Construction of a Cubist Girl Cap

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Technical Report No. UCB/EECS-2013-130
http://www.eecs.berkeley.edu/Pubs/TechRpts/2013/EECS-2013-130.html

July 11, 2013

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Abstract

Boy's surface is the simplest and most symmetrical way of making a compact model of the projective plane in R^3 without any singular points. This surface has 3-fold rotational symmetry and a single triple point from which three loops of intersection lines emerge. It turns out that there is a second, homeomorphically different way to model the projective plane with the same set of intersection lines, though it is less symmetrical. There seems to be only one such other structure beside Boy's surface, and it thus has been named Girl's surface. This alternative, finite, smooth model of the projective plane is more difficult to understand. In an effort to gain more insight into the geometry of this surface, various paper models have been constructed. The C_2–symmetric, "cubist" version of an open-ended Girl cap, with polyhedral facets primarily parallel to three rectilinear coordinate planes, seems particularly well suited to gain full understanding of this intriguing surface.

Introduction

There are many highly symmetrical, finite models of the projective plane, such as the cross cap surface or Steiner's Roman surface [10], but they typically have singular points with infinitely high curvature (Whitney umbrellas). In 1901, Werner Boy, a PhD student advised by David Hilbert, found a way to immerse the projective plane smoothly in 3-space, with no creases or corners or singularities [3]. This surface has 3-fold rotational symmetry and comes in two mirror-image forms that cannot be smoothly transformed into one another via a regular homotopy. Figure 1a shows what we call the right-handed version, because when we puncture this surface it becomes regular homotopically equivalent to a right-twisting Möbius band.

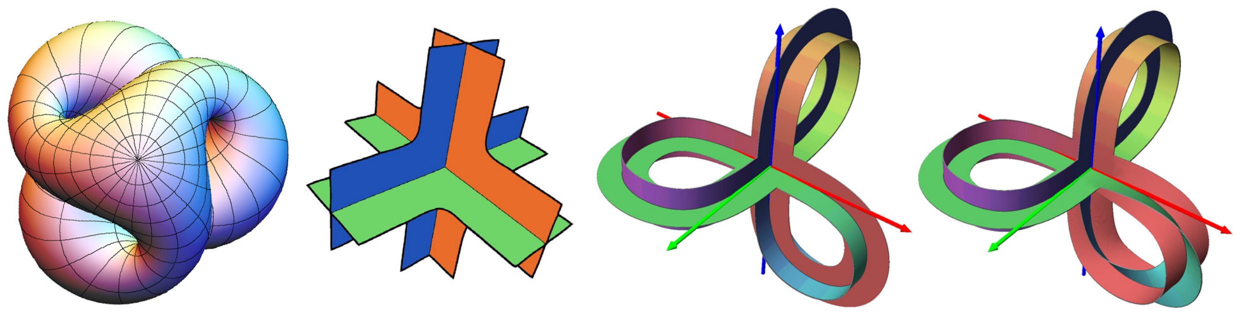
More than a century later, Goodman and Kossowski [4] investigated whether there might exist other smooth immersions of the projective plane in 3-space with a single triple point and with an equally simple intersection set. They found exactly one other immersion with the same topological complexity but with quite different geometrical properties. While it is now clear that Apéry knew of it and refers to it in his book *Models of the Real Projective Plane* [1], it had gone unnoticed in the mathematical community – perhaps because there was no smooth depiction of this surface. In 2009 Goodman and Kossowski modeled and presented this alternate surface; and since they could prove that there was only this one alternative to Boy's surface, they dubbed it *Girl's surface*.

I first learned about this surface through Sue Goodman in early 2013, and we decided to give it a broader exposure by presenting a paper at the Bridges 2013 conference in Enschede. We aimed to make our description understandable also to non-mathematicians by relying on a variety of virtual and physical visualization models. Making these models turned out to more challenging than I had anticipated. Several months had elapsed before I had realized a complete "*Cubist Model of a Girl Cap*" with which I could prove to my own satisfaction that I really understood the geometry of the Girl surface.

This report documents some of the steps that I took to make visualization models of this surface and gradually gaining an ever more thorough understanding of this surface and its relation to Boy's surface. In this report I will start out with a basic introduction to the problem as presented in our Bridges paper [5] and then fill in the details of my modeling activities. The goal is to leave a more detailed report on a design & development effort for a very specific, challenging problem, and to document various visualization and "rapid-prototyping" techniques used in this process.

Models of the Projective Plane

Any model of the projective plane in 3-dimensional space must intersect itself [9]. It is the structure of the intersection lines that we focus on. We start from the simplest possible intersection set for a smooth model of the projective plane in 3-dimensional space. It is known [2] that this intersection set must comprise a triple point: a point that locally appears as the intersection of three planes (Fig.1b). Six intersection-line branches emanate from this point. They form three loops circling back to the triple point. If we do this in the simplest, most symmetrical, and least twisted way (Fig.1c), we obtain the complete intersection set for Boy's surface. We are now looking for another model of the projective plane with a connected intersection set and with exactly one triple point that is homeomorphically distinct from Boy's surface – which means that there is not a one-to-one correspondence between the two models for every surface region bounded by an intersection line. Goodman and Kossowski [4] found that there is only *one* other way: *Girl's surface*. It turns out that for this surface, the neighborhood along one of the three intersection loops is twisted through 180° as is shown in the lower right branch of Figure 1d.

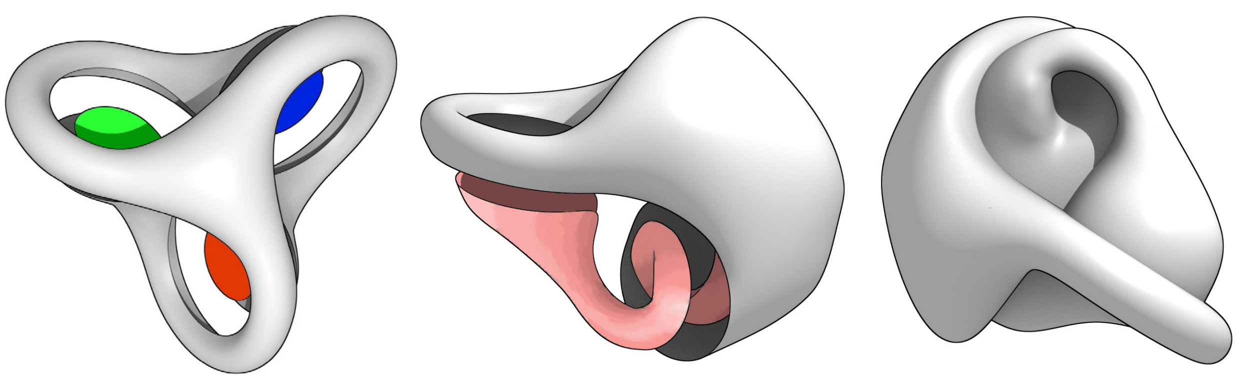


(a) (b) (c) (d)

Figure 1: *(a) r-Boy Surface; (b) triple point; (c) r-Boy and (d) l-Girl intersection-line neighborhoods.*

Constructing Girl's Surface – Starting from the Intersection-line Structure

Tracing around the rims in Figure 1c, we find four complete circuits that close back onto themselves (see also Fig.9a). Three of them lie inside the intersection-line lobes, and they can be readily filled in with disks (colored green, blue, and red). The fourth one (grey) winds around the outside in a rather complicated fashion; but it too can be extended and capped off without introducing any new self-intersections (Fig.2a). If we close all the gaps and merge the four "disks" in the three intersection line loops, we obtain Boy's surface (Fig.1a).

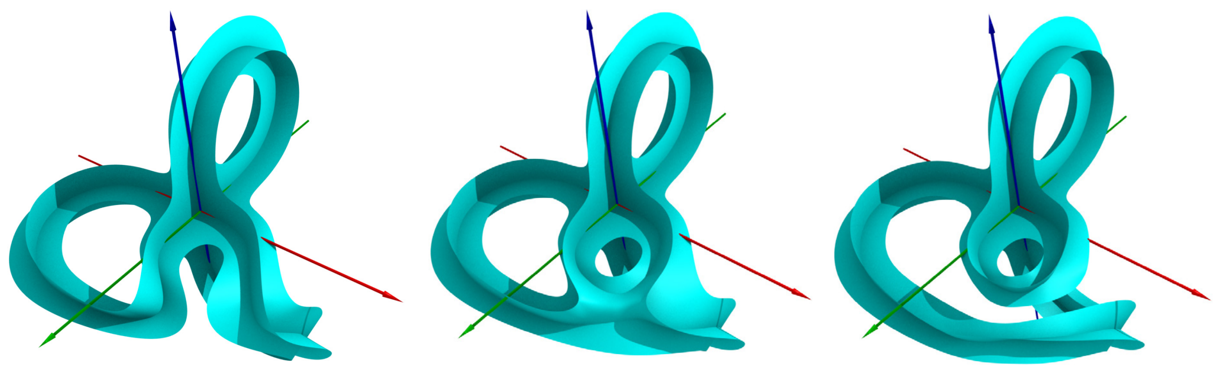


(a) (b) (c)

Figure 2: *Capping off the rim circuits: (a) of Fig.1c, (b) of Fig.1d; (c) the resulting l-Girl surface.*

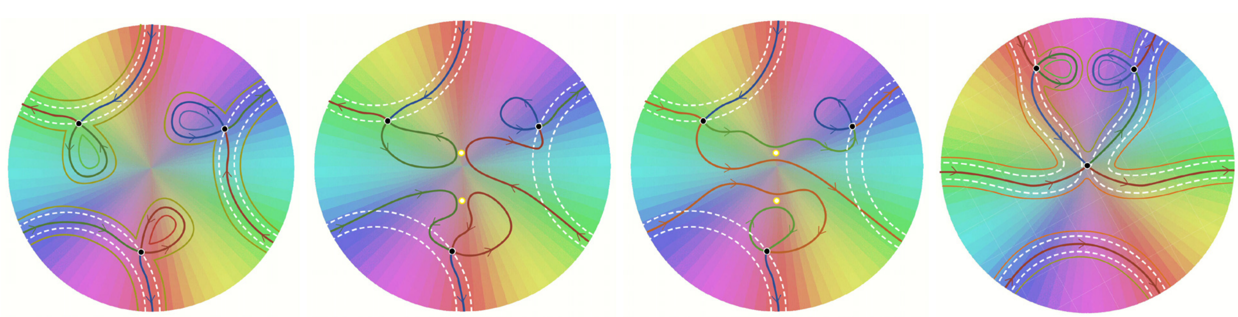
But what happens if we connect one or more of the intersection-line lobes with a half-twist back to the triple point as shown in Figure 1d for the bottom lobe? Tracing around the rim of this figure, we can see that there are still four complete circuits; two of which (the two upper ears) can still be capped off with the original local membranes (green and blue). Capping off the other two circuits results in more convoluted surfaces, which, however, are still topologically equivalent to disks (Fig.2b). If we now close the gaps along the intersection lines, the result is Girl's surface (Fig.2c). It is homeomorphically distinct from Boy's surface: There, each of the three intersection-line loops bounds a simple disk on the surface. This is not the case for Girl's surface: The loop with the half-twist does not serve as the rim of a disk on Girl's surface! The special behavior of this lobe also destroys the 3-fold symmetry of Boy's surface. For more models of Girl's surface, see [6].

Every smooth model of the projective plane can be transformed by a regular homotopy into either a right-handed version of Boy's surface (Fig.1a) or into its mirror image (*l-Boy*). So, what might such a deformation sequence look like for Girl's surface? Figure 3 shows a few crucial stills from an animation (programmed by Alex Mellnik [8]) of the complete transformation of an r-Boy surface into an r-Girl surface; it shows the point where two intersection lines bulge out and touch and form two new circuits – a small local one, and large, contorted one combining the remnants of the two merging loops.



