$$= \frac{1}{2}\log_2\left(1 + \frac{1-d_1}{d_1} + \frac{1-d_2}{d_2} + \frac{\sigma_{y_3}^2 - d_3}{d_3}\right. \tag{134}$$

$$\left. + \frac{(1-\rho^2)(1-d_1)(1-d_2)}{d_1 d_2} + \frac{c^2(1-\rho^2)}{\sigma_{y_3}^2}\frac{(1-d_1)(\sigma_{y_3}^2 - d_3)}{d_1 d_3} + \frac{1-\rho^2}{\sigma_{y_3}^2}\frac{(1-d_2)(\sigma_{y_3}^2 - d_3)}{d_2 d_3}\right), \tag{135}$$

$$I\left(y_1, y_2; u_1, u_2\right) = \frac{1}{2}\log_2\left(1 + \frac{1-d_1}{d_1} + \frac{1-d_2}{d_2} + \frac{(1-\rho^2)(1-d_1)(1-d_2)}{d_1 d_2}\right), \tag{136}$$

and

$$I\left(y_2; u_2\right) = \frac{1}{2}\log_2\frac{1}{d_2}. \tag{137}$$

Lemma 13 *We have*

(i) $\bar{R}_1^{(6)} + \bar{R}_2^{(6)} + \bar{R}_3^{(6)} - \phi(1,2,3) \leq 3.28$ *bits per sample.*

(ii) $\bar{R}_1^{(6)} + \bar{R}_3^{(6)} - \phi(1,3) \leq \frac{1}{2}\log_2 7$ *bits per sample.*

(iii) *If* $\frac{c^2(1-\rho^2)}{d} \geq 1$, $\bar{R}_1^{(6)} + \bar{R}_2^{(6)} + 2\bar{R}_3^{(6)} - \psi \leq 6.82$ *bits per samples.*

(iv) *If* $\frac{c^2(1-\rho^2)}{d} \leq 1$, $\bar{R}_3^{(6)} \leq \frac{1}{2}\log_2 9$ *bits per samples.*

Proof: Refer to Appendix I ■

5) Achievable Scheme Corresponding to $\bar{P}_2$ and $\bar{P}_3$ in Fig. 14: Corresponding to $\bar{P}_2$ and $\bar{P}_3$, we introduce the achievable rate vector $\bar{P}_2^G = (\bar{R}_1^{(2)}, \bar{R}_2^{(2)}, 0)$ and $\bar{P}_2^G = (\bar{R}_1^{(3)}, \bar{R}_2^{(3)}, 0)$ as follows. Here, we quantize source y_j to v_j, with quadratic distortion d_j, $j = 1, 2$, where,

$$d_1 = \frac{d}{4}, \tag{138}$$

$$d_2 = \min\{\frac{d}{4c^2}, 1\}. \tag{139}$$

Then we use Slepian-Wolf scheme to report the quantized version of y_1 and y_2 to the decoder. It is easy to see that, with vanishing probability of error, the decoder can reconstruct y_3 with quadratic distortion $d_3 = d$. The test channels for the quantization part are as follows:

$$v_j = \eta_j y_j + w_j, \tag{140}$$

where $w_j \sim \mathcal{N}(0, 1-\eta_j^2)$, and z_j is independent of y_j, and

$$\eta_1 = \sqrt{1-d_1}, \tag{141}$$

$$\eta_2 = \sqrt{1-d_2}. \tag{142}$$

Then, we have

$$\bar{R}_1^{(2)} = I\left(y_1, y_2; u_1, u_2 | u_2\right) = I\left(y_1, y_2; u_1, u_2\right) - I\left(y_2; u_2\right), \tag{143}$$

$$\bar{R}_2^{(2)} = I\left(y_2; u_2\right), \tag{144}$$

and

$$\bar{R}_2^{(3)} = I\left(y_1, y_2; u_1, u_2 | u_1\right) = I\left(y_1, y_2; u_1, u_2\right) - I\left(y_1; u_1\right), \tag{145}$$

$$\bar{R}_1^{(3)} = I\left(y_1; u_1\right), \tag{146}$$

where

$$I\left(y_1,y_2;u_1,u_2\right)=\frac{1}{2}\log_2\left(1+\frac{1-d_1}{d_1}+\frac{1-d_2}{d_2}+\frac{(1-\rho^2)(1-d_1)(1-d_2)}{d_1d_2}\right), \tag{147}$$

$$I\left(y_2;u_2\right)=\frac{1}{2}\log_2\frac{1}{d_2}, \tag{148}$$

$$I\left(y_1;u_1\right)=\frac{1}{2}\log_2\frac{1}{d_1}. \tag{149}$$

Lemma 14 *If* $\frac{c^2(1-\rho^2)}{d}\geq 1$*, then*

(i) $\bar{R}_1^{(2)}+\bar{R}_2^{(2)}-\psi=\bar{R}_1^{(3)}+\bar{R}_2^{(3)}-\psi\leq 5.66$ *bits per sample.*

(ii) $\bar{R}_1^{(2)}-\phi(1,3)\leq 2.16$ *bits per sample.*

(ii) $\bar{R}_2^{(3)}-\phi(2,3)\leq 2.16$ *bits per sample.*

Proof: Refer to Appendix J. ∎

C. Bounded Gap Result

From the results detailed in the pervious subsection, we have the following conclusions.

Theorem 15 *If* $c\leq\min\{\rho,\frac{1}{2\rho}\}$*, and* $\frac{c^2(1-\rho^2)}{d}\geq 1$*, then,*

- $\bar{P}_1^G=(0,0,\bar{R}_3^{(1)})$ *has zero gap from the outer-bounds* $R_1=0$*,* $R_2=0$*, and* $R_1+R_2+R_3=\phi(1,2,3)$*.*
- $\bar{P}_2^G=(\bar{R}_1^{(2)},\bar{R}_2^{(2)},0)$ *has a bounded gap from the outer-bounds* $R_1+R_3=\phi(1,3)$*,* $R_1+R_2+2R_3=\psi$*, and* $R_3=0$*.*
- $\bar{P}_3^G=(\bar{R}_1^{(3)},\bar{R}_2^{(3)},0)$ *has a bounded gap from the outer-bounds* $R_2+R_3=\phi(2,3)$*,* $R_1+R_2+2R_3=\psi$*, and* $R_3=0$*.*
- $\bar{P}_4^G=(0,\bar{R}_2^{(4)},\bar{R}_3^{(4)})$ *has a bounded gap from the outer-bounds* $R_1+R_3=\phi(1,3)$*,* $R_1+R_2+R_3=\phi(1,2,3)$*, and* $R_1=0$*.*
- $\bar{P}_5^G=(\bar{R}_1^{(5)},0,\bar{R}_3^{(5)})$ *has a bounded gap from the outer-bounds* $R_2+R_3=\phi(2,3)$*,* $R_1+R_2+R_3=\phi(1,2,3)$*,* $R_1+R_2+2R_3=\psi$*, and* $R_2=0$*.*
- $\bar{P}_6^G=(\bar{R}_1^{(6)},\bar{R}_2^{(6)},\bar{R}_3^{(6})$ *has a bounded gap from the outer-bounds* $R_1+R_3=\phi(1,3)$*,* $R_1+R_2+R_3=\phi(1,2,3)$*,* $R_1+R_2+2R_3=\psi$*.*

Therefore, the convex hull of the achievable rate vectors $\bar{P}_i^G$*,* $i=1,\ldots,6$*, is within a bounded gap from the outer-bounds formed by the cut-set outer-bounds* (33)-(39)*, and also the outer-bound* (40)*.*

Theorem 16 *If* $c\leq\min\{\rho,\frac{1}{2\rho}\}$*, and* $\frac{c^2(1-\rho^2)}{d}\leq 1$*, then,*

- $\bar{P}_1^G=(0,0,\bar{R}_3^{(1)})$ *has a zero gap from the outer-bounds* $R_1=0$*,* $R_2=0$*, and* $R_1+R_2+R_3=\phi(1,2,3)$*.*

- $\bar{P}_4^G = (0, \bar{R}_2^{(4)}, \bar{R}_3^{(4)})$ *has a bounded gap from the outer-bounds* $R_1 + R_3 = \phi(1,3)$, $R_1 + R_2 + R_3 = \phi(1,2,3)$, *and* $R_1 = 0$.
- $\bar{P}_5^G = (\bar{R}_1^{(5)}, 0, \bar{R}_3^{(5)})$ *has a bounded gap from the outer-bounds* $R_1 + R_2 + R_3 = \phi(1,2,3)$, $R_2 = 0$, *and* $R_3 = 0$.
- $\bar{P}_6^G = (\bar{R}_1^{(6)}, \bar{R}_2^{(6)}, \bar{R}_3^{(6})$ *has a bounded gap from the outer-bounds* $R_1 + R_3 = \phi(1,3)$, $R_1 + R_2 + R_3 = \phi(1,2,3)$, *and* $R_3 = 0$.

