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A Three-Stage Colonel Blotto Game with Applications to Cyber-Physical Security

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Abstract

We consider a three-step three-player complete information Colonel Blotto game in this paper, in which the first two players fight against a common adversary. Each player is endowed with certain amount of resources at the beginning of the game, and the number of battlefields on which a player and the adversary fights is specified. The first two players are allowed to form a coalition if it improves their payoffs. In the first stage, the first two players may add battlefields and incur costs. In the second stage, the first two players may transfer resources among each other. The adversary observes this transfer, and decides on the allocation of its resources to the two battles with the players. At the third step, the adversary and the other two players fight on the

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updated number of battlefields and receive payoffs. We characterize the subgame-perfect Nash equilibrium (SPNE) of the game in various parameter regions. In particular, we show that there are certain parameter regions in which if the players act according to the SPNE strategies, then (i) one of the first two players add battlefields and transfer resources to the other player (a coalition is formed), (ii) there is no addition of battlefields and no transfer of resources (no coalition is formed). We discuss the implications of the results on resource allocation for securing cyber physical systems.

1 Introduction

Colonel Blotto game models a scenario in which two players having certain resource levels fight over a finite number of battlefields. The players decide on the amount of resource they deploy on each battlefield in order to maximize their payoff.

The case of players with symmetric resources and three battlefields was solved by Borel and Ville in [1] in 1938. Gross and Wagner [2] generalized the result of symmetric resources to an arbitrary number of battlefields; they also derived a Nash equilibrium of the game when there are two battlefields and asymmetric resources among the players. Until 2006, two-players asymmetric-resource Colonel Blotto game with more than two battlefields remained unsolved. Roberson [3] completely characterized the equilibria for this case in 2006.

Following the fundamental work of [3] and [4], many interesting theoretical extensions followed, and numerous papers targeting specific domain of application are published; see for example [5], [6, 7], [8], [9] among many others. There have also been experimental studies for various applications in [10], [11], [12] among many others. One of the first experimental studies that looked into network infrastructures is [13]. Another interesting experimental paper is [14] where the authors study social interactions using a Facebook application called "Project Waterloo", which allows users to invite both friends and strangers to play Colonel Blotto against them.

Two interesting variations of Colonel Blotto game are considered in [6] and [7]. Kovenock et. al. in [6] include an extra stage to Blotto game, during which both players could add extra battlefields to the initial number of battlefields by paying a cost. The authors investigate the parameters of the game for which additional battlefields are added in equilibrium. In [7], Kovenock and Roberson study a two-stage game in which a common adversary is engaged into two Colonel Blotto games with two separate, and seemingly unrelated players (possible allies). At the first stage of the game, unilateral transfers between the players (except the adversary) are allowed. The authors demonstrate that a positive transfer occurs (or in other words, a coalition is formed) for a range of parameter configurations. However, the authors do not compute the Nash equilibrium for the two-stage game.

In this paper, motivated by the models of [6] and [7], we consider a three-stage three-player game in which the first two players could do both: (i) add battlefields at the first stage and (ii) transfer resource among each other (if it improves the expected payoffs to both players) at the second stage. The third player is the adversary, who observes the updated number of battlefields and the amount of resource transferred among the players, and then allocates its resource to the battles with two players at the second stage. In the third stage, each player among the first two players fights against the adversary on the updated number of battlefields. Our main contribution is that we compute the subgame-perfect Nash equilibrium (SPNE) of the three-stage game in certain regions of the parameter space.

Such settings allow us to gain insights into resource allocation decisions of network parties, facing a common adversary with known resources. For example, consider two resource-constrained networks of servers and a resource-constrained hacker who wants to access the servers. The options available to the network operators are to invest in additional servers (equivalent to adding battlefields in our model) and share resources among each-other for securing the servers. The hacker observes the security level of each network and decides on the amount of resource it deploys to hack each of the servers of the two networks. The main questions we would like to answer in this setting are (i) When is it better for network operators to add additional servers? (ii) When should the network operators share their resources to make their network more secure and improve their expected payoffs? (iii) What is the behavior of the hacker in such a scenario?

1.1 Outline of the paper

We formulate the three-stage Colonel Blotto game in Section 2. In Section 3, we recall the results about Nash equilibrium and expected payoffs to the players of the static Colonel Blotto game from [3]. Thereafter, we compute the SPNE of the three-stage game in Section 4 in certain parameter regions of the game. We conclude our discussion in Section 5.

1.2 Notations

For a natural number N, we use $[N]$ to denote the set $\{1, \ldots, N\}$. $\mathbb{R}_+$ and $\mathbb{Z}_+$ respectively denote the set of all non-negative real numbers and integers. Let $\mathcal{X}_i, i \in [N]$ be non-empty spaces and consider $x_1 \in \mathcal{X}_1, \ldots, x_N \in \mathcal{X}_N$. Then, $x_{1:N}$ denotes the set $\{x_1, \ldots, x_N\}$ and $\mathcal{X}_{1:N}$ denotes the product space $\mathcal{X}_1 \times \cdots \times \mathcal{X}_N$.

2 Problem Formulation

We now formulate the three-stage three-player Colonel Blotto game in this section. This is a complete information game, that is, at second and third stages of the game, the actions taken by the players in the previous stages are common knowledge. The first two players fight against an adversary, call it A. Hereafter, we use Player 3 and A interchangeably for the adversary. Players 1 and 2 are allowed to transfer resources among each other at the second stage of the game if it improves their payoff. This act of transferring a positive amount of resource can be thought of as a formation of coalition among the players.

The initial endowment of Player $i \in \{1, 2\}$ and the adversary, respectively, are denoted by β_i and α. The initial number of battlefields on which the battle between Player $i \in \{1, 2\}$ and the adversary will take place is denoted by n_i, and each battlefield carries a payoff denoted by $v_i > 0$. We assume that $n_i \geq 3$ for $i \in \{1, 2\}$.

2.1 Information Structures and Strategies of the Players

The model of the game and parameters of the game are common knowledge among the players before the game begins. At the first stage, based only on the model and the parameters of the game, Player $i \in \{1, 2\}$ decides on a non-negative integer $m_i \in \mathbb{Z}_+$, which denotes the number of battlefields Player i wants to add to the existing battlefields, and pays a cost cm_i^2, where we assume $c > 0$. The adversary does not take any action at the first stage.