(a) (b) (c)

Figure 3: Deformation of the intersection neighborhood from Boy's surface to Girl's surface

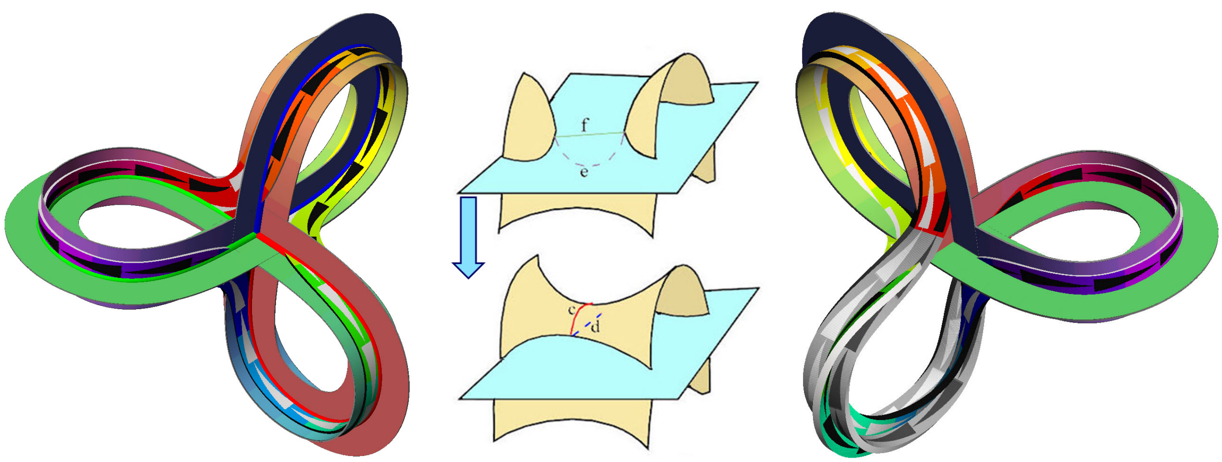


(a) (b) (c) (d)

Figure 4: *Intersection-line loops shown on the domain map of the projective plane: (a) Boy surface; (b) before and (c) after the merger of the red and green loops; (d) cleaned-up pattern for Girl surface. (To know what orientation of surface this represents, one would have to specify in what direction the Möbius flips occur as one steps across the perimeter.)*

This merging and recombination of intersection loops is shown schematically on the domain map of the projective plane (Fig.4a-d), which can be seen as a disk on which opposite points are identified (i.e., connected by going through infinity). The rainbow coloring shows this implied connectivity and indicates which perimeter points are connected to one another. The darker lines on these diagrams mark the three intersection-line loops, color coded to match the loopy tubes in Figure 5a. Every intersection line element shows up twice in each diagram, because two different points of the surface meet in those lines. The triple point, marked with a black dot, shows up three times. Also shown in these diagrams in white dashed outlines is a Möbius band embedded in the Boy surface. It mostly follows the intersection lines, and it almost touches the triple point three times. This Möbius band is also shown in the intersection-line neighborhood model (Fig.5a). As shown, it makes three left-handed 180° flips, which makes it regular homotopically equivalent to a simple right-twisting Möbius band; this shows that we are dealing here with an r-Boy surface.

This topological change is made possible by letting one surface region pass through a saddle shape formed by a different surface region, as schematically indicated in Figure 5b. Once the intersection neighborhoods have separated, we have again three loops that can be simplified and forced into a more symmetrical configuration. However, it turns out that one of the newly formed loops (the bottom one) now has 180° twist in it. Moreover, the chirality of the overall configuration has changed (Fig.5c) and the three "propeller blades" of Figure 1c are now angled in the opposite sense! Thus the result of the described regular homotopy move is a mirror image of what is shown in Figure 1d. In other words, when we cut and twist one of the intersection lobes as indicated in Figure 1d, and then complete the surface by capping off all circuits with individual disks, we obtain a left-handed Girl surface, which could then be transformed through the regular homotopy move described above into a left-handed Boy surface.



(a) (b) (c)

Figure 5: *(a) r-Boy intersection neighborhood with embedded Möbius band. (b) Detail of the saddle switch-over. (c) r-Girl intersection neighborhood with embedded Möbius band; (the neighborhood bands of the red loop have been eliminated to better show the two branches of the Möbius band in that loop).*

It is natural to ask: What will happen, if in Figure 1d we twist not just one lobe through 180°, but perhaps two, or all three of them – or if we apply the loop-combining, regular homotopy move shown in Figure 3 to the intersection neighborhood more than once? When brutally twisting intersection-line loops, we first have to wonder whether we still obtain models of the projective plane, and, if the answer is yes, what the resulting intersection-line structure looks like. When applying additional regular homotopy moves, we are at least sure that we maintain a valid model for the projective plane; we then only have to investigate whether additional intersection-line loops, and perhaps even additional triple points, will be generated.

The full discussion of all these issues is beyond the scope of this paper. Answers can be found in [4] and [6]. The argument that Girl's surface is the only alternative emerges from of a case-by-case study of other possibilities to connect the arcs in Figure 1b. The basic approach is as follows. A formula of

Izumiya-Marar [7] tells us that the Euler characteristic of any smooth space model of the projective plane with a single triple point is: χ (projective plane) + # triple points = 1 + 1 = 2. Taking the triple point as one vertex and the intersection-line arcs as three edges, we see that the remainder of the surface must be four disk faces so that $\chi = V - E + F = 1 - 3 + 4 = 2$. Therefore, after the arcs are connected, the rim of the neighborhoods must consist of exactly four complete circuits, and we must be able to cap off each circuit with a disk in such a way that we create no new self-intersections of the surface. Each case is examined: whether opposite or adjacent arcs are connected, whether the connections are made with or without twists, and with or without knots. Only by connecting the arcs as in Boy's surface or as described above for Girl's surface, will we get exactly four complete circuits on the rim that can be capped off by disks without creating any new self-intersections.

In summary, the discussion above conveys a couple of conceptual pathways by which we can obtain a Girl surface: We can either start from the twisted loop configuration in Figure 1d and cap off the intersection lines with four topological disks (two of which are rather convoluted) without producing additional intersections. Alternatively, we can start from a Boy surface and apply the regular homotopy transformation that creates the saddle switchover indicated in Figure 5b to obtain a Girl surface. The relationship among those surfaces and their mirror images is captured in Figure 6.

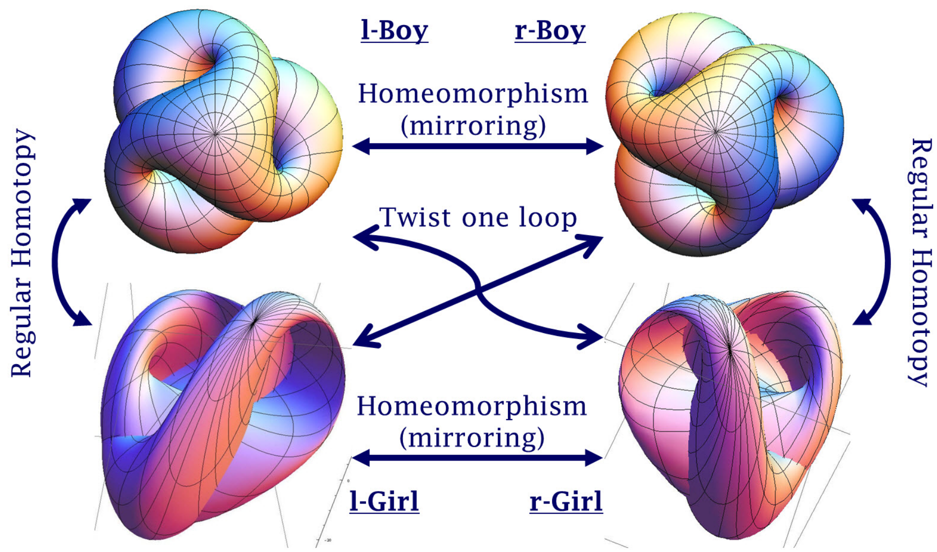


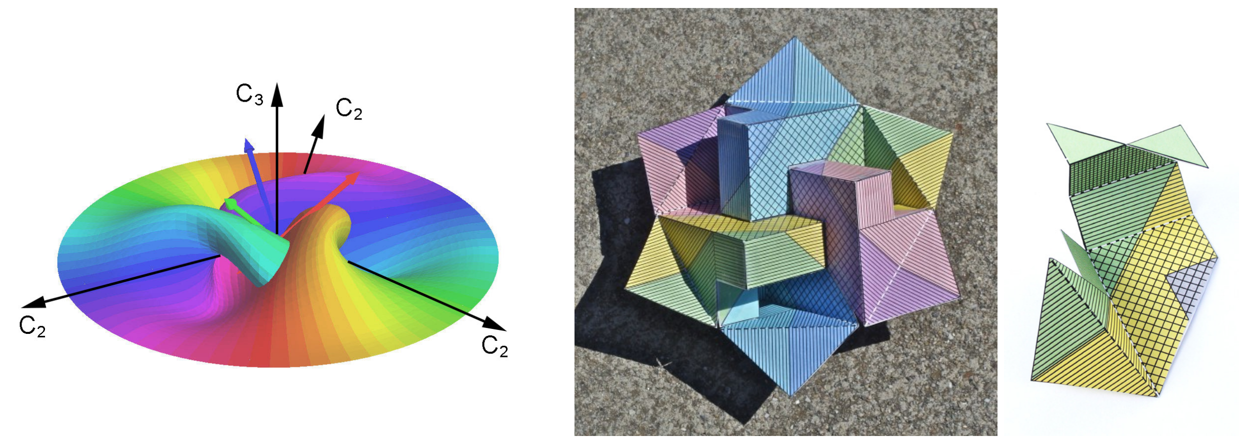
Figure 6: *Different simple, smooth, compact models of the projective plane and their relationships.*

However, both processes outlined above do not immediately describe the geometry of the resulting surface. Even the rather symmetrical Boy surface is difficult to understand when first encountered. Girl's surface is yet more difficult to understand even for seasoned mathematicians. The lack of symmetry and the fact that much of the crucial symmetry is hidden inside the dominant bulging shell, make it difficult to fully understand the geometry and connectivity of this single-sided surface. We need to develop visualization models that are more revealing. This can be done by puncturing these surfaces in a suitable place and then opening them up by stretching the perimeter of the puncture into a large circular outer rim. This process allows us to increase the symmetry of the resulting model and it also reveals much of the previously hidden intersection geometry.

Open Symmetrical Models of Boy Cap and Girl Cap

If we puncture Boy's surface (Fig.1a) at its center of 3-fold symmetry and then open that hole into a large circular rim, we obtain a *Boy cap*, which is topologically equivalent to a Möbius band. If we expand the puncture rim into a large "equator" and then distribute the six tunnel entrances symmetrically above and below this equatorial plane, we can obtain a surface with 6-fold D_3 symmetry, where the three C_2 rotation axes lie in that equatorial plane (Fig.7a). This open structure has the advantage that there are no features hiding inside a closed shell. Every part of this surface is clearly visible from one side or the other; thus it is easier to understand.

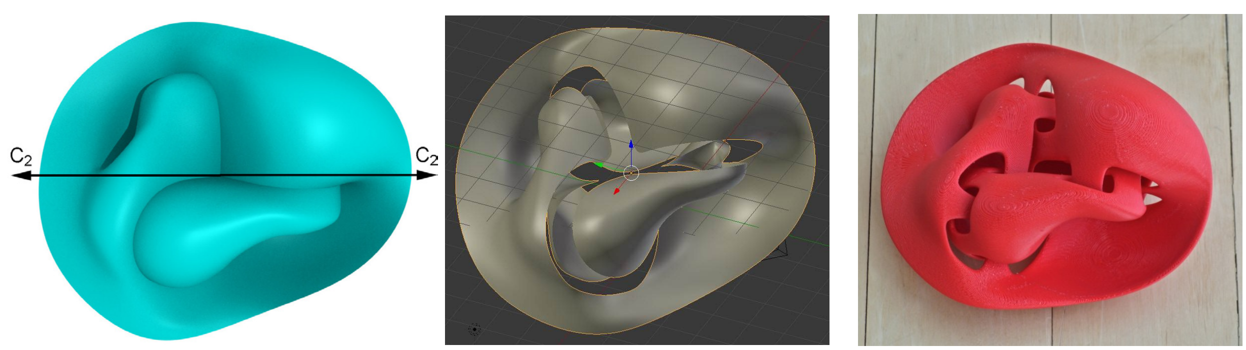
Understanding of such a surface can be enhanced even further by making a paper model. To make this task as easy as possible, we have designed a "cubist" version in which most vertices lie on an integer grid, and most faces join with 90° angles (Fig.7b). A discrete 6-band rainbow coloring has been applied, so that it is easy to see how opposite points on the equatorial circle connect to one another via a helical path with 180° of twist through the tunnels of the Boy cap. The whole paper model has been constructed from six copies of the template shown in Figure 7c. The net of this template is provided in the Appendix. To obtain a model with color "on both sides", you may want to print appropriate pairs of pages in the Appendix back-to-back in registered form, or print them single-sided and then glue corresponding sheets back-to-back, while trying to keep the patterns as nicely registered as possible.



(a) (b) (c)

Figure 7: *Symmetrical, open-rim r-Boy cap: (a) virtual B-spline model, (b) paper model, (c) template.*

An open model is even more valuable for Girl's surface, so that the special twisted lobe can be inspected from both sides. Such a Girl cap is depicted in different ways in Figures 8a-c. The 3-fold symmetry has been broken because there is only one twisted intersection loop. However, we can at least preserve one of the C_2 symmetry axes – the one that goes through that twisted lobe.



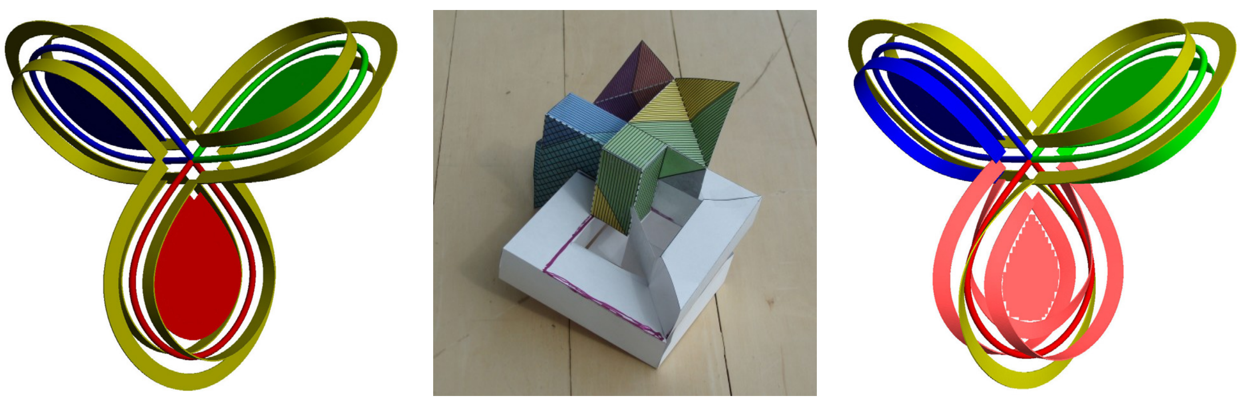
(a) (b) (c)

Figure 8: *Symmetrical, open-rim Girl cap: (a) virtual subdivision model, (b) with intersection-line neighborhood removed, (c) a physical model made on a Fused Deposition Modeling machine.*

To fully comprehend the geometry of this surface, I set out to also make a “cubist” paper model for this Girl cap, where the polyhedral faces forming the crucial central geometry should be lying exclusively on an integer grid of planes perpendicular to the three coordinate axes of Euclidean space. However, I found it rather difficult to capture the tight helical twisting of the surface, as it squeezes through the twisted intersection loop, using only 90°-angular facets; it can produce a whole lot of un-attractive and confusing “stair-casing.”

