Contention Bounds for Combinations of Computation
Graphs and Network Topologies

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ABSTRACT

Network topologies can have significant effect on the costs of algorithms due to inter-processor communication. Parallel algorithms that ignore network topology can suffer from contention along network links. However, for particular combinations of computations and network topologies, costly network contention may inevitably become a bottleneck, even for optimally designed algorithms. We obtain a novel contention lower bound that is a function of the network and the computation graph parameters. To this end, we compare the communication bandwidth needs of subsets of processors and the available network capacity (as opposed to per-processor analysis in most previous studies). Applying this analysis we improve communication cost lower bounds for several combinations of fundamental computations on common network topologies.

Categories and Subject Descriptors

F.2.1 [Analysis of Algorithms and Problem Complexity]: Numerical Algorithms and Problems—Computations on matrices

General Terms


Keywords

Network topology, Communication-avoiding algorithms, Strong scaling, Communication costs.

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1. INTRODUCTION

Good connectivity of the inter processor network is necessary for fast execution of parallel algorithms. Insufficient graph-expansion of the network provably slows down specific parallel algorithms that are communication intensive. While parallel algorithms that ignore network topology can suffer from contention along network links, for particular combinations of computations and network topologies, costly network contention may be inevitable, even for optimally designed algorithms. In this paper we obtain novel lower bounds on such contention cost, and point to cases where this cost is a performance bottleneck.

We use a variant of the distributed-memory communication model (cf, [16, 19, 11]), where the bandwidth-cost of an algorithm is proportional to the number of words communicated by one processor (we omit the latency cost / message count discussion from this work). As in the distributed-memory communication model we have $P$ processors and a local memory of size $M$ for each processor. However, here, we do not assume all-to-all connectivity, but rather some network graph $G_{Net}$ with $P$ vertices. In this work we assume all edges (network links) have the same bandwidth, and the nodes of the network are both processors and routers (i.e. a direct network, where no node is solely a router). We ignore processor injection rates in this model.

Most previous communication cost lower bounds for parallel algorithms utilize per-processor analysis. That is, the lower bounds establish that some processor must communicate a given amount of data. These include classical matrix multiply, direct and iterative linear algebra algorithms, FFT, Strassen and Strassen-like fast algorithms, graph related algorithms, $N$-body, sorting, and others (cf. [3, 25, 23, 32, 26, 11, 9, 15, 22, 8, 28, 35, 21, 34]).

By considering the network graphs, we introduce communication lower bounds for certain computations and networks that are tighter than what was previously known. We bound from below the number of words communicated between a subset of processors and the rest of the processors for a given parallel algorithm (defined by a computation graph and work assignment to the processors), and divide it by the number of words that the network is capable of communicating simultaneously between that subset of processors and the rest of the graph. This relates to the contention cost of the algorithm, which we specify in Definition 2.2. Applying the main theorem we improve (i.e., increase) communication cost lower bounds for several combinations of fundamental
computations on common network topologies. Note that we inherits any assumptions made in the original per-processor lower bounds, e.g., no recomputation. These contention bounds may suggest directions for hardware/network design tailored for heavily used computation kernels and may assist when scheduling users’ applications on (a subset of) a supercomputer.

2. CONTENTION LOWER BOUND

In this section we state our main result, which translates per-processor bandwidth cost lower bounds to contention cost lower bounds. The following definitions differentiate these costs.

Definition 2.1. Let a parallel algorithm be run on a parallel distributed-memory machine with P processors. The per-processor bandwidth cost \( W_{\text{proc}} \) is the maximum over processors \( 1 \leq p \leq P \) of the number of words sent or received by processor \( p \).

Observe that for \( W_{\text{proc}} \) we can plug in two types of per-processor lower bounds: memory-independent \( W_{\text{proc}}(P, N) \) (cf. [9]) and memory-dependent \( W_{\text{proc}}(P, M, N) \) (cf. [26, 11, 12, 21]) where \( N \) is the input and output data size.