The actions taken by the players at the first stage are common knowledge at the second stage. The second stage consists of two time steps. At the first time step, the first two players may choose to transfer resources among each other. We let $t_{i,j} : \mathbb{Z}_+^2 \to [0, \beta_i]$ denote the strategy of Player i, which is the amount of resource resource Player i transfers to Player $j \neq i$, $i, j \in \{1, 2\}$ as a function of the number of battlefields added by both players at the first stage of the game. We assume that $t_{i,i} \equiv 0$ for $i \in \{1, 2\}$. We use r_i to denote the amount of resource available to Player $i \in \{1, 2\}$ after the redistribution of resources. This is given by

$$r_i(t_{i,j}, t_{j,i}) = \beta_i + \sum_{j=1}^{2} (t_{j,i} - t_{i,j}). \tag{1}$$

The actions of the first two players (that is, the amount of resources transferred) are observed by the adversary at the second step at this stage of the game. The adversary then decides on the amount of resource it allocates to the battle with each player. In particular, the adversary decides on two functions $\alpha_i : \mathbb{Z}_+^2 \times [0, \beta_1] \times [0, \beta_2] \to [0, \alpha]$ subject to the constraint that $\alpha_1(m_{1:2}, t_{1,2}, t_{2,1}) + \alpha_2(m_{1:2}, t_{1,2}, t_{2,1}) \leq \alpha$.

We now consider the third stage of the game in which the adversary engages in two battles against the first two players. At this stage, the players (including the adversary) know the number of battlefields added, the transfer among the first two players and the adversary's allocation of the resource to each battle. Given this information, each player needs to decide on the amount of resource it deploys on each battlefield. Thus, the final stage of the game consists of two static Colonel Blotto games.

For a given triple $\alpha_i, r_i \in \mathbb{R}_+$ and $m_i \in \mathbb{Z}_+$, let us define the sets

$$\mathcal{A}_i(\alpha_i, m_i) := \left\{ \{\alpha_{i,k}\}_{k=1}^{n_i+m_i} \subset \mathbb{R}_+ : \sum_{k=1}^{n_i+m_i} \alpha_{i,k} = \alpha_i \right\},$$
$$\mathcal{B}_i(r_i, m_i) := \left\{ \{\beta_{i,k}\}_{k=1}^{n_i+m_i} \subset \mathbb{R}_+ : \sum_{k=1}^{n_i+m_i} \beta_{i,k} = r_i \right\}.$$

At the final stage, the adversary fights against Player $i \in \{1, 2\}$ on $n_i + m_i$ number of battlefields, where the resource levels of Player i and the adversary, respectively, are r_i and α_i. It is well known that if the number of battlefields is greater than two, then Nash equilibrium of the players in static Colonel

Blotto game exists only in mixed strategies [3]. Therefore, the action space of Player $i \in \{1, 2\}$ is $\wp(\mathcal{B}_i(r_i, m_i))$, whereas the action space of the adversary is $\prod_{i=1}^{2} \wp(\mathcal{A}_i(\alpha_i, m_i))$.

Henceforth, we use γ^i to denote the strategy of Player i, which is defined as follows:

$$\begin{aligned} \gamma^i &:= \{m_i, t_{i,1}, \ldots, t_{i,N}, \mu_i\}, i \in \{1, 2\} \\ \gamma^A &:= \{\alpha_1, \ldots, \alpha_N, \nu_1, \nu_2\}, \end{aligned}$$

where $\mu_i : \mathbb{Z}_+^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \wp(\mathcal{B}_i(r_i, m_i))$ and $\nu_i : \mathbb{Z}_+^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \wp(\mathcal{A}_i(\alpha_i, m_i))$, respectively, denote the strategies of Player $i \in \{1, 2\}$ and the adversary in the battle between them at the third stage of the game. Thus, each γ^i is a collection of functions and the set of all such γ^is is denoted by Γ^i. For ease of exposition, we drop the arguments of all functions $t_{i,j}$, r_i, α_i, μ_i, and ν_i $i, j \in \{1, 2\}$ in subsequent discussions, and use the same notation to denote the actions taken by the player.

2.2 Payoff Functions of the Players

At the third stage of the game, let us use $\beta_{i,k}$ and $\alpha_{i,k}$ to denote, respectively, the amount of resource Player i and adversary deploy on battlefield $k \in [n_i+m_i]$. On every battlefield $k \in [n_i+m_i]$, the player who deploys maximum amount of resource wins and receives a payoff v_i. In case of a tie, the players share the payoff equally[1]. We let $p_{i,k}(\beta_{i,k}, \alpha_{i,k})$ denote the payoff that Player i receives on the battlefield k, and it is given by

$$p_{i,k}(\beta_{i,k}, \alpha_{i,k}) = \begin{cases} v_i & \beta_{i,k} > \alpha_{i,k}, \\ \frac{v_i}{2} & \beta_{i,k} = \alpha_{i,k}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \in \{1, 2\}$ and $k \in \{1, \ldots, n_i+m_i\}$. The adversary's payoff on a battlefield k in the battle with Player i is

$$p_{i,k}^A(\beta_{i,k}, \alpha_{i,k}) = v_i - p_{i,k}(\beta_{i,k}, \alpha_{i,k}).$$

[1]It should be noted that if players play according to the Nash equilibrium strategies on the battlefields, then the case of both players having equal resource on a battlefield has a measure zero. Therefore, in equilibrium, the tie breaking rule does not affect the equilibrium expected payoffs.

We use π_i to denote the expected cost functional of Player i as a function of the strategies of all players. This is given by

$$\pi_i(\gamma^{1:3}) = \mathbb{E}\left[\sum_{k=1}^{n_i+m_i} p_{i,k}(\beta_{i,k}, \alpha_{i,k})\right] - cm_i^2, \quad i \in \{1, 2\},$$
$$\pi_3(\gamma^{1:3}) = \mathbb{E}\left[\sum_{i=1}^{N} \sum_{k=1}^{n_i+m_i} p_{i,k}^A(\beta_{i,k}, \alpha_{i,k})\right],$$

where the expectation is taken with respect to the probability induced on the random variables $\{\beta_{i,k}, \alpha_{i,k}\}_{i,k}$ by the choice of strategies of the players in the game. The model of the game and the payoff functions are common knowledge among the players. The Colonel Blotto game formulated above is referred to as $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$.

A three-tuple of strategies $\{\gamma^{1\star}, \gamma^{2\star}, \gamma^{3\star}\}$ is said to form a Nash equilibrium if

$$\pi_i(\gamma^{1:3\star}) \quad \geq \quad \pi_i(\gamma^i, \gamma^{-i\star}),$$

for all possible $\gamma^i \in \Gamma^i$, $i \in [3]$. Since this is a game of perfect information with stagewise additive payoff functions, we can compute the subgame-perfect Nash equilibrium (SPNE) of the game. SPNE of a complete information game is a refinement of Nash equilibria of the game, which can be computed using a backward inductive algorithm. We refer the reader to [15, p. 72] and [16, Definition 5.14, p. 250] for the precise definition and properties of SPNE of complete information games.