I started my model building effort with a copy of the cubist Boy cap, since, as indicated by Figures 1c and 1d as well as by Figures 2a and 2b, the upper half of the model would remain pretty much the same. The differences are captured in Figure 9: The blue and green disks remain the same. The complicated yellow outer annulus of the Boy surface (Fig.9a) stays the same in the central upper parts, but becomes simpler in the lower half, as it occupies only one of the four rails that accompany the red intersection-line loop (Fig.9c). On the other hand, the red disk of the Boy surface (Fig.9a) becomes much more convoluted and connects to the blue and green collars in Figure 9c.

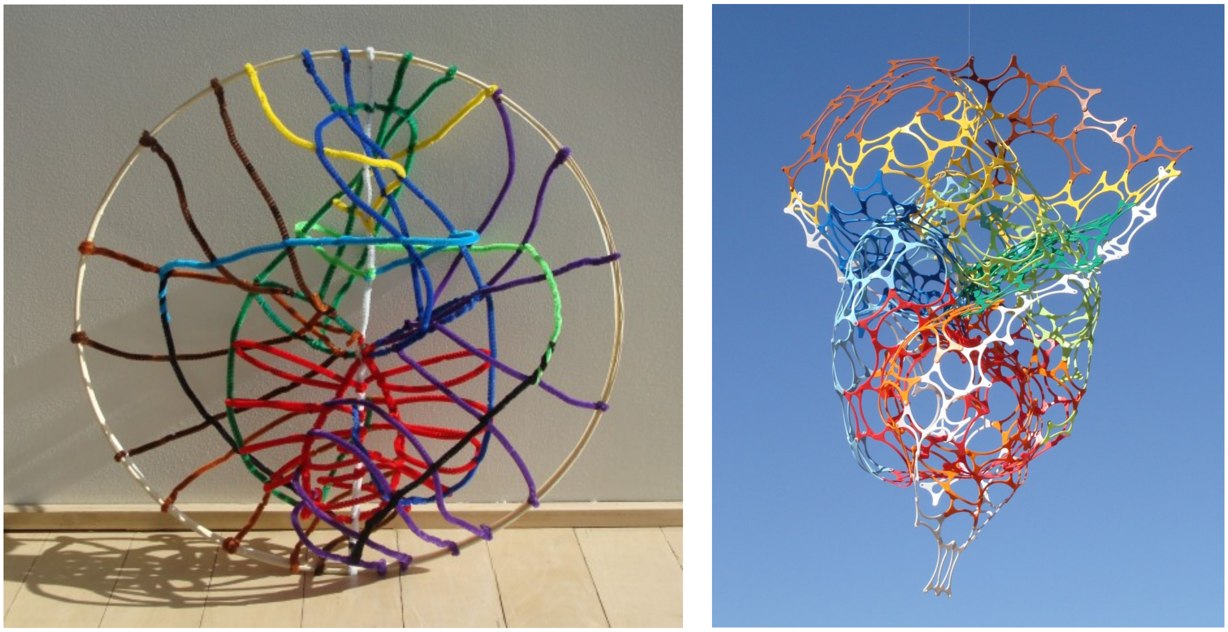
Cutting away the lower part in a Boy cap paper model (Fig.9b), I readily saw how the construction of the lower half had to get started. From the half-lemniscate collars below the green and blue disks two tubes had to be extruded; they first would sweep along the outside of the red intersection line loop and then gradually turn towards the inner side of the loop. At the bottom of the loop they would come in contact with one another, wind around each other, and finally emerge into the open “on the other side” as did the Boy-type tunnels in the upper half of the model. In this process, the side of the disk that formed the inside of the green collar would become the outside of the blue collar, and vice versa.



(a) (b) (c)

Figure 9: *(a) Intersection neighborhood of a Boy surface with the three simple inner disks instantiated, and the outer (yellow) annulus shown by its rim neighborhood; (b) modified cubist Boy cap model with a beginning of two extruded tubes starting from the blue and green collars; (c) intersection neighborhood of a Girl surface with its two simple inner disks instantiated, and with the new surface replacing the red disk outlined by its rim neighborhood;*

The virtual models of a C_2-symmetrical Girl cap provided by Alex Mellnik (Fig.8a,b), gave me a good overall idea what this surface looks like, and based on that understanding I started to make my own models of a Girl cap employing various materials. Figure 10a shows a model built from pipe-cleaners, nicely suspended in a wire ring cut from an old lamp shade. The model depicted in Figure 10b has been assembled from *Flexeez* plastic parts, which are available from Toys-R-Us. However, in both models I was not completely clear about the detailed geometry inside the red intersection loop. I remained somewhat puzzled about how exactly the green and blue tubes should join into one another while turning inside out. I had even more difficulties visualizing how all this could be realized with simple polyhedral facets parallel to the three Euclidean coordinate planes. Thus I started to concentrated on the inside of the red loop.



(a) (b)

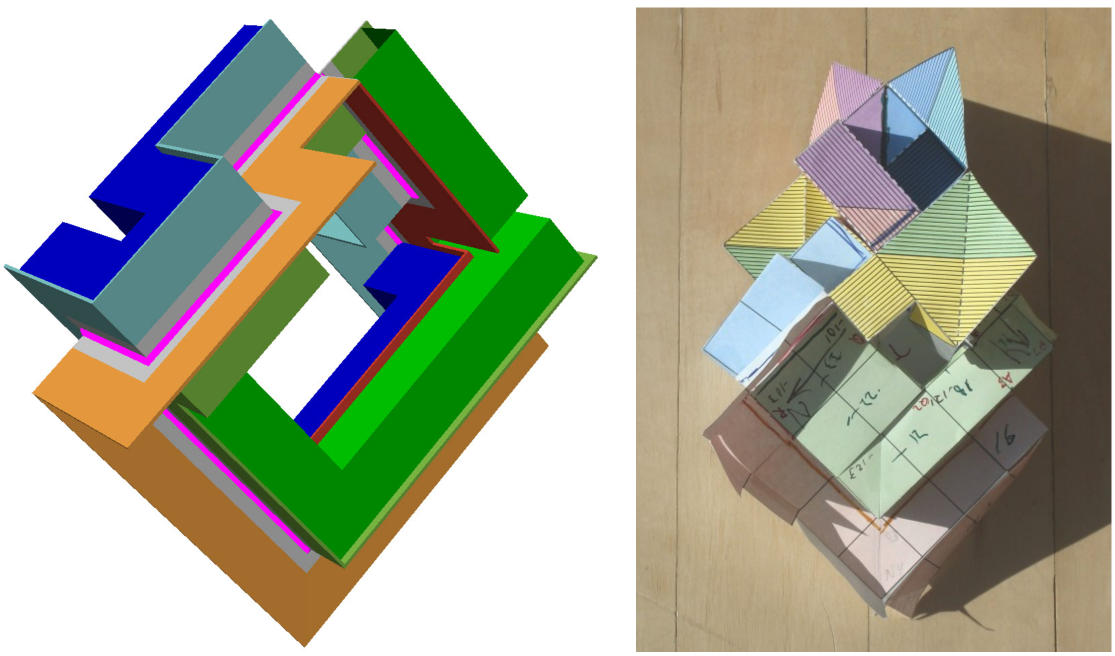
Figure 10: *Complete models of an open-rim Girl cap: (a) constructed from pipe-cleaners, (b) assembled from FLexeez parts from Toys-R-Us.*

Studying the models generated by Alex Mellnik (Fig.8a,b), I realized that, as we travel along the C_2–symmetry axis, the surface makes a total of <u>three</u> 180° torsional flips from one perimeter point to the opposite one. With the model shown in Figure 11a I tried to see how a sequence of such twists could be realized most compactly within cubist constraints. Also based on Mellnik's model, I created a paper-strip model of the geometry inside the red loop (Fig.11b). This made it clearer how the remnants of the red Boy disk near the triple point would be contorted so that they can be connected smoothly into the new blue/green surface (Fig.11c).

(a) (b) (c)

Figure 11: *Paper models to study the central geometry in the twisted lobe of Girl's cap: (a) focusing on the twisting along the C_2-symmetry axis, (b) examining the merging of the two extrusions starting at the blue and green collars in Figure 9c, (c) visualizing the resulting shape of the corner of the red surface near the triple point.*

Once I had gained clarity about the internal connectivity of the various surface pieces, I started to try to model them within the chosen cubist constraints. In particular, I tried to build the whole central geometry from 1″ square planar tiles that would join at 90° angles. The crucial focus had to be on the red line forming the intersection curve in this third twisted lobe. Along this line there had to be a contiguous crossing of four flanges meeting at 90° angles: one yellow flange used by the outer annulus; two flanges of the green half-lemniscate tube; two flanges of the blue tube – but with one of the flanges shared by the two “tubes.” The red intersection line had to be routed in such a way that a natural, torsion-minimizing sweep along this path would keep all four flanges in the proper orientation. After several futile tries, I found the satisfactory solution shown in Figure 12a. Based on this rectilinear sweep path, I could then construct a complete model of the twisted half of Girl’s cap using the 1″ square paper tiles (Fig.12b). Since I tried to keep the red loop as compact as possible, I still had to deal with some conflicts and constraints on the inside of the loop, and I was not sure that I had connected all the tiles so as to form a topological disk bounded by the red intersection line.



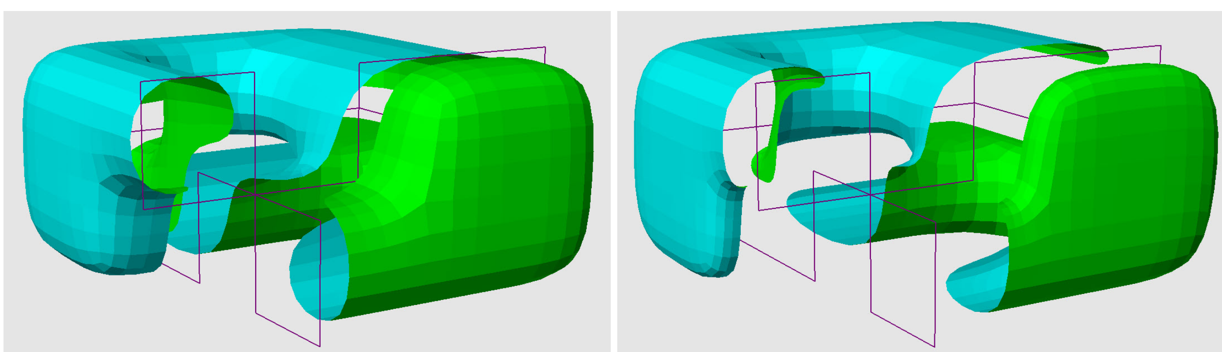
(a) (b)

Figure 12: *Forming a cubist intersection line for the twisted lobe of an open-rim Girl cap: (a) virtual sweep model extruding a profile of four flanges joining at right angles and including extrusions of the blue and green collars in Figure 9c; (b) paper model constructed from square tiles along the sweep path depicted on the left.*

At this point it was time to involve the computer again to make a complete cubist model. In a somewhat painstaking exercise, I typed the coordinates and connectivity of half the paper model into the computer, creating a partial polyhedral model that could be subjected to refinement and smoothing by subdivision. I chose the approximating Catmull-Clark subdivision scheme, and thus constructed mostly quadrilateral facets, except along the diagonal symmetry axis, where I ended up with a few triangles. The subdivision process smoothes the surface, softens and shrinks the rim of the surface, and opens up any internal cracks where tiles were not properly joined – possibly because they had the wrong orientations. This allowed me to quickly fix a few glaring errors. Subsequently, I was able to merge the surface with a copy rotated by 180° around the symmetry axis and with reversed normal vectors. This combined surface was also pulled back from the intersection line by a parameterized gap width, which could be adjusted interactively. Now the subdivision process revealed quickly and conclusively whether the resulting surface element was topologically equivalent to a disk. This analysis was carried out separately for the outer annulus, colored in orange and yellow (Fig.13 and 15) and for the inner disk, colored in green and cyan (Fig.14). Figure 14b demonstrates the capability of opening up the gaps along the intersection lines to obtain more transparency and to see better into the inside of the model. It should be noted, that along the outer perimeter of the orange/yellow annulus I deviated from the cubist orientation (as I also did for the cubist Boy surface in Figure 7b) in order to obtain a smooth, planar rim on the complete Girl's cap.