Definition 2.2. Let a parallel algorithm be run on a parallel distributed-memory machine with network graph \( G_{\text{Net}} \). For the corresponding per-processor lower bound deducing that the processors in \( s_t \) require at least \( W_{\text{proc}}(P/t, M \cdot t, N) \) words to be sent/received to the processors outside \( s_t \) throughout the running of the algorithm. At most \( d \cdot t \cdot h_t(G_{\text{Net}}) \) edges connect \( s_t \) to the rest of the graph. Hence at least one edge communicates at least \( \frac{W_{\text{proc}}(P/t, M \cdot t, N)}{d \cdot t \cdot h_t(G_{\text{Net}})} \) words. Since \( t \) is a free parameter, we can pick it to maximize \( W_{\text{link}}(P, M, N) \), and the theorem follows. □

Note that the memory-independent contention lower bound, \( W_{\text{link}} = W_{\text{link}}(P, N) \), follows.

\[ h_s(G) = \min_{S \subseteq V, |S| \leq s} \frac{|E(S, V \setminus S)|}{|E(S)|} \]

where \( E(S) \) is the set of edges that have at least one endpoint in vertex subset \( S \) and \( E(S, S) \) is the set of edges with only one endpoint in \( S \). The cardinality of a set \( S \) is represented by \( |S| \), the set of d-regular graphs, \( |E(S)| \leq d|S| \).

Theorem 2.3. Consider a distributed-memory machine with P processors, each with local memory of size \( M \), and an inter-processor network graph \( G_{\text{Net}} \). Given a computation with input and output data size \( N \), and lower bound on the memory-dependent per-processor bandwidth cost \( W_{\text{proc}}(P, M, N) \), for all algorithms that distribute the workload so that every processor performs \( \Omega(1/P) \) of the computation, and distributing the input and output data such that every processor stores \( O(1/P) \) of the data, the memory-dependent contention cost \( W_{\text{link}}(P, M, N) \) is bounded below by

\[ W_{\text{link}}(P, M, N) \geq \max_{t \in T} \frac{W_{\text{proc}}(P/t, M \cdot t, N)}{d \cdot t \cdot h_t(G_{\text{Net}})} \]

where

\[ T = \{ t : 1 \leq t \leq P/2, \exists S \subseteq V \text{s.t. } |S| = t \text{ and } h_t(G) = |E(S, V \setminus S)|/|E(S)| \}. \]

Proof. Consider a partitioning of the \( P \) processors into \( P/t \) subsets of size \( t \) (w.l.o.g., \( P \) is divisible by \( t \)), where at least one of the subsets \( s_t \) is connected to the rest of the network graph with at most \( d \cdot t \cdot h_t(G_{\text{Net}}) \) edges. \(^1\)

The existence of such a set \( s_t \) is guaranteed by the definition of \( h_t(G_{\text{Net}}) \) and \( T \). Then \( s_t \) has a total of \( M \cdot t \) local memory.

By the workload distribution assumption, the processors in \( s_t \) perform a fraction \( \Omega(t/P) \) of the flops, and by the data distribution assumption, \( s_t \) has local access to fraction \( O(t/P) \) of the input/output. Hence we can emulate this computation by a parallel machine with \( P/t \) processors, each with \( M \cdot t \) local memory (see Figure 1), and apply the corresponding per-processor lower bound deducing that the processors in \( s_t \) require at least \( W_{\text{proc}}(P/t, M \cdot t, N) \) words to be sent/received to the processors outside \( s_t \) throughout the running of the algorithm. At most \( d \cdot t \cdot h_t(G_{\text{Net}}) \) edges connect \( s_t \) to the rest of the graph. Hence at least one edge communicates at least \( \frac{W_{\text{proc}}(P/t, M \cdot t, N)}{d \cdot t \cdot h_t(G_{\text{Net}})} \) words. Since \( t \) is a free parameter, we can pick it to maximize \( W_{\text{link}}(P, M, N) \), and the theorem follows. □

Note that the memory-independent contention lower bound, \( W_{\text{link}} = W_{\text{link}}(P, N) \), follows.

Figure 1: Computation of \( t = 4 \) processors on a 16-processor machine can be emulated as the computation of one processor on a 4-processor machine.

3. PRELIMINARIES

3.1 Per-Processor Lower Bounds

Before deriving bounds on link contention, we review the per-processor communication bounds for several classes of algorithms.

Classical Linear Algebra.

Most classical direct linear algebra computations can be specified by three nested loops, and for dense \( n \times n \) matrices, the number of flops performed is \( \Theta(n^3) \).\(^2\) Informally, such computations, which include matrix multiplication, Cholesky and LU decompositions, and many others, can be defined by

\[ C_{i,j} = f_{ij}(g_{ijk}(A_{ik}, B_{kj})) \]

\[ 1 \leq i, j \leq n \quad (1) \]

where \( f \) and \( g \) are sets of functions particular to the computation. For example, in the case of classical matrix multiplication, \( f_{ij} \) is a summation and \( g_{ij,k} \) is a scalar multiplication for all \( i, j, k \). For a more formal definition, see [7, Definition

\[^1\]Note that \( s_t \) is connected to the rest of the network graph with exactly \( d \cdot t \cdot h_t(G_{\text{Net}}) \) edges only when \( |E(S)| = d|S| \).