Therefore, the convex hull of the achievable rate vectors $\bar{P}_i^G$, $i = 1,4,5,6$, *is within a bounded gap from the outer-bound formed by the cut-set outer-bounds* (33)-(39).

VIII. Rate Distortion Region for $\rho \leq c$.

Since this case is well-understood, we directly explain the result for the Gaussian sources. Here we consider two cases, where $\rho \leq \frac{1}{2}$ and $\rho \geq \frac{1}{2}$.

A. Case One: $\rho \geq \frac{1}{2}$

Obviously in this case, the rate vector $\hat{P}_1^G = (0, 0, \hat{R}_3^{(1)})$ is achievable, where,

$$\hat{R}_3^{(1)} = \frac{1}{2} \log_2 \frac{\sigma_{y_3}^2}{d}. \tag{150}$$

In addition, in [7], it is shown that $\hat{P}_2^G = (\hat{R}_1^{(2)}, \hat{R}_2^{(2)}, 0)$ is achievable, where

$$\hat{R}_1^{(2)} = \hat{R}_2^{(2)} = \frac{1}{2} \log_2 \frac{2\sigma_{y_3}^2}{d}. \tag{151}$$

Lemma 17 *If* $\rho \geq \frac{1}{2}$

(i) $\hat{R}_3^{(1)} - \phi(1,3) \leq \frac{1}{2} \log_2 \frac{4}{3}$ *bits per sample.*

(ii) $\hat{R}_3^{(1)} - \phi(2,3) \leq 1$ *bits per sample.*

(iii) $\hat{R}_1^{(2)} - \phi(1,3) \leq \frac{1}{2} \log_2 \frac{8}{3}$ *bits per sample.*

(iv) $\hat{R}_2^{(2)} - \phi(2,3) \leq 1.5$ *bits per sample.*

Proof: Please refer to Appendix K. ■

Therefore, refereing to Fig. 15, we have,

Theorem 18 *If* $c \geq \rho$ *and* $\rho \geq \frac{1}{2}$, *then*

- $\hat{P}_1^G$ *has a bounded gap from the outer-bounds* $R_1 + R_3 = \phi(1,3)$, $R_2 + R_3 = \phi(2,3)$, $R_1 = 0$, *and* $R_2 = 0$.
- $\hat{P}_2^G$ *has a bounded gap from the outer-bounds* $R_1 + R_3 = \phi(1,3)$, $R_2 + R_3 = \phi(2,3)$, *and* $R_3 = 0$.

Therefore, the convex hull of the achievable rate vectors $\hat{P}_i$, $i = 1,2$, *is within a bounded gap from the outer-bound formed by the cut-set outer-bounds* (33)-(38).

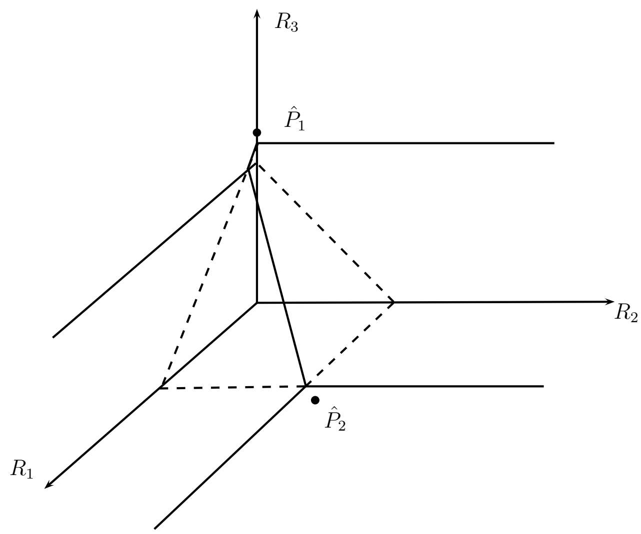

Fig. 15. Outer-Bounds for Case $\rho \leq c$ and $\rho \geq 0.5$

B. Case Two: $\rho \leq \frac{1}{2}$

Ignoring the advantage of the correlation between y_1 and y_2, one can easy see that $\widetilde{P}_2^G = (\widetilde{R}_1^{(2)}, \widetilde{R}_2^{(2)}, 0)$ is achievable, where

$$\widetilde{R}_1^{(2)} = \frac{1}{2}\log_2 \frac{4}{d}, \tag{152}$$

$$\widetilde{R}_2^{(2)} = \frac{1}{2}\log_2^+ \frac{4c^2}{d}. \tag{153}$$

Lemma 19 *If* $\rho \leq \frac{1}{2}$

(i) $\widetilde{R}_1^{(2)} - \phi(1,3) \leq \frac{1}{2}\log_2 \frac{16}{3}$ *bits per sample.*

(ii) $\widetilde{R}_2^{(2)} - \phi(2,3) \leq \frac{1}{2}\log_2 \frac{16}{3}$ *bits per sample.*

Proof: Directly follows. ■

In addition, we consider the rate vector $\widetilde{P}_5^G = (\widetilde{R}_1^{(5)}, 0, \widetilde{R}_2^{(5)})$, the same as P_5^G in Subsection V-D.

Lemma 20 *If* $\rho \leq \frac{1}{2}$*, then we have*

(i) $\widetilde{R}_3^{(5)} - \phi(2,3) \leq 1$ *bits per samples.*

(ii) $\widetilde{R}_1^{(5)} + \widetilde{R}_3^{(5)} - \phi(1,3) \leq 1$ *bits per samples.*

Proof: Refer to Appendix L. ■

In addition, consider the achievable rate vector $\tilde{P}_1^G = (0, 0, \widetilde{R}_3^{(1)})$, where

$$\widetilde{R}_3^{(1)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d}. \tag{154}$$

Lemma 21 *If* $\rho \leq \frac{1}{2}$

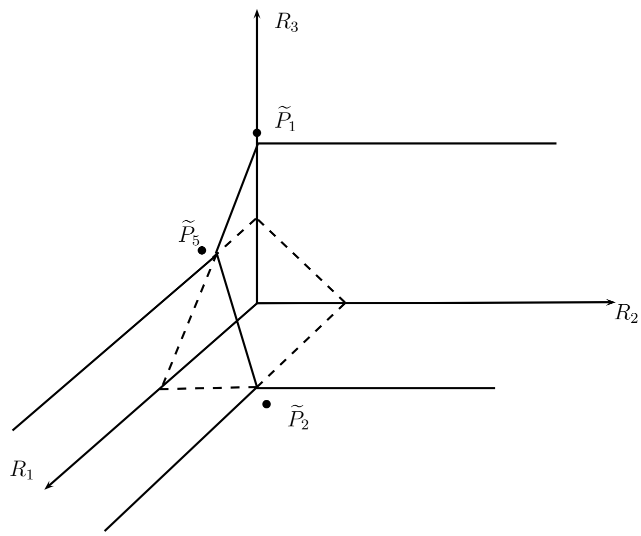

Fig. 16. Outer-Bounds for Case $\rho \leq c$ and $\rho \leq 0.5$

(i) $\widetilde{R}_3^{(1)} - \phi(1,3) \leq \frac{1}{2}$ *bits per sample.*

Proof:

$$\begin{aligned}
\widetilde{R}_3^{(1)} &- \phi(1,3) \\
&\leq \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{(1-\rho^2)}{d} \\
&\leq \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{(1-\rho^2)} \\
&\overset{(a)}{\leq} \frac{1}{2}\log_2 \frac{(1+c)}{1+\rho} \leq 0.5,
\end{aligned}$$

■

Then, refereing to Fig. 16, we have,

Theorem 22 *If* $c \geq \rho$ *and* $\rho \leq \frac{1}{2}$, *then*

- $\tilde{P}_1^G$ *has a bounded gap from the outer-bounds* $R_1 + R_3 = \phi(1,3)$, $R_1 = 0$, *and* $R_2 = 0$.
- $\tilde{P}_5^G$ *has a bounded gap from the outer-bounds* $R_1 + R_3 = \phi(1,3)$, $R_2 + R_3 = \phi(2,3)$, *and* $R_2 = 0$.
- $\tilde{P}_2^G$ *has a bounded gap from the outer-bounds* $R_1 + R_3 = \phi(1,3)$, $R_2 + R_3 = \phi(2,3)$, *and* $R_3 = 0$.