(a) (b) (c)

Figure 13: *Outer annular component of a symmetrical, open-rim Girl cap: (a) polyhedral input model, (b) after one level of Catmull-Clark subdivision, (c) after three levels of subdivision with smooth shading.*

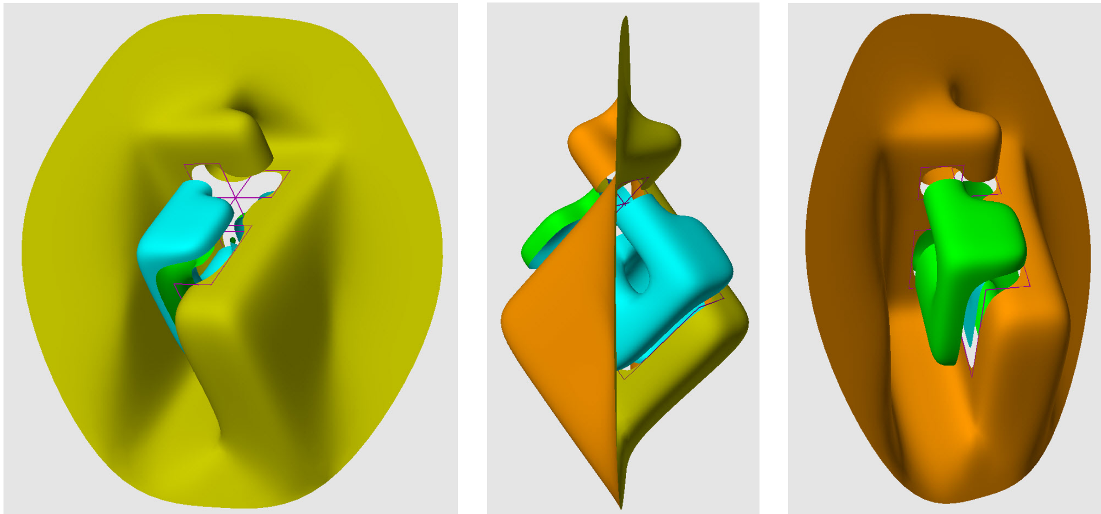


(a) (b)

Figure 14: *Inner twisted component of a symmetrical, open-rim Girl cap: (a) polyhedral model after two levels of Catmull-Clark subdivision, (b) same, with a wider separation gap along intersection lines.*

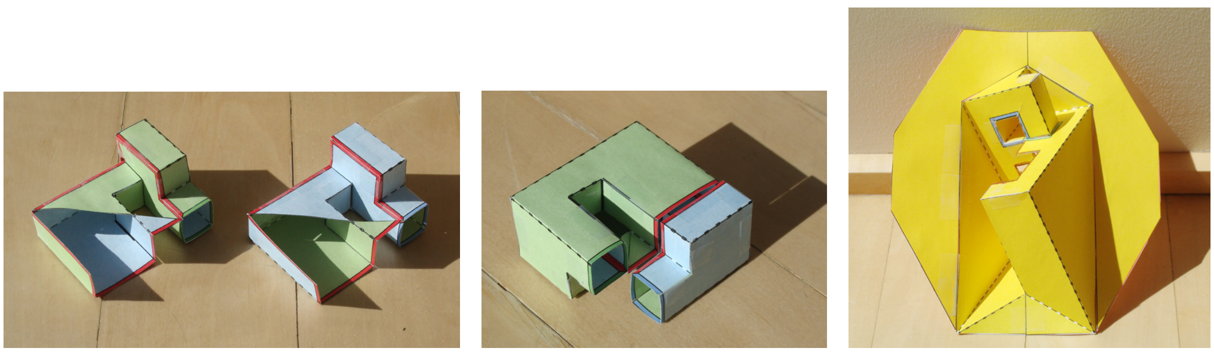
Finally, in Figure 15 the two surface elements are put together and are displayed from different directions with a relatively wide gap along the intersection lines, so that more of the inner geometry gets revealed. This model also allowed me to check whether there were any simple ways to realize the cubist geometry more compactly or with fewer "contortions." I could not find any such opportunities.

With the model checked to my satisfaction, I could now use the polyhedral starting shapes to make templates for a cubist paper model of Girl's surface (Fig.16, 17). Since I don't want the model to fall apart, I keep all surface components connected to the intersection lines. To provide some visibility into the inner core of the model, one may cut a couple of circular windows into the surface. The templates for this model can be found in the Appendix.



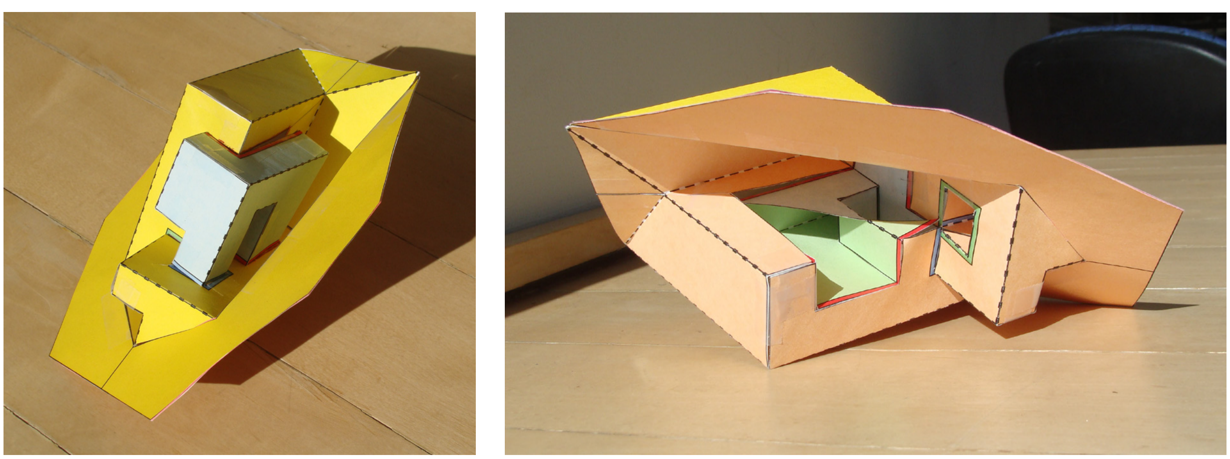
(a) (b) (c)

Figure 15: *Both, inner and outer components of a symmetrical, open-rim Girl cap:*
(a) "front" side, (b) side view, (c) oblique back-side view.



(a) (b) (c)

Figure 16: *Cubist paper model of an open-rim Girl cap:*
(a) the two halves of the inner blue-green disk, (b) the two halves combined; (c) the outer annulus.



(a) (b)

Figure 17: *Cubist paper model of an open-rim Girl cap:*
(a) blue half-tube inserted into the outer annulus, (b) assembly seen from the other side.

Summary and Conclusion

It was only after having completed the "cubist" version of Girl's cap that I felt that I was truly understanding the geometry of Girl's surface. Looking over my own shoulder during the process of developing this special paper model provided an interesting glimpse of the non-linear thinking processes that occur even in a relatively simple design task, where the goal was clearly defined, yet the method on how to get there was wide open. By writing down some of the difficulties I encountered and some of the thinking processes that allowed me to overcome these hurdles, I hope to provide some inspiration for other people struggling with difficult design tasks.

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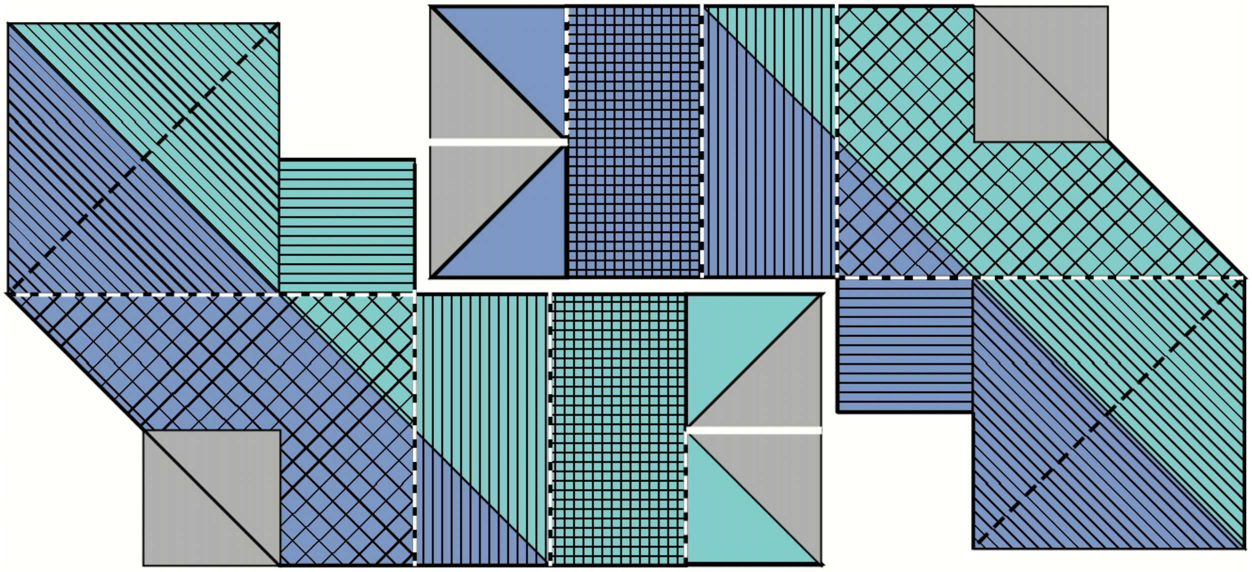
APPENDIX

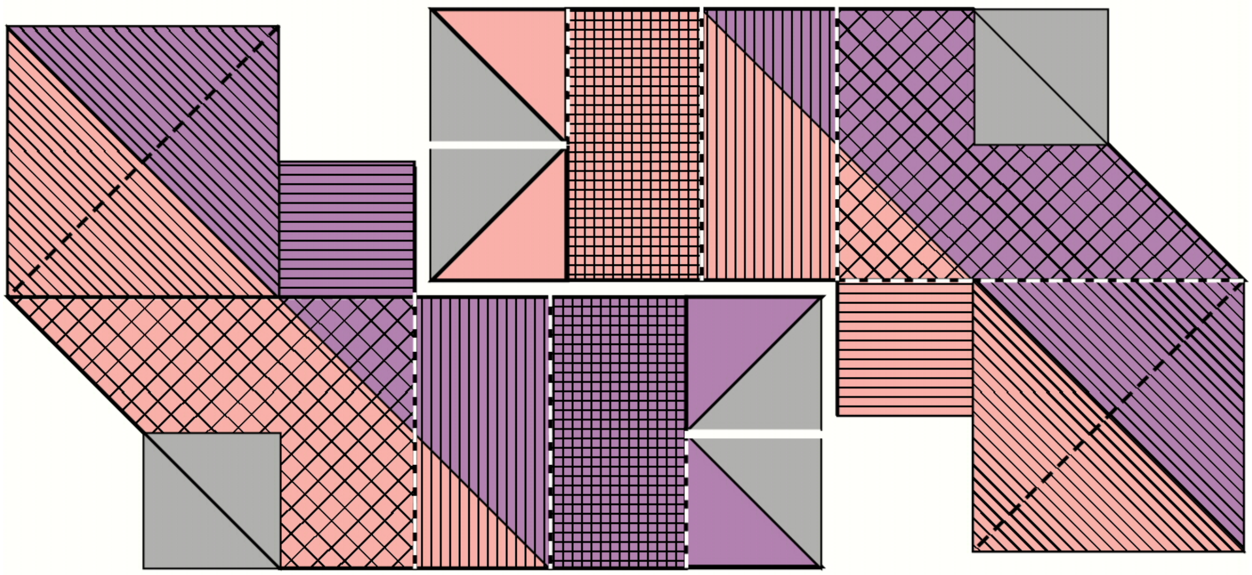
The following pages contain templates for making the cubist paper models of the open-rimmed Boy and Girl caps. Pages 13 and 14, when printed double-sided, should provide six polyhedral pieces with matching colors on both sides. They should be cut out and then folded so that black dashed lines become mountain-folds and white dashed lines become valley-folds (Fig.7c).