\[^2\]For matrix computations, we denote the size of the input/output to be \( N = \Theta(n^2) \).
For such computations, we have the following lower bound:

**Theorem 3.1** ([11], [26]). Consider an algorithm performing a computation of the form given by equation (1) on $P$ processors, each with local memory of size $M$, and assume one copy of the input data is initially distributed across processors and the computation is load balanced. Then the number of words some processor must communicate is at least

$$W_{\text{proc}}(P,M,N) = \Omega \left( \frac{n^3}{PM^{1/2}} \right).$$

Note that the local memory size $M$ appears in the denominator of the expression above, which is why we refer to it as the memory-dependent bound. Additionally, such computations also inherit a memory-independent lower bound:

**Theorem 3.2** ([9]). Consider an algorithm performing a computation of the form given by equation (1) on $P$ processors, and assume just one copy of the input data is initially distributed across processors and the computation is load balanced. Then the number of words some processor must communicate is at least

$$W_{\text{proc}}(P,N) = \Omega \left( \frac{n^2}{P^{2/3}} \right).$$

**Strassen-like Matrix Multiplication.**

Similar lower bounds exist for Strassen’s matrix multiplication and similar algorithms, though the proof techniques differ substantially. Informally, we use the term “Strassen-like” to refer to algorithms that recursively multiply matrices according to a base-case computation. For square algorithms, this corresponds to multiplying $m_0 \times m_0$ matrices with $m_0$ scalar multiplications, where $m_0$ and $m_0$ are constants. Using recursion, this results in a square matrix multiplication flop count of $\Theta(n^{\omega_0})$ where $\omega_0 = \log_{m_0} m_0$. Note that additional technical assumptions are required for the communication lower bounds to apply and that Strassen-like algorithms may have a rectangular base case; see [12, Section 5.1] for more details. The memory-dependent communication lower bound for Strassen-like algorithms is:

**Theorem 3.3** ([12, Corollary 1.5]). Consider a Strassen-like matrix multiplication algorithm that requires $\Theta(n^{\omega_0})$ total flops. Suppose a parallel algorithm performs the computation using $P$ processors (each with local memory of size $M$), load balances the flops, and performs no redundant computation. Then the number of words some processor must communicate is at least

$$W_{\text{proc}}(P,M,N) = \Omega \left( \frac{n^{\omega_0}}{PM^{\omega_0/2 - 1}} \right).$$

Additionally, such computations also inherit a memory-independent lower bound:

**Theorem 3.4.** Suppose a parallel algorithm performs a Strassen-like matrix multiplication algorithm requiring $\Theta(n^{\omega_0})$ flops, load balances the computation across $P$ processors, and performs no redundant computation. Then under some technical assumptions (see [12]) the number of words some processor must communicate is at least

$$W_{\text{proc}}(P,N) = \Omega \left( \frac{n^2}{P^{2/3 \omega_0}} \right).$$

**Proof.** Identical to Theorem 2.1 in [9], with $\omega_0$ replacing $\log_7$. ☐

**Programs Referencing Arrays.**

The model defined in Equation (1) encompasses most direct linear algebra computations, but lower bounds can be obtained for a more general set of computations. In particular, Christ et al. [21] consider programs of the following form:

$$\text{inner\_loop}(\mathbb{I}, (A_1, \ldots, A_m), (\phi_1, \ldots, \phi_m))$$

where $\mathbb{I} \subset \mathbb{Z}^d$, in some order,

$$\text{inner\_loop}(\mathbb{I}, (A_1, \ldots, A_m), (\phi_1, \ldots, \phi_m)) = \Omega \left( \frac{|\mathbb{I}|}{P^{m \cdot \text{HBL}}} \right),$$

where $\text{HBL}$ is the d-dimensional space of integers and $\text{inner\_loop()}$ represents a computation involving arrays $A_1, \ldots, A_m$ of dimensions $d_1, \ldots, d_m$ that are referenced by the corresponding subscripts $\phi_1(\mathbb{I}), \ldots, \phi_m(\mathbb{I})$ where $\phi_i$ are affine maps $\phi_i : \mathbb{Z}^d \to \mathbb{Z}^d$. For example, matrix-matrix multiplication has $(A_1, A_2, A_3) = (A, B, C)$, $\phi_1(\mathbb{I}) = \phi_1(i_1, i_2, i_3) = (i_1, i_3)$, $\phi_2(\mathbb{I}) = \phi_2(i_1, i_2, i_3) = (i_2, i_3)$, $\phi_3(\mathbb{I}) = \phi_3(i_1, i_2, i_3) = (i_1, i_2)$ and the function inner\_loop() is defined as $A_1(\phi_3(\mathbb{I})) = A_1(\phi_1(\mathbb{I})) + A_3(\phi_2(\mathbb{I}))$.