2.3 Research Questions and Solution Approach

We want to investigate the conditions under which in the game defined above, a coalition is formed in which the players transfer resources, or add additional battlefields. In particular, we want to know when

1. There is a positive transfer from one player to another. Note that in this scenario, the transfer should *increase or maintain* the payoffs to both, the donating player as well as the player who accepts the donation.

2. There is no transfer among the players at the second stage.

3. The adversary allocates all its resource to fight only one player.

4. The players have an incentive to add new battlefields.

We first recall some preliminary results on the two-player static Colonel Blotto game from [3]. Solving the general problem formulated above is somewhat difficult due to discontinuity of expected payoff functions as a function of endowments of the players in the static game. Therefore, we restrict our attention to a subset of all possible parameter regions in order to keep the analysis tractable. We compute the parameter regions that feature the scenarios listed above in Section 4.

Our analysis of the subgame starting at the second stage is similar to the one considered in [7]. However, the authors in [7] do not compute the Nash equilibrium of the game; they restricted their attention to computing the best response strategies of the players. We derive here the SPNE of the game formulated above. Furthermore, we have been unable to verify some of the assertions made in [7]. Therefore, in this paper, we state the complete proof of all the results presented in this paper, so that our treatment is self contained.

3 Preliminary Results on Static Two-Players Colonel Blotto Game

Consider a two-players static Colonel Blotto game with n battlefields. We let r_i denote the resource of Player i. Define $\mathcal{R}_i := \{a \in \mathbb{R}_+^n : \sum_{k=1}^n a_k \leq r_i\}$ and let $\partial\mathcal{R}_i$ be the boundary of the region $\mathcal{R}_i$. Then, the strategy of Player i is a joint measure over the space $\mathcal{R}_i$. Let $\mu_i \in \wp(\mathcal{R}_i)$ be the strategy of Player i. Then, we let $\mathrm{Pr}_{\#}^k\mu_i$ denote the marginal of μ_i on the k^{th} battlefield.

A player wins a battlefield if he deploys strictly larger amount of resource as compared with the other player on that battlefield. In case of a tie (both players deploying equal resources), the payoff is equally divided between the players. The payoff of winning a battlefield is given by v. Due to the cost functionals of the players, for a given strategy pair (μ_1, μ_2) of the players, the expected cost to Player i on battlefield $k \in [n]$ is dependent solely on the marginal distributions $(\mathrm{Pr}_{\#}^k\mu_1, \mathrm{Pr}_{\#}^k\mu_2)$.

The resources available to the players and the number of battlefields are common knowledge among the players. We call this Colonel Blotto game as $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$. We now recall the following result from [3].

Theorem 1 *Let $\Psi_i : \wp(\mathcal{R}_i) \to 2^{\wp(\mathcal{R}_i)}$ denote a set-valued map (correspondence), defined as*

$$\Psi_i(\mu_i) := \Big\{ \tilde{\mu}_i \in \wp(\mathcal{R}_i) : \mathit{supp}(\tilde{\mu}_i) = \mathit{supp}(\mu_i) \text{ and } Pr^k_{\#}\tilde{\mu}_i = Pr^k_{\#}\mu_i \text{ for all } k \in [n] \Big\}.$$

For the static Colonel Blotto game $\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$ with $n \geq 3$, there exists a Nash equilibrium $(\mu_1^\star, \mu_2^\star)$ with unique payoffs to each player. Let $\mu_{i,k}^\star := Pr^k_{\#}\mu_i^\star$ for $i \in \{1, 2\}$ and $k \in [n]$. Any Nash equilibrium of the game satisfies following properties:

1. *For Player i, $Pr^1_{\#}\mu_i^\star = Pr^2_{\#}\mu_i^\star = \cdots = Pr^n_{\#}\mu_i^\star$.*
2. *If $(\mu_1^\star, \mu_2^\star)$ is a Nash equilibrium of the game, then $\mathit{supp}(\mu_i^\star) = \partial\mathcal{R}_i$ for $i \in \{1, 2\}$.*
3. *If $(\tilde{\mu}_1^\star, \tilde{\mu}_2^\star)$ is any other Nash equilibrium of this game, then $Pr^k_{\#}\tilde{\mu}_i^\star = \mu_{i,k}^\star$.*
4. *If $(\tilde{\mu}_1, \tilde{\mu}_2)$ is a set of strategies such that $\mathit{supp}(\tilde{\mu}_i) = \mathit{supp}(\mu_i^\star) = \partial\mathcal{R}_i$ and $Pr^k_{\#}\tilde{\mu}_i = \mu_{i,k}^\star$, then $(\tilde{\mu}_1, \tilde{\mu}_2)$ is a Nash equilibrium of the game $\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$.*

Thus, $(\tilde{\mu}_1, \tilde{\mu}_2)$ is a Nash equilibrium if and only if $\tilde{\mu}_i \in \Psi_i(\mu_i^\star)$ for both players $i \in \{1, 2\}$.

Proof: See [3] for a proof of the above results. ∎

We let $\mathtt{NE}(\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$ denote the set of all Nash equilibria of the static Colonel Blotto game $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$. It should be noted that even though there are several Nash equilibria of the game $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$, the marginals on all battlefields of all equilibrium strategies of Player i are the same.

Lemma 2 *For the static Colonel Blotto game $\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$ with $n \geq 3$, suppose that r_1 and r_2 are such that $\frac{1}{n-1} \leq \frac{r_1}{r_2} \leq n - 1$. Then, the* *payoff functions of the players under Nash equilibrium* $(\mu_1^\star, \mu_2^\star)$ *are given by*

$$
\begin{aligned}
&P^1(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)) \\
&= \begin{cases}
nv\left(\frac{2}{n} - \frac{2r_2}{n^2 r_1}\right) & \text{if } \frac{1}{n-1} \leq \frac{r_1}{r_2} < \frac{2}{n} \\
nv\left(\frac{r_1}{2r_2}\right) & \text{if } \frac{2}{n} \leq \frac{r_1}{r_2} \leq 1 \\
nv\left(1 - \frac{r_2}{2r_1}\right) & \text{if } 1 \leq \frac{r_1}{r_2} \leq \frac{n}{2} \\
nv\left(1 - \frac{2}{n} + \frac{2r_1}{n^2 r_2}\right) & \text{if } \frac{n}{2} < \frac{r_1}{r_2} < n-1
\end{cases}, \\
&P^2(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)) \\
&\qquad = nv - P^1(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)).
\end{aligned}
$$

If $r_1 = 0$, *then* $P^1(\boldsymbol{SCB}(\{1, 0\}, \{2, r_2\}, n, v)) = 0$.