Therefore, the convex hull of the achievable rate vectors $\tilde{P}_i^G$, $i = 1,2,3$, *is within a bounded gap from the outer-bound formed by the cut-set outer-bounds* (33), (34), (34),(37), *and* (38).

Appendix A
Proof of Lemma 3

Part (i):

If $d \leq 1-\rho^2$,

$$\begin{aligned} R_1^{(2)} - \phi(1,3) =& \frac{1}{2}\log_2^+ \frac{2\sigma_{y_3}^2 d + 6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d} + \log_2 3 - \frac{1}{2}\log_2 \frac{1-\rho^2}{d} \\ \leq& \frac{1}{2}\log_2^+ \frac{2\sigma_{y_3}^2 (1-\rho^2) + 6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d} + \log_2 3 - \frac{1}{2}\log_2 \frac{1-\rho^2}{d} \\ \leq& \frac{1}{2}\log_2 8 + \log_2 3. \end{aligned}$$

On the other hand, if $d \geq 1-\rho^2$,

$$\begin{aligned} R_1^{(2)} - \phi(1,3) =& \frac{1}{2}\log_2^+ \frac{2\sigma_{y_3}^2 d + 6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d} + \log_2 3 \\ & \frac{1}{2}\log_2^+ \frac{2\sigma_{y_3}^2 d + 6d(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d} + \log_2 3 \\ \leq& \frac{1}{2}\log_2 8 + \log_2 3. \end{aligned}$$

Part (ii):

Since we assume that $1-c \leq \sqrt{1-\rho}$, then it is easy to see that

$$\sigma_{y_3}^2 = 1 + c^2 - 2c\rho = (1-c)^2 + 2c(1-\rho) \leq (1-\rho)(1+2c). \tag{155}$$

Then, we have

$$\begin{aligned} R_2^{(2)} - \phi(2,3) =& \frac{1}{2}\log_2^+ \frac{3\sigma_{y_3}^2}{d} + \log_2 3 - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho^2)}{d} && (156) \\ \leq& \frac{1}{2}\log_2 \frac{3\sigma_{y_3}^2}{c^2(1-\rho^2)} + \log_2 3 && (157) \\ \stackrel{(a)}{\leq}& \frac{1}{2}\log_2 \frac{3(1+2c)}{c^2(1+\rho)} + \log_2 3 && (158) \\ \stackrel{(b)}{\leq}& \frac{1}{2}\log_2 \frac{3(1+2c)}{c^2+0.5c} + \log_2 3, && (159) \\ \stackrel{(c)}{\leq}& \frac{1}{2}\log_2 12 + \log_2 3, && (160) \end{aligned}$$

where (a) follows from (155) and (b) relies on the fact that $c\rho \geq 0.5$, and (c) is based on $c \geq \frac{0.5}{\rho} \geq 0.5$.

Part (iii): Using the proof of part (i) it is easy to see that

$$R_1^{(2)} - \frac{1}{2}\log_2^+ \frac{1-\rho}{d} \leq \frac{1}{2}\log_2 8 + \log_2 3.$$

If $d \leq c^2(1-\rho)$, then

$$\begin{aligned}
& R_2^{(2)} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \\
&= \log_2 3 + \frac{1}{2}\log_2^+ \frac{3\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2 \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \\
&\leq \log_2 3 + \frac{1}{2}\log_2 \frac{3\sigma_{y_3}^2}{c^2(1-\rho)} - \frac{1}{2}\log_2 \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \\
&\overset{(a)}{\leq} \log_2 3 + \frac{1}{2}\log_2 \frac{3\sigma_{y_3}^2}{c^2(1-\rho)} - \frac{1}{2}\log_2 \frac{\rho(1-c)^2}{(c\sqrt{1-\rho}+\sqrt{(1-\rho)(1+c^2)})^2} \\
&= \log_2 3 + \frac{1}{2}\log_2 \frac{3\sigma_{y_3}^2}{c^2} - \frac{1}{2}\log_2 \frac{\rho(1-c)^2}{(c+\sqrt{1+c^2})^2} \\
&\overset{(b)}{\leq} \log_2 3 + \frac{1}{2}\log_2 \frac{9(c+\sqrt{1+c^2})^2}{c^2\rho} \\
&\overset{(c)}{\leq} \log_2 3 + \frac{1}{2}\log_2 18(1+\sqrt{5})^2,
\end{aligned}$$

where (a) relies on assumption $d \leq c^2(1-\rho)$, and (b) is based on $1-c \geq \sqrt{1-\rho}$ and

$$\frac{\sigma_{y_3}^2}{(1-c)^2} = \frac{1+c^2-2c\rho}{(1-c)^2} = 1 + \frac{2c(1-\rho)}{(1-c)^2} \leq 1+2c \leq 3. \tag{161}$$

Moreover, (c) follows from the fact that $c \geq \frac{1}{2\rho} \geq \frac{1}{2}$, and $\rho \geq \frac{1}{2c} \geq \frac{1}{2}$.

Otherwise, if $d \geq c^2(1-\rho)$, then

$$R_2^{(2)} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{162}$$

$$= \log_2 3 + \frac{1}{2}\log_2^+ \frac{3\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2 \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{163}$$

$$= \log_2 3 + \frac{1}{2}\log_2 \left(\frac{3\sigma_{y_3}^2}{\rho(1-c)^2} \frac{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{d} \right) \tag{164}$$

$$\overset{(a)}{\leq} \log_2 3 + \frac{1}{2}\log_2 \left(18 \left(1 + \sqrt{\frac{(1-\rho)(1+c^2)}{d}} \right)^2 \right) \tag{165}$$

$$\overset{(b)}{\leq} \log_2 3 + \frac{1}{2}\log_2 \left(18 \left(1 + \sqrt{\frac{1+c^2}{c^2}} \right)^2 \right) \tag{166}$$

$$\overset{(c)}{\leq} \log_2 3 + \frac{1}{2}\log_2 18(1+\sqrt{5})^2, \tag{167}$$

where (a) is based on $\rho \geq \frac{1}{2c} \geq \frac{1}{2}$ and (161), (b) relies on $d \geq c^2(1-\rho)$, and (c) is based on the fact that $c \geq \frac{1}{2\rho} \geq \frac{1}{2}$.

Appendix B
Proof of Lemma 5

Part (i):

$$R_{22}^{(3)} - \phi(2,3) \tag{168}$$
$$=1 + \frac{1}{2}\log_2(2 + \frac{6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d}) - \log_2^+ \frac{c^2(1-\rho^2)}{d} \tag{169}$$
$$=1 + \frac{1}{2}\log_2\left(2 + \frac{6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d}\min\{1, \frac{d}{c^2(1-\rho^2)}\}\right) \tag{170}$$
$$=1 + \frac{1}{2}\log_2\left(2 + \frac{6(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 c^2}\right) \tag{171}$$
$$\leq 1 + \frac{1}{2}\log_2\left(2 + \frac{6}{c^2}\right) \overset{(a)}{\leq} 1 + \frac{1}{2}\log_2(2+12) = 1 + \frac{1}{2}\log_2 28, \tag{172}$$

where (a) is based on $c \geq \frac{1}{2\rho} \geq \frac{1}{2}$. Therefore,

$$R_2^{(3)} - \phi(2,3) = R_{21} + R_{22} + R_{23} - \phi(2,3) = \log_2 3 + 3 + \frac{1}{2}\log_2(28). \tag{173}$$

Part(ii):

Similar to part (i), it is easy to show that

$$R_{22}^{(3)} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} \tag{174}$$
$$=1 + \frac{1}{2}\log_2(2 + \frac{6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d}) - \log_2^+ \frac{c^2(1-\rho)}{d} \tag{175}$$
$$=1 + \frac{1}{2}\log_2\left(2 + \frac{6(1-\rho)(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 d}\min\{1, \frac{d}{c^2(1-\rho)}\}\right) \tag{176}$$
$$=1 + \frac{1}{2}\log_2\left(2 + \frac{6(\sigma_{y_3}^2 - d)}{\sigma_{y_3}^2 c^2}\right) \tag{177}$$
$$\leq 1 + \frac{1}{2}\log_2\left(2 + \frac{6}{c^2}\right) \leq 1 + \frac{1}{2}\log_2(2+24) = 1 + \frac{1}{2}\log_2 26. \tag{178}$$

