Maximum Achievable Gain of a Two Port Network

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1 Introduction

The maximum gain of amplifiers at high operating frequencies (millimeter-wave and terahertz) is severely limited by the cut-off frequency of the transistors. Given the two port parameters at the desired frequency, the maximum stable gain achievable by the device is fixed. However, by adding an external lossless passive network, the composite two port parameters of the circuit can be modified. Under these conditions, it is possible to achieve a much higher gain for the device if the two port parameters are chosen wisely. This report derives the maximum achievable gain of a two port network and discusses an embedding technique to achieve this desired gain. The rest of the report is organized as follows. Section 2 provides the expression for the maximum gain of a two port device and discusses the conditions required to satisfy the above. Section 3 gives the embedding procedure to achieve this maximum gain. A design example is discussed in Section 4 and concluding remarks are provided in Section 5.

2 Expression for Maximum Gain

In this section, we will derive the expression for the maximum achievable gain of a two port network. The gain of a two port network G is given by (1) [1] and the unilateral gain U is given by (2) [2],

$$G = \left|\frac{y_{21}}{y_{12}}\right| (K_f - \sqrt{{K_f}^2 - 1}) \tag{1}$$

$$U = \frac{|y_{21} - y_{12}|^2}{4[\Re(y_{11})\Re(y_{22}) - \Re(y_{12})\Re(y_{21})]} \tag{2}$$

where K_f is the stability factor. Following the convention in [3], we define the complex measure of reciprocity A as

$$A = \frac{y_{21}}{y_{12}} \tag{3}$$

We now derive the relationship between G, U and A as discussed in [3]. Even though the final expression has been discussed in this paper, the proof for this hasn't been derived in literature. From (2),

$$U = \frac{|y_{21} - y_{12}|^2}{4\Re(y_{11})\Re(y_{22}) - 2\Re(y_{12}y_{21} + y_{12}y_{21}^*)} \tag{4}$$

$$U = \frac{|y_{21} - y_{12}|^2}{2[2\Re(y_{11})\Re(y_{22}) - \Re(y_{12}y_{21})] - 2\Re(y_{12}y_{21}^*)} \tag{5}$$

The stability factor K_f is given as [1]

$$K_f = \frac{2\Re(y_{11})\Re(y_{22}) - \Re(y_{12}y_{21})}{|y_{12}y_{21}|} \tag{6}$$

Using (6) in (5), we get

$$U = \frac{|y_{21} - y_{12}|^2}{2|y_{12}y_{21}|K_f - 2\Re(y_{12}y_{21}^*)} \tag{7}$$

$$U = \frac{|y_{21} - y_{12}|^2}{|y_{12}y_{21}|(K_f - \sqrt{{K_f}^2 - 1}) + |y_{12}y_{21}|(K_f + \sqrt{{K_f}^2 - 1}) - 2\Re(y_{12}y_{21}^*)} \tag{8}$$

Using (1) in (8), we obtain

$$U = \frac{|y_{21} - y_{12}|^2}{|y_{12}|^2 G + \frac{|y_{21}|^2}{G} - 2\Re(y_{12}y_{21}^*)} \tag{9}$$

$$U = \frac{G|y_{21} - y_{12}|^2}{|y_{12}|^2 G^2 + |y_{21}|^2 - 2G\Re(y_{12}y_{21}^*)} \tag{10}$$

$$U = \frac{G|y_{21} - y_{12}|^2}{|y_{21}|^2 + |y_{12}|^2 G^2 - G(y_{12}y_{21}^*) - G(y_{12}^* y_{21})} \tag{11}$$

$$U = \frac{G|y_{21} - y_{12}|^2}{|y_{21} - y_{12}G|^2} \tag{12}$$

From the definition of A as in (3), we get

$$U = \frac{|A-1|^2}{|A-G|^2} G \tag{13}$$

which is the expression listed in [3]. We will now derive the maximum achievable gain from this network. Equation (13) can be rewritten as

$$\frac{G}{U} = \frac{|A|^2 + G^2 - G(A+A^*)}{|A|^2 + 1 - (A+A^*)} \tag{14}$$

Let $A = r\exp(j\theta)$. When the device is unconditionally stable, the maximum gain occurs at the point where $K_f = 1$ [1]. Under this condition, from (1) and (3), $G = r$. Using this in (14), we have

$$\frac{r}{U} = \frac{r^2 + r^2 - 2r^2\cos(\theta)}{r^2 + 1 - 2r\cos(\theta)} \tag{15}$$