Remark 1 *Note that for a fixed* r_2, n *and* v, $r_1 \mapsto P^1(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$ *is a concave monotonically increasing function in the parameter region* $\frac{1}{n-1} \leq \frac{r_1}{r_2} \leq n-1$. *This is also illustrated in Figure 1 for a specific set of parameters. Furthermore,* $r_1 \mapsto P^1(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$ *is a non-decreasing function on* $\mathbb{R}_+$. *This is a consequence of the result in [3].* □

4 SPNE of the Three-Stage Game

In this section, we compute the subgame-perfect Nash equilibrium for the game formulated in Section 2. The SPNE of a complete information game is computed using a recursive algorithm. First, the Nash equilibrium for the game at the final stage is computed. Then, at any stage before the final stage, the Nash equilibrium for the subgame starting at that stage is considered and Nash equilibrium is computed for that game.

In what follows, we use $t := t_{1,2} - t_{2,1}$ to denote the total amount transferred from Player 1 to Player 2 at the second stage. The value of t can take negative value if Player 2 transfers its resource to Player 1. We use $r_1 := r_1(t) = \beta_1 - t$ and $r_2 := r_2(t) = \beta_2 + t$ respectively to denote the resource levels of Player 1 and Player 2 after the transfer.

At the final stage, Player $i \in \{1, 2\}$ and the adversary play a static Colonel Blotto game on $n_i + m_i$ battlefields with resource levels r_i and α_i, respectively. The Nash equilibrium of the static Colonel Blotto game is given in Theorem 1. Thus, we have the following result.

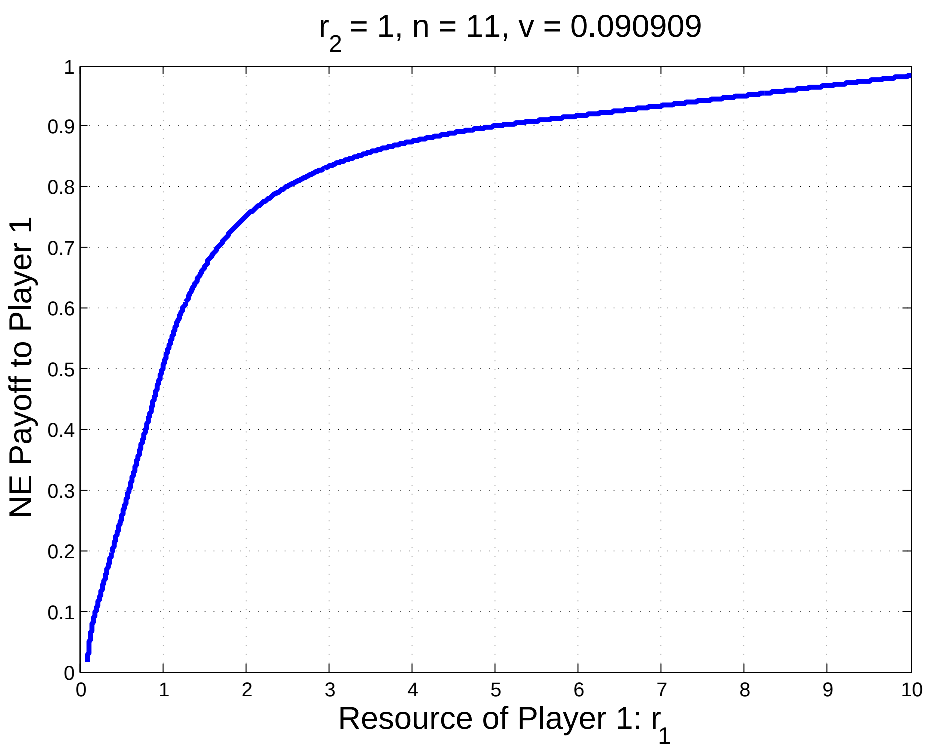

Figure 1: For a fixed resource r_2 of Player 2, the payoff to Player 1 is concave function of its endowment of resources r_1.

Lemma 3 *At the final stage, Player $i \in \{1,2\}$ and the adversary play a static Colonel Blotto game $\boldsymbol{SCB}(\{1, r_i\}, \{2, \alpha_i\}, n_i+m_i, v_i)$. Thus, the SPNE at the last stage is any pair of strategies $(\mu_i^\star, \nu_i^\star) \in \mathtt{NE}(\boldsymbol{SCB}(\{1, r_i\}, \{2, \alpha_i\}, n_i+m_i, v_i))$.*

As a consequence of the lemma above, we only need to compute the SPNE strategies of the players at the first two stages. We now restrict our analysis to a subset of all possible parameter regions in order to keep it tractable. In particular, we focus our attention to only those games in which if players act according to SPNE at the first two stages, then the ratio of r_i and α_i lies in the interval $(\frac{2}{n_i}, \frac{n_i}{2})$, either for both $i \in \{1,2\}$ or $\alpha_i = 0$ for some $i \in \{1,2\}$. With this simplification, there are only four possible cases:

1. $2/n_1 < \alpha_1/r_1 < 1$ and $2/n2 < \alpha_2/r_2 < 1$
2. $2/n_1 < r_1/\alpha_1 < 1$ and $2/n_2 < r_2/\alpha_2 < 1$
3. $2/n_1 < \alpha_1/r_1 < 1$ and $2/n_2 < r_2/\alpha_2 < 1$
4. $2/n_1 < r_1/\alpha_1 < 1$ and $2/n_2 < \alpha_2/r_2 < 1$

However, Case 4 above is just Case 3 with index of the players interchanged. Therefore, we compute the SPNE of the game here only for the first three cases. Toward this end, we first compute the reaction curve of the adversary at the second stage in the next subsection.

4.0.1 Preliminary Notation for Results

We now define some notation that we use throughout the rest of the paper.

$$\begin{aligned}
a_1(m_1, m_2, t) &:= \frac{\alpha}{1 + \sqrt{\frac{(n_2+m_2)v_2(\beta_2+t)}{(n_1+m_1)v_1(\beta_1-t)}}}, \\
a_2(m_1, m_2, t) &:= \alpha - a_1(m_1, m_2, t), \\
\lambda_1(m_1, m_2, t) &:= \sqrt{\frac{(n_2+m_2)v_2(\beta_1-t)(\beta_2+t)}{(n_1+m_1)v_1}},
\end{aligned}$$

$$d(m_1, m_2, t) := \begin{cases} \alpha & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} > \frac{(n_2+m_2)v_2}{\beta_2+t} \\ 0 & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} < \frac{(n_2+m_2)v_2}{\beta_2+t} \\ \alpha \ w.p.\ p \in (0,1) & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} = \frac{(n_2+m_2)v_2}{\beta_2+t} \end{cases}.$$

4.1 Best Response of the Adversary

We first compute the best response strategies (also called reaction curves [16]) of the adversary in the game.