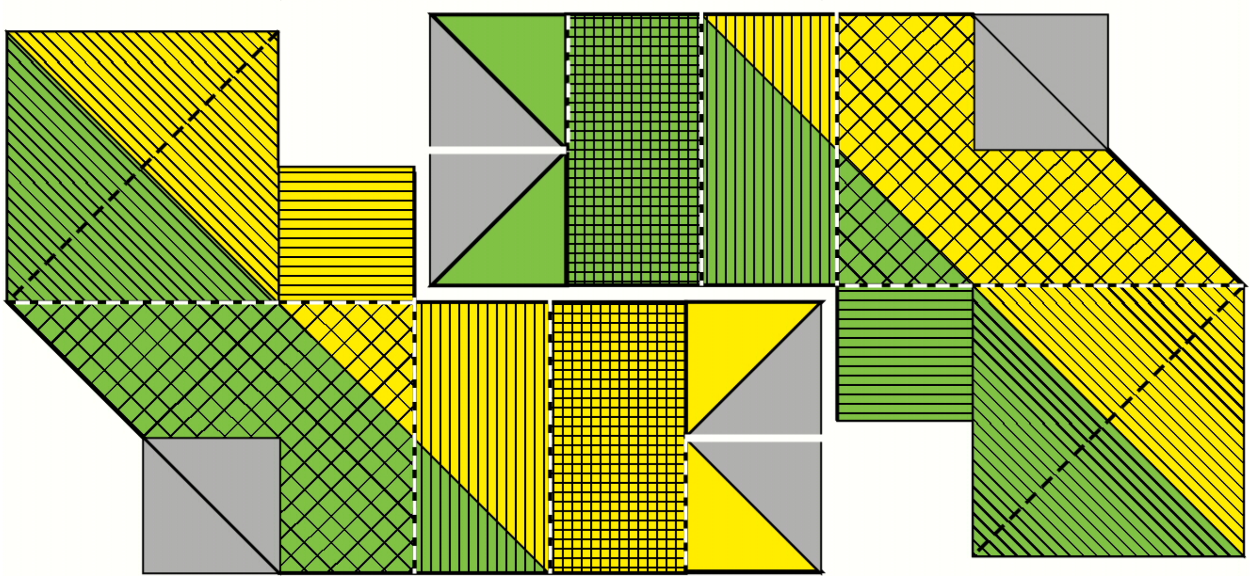
In a similar way, pages 15 and 16 provide the templates for the blue-green disk at the core of Girl's cap (Fig. 16a,b). Pages 17 and 18 provide templates or the "non-cubist outer part of the yellow-orange annulus surrounding the blue-green core. The cubist portion of this annulus is captured in the templates on pages 19 and 20.

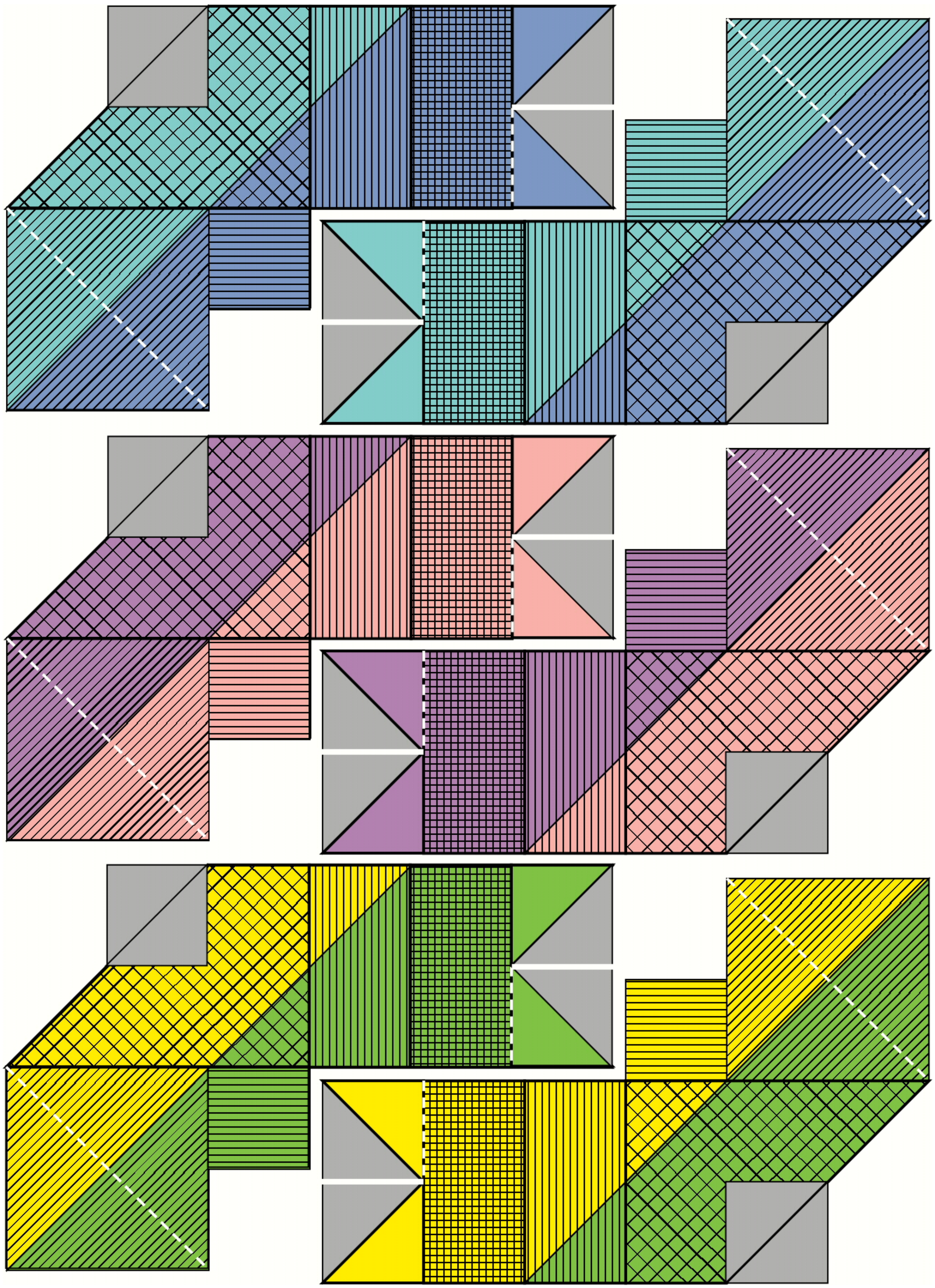
If one prints these templates single-sided (since the back-to-front registration on most printers is not well established), one should just cut the gross outlines around the whole cluster of templates on the same page and then glue these cut-outs back-to-back. Subsequently one can then do all the additional detailed trimming necessary on the doubled-up paper. The unit squares for the Girl cap are about 5/8″ on a side. Of course, one could uniformly enlarge all templates by a constant factor to obtain a larger model.

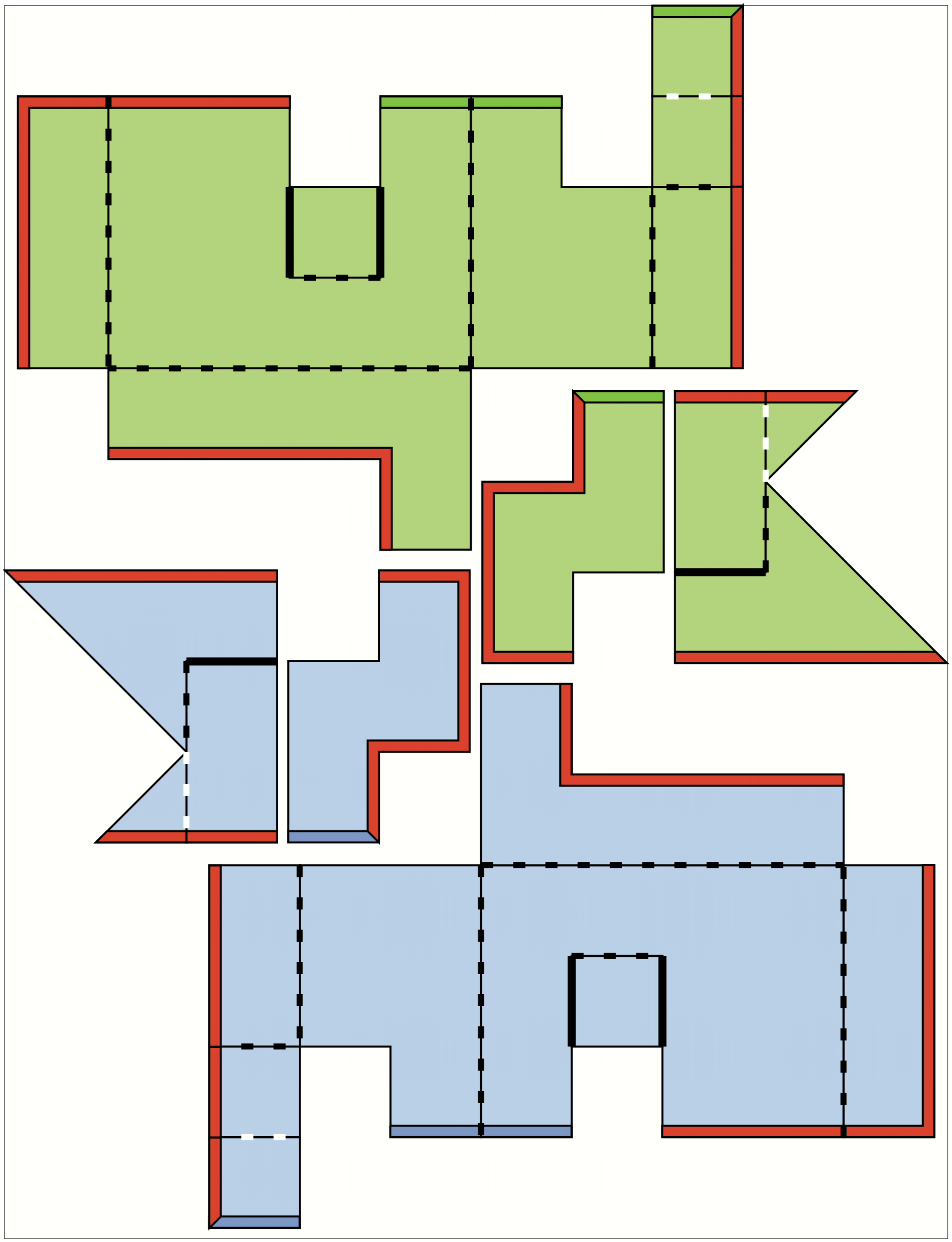
I am gluing together the folded templates with magic Scotch tape. I you want to use glue instead, you should add little extra stubs along the edges where two template facets need to join. Pictures of the completed models can be seen in Figures 16 and 17.