Because the work inside the loop is currently defined as a general function, the space of potential executions of inner\_loop() must be restricted in a manageable manner, or to “legal parallel executions” as defined in [21]. To express the lower bounds, we define a set of linear constraints on a vector of unknown scalars $(s_1, \ldots, s_m)$

$$\text{rank}(H) \leq \sum_{j=1}^m s_j \text{rank}(\phi_j(H)),$$

for all subgroups $H \subset \mathbb{Z}^d$, where rank$(H)$ is the cardinality of any maximal subset of Abelian group $H$ that is linearly independent.3 For such computations we have the following lower bound:

**Theorem 3.5** ([21]). Consider an algorithm performing a computation of the form given by equation (2) on $P$ processors, each with local memory of size $M$, and assume the input data is initially evenly distributed across processors. Then for any legal parallel execution and sufficiently large $|\mathbb{I}|/P^m$, the number of words some processor must communicate is at least

$$W_{\text{proc}}(P,M,N) = \Omega \left( \frac{|\mathbb{I}|}{P^{m \cdot \text{HBL}}} \right),$$

where $s_{\text{HBL}}$ is the minimum value of $\sum_{i=1}^m s_i$ subject to (3), assuming that this linear program is feasible (see [21]).

We restate the memory-independent bound from [21] for such computations (note that the formal proof has not yet appeared). For legal parallel executions of computations of the form (2) on $P$ processors, some processor must move

$$W_{\text{proc}}(P,N) = \Omega \left( \left( \frac{|\mathbb{I}|}{P^m} \right)^{1/s_{\text{HBL}}} - \frac{N}{P} \right)$$

words where $N$ is the sum of the sizes of arrays $\{A_i\}$ (assumed to be evenly distributed across processors) and $s_{\text{HBL}}$ is defined as in Theorem 3.5. In most cases, the negative

3 The rank of an Abelian group is analogous to the concept of the dimension of a vector space.
The degree of each vertex is \( D \). Bounds are tight, up to a constant factor. For a tighter lower bound on\( n \)-body communication network between cores [27]. In this section, we derive contention lower bounds by plugging the memory-dependent and memory-independent per-processor lower bounds [26, 12, 9, 21] into Theorem 2.3 and using the properties of \( D \)-dimensional tori. Table 1 summarizes these results. In the algebra that follows, we assume the network topology to be a \( D \)-dimensional torus or mesh.

**Direct Linear Algebra, Strassen, Strassen-like, \( O(n^2) \) n-body algorithms.**

We apply Theorem 2.3 to the relevant per-processor bounds given in Section 3.1. Let \( F \) denote the number of work operations (e.g. flops or loop iterations) of the different computations. The per-processor memory-bound lower bound is thus:

\[
W_{\text{proc}}(P, M, N) = \Omega \left( \frac{F}{P^\alpha M^{n-1}} \right)
\]

where \( \alpha = 3/2 \) for direct dense linear algebra, \( \alpha = \omega_0/2 \) for Strassen-like matrix multiplication, \( \alpha = 2 \) for the \( O(n^2) \) n-body problem. We next apply Theorem 2.3 to (5). By Lemma 3.7, for a \( D \)-dimensional torus, the denominators of the contention bounds in Theorem 2.3 and Expression (7) are \( 2D \cdot t \cdot \Theta(t^{-1/D}) \). Thus, the memory-dependent contention bound is:

\[
W_{\text{link}}(P, M, N) = \max_{t \in T} \Omega \left( \frac{F}{P^\alpha M^{\alpha-1}} \cdot t^{1-\alpha+1/D} \right)
\]

Note that \( t^{1-\alpha+1/D} \) is monotonic (in the given range), but that the exponent can be positive, negative or zero. If the exponent of \( t \) is negative or zero, then the expression is maximized at \( t = 1 \), reproducing the per-processor bound (up to a constant factor). If the exponent is positive, namely \( D \leq D_t = 1/(\alpha - 1) \), then the expression is maximized at...


\[ t = P/2, \text{ and we obtain a new and tighter bound}^4: \]

\[ W_{\text{link}}(P, M, N) = \Omega \left( \frac{F}{P^{\alpha - 1} D M^{\alpha - 1}} \right). \]  