$$r^2 - 2r[\cos(\theta) - U\cos(\theta) + U] + 1 = 0 \tag{16}$$

$$r = f(\theta) \pm \sqrt{f(\theta)^2 - 1} \tag{17}$$

where $f(\theta) = U - (U-1)\cos(\theta)$. As θ increases from 0 to π, the function $f(\theta)$ increases. The solution with the negative square root sign starts from unity and decays to zero as $f(\theta)$ increases. This would mean that $|y_{21}/y_{12}| \le 1$. However, we know that the device has gain higher than 0 dB at these frequencies and this condition isn't true. Using the positive square root, the gain is maximized when $f(\theta)$ is maximum and that occurs at $\theta = \pi$. Then, $f(\theta) = 2U - 1$. Using this in (17), we obtain

$$G_{MAX} = r_{\theta=\pi} = 2U - 1 + 2\sqrt{U(U-1)} \approx 4U \tag{18}$$

Therefore,

$$\frac{y_{21}}{y_{12}} = -G_{MAX} \tag{19}$$

This completes the second part of the proof for the result listed in [3]. Once must note that the maximum gain occurs when $K_f = 1$ or when the device is at the edge of stability. Also, the above optimization assumes that U is unchanged which is true under a lossless passive embedding. Using (6) and $K_f = 1$,

$$2\Re(y_{11})\Re(y_{22}) - \Re(y_{12}y_{21}) = |y_{12}y_{21}| \tag{20}$$

$$2\Re(y_{11})\Re(y_{22}) = G_{MAX}[|{y_{12}}^2| - \Re({y_{12}}^2)] \tag{21}$$

$$2\Re(y_{11})\Re(y_{22}) = 2G_{MAX}[\Im(y_{12})]^2 \tag{22}$$

Hence, we can now rewrite the required conditions for maximum gain as follows

$$\begin{aligned} \frac{y_{21}}{y_{12}} &= -G_{MAX} \\ \Re(y_{11})\Re(y_{22}) &= G_{MAX}[\Im(y_{12})]^2 \end{aligned} \tag{23}$$

where G_{MAX} is as calculated in (18).

3 Design of Embedding Network

The two port network block diagram is shown in Fig. 1. The y-parameters of the network is given as

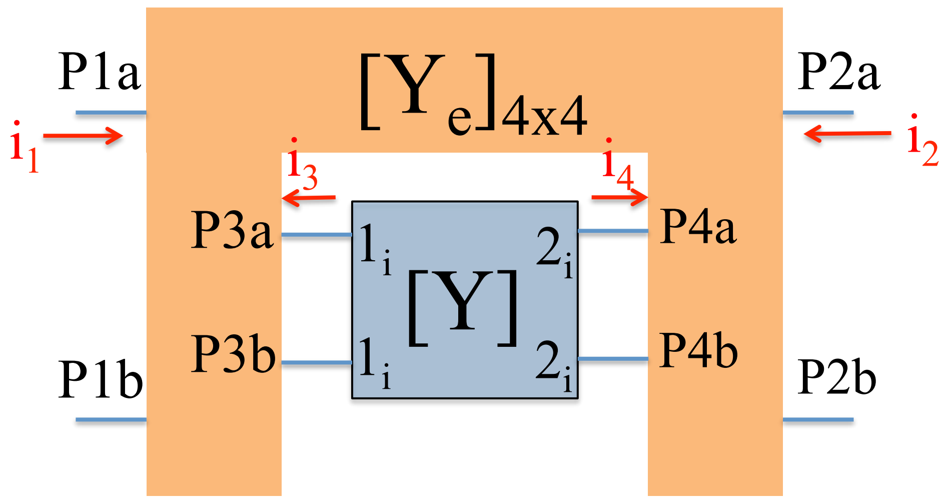

Figure 1: Two port network with passive embedding

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \tag{24}$$

The embedding network y-parameters are given as

$$\mathbf{Y}_e = j \begin{bmatrix} x_1 & b_1 & b_2 & b_3 \\ b_1 & x_2 & b_4 & b_5 \\ b_2 & b_4 & x_3 & b_6 \\ b_3 & b_5 & b_6 & x_4 \end{bmatrix} \tag{25}$$

This can be rewritten as

$$\mathbf{Y}_e = j \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_3 \\ \mathbf{A}_3{}^T & \mathbf{A}_2 \end{bmatrix} \tag{26}$$

Here $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are 2×2 matrices. The current voltage relationship between the intrinsic y-parameters is given as

$$\begin{bmatrix} -i_3 \\ -i_4 \end{bmatrix} = \mathbf{Y} \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} \tag{27}$$