Similarly, we have

$$R_{11}^{(3)}-\frac{1}{2}\log_2^+\frac{1-\rho}{d} \tag{179}$$

$$=\frac{1}{2}\log_2(2+\frac{6(1-\rho)(\sigma_{y_3}^2-d)}{\sigma_{y_3}^2 d})-\frac{1}{2}\log_2^+\frac{1-\rho}{d} \tag{180}$$

$$=\frac{1}{2}\log_2\left(2+\frac{6(1-\rho)(\sigma_{y_3}^2-d)}{\sigma_{y_3}^2 d}\min\{1,\frac{d}{1-\rho}\}\right) \tag{181}$$

$$=\frac{1}{2}\log_2\left(2+\frac{6(\sigma_{y_3}^2-d)}{\sigma_{y_3}^2}\right) \tag{182}$$

$$\leq\frac{1}{2}\log_2(2+6)\leq\frac{1}{2}\log_2 8. \tag{183}$$

From (79) and (86), we have,

$$R_{13}^{(3)}=\frac{1}{2}\log_2\frac{\sigma_{\Lambda_{\hat{C}}}^2}{\sigma_{\Lambda_M}^2}=\max\left\{\frac{1}{2}\log_2\frac{(\rho-c+\sigma_{\Lambda_M})^2}{\sigma_{\Lambda_M}^2},\frac{1}{2}\log_2\frac{\sigma_{\Lambda_C}^2}{\sigma_{\Lambda_M}^2}\right\}. \tag{184}$$

On the other hand, we have

$$\frac{1}{2}\log_2\frac{(\rho-c+\sigma_{\Lambda_M})^2}{\sigma_{\Lambda_M}^2}-\frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{185}$$

$$=\log_2\frac{\rho-c+\sigma_{\Lambda_M}}{\sigma_{\Lambda_M}}-\log_2^+\frac{\sqrt{\rho}(1-c)}{\sqrt{d}+\sqrt{(1-\rho)(1+c^2)}} \tag{186}$$

$$\leq\log_2\left(1+\frac{\rho-c}{\sqrt{2(1-\rho)+2q}}\frac{\sqrt{d}+\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\rho}(1-c)}\right) \tag{187}$$

$$\overset{(a)}{\leq}\log_2\left(1+\frac{1}{\sqrt{2(1-\rho)+2q}}\frac{\sqrt{d}+\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\rho}}\right) \tag{188}$$

$$\overset{(b)}{\leq}\log_2\left(1+\frac{1}{\sqrt{2(1-\rho)+\frac{2d}{3}}}\frac{\sqrt{d}+\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\rho}}\right) \tag{189}$$

$$=\log_2\left(1+\frac{\sqrt{d}}{\sqrt{\rho}\sqrt{2(1-\rho)+\frac{2d}{3}}}+\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\rho}\sqrt{2(1-\rho)+\frac{2d}{3}}}\right) \tag{190}$$

$$\overset{(c)}{\leq}\log_2\left(1+\sqrt{3}+\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\rho}\sqrt{2(1-\rho)}}\right) \tag{191}$$

$$=\log_2\left(1+\sqrt{3}+\frac{\sqrt{(1+c^2)}}{\sqrt{2\rho}}\right) \tag{192}$$

$$\overset{(d)}{\leq}\log_2\left(1+\sqrt{3}+\frac{\sqrt{(1+c^2)}}{\sqrt{2c}}\right)=\log_2\left(1+\sqrt{3}+\frac{\sqrt{5}}{2}\right), \tag{193}$$

where (a) is based on $\rho-c\leq 1-c$, (b) follows from $\frac{\sigma_{y_3}^2 d}{3(\sigma_{y_3}^2-d)}\geq\frac{d}{3}$, (c) is based on $\rho\geq\frac{1}{2c}\geq\frac{1}{2}$, and (d) is due to $c\leq\rho$.

In addition, we have

$$\frac{1}{2}\log_2\frac{\sigma^2_{\Lambda_C}}{\sigma^2_{\Lambda_M}}-\frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{194}$$

$$=\frac{1}{2}\log_2\frac{\sigma^2_{y_3}+3q}{2(1-\rho)+2q}-\frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{195}$$

$$\overset{(a)}{\leq}\frac{1}{2}\log_2\frac{\sigma^4_{y_3}}{(2(1-\rho))(\sigma^2_{y_3}-d)+\frac{2}{3}d\sigma^2_{y_3}}-\frac{1}{2}\log_2\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{196}$$

$$=\frac{1}{2}\log_2\left(\frac{\sigma^4_{y_3}}{(2(1-\rho))(\sigma^2_{y_3}-d)+\frac{2}{3}d\sigma^2_{y_3}}\frac{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{\rho(1-c)^2}\right) \tag{197}$$

$$\overset{(b)}{\leq}\frac{1}{2}\log_2\left(\frac{(1+2c)\sigma^2_{y_3}}{(2(1-\rho))(\sigma^2_{y_3}-d)+\frac{2}{3}d\sigma^2_{y_3}}\frac{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{\rho}\right) \tag{198}$$

$$=\frac{1}{2}\log_2\left(\frac{(1+2c)}{(2(1-\rho))(1-\frac{d}{\sigma^2_{y_3}})+\frac{2}{3}d}\frac{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{\rho}\right) \tag{199}$$

$$\overset{(c)}{\leq}\log_2\left(2\frac{\sqrt{d}+\sqrt{(1-\rho)(1+c^2)}}{\sqrt{(2(1-\rho))(1-\frac{d}{\sigma^2_{y_3}})+\frac{2}{3}d}}\right) \tag{200}$$

$$=\log_2\left(2\frac{\sqrt{d}}{\sqrt{(2(1-\rho))(1-\frac{d}{\sigma^2_{y_3}})+\frac{2}{3}d}}+2\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{(2(1-\rho))(1-\frac{d}{\sigma^2_{y_3}})+\frac{2}{3}d}}\right) \tag{201}$$

$$=\log_2\left(\sqrt{6}+2\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{(2(1-\rho))(1-\frac{d}{\sigma^2_{y_3}})+\frac{2}{3}d}}\right), \tag{202}$$

where (a) is due to the particular choice of q in (73), (b) follows from the assumption that $1-c\geq\sqrt{1-\rho}$, and

$$\sigma^2_{y_3}=1+c^2-2c\rho=(1-c)^2+2c(1-\rho)\geq(1-\rho)+2c(1-\rho)=(1-\rho)(1+2c). \tag{203}$$

Inequality (c) is based on $\frac{1+2c}{\rho}\geq\frac{1+2c}{c}\geq 4$, as $\rho\geq c\geq\frac{1}{2\rho}\geq\frac{1}{2}$.

Here we consider two cases where $d\leq\frac{\sigma^2_{y_3}}{2}$, or $d>\frac{\sigma^2_{y_3}}{2}$. If $d\leq\frac{1}{2}\sigma^2_{y_3}$, then $1-\frac{d}{\sigma^2_{y_3}}\geq\frac{1}{2}$, then we have,

$$\text{R.H.S (202)}\leq\log_2\left(\sqrt{6}+2\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{1-\rho+\frac{2}{3}d}}\right) \tag{204}$$

$$\leq\log_2\left(\sqrt{6}+2\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{1-\rho}}\right) \tag{205}$$

$$\leq\log_2\left(\sqrt{6}+2(1+c^2)\right) \tag{206}$$

$$\overset{(a)}{\leq}\log_2\left(\sqrt{2}(\sqrt{3}+2)\right), \tag{207}$$

where (a) is based on $\frac{1+c^2}{1+c}\leq 1$, for $c\leq 1$.

Otherwise, if $d > \frac{1}{2}\sigma_{y_3}^2$, then

$$\text{R.H.S (202)} \leq \log_2\left(\sqrt{6} + 2\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\frac{2}{3}d}}\right) \tag{208}$$

$$\leq \log_2\left(\sqrt{6} + 2\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\frac{1}{3}\sigma_{y_3}^2}}\right) \tag{209}$$

$$\overset{(a)}{\leq} \log_2\left(\sqrt{6} + 2\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\frac{1}{3}(1-\rho)(1+2c)}}\right) \tag{210}$$

$$\overset{(b)}{\leq} \log_2\left(\sqrt{6} + 2\sqrt{2}\right), \tag{211}$$

where (a) follows from (203), and (b) is based on $\frac{\sqrt{(1-\rho)(1+c^2)}}{\sqrt{\frac{1}{3}(1-\rho)(1+2c)}} \leq \sqrt{2}$. for $0.5 \leq c \leq 1$.