(7)

The per-processor memory-independent bound is

\[ W_{\text{proc}}(P, N) = \Omega \left( \frac{N}{P^{1/\alpha}} \right) \]

(8)

We next apply Theorem 2.3 to (8) and obtain:

\[ W_{\text{link}}(P, N) = \max_{t \in T} \Omega \left( \frac{N}{P^{1/\alpha}} \cdot t^{1/\alpha - 1 + 1/D} \right) \]

(9)

Again, \( t^{1/\alpha - 1 + 1/D} \) is monotonic and may be positive, negative or zero. If the exponent of \( t \) is negative or zero, then the expression is maximized at \( t = 1 \), reproducing the per-processor bound (up to a constant factor). If the exponent is positive, namely \( D \leq D_2 = \alpha/\alpha - 1 \), then the expression is maximized at \( t = P/2 \), and we obtain a new and tighter bound:

\[ W_{\text{link}}(P, N) = \Omega \left( \frac{N}{P^{1/\alpha - 1/D}} \right). \]

(10)

Table 1 presents the communication lower bounds for each of the computations described in Sections 3.1 on \( D \)-dimensional tori with the respective values of \( F \) and \( \alpha \).

**Programs that Reference Arrays.**

Note that if we assume that \( F = O(N^\alpha) \) in the memory-independent lower bound for programs that reference arrays with \( \alpha = s_{\text{HBL}} \), we arrive at the form of this bound used for the derivation of the direct linear algebra, Strassen, Strassen-like and \( O(n^2) \) n-body contention bounds. In general, this does not have to be the case for the set of programs defined by Expression 2 above.

According to Theorem 3.5, the memory-dependent per-processor bandwidth lower bound for programs defined by Expression 2 is

\[ W_{\text{proc}}(P, M, N) = \Omega \left( \frac{|Z|}{P M^{s_{\text{HBL}} - 1}} \right). \]

Similar to the derivation for the previous problems (albeit with \( \alpha = s_{\text{HBL}} \)), the bound becomes

\[ W_{\text{link}}(P, M, N) = \max_{1 \leq t \leq P/2} \Omega \left( \frac{|Z|}{P M^{s_{\text{HBL}} - 1}} \cdot t^{1-s_{\text{HBL}} + 1/D} \right) \]

which is maximized at either \( t = 1 \) (the per-processor bound), or \( t = P/2 \) (see Footnote 4). So, we obtain

\[ W_{\text{link}}(P, M, N) = \Omega \left( \frac{|Z|}{P M^{s_{\text{HBL}} - 1/D}} \right) \]

as a memory-dependent lower bound on contention. In a similar manner, we can derive a memory-independent contention lower bound. From Equation (4), the memory-independent per-processor bound is

\[ W_{\text{proc}}(P, N) = \Omega \left( \frac{|Z|}{P M^{s_{\text{HBL}} - 1}} \right) \]

assuming we drop the \( N/P \) term from the bound. At \( t = P/2 \) (as again we observe that the contention bound is maximized at either \( t = 1 \) or \( t = P/2 \)), we derive the memory-independent lower bound on contention

\[ W_{\text{link}}(P, N) = \Omega \left( \frac{|Z|}{P M^{s_{\text{HBL}} - 1/D}} \right). \]

**FFT/Sorting.**

As with the previous algorithms, we apply Theorem 2.3 to the relevant per-processor bound given in Section 3.1.

The per-processor memory-independent bound is thus

\[ W_{\text{proc}}(P, N) = \Omega \left( \frac{n \log(n)}{P \log(n/P)} \right). \]

(11)

We next apply this bound to Theorem 2.3 and obtain:

\[ W_{\text{link}}(P, N) = \max_{1 \leq t \leq P/2} \Omega \left( \frac{n \log(n)}{P \log(nt/P)} t^{1/D} \right) \]

(12)

\[ = \frac{n \log(n)}{P} \max_{1 \leq t \leq P/2} \Omega \left( \frac{t^{1/D}}{\log(nt/P)} \right). \]

Again, when \( t = 1 \) we obtain the original per-processor bound. Equation 12 has a stationary point at \( t = P C D / n \) (where \( C \) is the base of the logarithm), but via consideration of the second derivative wrt to \( t \), it can be shown that this point is a minima for all relevant values of \( n, P \) and \( D \). Thus, we can derive a memory-independent contention bound by setting \( t = P/2 \) (see Footnote 4):

\[ W_{\text{link}}(P, N) = \Omega \left( \frac{N}{P^{1/\alpha - 1/D}} \right) \]

(13)

as \( N = O(n) \).