Lemma 4 *Consider a game* $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$*. For a* $t \in [-\beta_2, \beta_1]$*, let* $r_1 = \beta_1 - t$ *and* $r_2 = \beta_2 + t$*. Fix* $m_1, m_2 \in \mathbb{Z}_+$*. The strategy of the adversary that maximizes its payoff is:*

1. *If* $\frac{2}{n_1+m_1} < \frac{\alpha}{\beta_1 - t} < 1$ *and* $\frac{2}{n_2} < \frac{\alpha}{\beta_2+t} < 1$*, then*

$$\alpha_1^*(m_1, m_2, t) = d(m_1, m_2, t).$$

2. *If* $\frac{2}{n_i+m_i} < \frac{r_i}{a_i(m_1,m_2,t)} < 1, \quad i = 1, 2$*, then*

$$\alpha_1^*(m_1, m_2, t) \quad = \quad a_1(m_1, m_2, t).$$

3. *If* $\frac{2}{n_1+m_1} < \frac{\alpha - \lambda_1(m_1,m_2,t)}{\beta_1 - t} < 1$ *and* $\frac{2}{n_2+m_2} < \frac{\beta_2+t}{\lambda_1(m_1,m_2,t)} < 1$*, then*

$$\alpha_1^*(m_1, m_2, t) = \alpha - \lambda_1(m_1, m_2, t).$$

Proof: Since Player i and the adversary are going to play a static Colonel Blotto game $\mathbf{SCB}(\{i, r_i\}, \{A, \alpha_i\}, n_i + m_i, v_i)$ at the final stage of the game, the expected payoff functions to the players are given by the result of Lemma 2 (that are dependent on the ratio r_i/α_i).

The reaction curve for the adversary is the best response strategy of the adversary given the strategy of the other two players. Towards this end, fix $m_{1:2}$ and t and define $e_i := (n_i + m_i)v_i$ for $i = 1, 2$. The expected payoff function to the adversary as a function of the adversary's allocation α_1 to the battle with Player 1 for the three cases, respectively, are

$$\begin{aligned}
\text{Case 1: } \pi_A(\alpha_1) &= \frac{e_1\alpha_1}{2(\beta_1 - t)} + \frac{e_2(\alpha - \alpha_1)}{2(\beta_2 + t)}, \\
\text{Case 2: } \pi_A(\alpha_1) &= e_1\left(1 - \frac{(\beta_1 - t)}{2\alpha_1}\right) \\
&\quad + e_2\left(1 - \frac{(\beta_2 + t)}{2(\alpha - \alpha_1)}\right), \\
\text{Case 3: } \pi_A(\alpha_1) &= \frac{e_1\alpha_1}{2(\beta_1 - t)} + e_2\left(1 - \frac{(\beta_2 + t)}{2(\alpha - \alpha_1)}\right).
\end{aligned}$$

In Cases 2 and 3, the payoff to the adversary π_A is a concave function of α_1, since the second derivative of π_A with respect to α_1 is strictly negative. One can set the first derivative of π_A to zero to get the optimal value of α_1 as a function of m_1, m_2, and t. The fact that $d(m_1, m_2, t)$ maximizes the payoff π_A in Case 1 can be verified easily. This completes the proof of the lemma.
■

We are now in a position to compute the SPNE of the game considered above. We first consider in the next subsection the case when the adversary has least amount of resources as compared to other players of the game. Thereafter, we consider other cases of the game.

4.2 Adversary with Least Resources

We now turn our attention to computing SPNE of the three-stage Colonel Blotto game formulated in Section 2. First, we consider a case when the adversary has the least amount of resources among all players. We show that if the parameters of the game satisfy certain assumptions, then there exists a family of SPNE in this game.

4.2.1 Preliminary Notation for Theorem 5

Let $\bar{m}_1 = \arg\max_{m_1 \in \mathbb{Z}_+} m_1 v_1 - c m_1^2$ and $\bar{m}_2 = \arg\max_{m_2 \in \mathbb{Z}_+} m_2 v_2 - c m_2^2$. Define

$$\begin{aligned}
\bar{t}_{1,2}(m_1, m_2) &= \frac{(n_2 + m_2)v_2\beta_1 - (n_1 + m_1)v_1\beta_2}{(n_1 + m_1)v_1 + (n_2 + m_2)v_2}, \\
\bar{t}_{2,1}(m_1, m_2) &= \frac{(n_1 + m_1)v_1\beta_2 - (n_2 + m_2)v_2\beta_1}{(n_1 + m_1)v_1 + (n_2 + m_2)v_2}, \\
\zeta_1 &= \bar{t}_{2,1}(0, \bar{m}_2) \qquad \zeta_2 = \bar{t}_{1,2}(\bar{m}_1, 0).
\end{aligned}$$

Theorem 5 *Consider a game* $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$ *with* $\alpha < \min\{\beta_1, \beta_2\}$ *and*

$\frac{2}{n_i} < \frac{\alpha}{\beta_i}$ *for* $i \in \{1,2\}$*. If the parameters of the game satisfy either*

$$\begin{aligned} &\frac{(n_1+\bar{m}_1)v_1}{\beta_1} < \frac{n_2 v_2}{\beta_2}, \quad \left(1-\frac{\alpha}{2(\beta_2+\zeta_2)}\right) v_2 < c, \\ &\frac{2}{n_1+\bar{m}_1} < \frac{\alpha}{\beta_1-\zeta_2} < 1, \quad \frac{2}{n_2} < \frac{\alpha}{\beta_2+\zeta_2} < 1, \end{aligned} \tag{2}$$

$$\begin{aligned} \textit{or} \quad &\frac{n_1 v_1}{\beta_1} > \frac{(n_2+\bar{m}_2)v_2}{\beta_2}, \quad \left(1-\frac{\alpha}{2(\beta_1+\zeta_1)}\right) v_1 < c, \\ &\frac{2}{n_2+\bar{m}_2} < \frac{\alpha}{\beta_2-\zeta_1} < 1, \quad \frac{2}{n_1} < \frac{\alpha}{\beta_1+\zeta_1} < 1, \end{aligned} \tag{3}$$

then there is a family of SPNEs for this game given by

$$\begin{aligned} &\alpha_1^\star(m_1,m_2) = d(m_1,m_2,t), \\ &t_{1,2}^\star(m_1,m_2) = \\ &\begin{cases} t \in [0, \bar{t}_{1,2}(m_1,m_2)) & \textit{if } \frac{(n_1+m_1)v_1}{\beta_1} < \frac{(n_2+m_2)v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases} \\ &t_{2,1}^\star(m_1,m_2) = \\ &\begin{cases} t \in [0, \bar{t}_{2,1}(m_1,m_2)) & \textit{if } \frac{(n_1+m_1)v_1}{\beta_1} > \frac{(n_2+m_2)v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} m_1^\star &= \begin{cases} \bar{m}_1 & \textit{if } \frac{(n_1+\bar{m}_1)v_1}{\beta_1} < \frac{n_2 v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases}, \\ m_2^\star &= \begin{cases} \bar{m}_2 & \textit{if } \frac{n_1 v_1}{\beta_1} > \frac{(n_2+\bar{m}_2)v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases}. \end{aligned}$$

Thus, along the equilibrium path, one player has an incentive to add battlefields and transfer some (or none) of its resource to the other player.