Therefore, we have

$$R_1^{(3)} + R_2^{(3)} - \psi \tag{212}$$

$$\leq 3 + \frac{1}{2}\log_2 26 + \frac{1}{2}\log_2 8 + \max\left\{\log_2\left(1 + \sqrt{3} + \sqrt{\frac{5}{4}}\right), \log_2\left(\sqrt{6} + 2\sqrt{2}\right)\right\} + 2\log_2 3 \tag{213}$$

$$= 12.42 \text{ bits per sample.} \tag{214}$$

Appendix C
Proof of Lemma 6

We note that if $1 - c \leq \sqrt{1-\rho}$, then $\sigma_{y_3}^2 = 1 + c^2 - 2c\rho = (1-c)^2 + 2c(1-\rho) \leq (1-\rho)(1+2c)$. Using this inequality, we can prove both parts, as follows.

Part (i):

We have

$$R_3^{(1)} - \phi(1,3) = \frac{1}{2}\log_2\frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+\frac{1-\rho^2}{d} \tag{215}$$

$$\leq \frac{1}{2}\log_2\frac{(1-\rho)(1+2c)}{d} - \frac{1}{2}\log_2^+\frac{1-\rho^2}{d} \tag{216}$$

$$\leq \frac{1}{2}\log_2\frac{(1+2c)}{1+\rho} \leq \frac{1}{2}\log_2 1.5, \tag{217}$$

where we use the fact that $0.5 \leq \frac{1}{2\rho} \leq c \leq \rho$.

Part (ii): We have

$$R_3^{(1)} - \phi(1,3) = \frac{1}{2}\log_2\frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+\frac{c^2(1-\rho^2)}{d} \tag{218}$$

$$\leq \frac{1}{2}\log_2\frac{(1-\rho)(1+2c)}{d} - \frac{1}{2}\log_2^+\frac{c^2(1-\rho^2)}{d} \tag{219}$$

$$\leq \frac{1}{2}\log_2\frac{(1+2c)}{c^2(1+\rho)} \leq \frac{1}{2}\log_2\frac{16}{3}, \tag{220}$$

where we use the fact that $0.5 \leq \frac{1}{2\rho} \leq c \leq \rho$.

Appendix D
Proof of Lemma 7

If $d \leq \sigma_{y_3}^2(1-\rho_{13}^2)$, then

$$R_1^{(5)} = \frac{1}{2}\log_2 \frac{1}{1-\rho_{13}^2}, \tag{221}$$

$$R_3^{(5)} = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^4)}{d}, \tag{222}$$

otherwise if $d \geq \sigma_{y_3}^2(1-\rho_{13}^2)$, then

$$R_1^{(5)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d}, \tag{223}$$

$$R_3^{(5)} = \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} + \rho_{13}^2\right). \tag{224}$$

Part (i):

It is easy to see that

$$\phi(2,3) = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d}. \tag{225}$$

Therefore, $d \leq \sigma_{y_3}^2(1-\rho_{13}^2)$,

$$R_3^{(5)} - \phi(2,3) = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^4)}{d} - \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} \leq \frac{1}{2}\log_2^+(1+\rho_{13}^2) \leq 1, \tag{226}$$

otherwise if $d \geq \sigma_{y_3}^2(1-\rho_{13}^2)$, then

$$R_3^{(5)} - \phi(2,3) = \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} + \rho_{13}^2\right) - \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} \tag{227}$$

$$= \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} + \rho_{13}^2\right) \leq \frac{1}{2}\log_2\left(1+\rho_{13}^2\right) \leq 1. \tag{228}$$

Part (ii):

It is easy to see that

$$\phi(1,2,3) = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2}{d}. \tag{229}$$

If $d \leq \sigma_{y_3}^2(1-\rho_{13}^2)$, then

$$\begin{aligned} R_1^{(5)} + R_3^{(5)} - &\phi(1,2,3) \\ &= \frac{1}{2}\log_2 \frac{1}{1-\rho_{13}^2} + \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^4)}{d} - \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} \leq \frac{1}{2}\log_2(1+\rho_{13}^2) \leq 1. \end{aligned}$$

Otherwise if $d \geq \sigma_{y_3}^2(1-\rho_{13}^2)$, then

$$R_1^{(5)} + R_3^{(5)} - \phi(1,2,3)$$
$$= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} + \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} + \rho_{13}^2 \right) - \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} = \frac{1}{2}\log_2^+ \left(1+\rho_{13}^2\right) \leq 1.$$

Part (iii):

From (40), we have,

$$\psi = \frac{1}{2}\log_2^+ \frac{1-\rho}{d} + \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} + \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}. \tag{230}$$

Then, we have

$$\frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d(1+\rho)}. \tag{231}$$

In addition,

$$\frac{1}{2}\log_2^+ \frac{(1-\rho)}{d} = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{dc^2(1+\rho)}. \tag{232}$$

Then, using the proof of part (i), it is easy to see that

$$R_3^{(5)} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} \leq 2, \tag{233}$$

and

$$R_3^{(5)} - \frac{1}{2}\log_2^+ \frac{(1-\rho)}{d} \leq 2. \tag{234}$$

Moreover, if $d \leq \sigma_{y_3}^2(1-\rho_{13}^2) = c^2(1-\rho^2)$, then

$$R_1^{(5)} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{235}$$
$$= \frac{1}{2}\log_2^+ \frac{1}{1-\rho_{13}^2} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{236}$$
$$= \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2}{c^2(1-\rho^2)} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{237}$$
$$\overset{(a)}{\leq} \frac{1}{2}\log_2^+ \frac{(1+2c)(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{c^2\rho(1-\rho^2)} \tag{238}$$
$$\overset{(b)}{\leq} \frac{1}{2}\log_2^+ \frac{(1+2c)(\sqrt{c^2(1-\rho^2)}+\sqrt{(1-\rho)(1+c^2)})^2}{c^2\rho(1-\rho^2)} \tag{239}$$
$$= \frac{1}{2}\log_2^+ \frac{(1+2c)(\sqrt{c^2(1+\rho)}+\sqrt{1+c^2})^2}{c^2\rho(1+\rho)} \overset{(c)}{\leq} 2.2, \tag{240}$$

where (a) is based on (161), (b) follows from the assumption $d \leq \sigma_{y_3}^2(1-\rho_{13}^2) = c^2(1-\rho^2)$, and (c) relies on $0.5 \leq \frac{1}{2\rho} \leq c \leq \rho$.

Moreover, if $d \geq \sigma_{y_3}^2 (1-\rho_{13}^2) = c^2(1-\rho^2)$, then

$$R_1^{(5)} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{241}$$

$$= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{242}$$

$$\stackrel{(a)}{\leq} \frac{1}{2}\log_2 \frac{(1+2c)(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{d\rho} \tag{243}$$

$$\stackrel{(b)}{\leq} \frac{1}{2}\log_2 \left(\frac{(1+2c)}{\rho} \left[1 + \sqrt{\frac{1+c^2}{c^2(1+\rho)}} \right]^2 \right) \leq 2.2, \tag{244}$$

where (a) is based on (161), (b) follows from the assumption $d \leq \sigma_{y_3}^2 (1-\rho_{13}^2) = c^2(1-\rho^2)$, and (c) relies on $0.5 \leq \frac{1}{2\rho} \leq c \leq \rho$.