**4.2 Analysis and Interpretation**

**Which bound dominates?**

Our first observation is that, for these computations, the memory-independent contention bound dominates the memory-dependent contention bound for many algorithms. In the cases of direct linear algebra, Strassen and Strassen-like, and the \( O(n^2) \) n-body problem we prove this by contradiction: if the memory-dependent contention bound dominates, then the problem is too large to be distributed across all the processors’ local memories. Thus, if

\[ \frac{F}{P^{\alpha - 1/D} M^{\alpha - 1}} > \frac{N}{P^{1/\alpha - 1/D}} \]

then, as \( F = O(N) \), we have

\[ N^{\alpha - 1} > \frac{P^{\alpha - 1} M^{\alpha - 1}}{D} \]

which is a contradiction as we assumed that \( N \leq PM \). For programs that reference arrays, the proof requires a bit more of the theoretical apparatus from [21] and is proven in Appendix B. We note that in practice the value of constants

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\(^4\) Note that there may not be a subset of the vertices of \( G_{N_e} \) that attains the small set expansion \( h_t(G_{N_e}) \) of size *exactly* \( P/2 \). However, the small set expansion of tori and meshes is attained for small sets of size \( P/c \) for some constant \( c \geq 2 \) (e.g. consider a sub-tori), hence the following contention analysis holds up to a constant factor.
may result in the memory-dependent contention bound being dominant, despite the asymptotic result.

For direct linear algebra, Strassen, Strassen-like and \(O(n^2)\) n-body algorithms, Figure 2 illustrates the relationships between the four types of bounds for a fixed computation, fixed problem size \(N\), and fixed local memory size \(M\), varying the number of processors \(P\) and the torus dimension \(D\). See Appendix A for the derivation of the expressions used in Figure 2.

Depending on the dimension of the torus and number of processors, the tightest bound may be one of the previously known per-processor bounds or the memory-independent contention bound. We first consider subdividing the vertical axis of Figure 2, which corresponds to the torus dimension \(D\). Intuitively speaking, the smaller \(D\) is, the more likely contention will dominate communication costs. For a given algorithm, we let \(D = \lfloor 1/(\alpha - 1) \rfloor\) be the maximum torus dimension such that the communication cost is dominated by contention for all input and machine parameters. Similarly, we let \(D = \lfloor \alpha/(\alpha - 1) \rfloor\) be the minimum torus dimension so that the communication cost is not dominated by the contention (at least not by the bound proved here). Note that for a combination of an algorithm and a \(D\)-dimensional torus such that \(D_1 < D < D_2\), either the per-processor memory-dependent bound or the memory-independent contention bound may dominate. See Table 2 for values of \(D_1\) and \(D_2\) for various matrix multiplication algorithms. In particular, note that for the classical algorithm, a 2D torus is not sufficient to avoid contention. While Cannon’s algorithm [18] does not suffer from contention on a 2D torus network, it is also not communication-optimal. The more communication-efficient “3D” algorithms [14, 4, 31, 36], which utilize extra memory and have the ability to strong scale perfectly, require a 3D torus to attain the per-processor lower bounds. For matrix multiplication algorithms with smaller exponents, the torus dimension requirements for remaining contention-free are even larger.