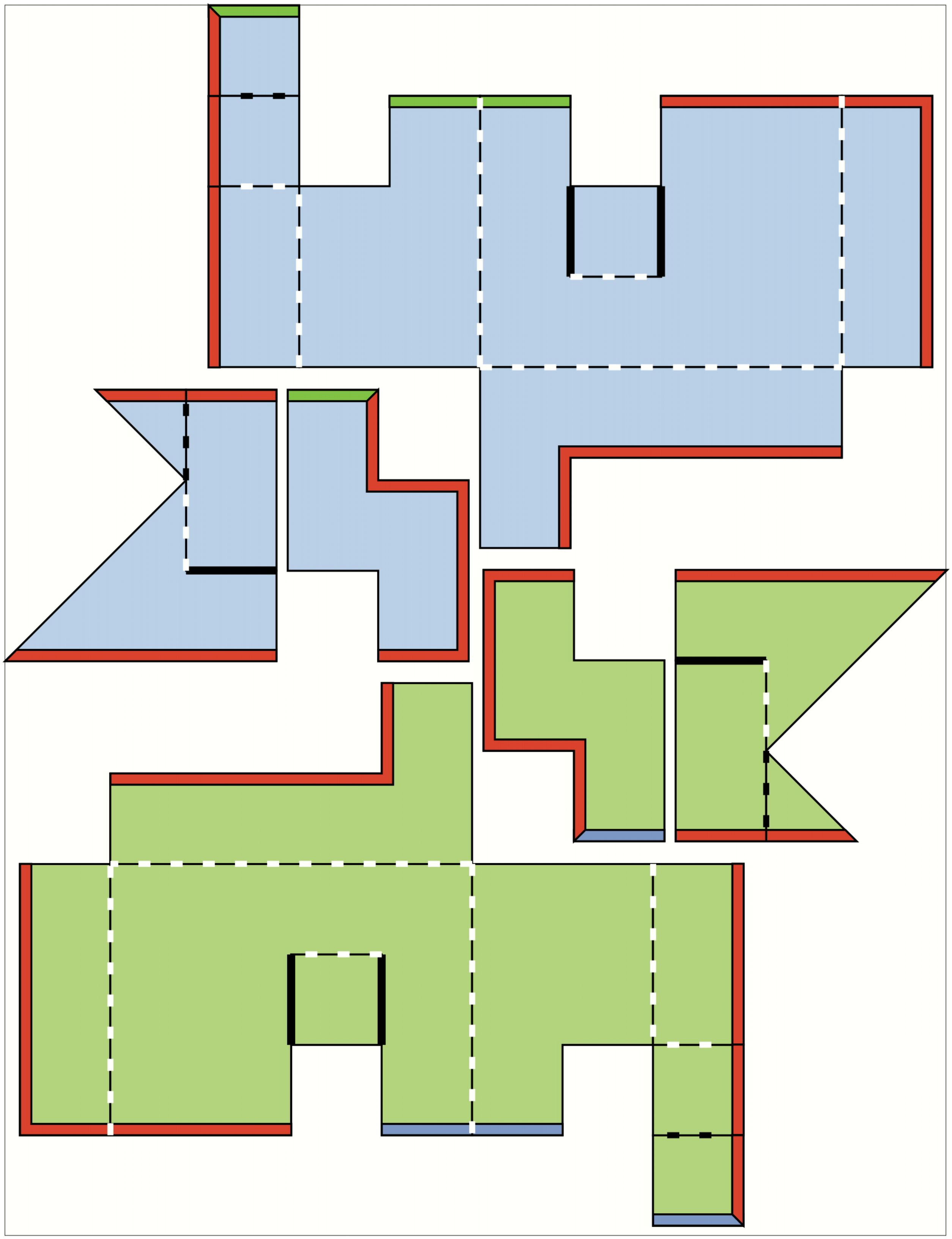


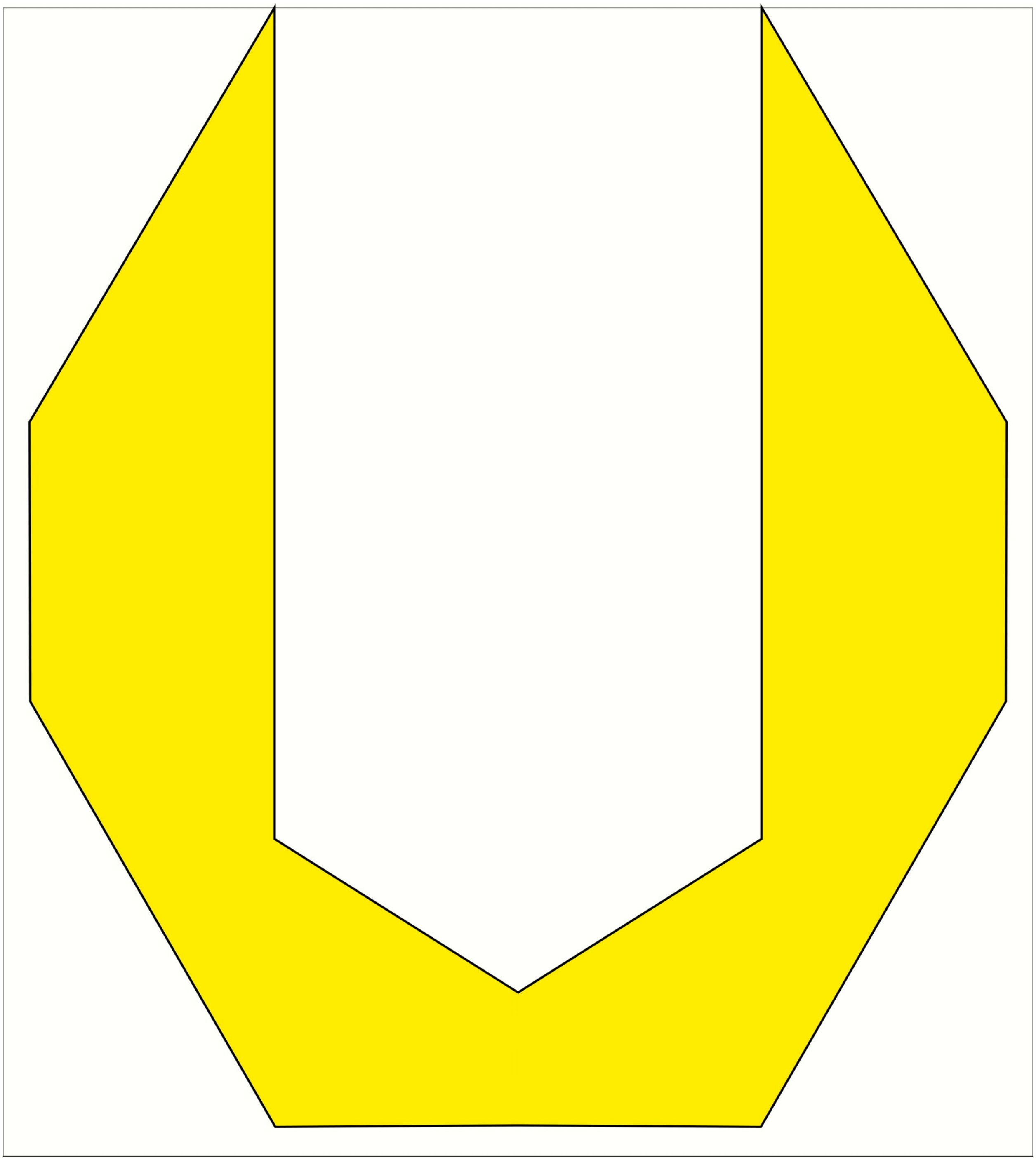


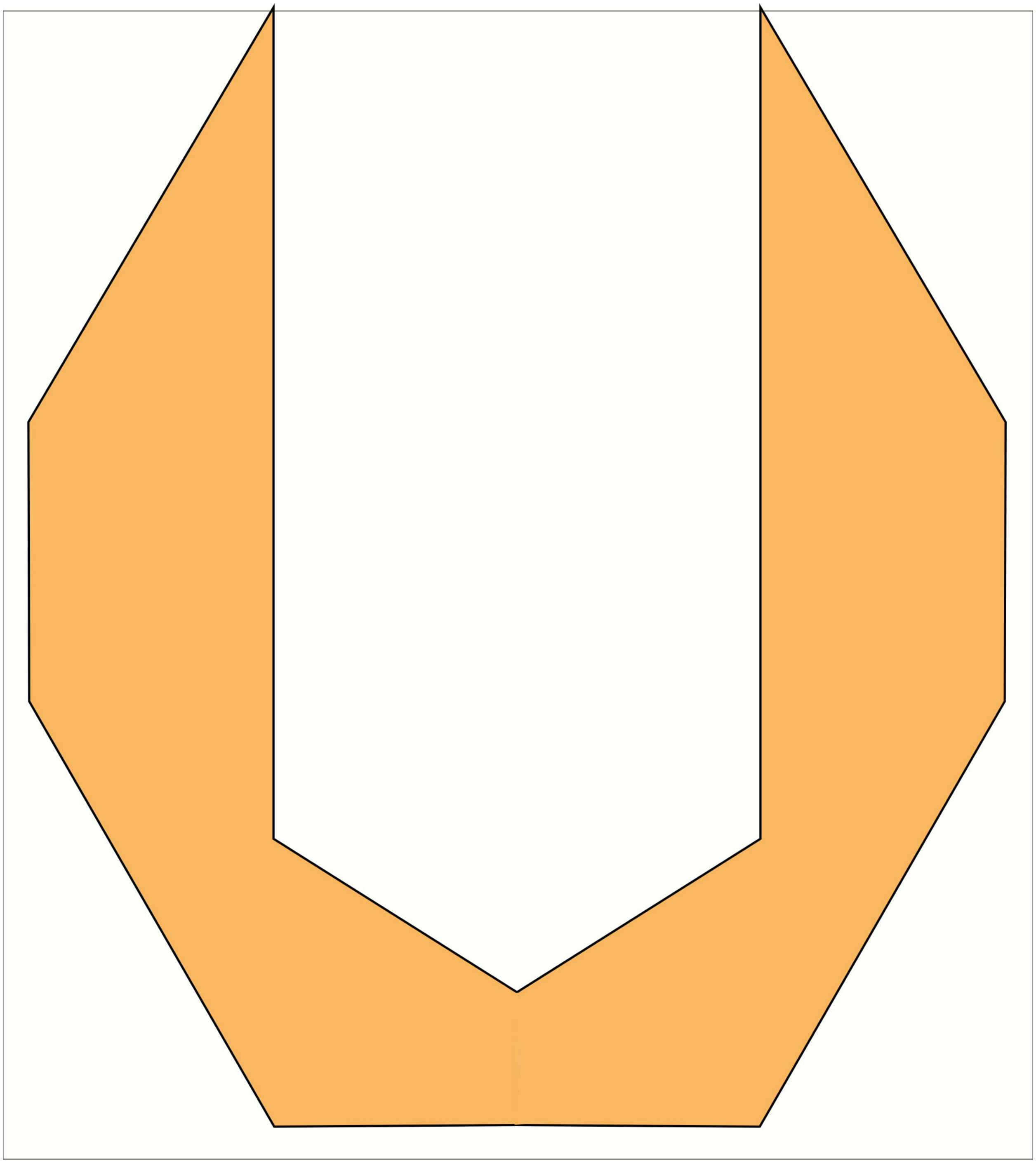


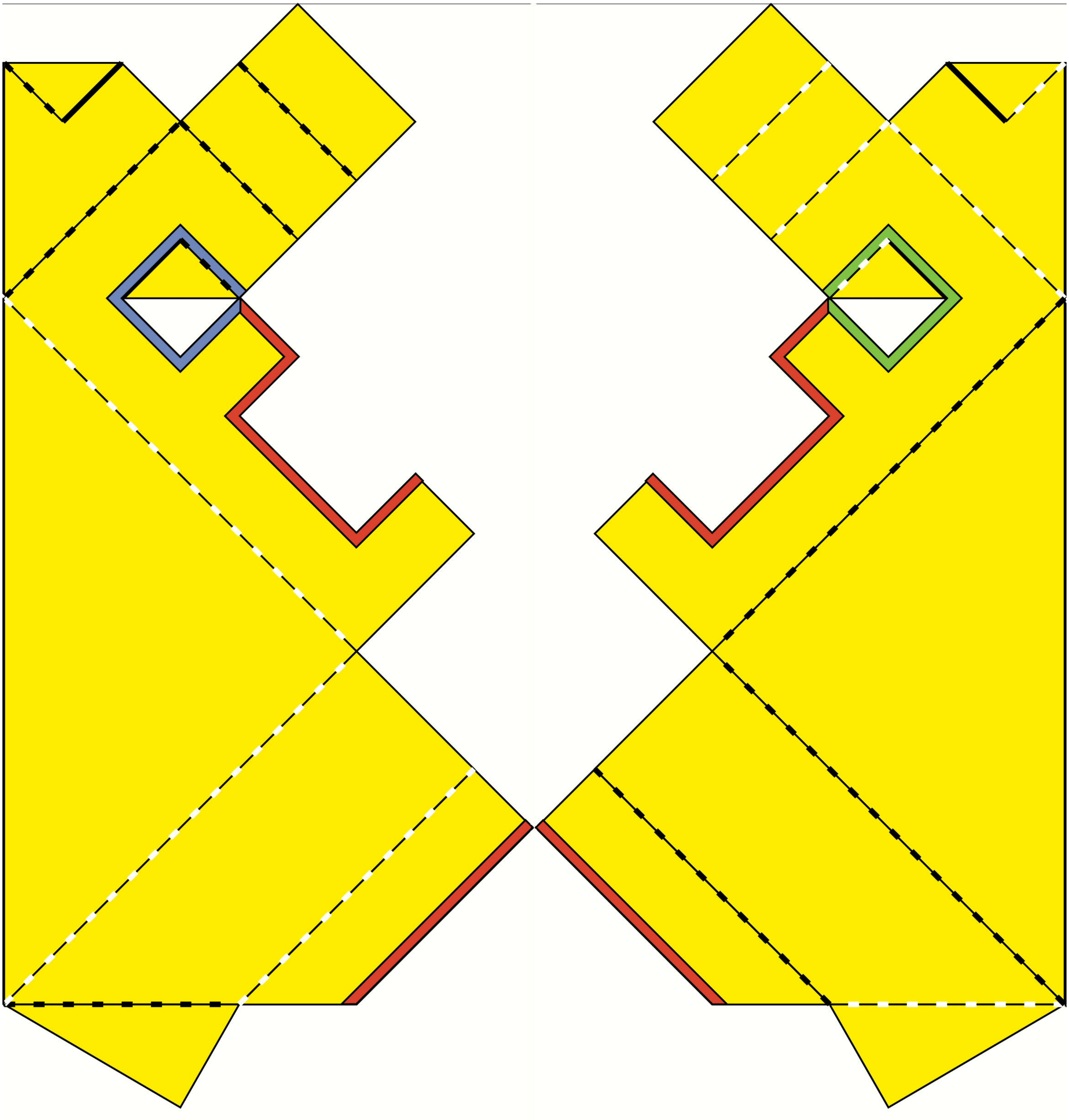


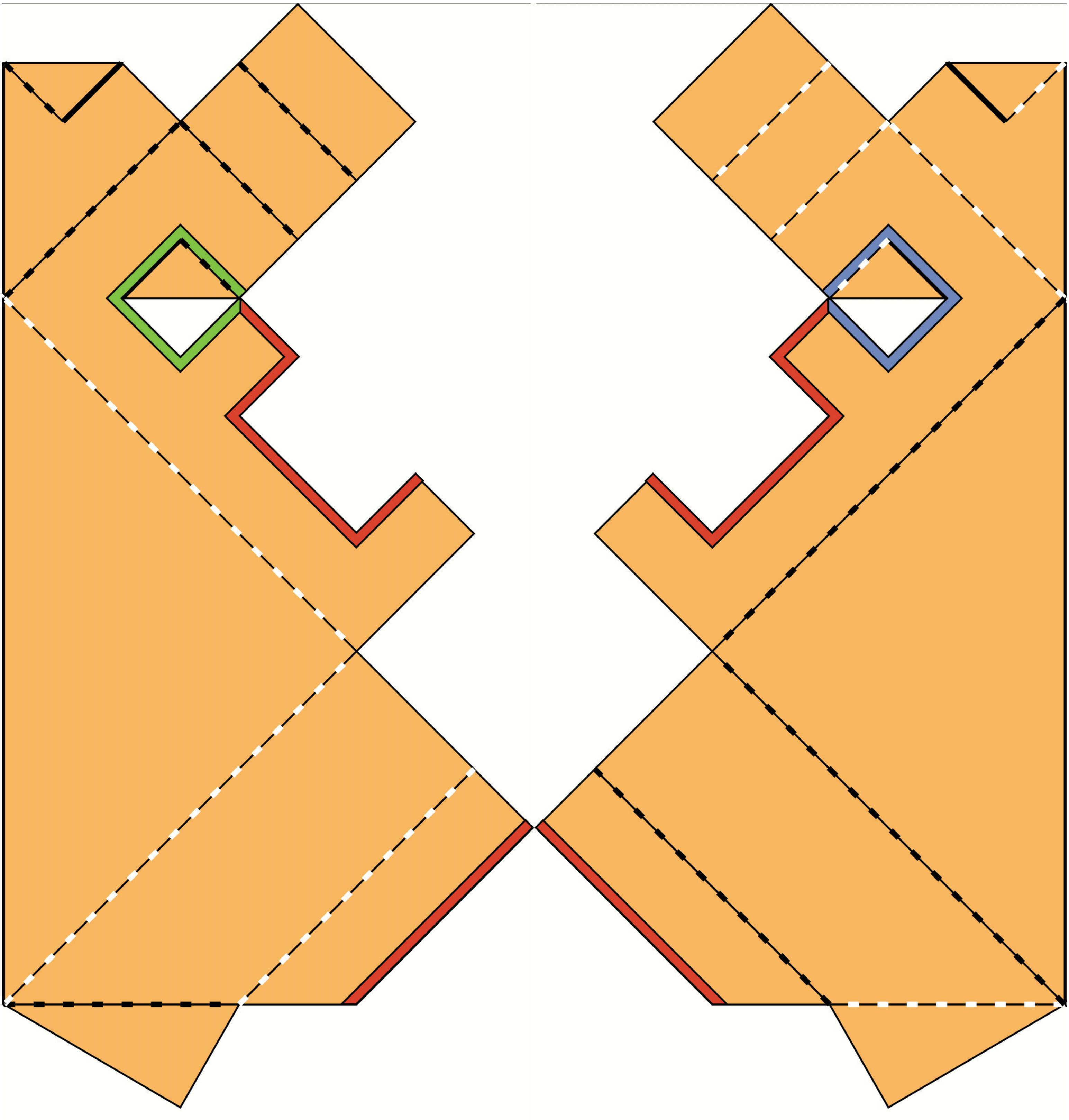












Epilogue

When Alex Mellnik saw my cubist model of the Girl cap, he remembered that Apéry also had a sketch of a cubist model of his second representation of the projective plane in his book [1]. As soon as I got a hold of the relevant figures (Fig.E1 and E2), I tried to construct the corresponding model.

1.2 The Real Projective Plane 23

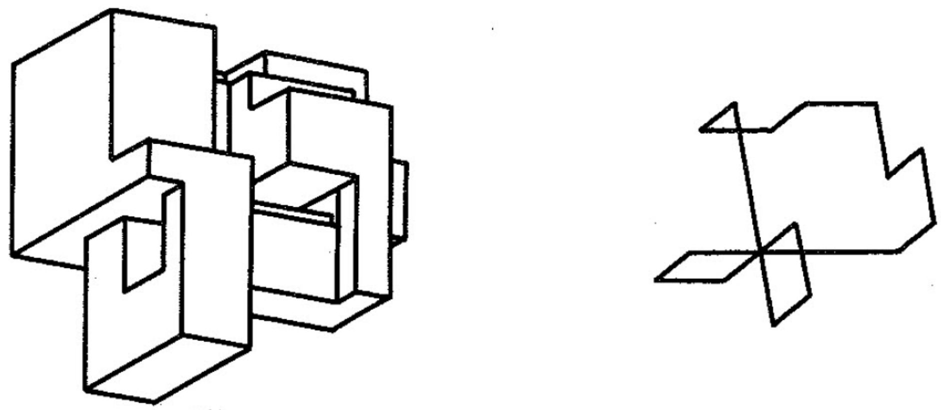


Fig. 16 Nonstandard combinatorial model of the projective plane in $\mathbb{R}^3$ corresponding to the construction of Fig. 14, and its self-intersection curve.

Figure E1: *Apéry's second polygonal model of the projective plane and its intersection loops.*

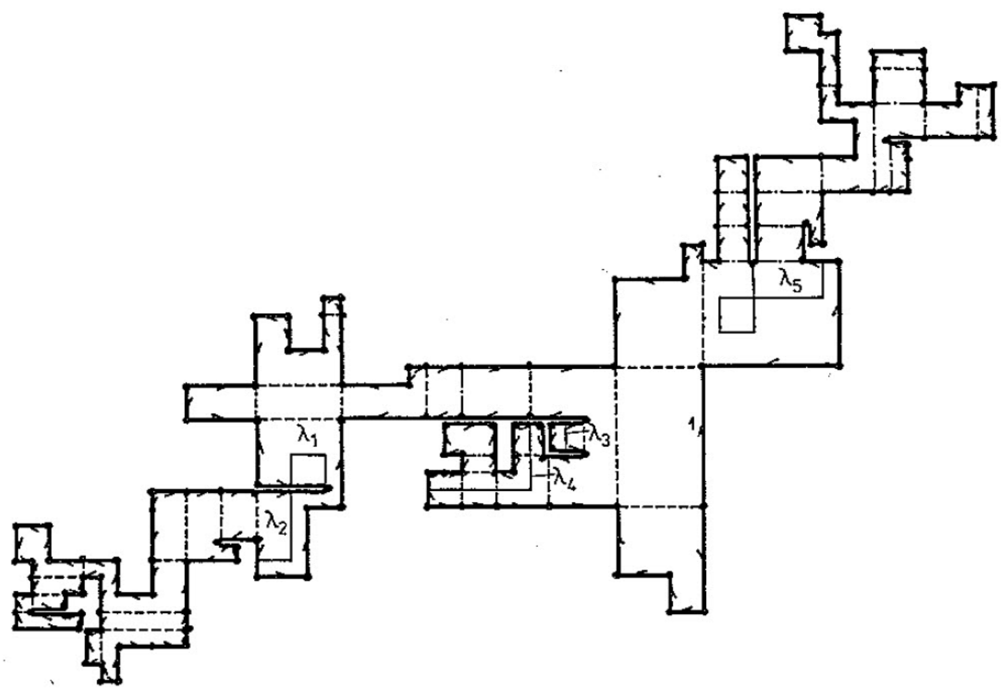


Fig. 14 Fold and glue as for Fig. 10. The self-intersection curve is the image of (cf. Fig. 15)

Figure E2: *Apéry's sketch of the net of his polyhedral model.*

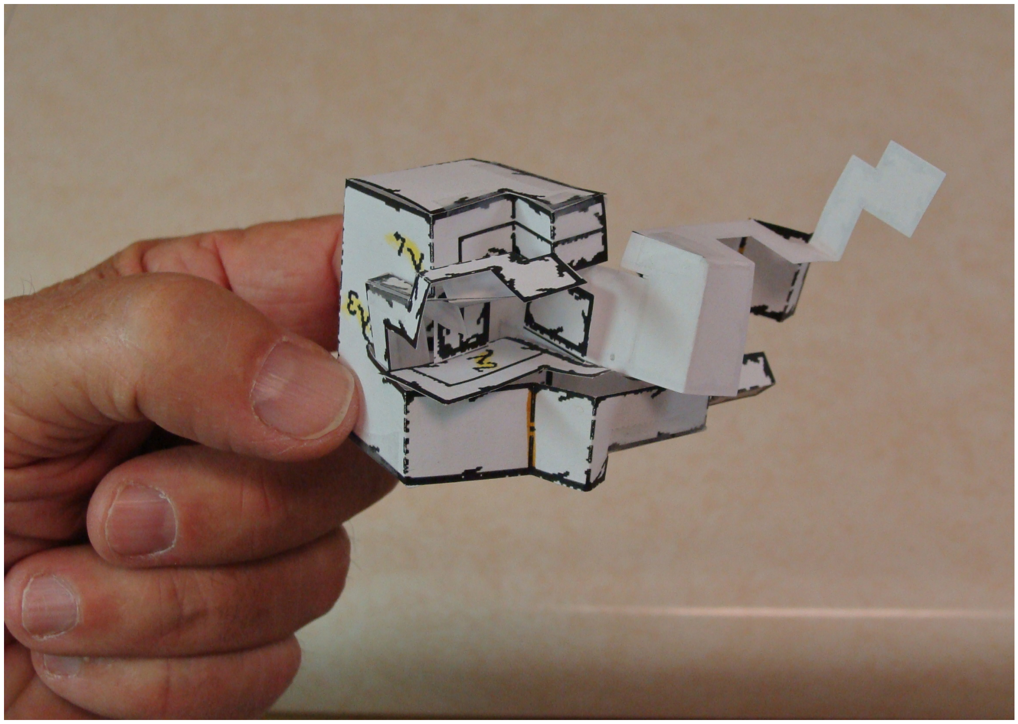


Figure E3: *My first attempt at folding up the net presented in Apéry's book.*