APPENDIX E
PROOF OF LEMMA 8

If $d \leq \sigma_{y_3}^2 (1-\rho_{13}^2)$, then

$$R_1^{(4)} = \frac{1}{2}\log_2 \frac{1}{1-\rho_{23}^2}, \tag{245}$$

$$R_3^{(4)} = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2 (1-\rho_{23}^4)}{d}, \tag{246}$$

otherwise if $d \geq \sigma_{y_3}^2 (1-\rho_{23}^2)$, then

$$R_1^{(4)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d}, \tag{247}$$

$$R_3^{(4)} = \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2 (1-\rho_{23}^2)}{d} + \rho_{23}^2 \right). \tag{248}$$

In addition, it is easy to show,

$$\phi(1,3) = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2 (1-\rho_{13}^2)}{d}. \tag{249}$$

Then, similar to the proof of Lemma 7, we can prove part (i) and part (ii). In addition, we can show that

$$R_3^{(4)} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} \leq 2, \tag{250}$$

and

$$R_3^{(4)} - \frac{1}{2}\log_2^+ \frac{(1-\rho)}{d} \leq 2. \tag{251}$$

Moreover, if $d \leq \sigma_{y_3}^2\left(1-\rho_{23}^2\right) = 1-\rho^2$, then

$$R_1^{(4)} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{252}$$

$$= \frac{1}{2}\log_2^+ \frac{1}{1-\rho_{13}^2} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{253}$$

$$= \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2}{1-\rho^2} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{254}$$

$$\stackrel{(a)}{\leq} \frac{1}{2}\log_2^+ \frac{(1+2c)(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{\rho(1-\rho^2)} \tag{255}$$

$$\stackrel{(b)}{\leq} \frac{1}{2}\log_2^+ \frac{(1+2c)(\sqrt{1-\rho^2}+\sqrt{(1-\rho)(1+c^2)})^2}{\rho(1-\rho^2)} \tag{256}$$

$$= \frac{1}{2}\log_2^+ \frac{(1+2c)(\sqrt{1+\rho}+\sqrt{1+c^2})^2}{\rho(1+\rho)} \stackrel{(c)}{\leq} 1.84, \tag{257}$$

where (a) is based on (161), (b) follows from the assumption $d \leq \sigma_{y_3}^2\left(1-\rho_{23}^2\right) = 1-\rho^2$, and (c) relies on $0.5 \leq \frac{1}{2\rho} \leq c \leq \rho$.

Moreover, if $d \geq \sigma_{y_3}^2\left(1-\rho_{23}^2\right) = 1-\rho^2$, then

$$R_1^{(4)} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{258}$$

$$= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{259}$$

$$\stackrel{(a)}{\leq} \frac{1}{2}\log_2 \frac{(1+2c)(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}{d\rho} \tag{260}$$

$$\stackrel{(b)}{\leq} \frac{1}{2}\log_2 \left(\frac{(1+2c)}{\rho}\left[1+\sqrt{\frac{1+c^2}{1+\rho}}\right]^2\right) \leq 1.84, \tag{261}$$

where (a) is based on (161), (b) follows from the assumption $d \leq \sigma_{y_3}^2\left(1-\rho_{23}^2\right) = 1-\rho^2$, and (c) relies on $0.5 \leq \frac{1}{2\rho} \leq c \leq \rho$.

Appendix F
An Inequality

Lemma 23 *If* $c \leq \min\{\rho, \frac{1}{2\rho}\}$ *and* $d \leq c^2(1-\rho^2)$*, then*

$$\frac{1}{2}\log_2 \frac{1}{(1-\rho)} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \leq 3. \tag{262}$$

Proof: First, we note that $c \leq \min\{\rho, \frac{1}{2\rho}\} \leq \frac{1}{\sqrt{2}}$. In addition, we have

$$d \leq c^2(1-\rho^2) \leq \frac{1}{2}(1-\rho^2) \leq 1-\rho. \tag{263}$$

If $\rho(1-c)^2 \geq (\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2$, then

$$\frac{1}{2}\log_2\frac{1}{(1-\rho)}-\frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{264}$$

$$\leq\frac{1}{2}\log_2\frac{1}{(1-\rho)}-\frac{1}{2}\log_2\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}, \tag{265}$$

$$=\frac{1}{2}\log_2\left[\frac{1}{\rho(1-c)^2}\left(\sqrt{\frac{d}{1-\rho}}+\sqrt{1+c^2}\right)^2\right] \tag{266}$$

$$\overset{(a)}{\leq}\frac{1}{2}\log_2\left[2\left(1+\sqrt{\frac{3}{2}}\right)^2\right]=1.654, \tag{267}$$

where (a) follows from the fact $c \leq \frac{\sqrt{2}}{2}$. In addition, if $\rho(1-c)^2 \geq (\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2$, then we have $\rho \geq (1-\rho)(1+c^2)$, and therefore, $\rho \geq \frac{1+c^2}{2+c^2} \geq \frac{1}{2}$.

On the other hand, if $\rho(1-c)^2 \leq (\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2$, and $\rho > 0.9$, then since $d \leq 1-\rho$ we have,

$$\rho(1-c)^2 \leq (1-\rho)(1+\sqrt{1+c^2})^2.$$

Therefore,

$$\frac{1}{2}\log_2\frac{1}{(1-\rho)}-\frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{268}$$

$$=\frac{1}{2}\log_2\frac{1}{1-\rho} \tag{269}$$

$$\leq\frac{1}{2}\log_2\frac{(1+\sqrt{1+c^2})^2}{\rho(1-c)^2}\overset{(a)}{\leq}3.00, \tag{270}$$

where (a) relies on $c \leq \frac{\sqrt{2}}{2}$.

In addition, if $\rho(1-c)^2 \leq (\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2$, and $\rho > 0.9$, then

$$\frac{1}{2}\log_2\frac{1}{(1-\rho)}-\frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{271}$$

$$=\frac{1}{2}\log_2\frac{1}{1-\rho}\leq 1.66. \tag{272}$$

■

Appendix G
Proof of Lemma 11

If $d \leq \sigma_{y_3}^2(1-\rho_{13}^2)$, then

$$\bar{R}_1^{(5)} =\frac{1}{2}\log_2\frac{1}{1-\rho_{13}^2}, \tag{273}$$

$$\bar{R}_3^{(5)} =\frac{1}{2}\log_2^+\frac{\sigma_{y_3}^2(1-\rho_{13}^4)}{d}, \tag{274}$$

otherwise if $d \geq \sigma_{y_3}^2 \left(1-\rho_{13}^2\right)$, then

$$\bar{R}_1^{(5)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d}, \tag{275}$$

$$\bar{R}_3^{(5)} = \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} + \rho_{13}^2\right). \tag{276}$$

Part (i):

The proof is the same as the proof of Lemma 7-Part (i).

Part (ii):

The proof is the same as the proof of Lemma 7-Part (ii).

Part (iii):

We note that $\sigma_{y_3}^2 \left(1-\rho_{13}^2\right) = c^2(1-\rho^2)$. Then,

$$d \leq \sigma_{y_3}^2 \left(1-\rho_{13}^2\right) = c^2(1-\rho^2) \leq \frac{1}{2}(1-\rho^2) \leq (1-\rho), \tag{277}$$

which follows from $c \leq \frac{\sqrt{2}}{2}$.

From (40), we have,

$$\psi = \frac{1}{2}\log_2^+ \frac{1-\rho}{d} + \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} + \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2}. \tag{278}$$

It is easy to see that,

$$\bar{R}_3^{(5)} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} \leq \frac{1}{2}\log_2^+ \frac{2c^2(1-\rho^2)}{d} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} \leq 1. \tag{279}$$

In addition, we have

$$\bar{R}_1^{(5)} + \bar{R}_3^{(5)} - \frac{1}{2}\log_2^+ \frac{1-\rho}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{280}$$

$$= \frac{1}{2}\log_2 \frac{1}{1-\rho_{13}^2} + \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2 \left(1-\rho_{13}^4\right)}{d} - \frac{1}{2}\log_2 \frac{1-\rho}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{281}$$

$$\overset{(a)}{\leq} \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{c^2(1-\rho^2)} + \frac{1}{2}\log_2^+ \frac{2c^2(1-\rho^2)}{d} - \frac{1}{2}\log_2 \frac{1-\rho}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{282}$$

$$= \frac{1}{2}\log_2 \frac{2\sigma_{y_3}^2}{(1-\rho)} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \overset{(b)}{\leq} 4 \tag{283}$$

where (a) follows from $1+\rho_{23}^2 \leq 2$, and (b) is based on Lemma 23 and also $\sigma_{y_3}^2 \leq 2$.