Proof: See Appendix 5. ■

Remark 2 *In the theorem above, if* $c > v_1$*, then* $\bar{m}_1 = 0$*. Similarly, if* $c > v_2$*, then* $\bar{m}_2 = 0$*.* □

In the theorem above, there are three important points to notice: The first point is that the adversary randomizes its action when $\frac{(n_1+m_1)v_1}{\beta_1-t} = \frac{(n_2+m_2)v_2}{\beta_2+t}$.

Suppose $\frac{(n_1+m_1)v_1}{\beta_1} < \frac{(n_2+m_2)v_2}{\beta_2}$. Then, as t increases, $\frac{(n_1+m_1)v_1}{\beta_1-t}$ increases while $\frac{(n_2+m_2)v_2}{\beta_2+t}$ decreases. The two quantities $\frac{(n_1+m_1)v_1}{\beta_1-t}$ and $\frac{(n_2+m_2)v_2}{\beta_2+t}$ become equal exactly when $t = \bar{t}_{1,2}(m_1, m_2)$. Thus, Player 1 will never transfer an amount equal to $\bar{t}_{1,2}(m_1, m_2)$ in this case since this action reduces its payoff. This is the reason why we see that Player 1 transfers any amount t in the interval $[0, \bar{t}_{1,2}(m_1, m_2))$ when playing according to the SPNE.

The second point to notice is that one player adds battlefields as well as transfer its resource to the other player if the value c is small enough. The third point to note is that the player transferring its resource to the other player is the one with minimum $\frac{(n_i+m_i)v_i}{\beta_i}$, and not necessarily the player who has maximum resource level β_i. This is contrary to our intuition. A take away from this result is that the rich player need not always be better off in a war as it may have more and/or highly valued battlefields to fight on. This phenomena is also illustrated numerically in Figure 2. In the red region, even though Player 1 has less resource than Player 2, Player 1 may choose to transfer some of its resource to Player 2. We now consider other scenarios in the next subsection.

4.3 Other Cases

In this subsection, we consider the case in which the adversary has comparable or more resource than that of the other players. The SPNE of the game in such a case is as follows.

4.3.1 Preliminary Notation for Theorem 6

$$\begin{aligned}
\bar{t}_1(m_1, m_2) &:= \frac{(\beta_1 - \beta_2)}{2} - \frac{(\beta_1 + \beta_2)}{2} \\
&\quad \times \sqrt{\frac{(n_1 + m_1)v_1}{(n_1 + m_1)v_1 + (n_2 + m_2)v_2}}, \\
w_1(m_1, m_2) &:= (n_1 + m_1)v_1 + \\
&\quad \sqrt{(n_1 + m_1)v_1((n_1 + m_1)v_1 + (n_2 + m_2)v_2)}, \\
\bar{m}_1 &:= \arg\max_{m_1 \in \mathbb{Z}_+} m_1 v_1 \left(1 - \frac{\alpha}{2(\beta_1 + \beta_2)}\right) - c m_1^2, \\
\zeta_1(m_1, m_2) &:= \frac{4(n_1 + m_1)v_1\alpha^2}{(n_2 + m_2)v_2(\beta_1 + \beta_2)^2}.
\end{aligned}$$

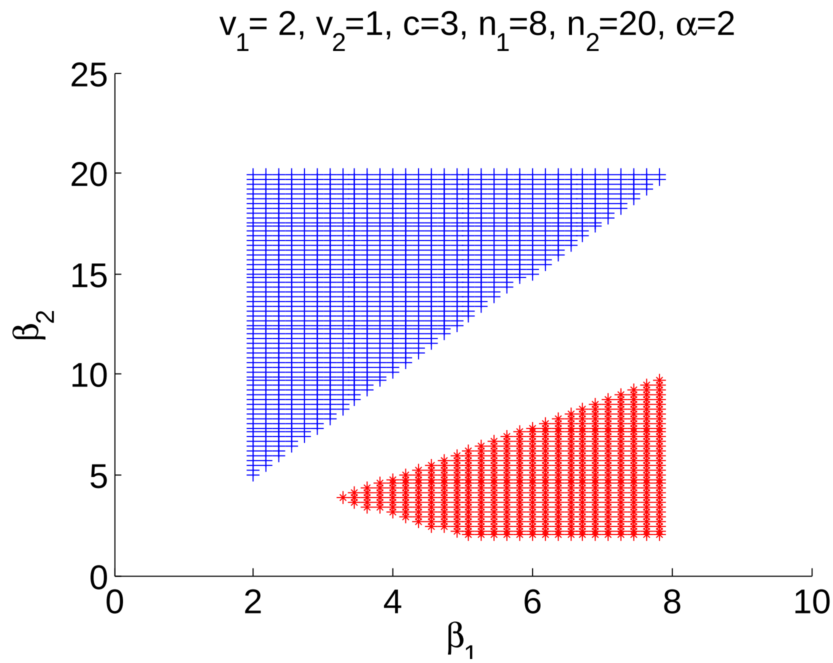

Figure 2: For fixed parameters $v_1 = 2, v_2 = 1$, $c = 3$, $n_1 = 8, n_2 = 20$, and $\alpha = 2$, Player 1 transfers to Player 2 in the red region, whereas Player 2 transfers to Player 1 in the blue region. There is no addition of battlefield by any player (see also Remark 2) in the colored region. In the white region, transfer may or may not occur. See Theorem 5 for complete characterization.