In a similar manner, the current voltage relationship between the embedded network y-parameters is given as

$$\begin{bmatrix} i_3 \\ i_4 \end{bmatrix} = j\mathbf{A}_3{}^T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + j\mathbf{A}_2 \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} \tag{28}$$

Using (27) in (28), we obtain

$$\begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = -(\mathbf{Y} + j\mathbf{A}_2)^{-1} j\mathbf{A}_3{}^T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{29}$$

The current voltage relationship between the embedded network y-parameters is given as

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = j\mathbf{A}_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + j\mathbf{A}_3 \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} \tag{30}$$

Using (29) in (30), we get

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = [j\mathbf{A}_1 + \mathbf{A}_3(\mathbf{Y} + j\mathbf{A}_2)^{-1}\mathbf{A}_3{}^T] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{31}$$

We denote the y-parameter relationship as Y_f, where Y_f is given as

$$\mathbf{Y}_f = j\mathbf{A}_1 + \mathbf{Y}_m \tag{32}$$

where

$$\mathbf{Y}_m = \mathbf{A}_3\mathbf{Y}_i\mathbf{A}_3{}^T \tag{33}$$

and

$$\mathbf{Y}_i = (\mathbf{Y} + j\mathbf{A}_2)^{-1} \tag{34}$$

Using (32) in (23), we obtain

$$\Re(y_{11f})\Re(y_{22f}) = G_{MAX}[\Im(y_{12f})]^2 \tag{35}$$

$$\frac{y_{21f}}{y_{12f}} = -G_{MAX} \tag{36}$$

We must note that $A_1 = \begin{bmatrix} x_1 & b_1 \\ b_1 & x_2 \end{bmatrix}$ and $A_3 = \begin{bmatrix} b_2 & b_3 \\ b_4 & b_5 \end{bmatrix}$.

Using (32), (35) can be written as

$$\Re(y_{11m})\Re(y_{22m}) = G_{MAX}[\Im(y_{12m}) + b_1]^2 \tag{37}$$

and (36) becomes

$$\frac{y_{21m} + jb_1}{y_{12m} + jb_1} = -G_{MAX} \tag{38}$$

Therefore, we calculate the value of b_1 as

$$b_1 = j\frac{y_{21m} + y_{12m}G_{MAX}}{1 + G_{MAX}} \tag{39}$$

Since b_1 is real,

$$\Re(y_{21m} + y_{12m}G_{MAX}) = 0 \tag{40}$$

Hence, using the value of b_1, we can rewrite (37) as

$$\Re(y_{11m})\Re(y_{22m}) = G_{MAX}\left[\Im(y_{12m}) - \frac{\Im(y_{21m}) + \Im(y_{12m})G_{MAX}}{1 + G_{MAX}}\right]^2 \tag{41}$$

$$\Re(y_{11m})\Re(y_{22m}) = \frac{G_{MAX}}{[1 + G_{MAX}]^2}[\Im(y_{12m} - y_{21m})]^2 \tag{42}$$

By using (33) and the value for $\mathbf{A}_3$, $\mathbf{Y}_m$ can be written as

$$\mathbf{Y}_m = \begin{bmatrix} {b_2}^2 y_{11i} + b_2 b_3 (y_{12i} + y_{21i}) + {b_3}^2 y_{22i} & b_2 b_4 y_{11i} + b_2 b_5 y_{12i} + b_3 b_4 y_{21i} + b_3 b_5 y_{22i} \\ b_2 b_4 y_{11i} + b_2 b_5 y_{21i} + b_3 b_4 y_{12i} + b_3 b_5 y_{22i} & {b_4}^2 y_{11i} + b_4 b_5 (y_{12i} + y_{21i}) + {b_5}^2 y_{22i} \end{bmatrix} \tag{43}$$

Using (43) in (40) and (42), we obtain

$$\Re(y_{11i}) + \frac{b_5}{b_4}\frac{\Re(y_{21i} + y_{12i}G_{MAX})}{1 + G_{MAX}} + \frac{b_3}{b_2}\frac{\Re(y_{12i} + y_{21i}G_{MAX})}{1 + G_{MAX}} + \frac{b_3 b_5}{b_2 b_4}\Re(y_{22i}) = 0 \tag{44}$$

$$\begin{aligned} \Re(y_{11i} + \frac{b_3}{b_2}(y_{12i} + y_{21i}) + \frac{{b_3}^2}{{b_2}^2} y_{22i}) \Re(y_{11i} + \frac{b_5}{b_4}(y_{12i} + y_{21i}) + \frac{{b_5}^2}{{b_4}^2} y_{22i}) \\ = \frac{G_{MAX}}{[1 + G_{MAX}]^2}\left[\Im[(\frac{b_5}{b_4} - \frac{b_3}{b_2})(y_{12i} - y_{21i})]\right]^2 \end{aligned} \tag{45}$$