Part (iv):

$$\bar{R}_3^{(5)} = \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} + \rho_{13}^2\right) = \frac{1}{2}\log_2^+ \left(\frac{c^2(1-\rho^2)}{d} + \rho_{13}^2\right) \leq \frac{1}{2}\log_2^+ \left(1+\rho_{13}^2\right) \leq 0.5. \tag{284}$$

APPENDIX H
PROOF OF LEMMA 12

If $d \leq \sigma_{y_3}^2 \left(1-\rho_{23}^2\right)$, then

$$\bar{R}_2^{(4)} = \frac{1}{2}\log_2 \frac{1}{1-\rho_{23}^2}, \tag{285}$$

$$\bar{R}_3^{(4)} = \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2 \left(1-\rho_{23}^4\right)}{d}, \tag{286}$$

otherwise if $d \geq \sigma_{y_3}^2 \left(1-\rho_{23}^2\right)$, then

$$\bar{R}_1^{(4)} = \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d}, \tag{287}$$

$$\bar{R}_3^{(4)} = \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{23}^2)}{d} + \rho_{23}^2\right). \tag{288}$$

Part (i):

The proof is the same as the proof of Lemma 8-Part (i).

Part (ii):

The proof is the same as the proof of Lemma 8-Part (ii).

APPENDIX I
PROOF OF LEMMA 13

Part (i):

$$\bar{R}_1^{(6)} + \bar{R}_2^{(6)} + \bar{R}_3^{(6)} - \phi(1,2,3) \tag{289}$$

$$= I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) - \phi(1,2,3) \tag{290}$$

$$\leq \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{\sigma_{y_3}^2}{d_3} + \frac{1-\rho^2}{d_1 d_2} + \frac{c^2(1-\rho^2)}{d_1 d_3} + \frac{1-\rho^2}{d_2 d_3}\right) - \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d_3} \tag{291}$$

$$\stackrel{(a)}{\leq} \frac{1}{2}\log_2\left(\frac{2}{\sigma_{y_3}^2} + 1 + \frac{3(1+\rho)}{\sigma_{y_3}^2}\right) \stackrel{(b)}{\leq} 3.28, \tag{292}$$

where (a) follows simply from (123)-(125), and (b) relies on

$$\sigma_{y_3}^2 = (1-c)^2 + 2c(1-\rho) \geq (1-c)^2 \geq \left(1 - \frac{\sqrt{2}}{2}\right)^2, \tag{293}$$

and (105).

Part (ii):

$$\bar{R}_1^{(6)} + \bar{R}_3^{(6)} - \phi(1,3) \tag{294}$$

$$= I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) - I(y_2; u_2) - \phi(1,3) \tag{295}$$

$$\leq \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{\sigma_{y_3}^2}{d_3} + \frac{1-\rho^2}{d_1 d_2} + \frac{c^2(1-\rho^2)}{d_1 d_3} + \frac{1-\rho^2}{d_2 d_3}\right) - \frac{1}{2}\log_2\frac{1}{d_2} - \frac{1}{2}\log_2^+\frac{1-\rho^2}{d} \tag{296}$$

$$\leq \frac{1}{2}\log_2\left(\frac{d_2}{d_1} + 1 + \frac{\sigma_{y_3}^2 d_2}{d} + \frac{1-\rho^2}{d_1} + \frac{c^2(1-\rho^2)d_2}{d_1 d_3} + \frac{1-\rho^2}{d}\right) - \frac{1}{2}\log_2^+\frac{1-\rho^2}{d} \overset{(a)}{\leq} \frac{1}{2}\log_2 7, \tag{297}$$

where (a) relies on (123)-(125).

Part (iii):

Let us assume that $d \leq c^2(1-\rho)$, then from (123)-(125), we have $d_1 = c^2(1-\rho)$ and $d_2 = 1-\rho$. In addition,

$$I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) \tag{298}$$

$$\leq \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{\sigma_{y_3}^2}{d_3} + \frac{1-\rho^2}{d_1 d_2} + \frac{c^2(1-\rho^2)}{d_1 d_3} + \frac{1-\rho^2}{d_2 d_3}\right) \tag{299}$$

$$\leq \frac{1}{2}\log_2\left(\frac{1}{c^2(1-\rho)} + \frac{1}{1-\rho} + \frac{\sigma_{y_3}^2}{d} + \frac{1+\rho}{c^2(1-\rho)} + \frac{1+\rho}{d} + \frac{1+\rho}{d}\right) \tag{300}$$

$$\leq \frac{1}{2}\log_2\left(\frac{4}{c^2(1-\rho)} + \frac{6}{d}\right) \leq \frac{1}{2}\log_2\left(\frac{10}{d}\right). \tag{301}$$

Moreover,

$$I\left(y_1, y_2; u_1, u_2\right) = \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1-d_2}{d_2} + \frac{(1-\rho^2)(1-d_1)(1-d_2)}{d_1 d_2}\right) \tag{302}$$

$$\geq \frac{1}{2}\log_2\left(\frac{1}{c^2(1-\rho)}\right). \tag{303}$$

Then

$$\bar{R}_1^{(6)} + \bar{R}_2^{(6)} + 2\bar{R}_3^{(6)} - \psi \tag{304}$$

$$= 2I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) - I\left(y_1, y_2; u_1, u_2\right) - \psi \tag{305}$$

$$\leq \log_2\left(\frac{10}{d}\right) - \frac{1}{2}\log_2\left(\frac{1}{c^2(1-\rho)}\right) \tag{306}$$

$$- \frac{1}{2}\log_2\frac{1-\rho}{d} - \frac{1}{2}\log_2\frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \tag{307}$$

$$\leq \frac{1}{2}\log_2\left(\frac{100}{1-\rho}\right) - \frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \overset{(a)}{\leq} 6.32, \tag{308}$$

where (a) follows from Lemma 23.

Now let us assume that $c^2(1-\rho) \leq d \leq c^2(1-\rho^2)$, then

$$d \leq c^2(1-\rho^2) \leq \frac{1}{2}(1-\rho^2) \leq 1-\rho, \tag{309}$$

which is based in (105). Therefore, from (123)-(125), we have $d_1 = d$ and $d_2 = 1-\rho$. Then, we can easily show that

$$I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) \leq \frac{1}{2}\log_2\left(\frac{10}{d}\right). \tag{310}$$

Moreover,

$$I\left(y_1, y_2; u_1, u_2\right) = \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1-d_2}{d_2} + \frac{(1-\rho^2)(1-d_1)(1-d_2)}{d_1 d_2}\right) \tag{311}$$

$$\geq \frac{1}{2}\log_2\frac{1}{d} \geq \frac{1}{2}\log_2\left(\frac{1}{c^2(1-\rho^2)}\right). \tag{312}$$

Then,

$$\bar{R}_1^{(6)} + \bar{R}_2^{(6)} + 2\bar{R}_3^{(6)} - \psi \tag{313}$$

$$= 2I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) - I\left(y_1, y_2; u_1, u_2\right) - \psi \tag{314}$$

$$\leq \log_2\left(\frac{10}{d}\right) - \frac{1}{2}\log_2\left(\frac{1}{c^2(1-\rho^2)}\right) \tag{315}$$

$$-\frac{1}{2}\log_2\frac{1-\rho}{d} - \frac{1}{2}\log_2^+\frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \tag{316}$$

$$\leq \frac{1}{2}\log_2\left(\frac{100(1+\rho)}{1-\rho}\right) - \frac{1}{2}\log_2^+\frac{\rho(1-c)^2}{(\sqrt{d}+\sqrt{(1-\rho)(1+c^2)})^2} \leq 6.82, \tag{317}$$

where (a) follows from Lemma 23.

Part (iv):

$$\bar{R}_3^{(6)} = I\left(y_1, y_2, y_3; u_1, u_2, u_3\right) - I\left(y_1, y_2; u_1, u_2\right) \tag{318}$$

$$\leq \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{\sigma_{y_3}^2}{d_3} + \frac{1-\rho^2}{d_1 d_2} + \frac{c^2(1-\rho^2)}{d_1 d_3} + \frac{1-\rho^2}{d_2 d_3}\right) - \frac{1}{2}\log_2\left(\frac{1}{d_1}\right) \tag{319}$$

$$\leq \frac{1}{2}\log_2\left(1 + \frac{d_1}{d_2} + \frac{\sigma_{y_3}^2 d_1}{d_3} + \frac{1-\rho^2}{d_2} + \frac{c^2(1-\rho^2)}{d_3} + \frac{d_1(1-\rho^2)}{d_2 d_3}\right) \tag{320}$$

$$\leq \frac{1}{2}\log_2\left(1 + 1 + \frac{\sigma_{y_3}^2}{1+\rho} + 1 + \rho + 1 + 1 + \rho\right) \leq \frac{1}{2}\log_2 9. \tag{321}$$

Appendix J
Proof of Lemma 14

Since $d \leq c^2(1-\rho^2)$, therefore, $d \leq c^2$, then, $d_2 = \frac{d}{4c^2}$.