Theorem 6 *Consider a game* $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$. *The SPNE of the game is given as*

1. *Assume* $c > \frac{\beta_1+\beta_2}{4\alpha} \max\{w_1(1,0) - w_1(0,0), v_2\}$ *and let* $\bar{t}_1 := \bar{t}_1(m_1, m_2)$. *If* $\frac{2}{n_i+m_i} < \frac{r_i(t)}{a_i(m_1,m_2,t)} < 1,\ i = 1, 2,$ *then*

$$\begin{aligned}
\alpha_1^\star(m_1, m_2, t) &= a_1(m_1, m_2, t), \\
t_{1,2}^\star(m_1, m_2) &= \begin{cases} \bar{t}_1 & \text{if } \frac{\beta_1-\beta_2}{2\beta_1\beta_2} > \sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}} \\ 0 & \text{otherwise} \end{cases} \\
t_{2,1}^\star(m_1, m_2) &= 0, \qquad m_1^\star = m_2^\star = 0.
\end{aligned}$$

2. *If* $c > \frac{(\beta_1+\beta_2)v_2}{4\alpha}$, $\frac{2}{n_1+m_1} < \frac{\alpha-\lambda_1(m_1,m_2,t)}{(\beta_1-t)} < 1$, *and* $\frac{2}{n_2+m_2} < \frac{(\beta_2+t)}{\lambda_1(m_1,m_2,t)} < 1$, *then*

$$\alpha_1^\star(m_1,m_2,t) = \alpha - \lambda_1(m_1,m_2,t),$$
$$t_{1,2}^\star(m_1,m_2) =$$
$$\begin{cases} \frac{\beta_1-\zeta_1(m_1,m_2)\beta_2}{\zeta_1(m_1,m_2)+1} & \textit{if } \frac{\beta_1+\beta_2}{2\alpha} > \sqrt{\frac{(n_1+m_1)v_1\beta_2}{(n_2+m_2)v_2\beta_1}} \\ 0 & \textit{otherwise.} \end{cases}$$
$$t_{2,1}^\star(m_1,m_2) = 0, \qquad m_1^\star = \bar{m}_1, \qquad m_2^\star = 0.$$

Proof: See Appendix 5. ∎

Remark 3 *If* $c > v_1\left(1 - \frac{\alpha}{2(\beta_1+\beta_2)}\right)$, *then* $\bar{m}_1 = 0$, *which implies* $m_1^\star = 0$ *in the Case 2 above.* □

In the theorem above, there is a unique transfer among the players, and therefore, unique SPNE. The first case is that of adversary having a significantly larger amount of resources than the sum of the resources of the other two players. It is easy to note that Player 1 transfers to Player 2 in this case when $\frac{\beta_1-\beta_2}{2\beta_1\beta_2} > \sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}} > 0$, which implies that $\beta_1 > \beta_2$. Thus, the rich player transfers resource to the poor player. Furthermore, the first two players do not add battlefields if the value of c is high enough.

In the second case above, the adversary has a comparable resource level with respect to the resource levels of the other two players. In this case, like in the first case, the rich player transfers resource to the poor player. This can be shown as follows: $\alpha > \lambda_1(m_1,m_2,t)$ implies

$$\frac{\beta_1+\beta_2}{2\alpha} < \sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}}.$$

Also, Player 1 transfers resource to Player 2 if

$$\frac{\beta_1+\beta_2}{2\alpha} > \sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}}\sqrt{\frac{\beta_2}{\beta_1}}.$$

Both inequalities can be satisfied simultaneously if, and only if, $\beta_1 > \beta_2$. Thus, only rich player donates to the poor player in this case. A graphical

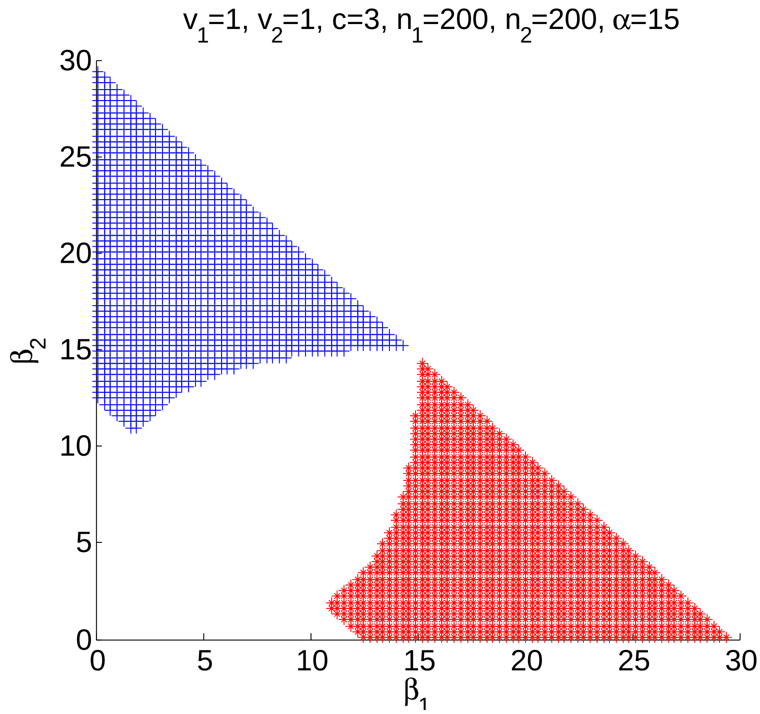

Figure 3: For fixed parameters $v_1 = 1, v_2 = 1$, $c = 3$, $n_1 = 200, n_2 = 200$, and $\alpha = 15$, Player 1 transfers to Player 2 in the red region, whereas Player 2 transfers to Player 1 in the blue region. There is no addition of battlefield by any player (see also Remark 3) in the colored region. In the white region, transfer may or may not occur. See Theorem 6 Case 2 for complete characterization. Note that the graph is symmetric around $\beta_1 = \beta_2$ line because $n_1 = n_2$ and $v_1 = v_2$.

representation of when a transfer occurs and who transfers whom is given in Figure 3.

In both the cases above, it should be noted that even though both players fight with the adversary, there is a positive transfer of resource from the rich player to poor player. This result was also reported in [7], but the authors did not compute the equilibrium behavior. Even though for every player, higher resource implies higher payoff (see Figure 1), we see that a positive transfer takes place because the adversary observes the amount of resource transferred and changes its allocation appropriately in order to maximize its payoff.

In Case 2 above, Player 1 adds battlefields as well as transfer its resource to Player 2. This is similar to the behavior we saw in Theorem 5, where the adversary had least resources among all players. In Theorem 5, the donating player may choose not to donate any resource; on the contrary, in Theorem 6, we saw that the donating player must donate a unique positive amount of resource to the other player. It should also be noted that in all the cases above, if the value of c was small, then adding battlefields is beneficial to Players 1 and 2. We consider the case of c large enough here for ease of exposition.

We now have a qualitative picture of the behavior of resource-constrained players who could be attacked by a common resource-constrained adversary. Going back to the example of network operators and the hacker stated in Section 1, we now know the parameter regions where it is beneficial for the network operators to form a coalition for sharing resources. If the cost of adding additional servers is small enough, it is in the best interest of network operators to add more servers. We also know the amount of resource the hacker will allocate to hack the servers in each network. In particular, if the hacker has very little resource as compared to the network operators, then the hacker attacks only one of the networks (see Theorem 5 for details). If the hacker has comparable or more resources than the network operators, the hacker divides its resource into two parts, where each part is used to attack each of the network operators (see Theorem 6 for details).