My first attempt was not very successful. First the net was too small, and I had constructed some of the polyhedral elements in the wrong order; I had assembled some of the larger, easier-to-understand boxes, before I had completed some of the tubular structures that were supposed to go inside. But this attempt still convinced me that this net would actually form the depicted cubist model. It also gave me a good understanding of Apéry's model sketch and allowed me to color his drawing based on the orientation of the polygonal facets. I also marked visible intersection line segments and aligned the intersection diagram with the surface model (Fig. E4).

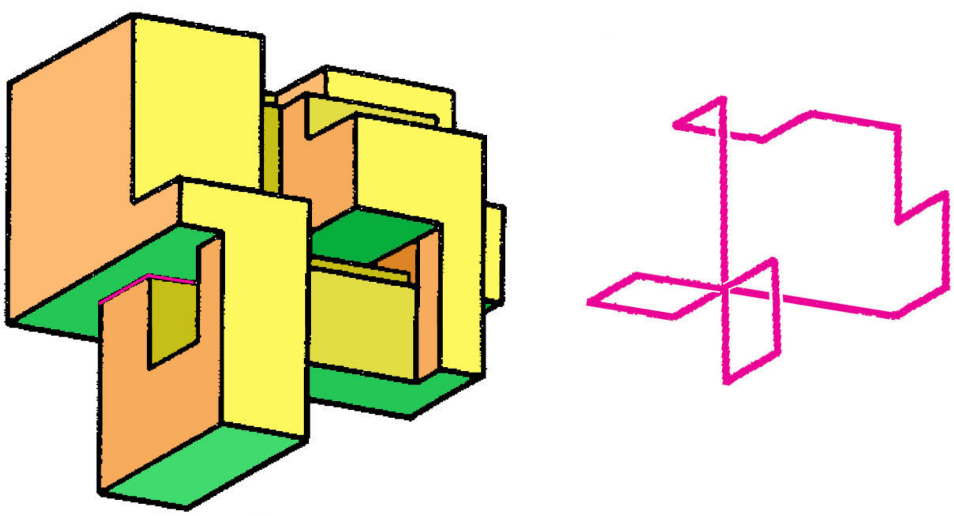


Figure E4: *Color-enhanced Apéry model with marked parts of the self-intersection lines.*

In addition, this exercise gave me a much better understanding of the net and allowed me to clearly distinguish between mountain-folds (green) and valley-folds (orange), as well as to enhance the self-intersection lines (magenta in Fig.E5). I then made an enlarged version of this net and broke it into three parts for easier printing and easier assembly. The assembly of those three components was then rather straight forward (Fig.E6).