| Table 1: Per-processor bounds (\(W_{\text{proc}}\)) [26, 11, 9, 12, 15] vs. the new contention bounds (\(W_{\text{link}}\)) on a \(D\)-dimensional torus for classical linear algebra, fast matrix multiplication, \(O(n^2)\) n-body, Fast Fourier Transform (FFT) and a general set of programs that reference arrays. |
|---|---|---|
| Direct Linear Algebra | Memory Dependent | Memory Independent |
| \(W_{\text{proc}}\) | \(\Omega\left(\frac{n^3}{P M^{1/7}}\right)\) | \(\Omega\left(\frac{n^2}{P^{6/7}}\right)\) |
| \(W_{\text{link}}\) | \(\Omega\left(\frac{n^2}{P^{2/7} M^{1/7}}\right)\) | \(\Omega\left(\frac{n^2}{P^{1/7}}\right)\) |
| Strassen and Strassen-like | Memory Dependent | Memory Independent |
| \(W_{\text{proc}}\) | \(\Omega\left(\frac{n^3}{P M^{0.25}}\right)\) | \(\Omega\left(\frac{n^2}{P^{0.25}}\right)\) |
| \(W_{\text{link}}\) | \(\Omega\left(\frac{n^2}{P^{0.5} M^{0.25}}\right)\) | \(\Omega\left(\frac{n^2}{P^{1/7}}\right)\) |
| \(O(n^2)\) n-body | Memory Dependent | Memory Independent |
| \(W_{\text{proc}}\) | \(\Omega\left(\frac{n^2}{P^{1/3}}\right)\) | \(\Omega\left(\frac{n}{P^{1/3}}\right)\) |
| \(W_{\text{link}}\) | \(\Omega\left(\frac{n^2}{P^{1/3} M^{1/3}}\right)\) | \(\Omega\left(\frac{n}{P^{1/3}}\right)\) |
| FFT/Sorting | Memory Dependent | Memory Independent |
| \(W_{\text{proc}}\) | \(\Omega\left(\frac{1}{P^{1/4} M^{1/4}}\right)\) | \(\Omega\left(\frac{1}{P^{1/4}}\right)\) |
| \(W_{\text{link}}\) | \(\Omega\left(\frac{1}{P^{1/4} M^{1/4}}\right)\) | \(\Omega\left(\frac{1}{P^{1/4}}\right)\) |
| Programs Referencing Arrays | Memory Dependent | Memory Independent |
| \(W_{\text{proc}}\) | \(\Omega\left(\frac{1}{P^{1/3} M^{1/3}}\right)\) | \(\Omega\left(\frac{1}{P^{1/3}}\right)\) |
| \(W_{\text{link}}\) | \(\Omega\left(\frac{1}{P^{1/3} M^{1/3}}\right)\) | \(\Omega\left(\frac{1}{P^{1/3}}\right)\) |

**Range of perfect strong scaling.**

We next consider subdividing the horizontal axis of Figure 2, which corresponds to the number of processors \(P\). Because Figure 2 shows a fixed problem size, increasing \(P\) (moving to the right) corresponds to “strong scaling.” We differentiate between whether or not the computation has the possibility of strong scaling perfectly: that is, for a fixed problem size, increasing the number of processors by a constant factor reduces the communication costs (and running time) by the same constant factor. Note that of the bounds, the memory-dependent per-processor bound (Equation (5)) exhibits this possibility of perfect strong scaling, as \(P\) appears in the denominator with an exponent of 1. However, as \(P\) increases, one of the memory-independent bounds eventually dominates and perfect strong scaling is no longer possible. See [9] for a discussion of this behavior given only per-processor bounds.

For direct linear algebra, Strassen-like methods and the \(O(n^2)\) n-body problem, when \(D \geq D_2\) and \(P \leq (F/N M^{\alpha - 1})^{\frac{\alpha}{\alpha - 1}}\), then the memory-dependent per-processor bound dominates. When this happens, we have a perfect strong scaling range. For values of \(P\) beyond this range, the communication cost is dominated by the memory-independent per-processor bound (see [9] for further discussion). When \(D_1 < D < D_2\), a smaller strong-scaling ranges exists for \(P \leq (F/N M^{\alpha - 1})^D\); for values of \(P\) beyond this range, the communication cost bound is dominated by contention. If \(D \leq D_1\), then the contention bounds always dominate and there is no strong-scaling range. A similar analysis can demonstrate such a region of perfect strong scaling in runtime for programs that reference arrays.

Figure 3 shows this behavior for Strassen’s matrix multi-
\[
P = \left( \frac{E}{NM^{\alpha-1}} \right)^{\alpha/(\alpha-1)}
\]

\[
P = \left( \frac{E}{NM^{\alpha-1}} \right)^D
\]

\[
P = \left( \frac{E}{NM^{\alpha-1}} \right)^{1/(\alpha-1)}
\]

Figure 2: Relationship between the per-processor and contention communication lower bounds for direct linear algebra, Strassen/Strassen-like and the \(O(n^2)\) n-body problems.

![Diagram](image)

**Table 2:** Torus dimensions so that communication cost is either always contention bound \((D \leq [D_1])\) or never contention bound \((D \geq [D_2])\) for a selection of matrix multiplication algorithms. The assertions regarding the last three algorithms are under some technical assumptions / conjecture, see [12].

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>(\omega_0)</th>
<th>(D_1)</th>
<th>(D_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Strassen [38]</td>
<td>(\approx 2.81)</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Schönhage [33]</td>
<td>(\approx 2.55)</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Strassen [39]</td>
<td>(\approx 2.48)</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Vassilevska [40]</td>
<td>(\approx 2.372)</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

5. **FUTURE RESEARCH**

**Other Networks.**

In this work, we exclusively address link contention bounds for tori and mesh networks. We suspect that results for hypercubes and certain indirect networks (e.g., fat trees) should follow easily. For indirect networks, a method for integrating router nodes into the model of computation is needs to be defined. Indirect topologies are common in datacenters as well as on-chip networks, so such an extension of the contention bounds for direct networks would be useful.