5 Conclusion

In this paper, we formulated a three-stage Colonel Blotto game and computed the subgame-perfect Nash equilibrium of the game in various parameter regions. We showed that under some sufficient conditions, the players may have an incentive to add battlefields or form a coalition as it improves their expected payoffs. We also showed that when the adversary has least amount of resource among all players, then the adversary fights against only one of the two players. In several instances of the game, counterintuitive behavior of the players emerge, wherein one player transfers resource to the other player even though both players engage in battles against the adversary. In certain parameter regions, the player with minimum resource among the first two players may transfer some of its resource to the other player. For the future, we would like to extend the analysis to multi-player version of this game.

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Appendix

A1. Proof of Theorem 5

The best response strategy of the adversary is given in Lemma 4. Assume that (2) is satisfied. Fix m_1 and m_2, and assume that the ratio of α and $\beta_1 - \bar{t}_{1,2}(m_1, m_2)$ lies in the interval $(\frac{2}{n_1+m_1}, 1)$ and $\frac{(n_1+m_1)v_1}{\beta_1} < \frac{(n_2+m_2)v_2}{\beta_2}$. For this set of parameters, $\alpha_1^\star = 0$, or in other words, the adversary does not fight Player 1. If Player 1 transfers any amount t of its resource in the interval $[0, \bar{t}_{1,2}(m_1, m_2))$, then $\frac{(n_1+m_1)v_1}{\beta_1 - t}$ is strictly less than $\frac{(n_2+m_2)v_2}{\beta_2 + t}$ after the transfer takes place. This maintains the payoff of Player 1, so it is indifferent to transfer. Any positive transfer to Player 2 improves the payoff of Player 2, so it accepts the transfer. The case of (3) can also be done using the same arguments with the roles of Players 1 and 2 reversed.

Now, consider the total payoff functionals of the players at the first stage. Maximizing the expected payoff functionals of Players 1 and 2 over m_1 and m_2 given $\alpha_1^\star, t_{1,2}^\star$ and $t_{2,1}^\star$, we get the result. In particular, in the case when (2) is satisfied, Player 1 does not fight against the adversary at the third stage. Thus, adding $\bar{m}_1$ battlefields improves its payoff. On the other hand, since $\left(1 - \frac{\alpha}{2(\beta_2+\zeta_2)}\right) v_2 < c$, Player 2’s payoff reduces if it adds any battlefield. The case of (3) using similar arguments.

The sufficient conditions on the parameters ensures that any player in $\{1, 2\}$ and the adversary's allocation have appropriate ratios if all players act according to the SPNE. The proof of the theorem is complete.

A2. Proof of Theorem 6

The best response of strategy of the adversary remains the same as in Lemma 4. We only need to compute the best response strategies of the first two players for the given strategy of the adversary. Let $\phi_1(t)$ be the payoff to Player 1 if it transfers t amount of resource to Player 2.

1. For this case, $\phi_1(t)$ is given by

$$\phi_1(t) = \frac{(n_1 + m_1)v_1}{2} \frac{(\beta_1 - t)}{a_1(m_1, m_2, t)}.$$

One can check that $\frac{d^2\phi_1}{dt^2} < 0$ and

$$\frac{d\phi_1}{dt} = \frac{(n_1 + m_1)v_1}{2\alpha} \Bigg(-1 + \frac{1}{2}\sqrt{\frac{(n_2 + m_2)v_2}{(n_1 + m_1)v_1}} \times \frac{(\beta_1 - \beta_2 - 2t)}{\sqrt{(\beta_1 - t)(\beta_2 + t)}} \Bigg).$$

Furthermore, if $\frac{\beta_1 - \beta_2}{\beta_1\beta_2} > 2\sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}}$, then $\frac{d\phi_1}{dt} > 0$, that is, it is beneficial for Player 1 to transfer its resource. One can similarly compute the expected payoff to Player 2 as a function of t and show that it is beneficial for Player 2 to accept the transfer. Since ϕ_1 is a concave function, we can set the derivative to zero to compute the transfer that achieves the maximum. This gives us the value of optimal amount of resource $t^\star_{1,2}$ as a function of m_1 and m_2 that is transferred to Player 2.

At the first stage, the expected payoff function of Player 1 and 2, respectively, are

$$\frac{\beta_1 + \beta_2}{4\alpha} w_1(m_1, m_2) - cm_1^2,$$
$$(n_2 + m_2)v_2 \frac{\beta_1 + \beta_2}{4\alpha} - cm_2^2.$$

Now, if $c > \frac{\beta_1+\beta_2}{4\alpha} \max\{w_1(1, 0) - w_1(0, 0), v_2\}$, then not adding any battlefield yields maximum payoff to both players. Thus, $m_1^\star = m_2^\star = 0$.

2. For this case, $\phi_1(t)$ is given by

$$\phi_1(t) = (n_1 + m_1)v_1 \left(1 - \frac{\alpha - \lambda_1(m_1, m_2, t)}{2(\beta_1 - t)}\right).$$

The derivative of ϕ_1 with respect to t is given by

$$\begin{aligned} \frac{d\phi_1}{dt} &= -\frac{(n_1 + m_1)v_1\alpha}{2(\beta_1 - t)^2}\Bigg(1 - \frac{(\beta_1 + \beta_2)}{2\alpha} \\ &\quad \times \sqrt{\frac{(n_2 + m_2)v_2}{(n_1 + m_1)v_1}}\sqrt{\frac{\beta_1 - t}{\beta_2 + t}}\Bigg), \\ \implies \frac{d\phi_1}{dt} &\begin{cases} > 0 & \text{if } t < t^\star_{1,2}(m_1, m_2) \\ = 0 & \text{if } t = t^\star_{1,2}(m_1, m_2) \\ < 0 & \text{if } t > t^\star_{1,2}(m_1, m_2) \end{cases}. \end{aligned}$$

Thus, if $t^\star_{1,2}(m_1, m_2)$ is positive, then Player 1 transfers $t^\star_{1,2}(m_1, m_2)$ amount of resource to Player 2. One can compute the expected payoff to Player 2 as a function of t and show that it is beneficial for Player 2 to accept the transfer.

At the first stage, the expected payoff function of Players 1 and 2, respectively, are

$$\begin{gathered} (n_1 + m_1)v_1 - cm_1^2 + \frac{1}{8\alpha(\beta_1 + \beta_2)} \times \\ \left((n_2 + m_2)v_2(\beta_1 + \beta_2)^2 - 4(n_1 + m_1)v_1\alpha^2\right), \\ (n_2 + m_2)v_2\frac{\beta_1 + \beta_2}{4\alpha} - cm_2^2. \end{gathered}$$

Now, if $c > \frac{(\beta_1+\beta_2)v_2}{4\alpha}$, then $m_1^\star = \bar{m}_1$ and $m_2^\star = 0$ maximizes the expected payoffs.

The proof of the theorem is thus complete.