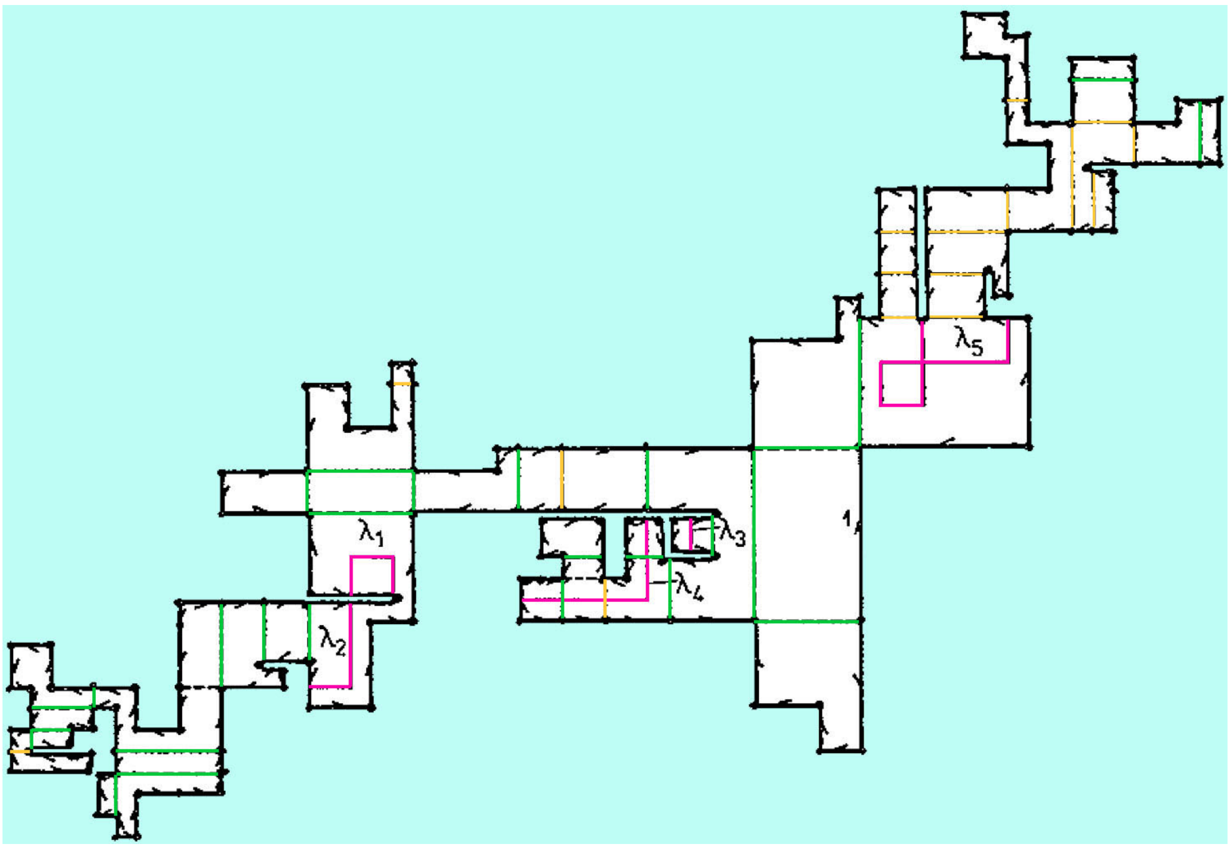
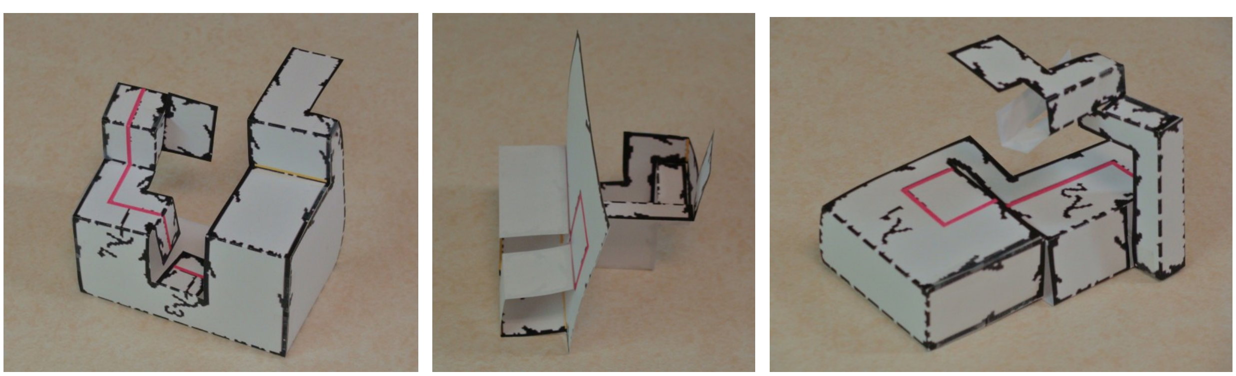


Figure E5: *Apéry net with color-coded fold lines and intersection line segments.*



(a) (b) (c)

Figure E6: *The three individually folded-up components.*

Figure E7 shows components E6a and E6c merged and glued together from two different views. This step was not too tricky; but adding the third component (Fig.E6b) into the assembly was really difficult (Fig.E8). Fortunately, paper is really flexible, and torn paper can readily be taped again!

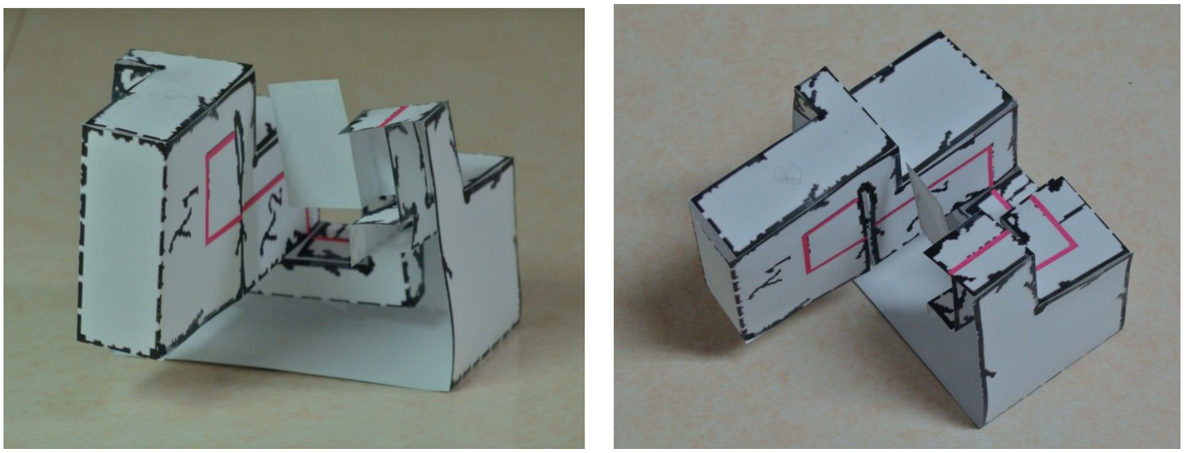


Figure E7: *Components E6a and E6b merged.*

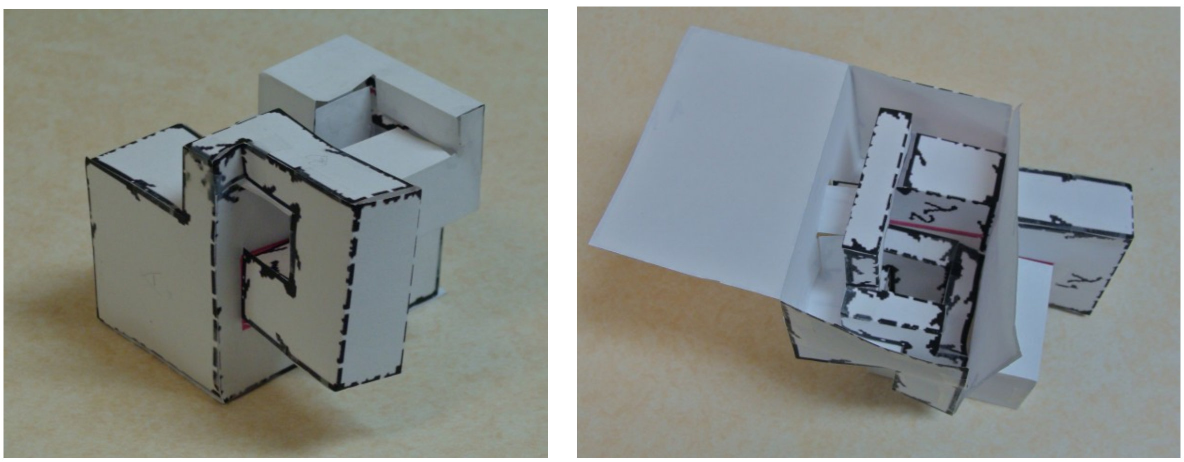


Figure E8: All three components merged. On the right the model has been turned upside down and the bottom face has been opened to show how much of the geometry is hidden inside this big box.

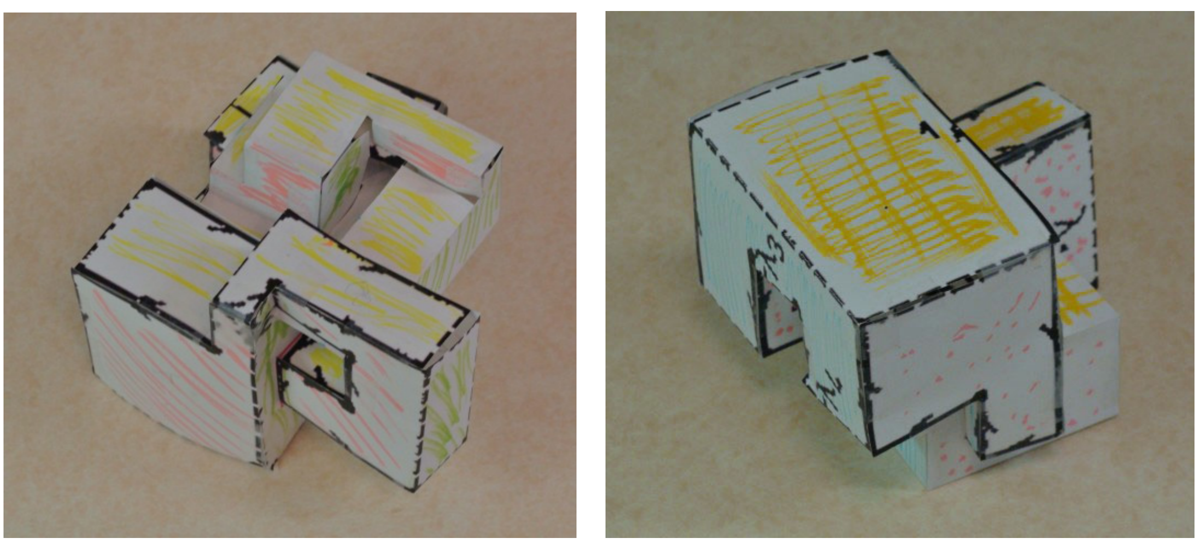


Figure E9: *Faces of the complete model are color coded based on their orientation.*

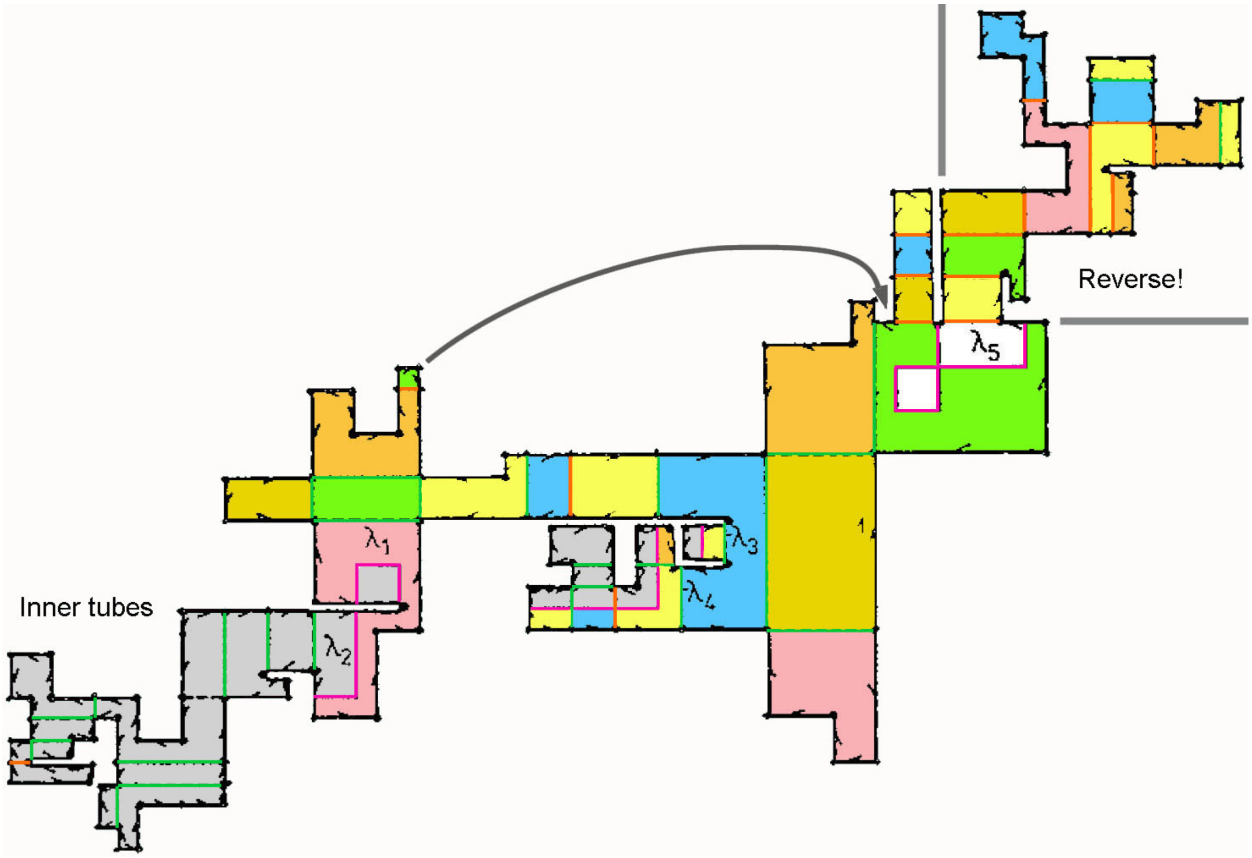


Figure E10: *The net colored in accordance with the coloring of the model in Figure E9.*

On this model I marked all visible faces with 6 different colors, based on their orientation in space and then transferred those colored areas back onto the net (Fig.E10). Gray denotes inner, invisible surfaces. There are a few additional improvements that I would like to see in this net, in order to make it easier to construct a paper model of this surface: First the geometry needs to be cleaned up so that all edges actually line up on an integer grid; the wide gaps between adjacent, independent faces are not appropriate. In the upper right hand corner, the geometry should be reversed so that the colors printed on only one side of the paper will appear on the visible side of the model. Moreover, the net should be partitioned more logically, so that faces that "naturally" seem to belong together, as for instance the four walls of a prismatic tube, are kept together. This would also entail avoiding to split any connected coplanar regions, as occurs now with the largest green face, where indicated by the curved grey arrow. All this should then make the model much easier to assemble.

Because one thing one thing has become absolutely clear to me: The best way to understand Girl's surface is to build a paper model!