![Diagram](image)

Figure 3: Communication bounds for Strassen’s algorithm on \(D\)-dimensional tori. The lower plot is log-log, while the upper is linear on the y-axis. Horizontal lines in the lower plot correspond to perfect strong scaling.
Applicability.

A network may have expansion sufficiently large to preclude the use of our contention bound on a given computation, yet the contention may still dominate the communication cost. This calls for further study on how well computations and networks match each other. Similar questions have been addressed by Leiserson and others [13, 24, 29], and had a large impact on the design of supercomputer networks. In particular, a parallel computer that uses a fat tree communication network can simulate any other routing network, at the cost of at most polylogarithmic slowdown.

Communication Efficient Algorithms.

Some parallel algorithms are network aware, and attain the per-processor communication lower bounds, when network graphs allow it (cf. [36] for classical matrix multiplication on 3D torus). Many algorithms are communication optimal when all-to-all connectivity is assumed, but their performance on other topologies has not yet been studied. Are there algorithms that attain the communication lower bounds for any realistic network graph (either by auto tuning, or by network-topology-oblivious tools)?

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6. REFERENCES


APPENDIX

A. DERIVATION OF FIGURE EXPRESSIONS

- **Equivalence point for per-processor bounds**

  We set the per-processor bounds equal to each other, and solve for $P$:

  $$\frac{F}{PM^{\alpha-1}} = \Theta \left( \frac{N}{P^{1/\alpha}} \right)$$

  $$P = \Theta \left( \frac{F}{NM^{\alpha-1}} \right)^{\alpha/(\alpha-1)}$$

- **Equivalence point for contention bounds**

  We set the contention bounds equal to each other, and solve for $P$:

  $$\frac{F}{P^{\alpha-1/\beta}M^{\alpha-1}} = \Theta \left( \frac{N}{P^{1/\beta}} \right)$$

  $$P = \Theta \left( \frac{F}{NM^{\alpha-1}} \right)^{1/(\alpha-1)}$$
• Equivalence point for the memory-dependent per-processor and memory-independent contention bounds

We set the memory-dependent per-processor and memory-independent contention bounds equal to each other, and solve for $P$ as a function of $D$:

$$\frac{F}{PM^{s-1}} = \Theta\left(\frac{N}{P^{1-1/D}}\right)$$

$$P = \Theta\left(\frac{F}{NM^{s-1}}\right)^D$$

**B. DOMINANCE OF MEMORY-INDEPENDENT CONTENTION BOUND**

**Claim B.1.** Let Alg be an algorithm performing a computation of the form given by equation (2) on $P$ processors, each with local memory of size $M$, and assume the input data is initially evenly distributed across processors. Then,

$$\frac{|Z|^{1/HBL}}{M} \leq \sum_{j=1}^{m} \frac{|\phi_j(Z)|}{M}.$$  

As the minimum number of processors required to hold the problem is the right-hand side of this inequality, we conclude that the memory-independent contention bound dominates the memory-dependent contention bound as the two bounds are equivalent when $P = |Z|^{1/HBL}/M$.

**Proof.** To begin a proof, the HBL bound discussed in Christ et al. [21], states (with certain assumptions) that

$$|Z| \leq \prod_{j=1}^{m} |\phi_j(Z)|^{s_j}.$$  

To detail an argument from Section 2 of [21], we present several greater upper bounds on $|Z|$ that will allow us to demonstrate the desired result:

$$|Z| \leq \prod_{j=1}^{m} |\phi_j(Z)|^{s_j} \leq \prod_{j=1}^{m} \left(\max_{j=1}^{m} |\phi_j(Z)|\right)^{s_j}$$

$$= \left(\max_{j=1}^{m} |\phi_j(Z)|\right)^{\sum_{j=1}^{m} s_j} = \left(\max_{j=1}^{m} |\phi_j(Z)|\right)^{s_{HBL}}$$

As $\max_{j=1}^{m} x_j \leq \sum_{j=1}^{m} x_j$ if all $x_j \geq 0$,

$$|Z| \leq \left(\max_{j=1}^{m} |\phi_j(Z)|\right)^{s_{HBL}} \leq \left(\sum_{j=1}^{m} |\phi_j(Z)|\right)^{s_{HBL}}$$

which proves the desired inequality if we take $s_{HBL}$th root of both sides and divide by $M$.  \[\square\]