Part (i):

$$d \leq c^2(1-\rho^2) \leq \frac{1}{2}(1-\rho^2) \leq 1-\rho, \tag{322}$$

where we use (105).

Let us assume that $d \leq c^2(1-\rho)$, then we have

$$\bar{R}_1^{(2)} + \bar{R}_2^{(2)} - \psi = I\left(y_1, y_2; u_1, u_2\right) - \psi \tag{323}$$

$$\leq \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{1-\rho^2}{d_1 d_2}\right) - \frac{1}{2}\log_2^+ \frac{1-\rho}{d} \tag{324}$$

$$- \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \tag{325}$$

$$\leq \frac{1}{2}\log_2\left(\frac{4}{d} + \frac{4c^2}{d} + \frac{16c^2(1-\rho^2)}{d^2}\right) - \frac{1}{2}\log_2 \frac{1-\rho}{d} \tag{326}$$

$$- \frac{1}{2}\log_2 \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \tag{327}$$

$$\overset{(a)}{\leq} \frac{1}{2}\log_2\left(\frac{40c^2(1-\rho)}{d^2}\right) - \frac{1}{2}\log_2 \frac{1-\rho}{d} \tag{328}$$

$$- \frac{1}{2}\log_2 \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \tag{329}$$

$$\overset{(b)}{=} \frac{1}{2}\log_2\left(\frac{40}{1-\rho}\right) - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \leq 5.66, \tag{330}$$

where (a) follows from the assumption that $d \leq c^2(1-\rho)$, and (b) is based on Lemma 23.

On the other hand, if $c^2(1-\rho) \leq d \leq c^2(1-\rho^2)$, then we have

$$\bar{R}_1^{(2)} + \bar{R}_2^{(2)} - \psi = I\left(y_1, y_2; u_1, u_2\right) - \psi \tag{331}$$

$$\leq \frac{1}{2}\log_2\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{1-\rho^2}{d_1 d_2}\right) - \frac{1}{2}\log_2^+ \frac{1-\rho}{d} \tag{332}$$

$$- \frac{1}{2}\log_2^+ \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \tag{333}$$

$$\leq \frac{1}{2}\log_2\left(\frac{4}{d} + \frac{4c^2}{d} + \frac{16c^2(1-\rho^2)}{d^2}\right) - \frac{1}{2}\log_2 \frac{1-\rho}{d} \tag{334}$$

$$- \frac{1}{2}\log_2 \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \tag{335}$$

$$\overset{(a)}{\leq} \frac{1}{2}\log_2\left(\frac{40c^2(1-\rho^2)}{d^2}\right) - \frac{1}{2}\log_2 \frac{1-\rho}{d} \tag{336}$$

$$- \frac{1}{2}\log_2 \frac{c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \tag{337}$$

$$\overset{(b)}{=} \frac{1}{2}\log_2\left(\frac{80}{1-\rho}\right) - \frac{1}{2}\log_2^+ \frac{\rho(1-c)^2}{(\sqrt{d} + \sqrt{(1-\rho)(1+c^2)})^2} \leq 6.16, \tag{338}$$

where (a) follows from the assumption that $d \leq c^2(1-\rho^2)$, and (b) is based on Lemma 23.

Part (ii):

In part (i), we proved that

$$I\left(y_1, y_2; u_1, u_2\right) \leq \frac{1}{2}\log_2 \frac{80c^2(1-\rho)}{d^2} \tag{339}$$

$$\bar{R}_1^{(2)} - \phi(1,3) = I\left(y_1, y_2; u_1, u_2\right) - I\left(y_2; u_2\right) - \psi(1,3) \tag{340}$$

$$\leq \frac{1}{2}\log_2 \frac{80c^2(1-\rho)}{d^2} - \frac{1}{2}\log_2 \frac{4c^2}{d} - \frac{1}{2}\log_2^+ \frac{1-\rho^2}{d} \tag{341}$$

$$\leq \frac{1}{2}\log_2 \frac{20(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{1-\rho^2}{d} \leq \frac{1}{2}\log_2 20 = 2.16. \tag{342}$$

Part (iii):

$$\bar{R}_2^{(1)} - \phi(2,3) = I\left(y_1, y_2; u_1, u_2\right) - I\left(y_1; u_1\right) - \psi(2,3) \tag{343}$$

$$\leq \frac{1}{2}\log_2 \frac{80c^2(1-\rho)}{d^2} - \frac{1}{2}\log_2 \frac{4}{d} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho^2)}{d} \tag{344}$$

$$\leq \frac{1}{2}\log_2 \frac{20c^2(1-\rho)}{d} - \frac{1}{2}\log_2^+ \frac{1-\rho^2}{d} \leq \frac{1}{2}\log_2 20 = 2.16. \tag{345}$$

APPENDIX K
PROOF OF LEMMA 17

We note that if $c \geq \rho$, then,

$$\sigma_{y_3}^2 = 1 + c^2 - 2c\rho = (1-c)^2 + 2c(1-\rho) \leq (1-\rho)(1-c) + 2c(1-\rho) = (1-\rho)(1+c) \leq 2(1-\rho). \tag{346}$$

Part (i):

$$\hat{R}_3^{(1)} - \phi(1,3) \tag{347}$$

$$= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{(1-\rho^2)}{d} \tag{348}$$

$$= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{(1-\rho^2)}{d} \tag{349}$$

$$\leq \frac{1}{2}\log_2 \frac{(1+c)(1-\rho)}{d} - \frac{1}{2}\log_2 \frac{(1-\rho^2)}{d} \tag{350}$$

$$\leq \frac{1}{2}\log_2 \frac{(1+c)}{1+\rho} \leq \frac{1}{2}\log_2 \frac{4}{3}. \tag{351}$$

Part (ii):

$$\hat{R}_3^{(1)} - \phi(2,3) \tag{352}$$

$$= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho^2)}{d} \tag{353}$$

$$= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} - \frac{1}{2}\log_2^+ \frac{c^2(1-\rho^2)}{d} \tag{354}$$

$$\leq \frac{1}{2}\log_2 \frac{(1+c)(1-\rho)}{d} - \frac{1}{2}\log_2 \frac{c^2(1-\rho^2)}{d} \tag{355}$$

$$\leq \frac{1}{2}\log_2 \frac{(1+c)}{c^2(1+\rho)} \leq 1. \tag{356}$$

Proofs of parts (iii) and (iv) are the same as the proof of parts (i) and (ii).

APPENDIX L
PROOF OF LEMMA 20

Part (i) follows from the proof of Lemma 7, part (i).

Part (ii):

If $d \leq \sigma_{y_3}^2 \left(1-\rho_{13}^2\right)$, then

$$\begin{aligned}
\widetilde{R}_1^{(5)} + \widetilde{R}_3^{(5)} - \phi(1,3) \\
&= \frac{1}{2}\log_2 \frac{1}{1-\rho_{13}^2} + \frac{1}{2}\log_2^+ \frac{\sigma_{y_3}^2\left(1-\rho_{13}^4\right)}{d} - \frac{1}{2}\log_2^+ \frac{(1-\rho^2)}{d} \\
&\leq \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2(1+\rho_{13}^2)}{(1-\rho^2)} \\
&\stackrel{(a)}{\leq} \frac{1}{2}\log_2 \frac{(1+c)(1+\rho_{13}^2)}{1+\rho} \leq 1,
\end{aligned}$$

where (a) follows from (346).

Otherwise if $d \geq \sigma_{y_3}^2 \left(1-\rho_{13}^2\right)$, then

$$\begin{aligned}
&\widetilde{R}_1^{(5)} + \widetilde{R}_3^{(5)} - \phi(1,3) \\
&\leq \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{d} + \frac{1}{2}\log_2^+ \left(\frac{\sigma_{y_3}^2(1-\rho_{13}^2)}{d} + \rho_{13}^2\right) - \frac{1}{2}\log_2 \frac{(1-\rho^2)}{d} \\
&= \frac{1}{2}\log_2 \frac{\sigma_{y_3}^2}{(1-\rho^2)} + \frac{1}{2}\log_2^+ \left(1+\rho_{13}^2\right) \\
&\stackrel{(a)}{\leq} \frac{1}{2}\log_2 \frac{(1+c)(1+\rho_{13}^2)}{1+\rho} \leq 1,
\end{aligned}$$

where (a) follows from (346).

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