We can now discuss the design procedure to achieve the required embedding network as follows:

1. Start with the given y-parameter $\mathbf{Y}$.
2. Calculate Mason's Unilateral Gain (U) and calculate G_{MAX} from (18).
3. We can add matrix $\mathbf{A}_2$ if required.
4. We can make $b_2 = b_4 = 1$ and solve b_3 and b_5 from (44) and (45).
5. Calculate $\mathbf{Y}_m$ and solve for b_1 from (39).
6. If the component values are not reasonable to implement, b_2 and b_4 can be scaled. It is the ratio b_3/b_2 and b_5/b_4 that is required to satisfy (44) and (45) and not the absolute values of b_2, b_3, b_4 and b_5.
7. x_1, x_2, x_3 and x_4 can be any value as long as they satisfy the requirements for a passive lossless network.

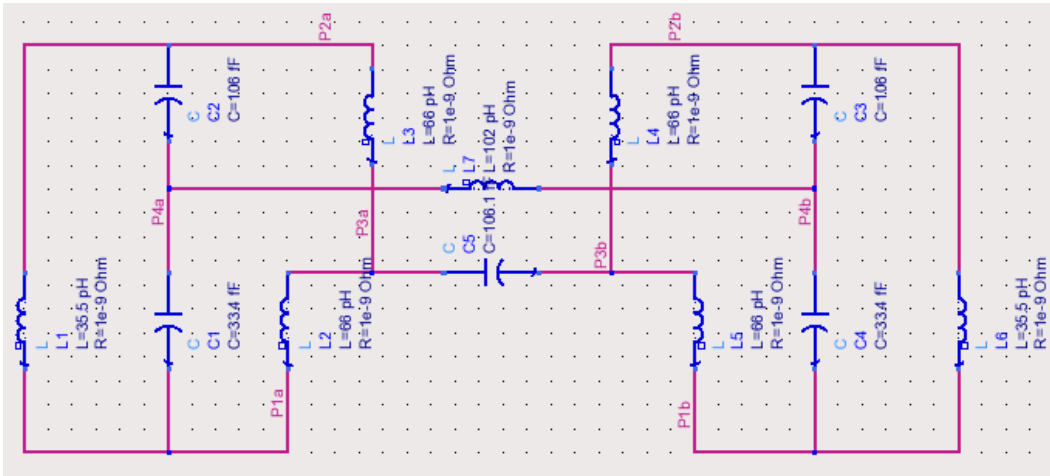

Figure 2: Lossless passive embedding circuit

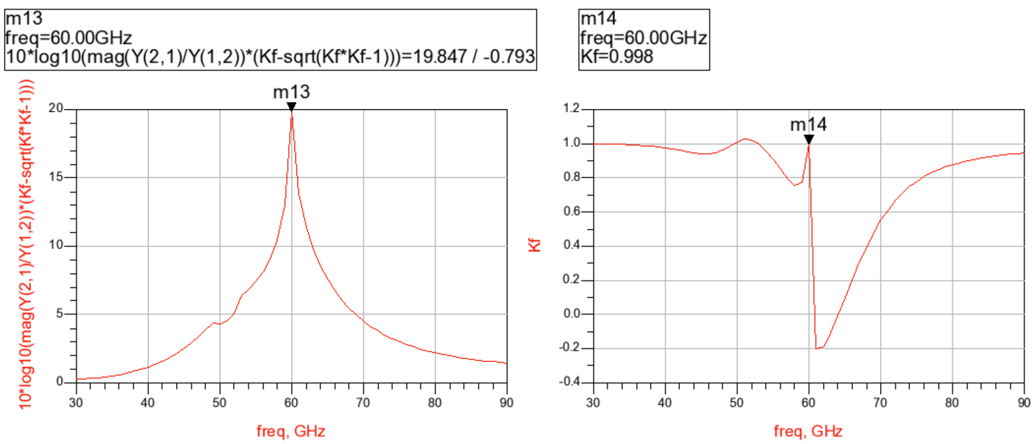

Figure 3: Simulation results: Gain and Stability factor (K_f)

4 Design example

We now present a design example at 60 GHz. The simulated y-parameters for a transistor cell at 60 GHz is given as

$$\mathbf{Y} = \begin{bmatrix} 1.01 \times 10^{-3} + j1.31 \times 10^{-2} & -2.47 \times 10^{-4} - j2.95 \times 10^{-3} \\ 3.76 \times 10^{-2} - j7.58 \times 10^{-3} & 5.36 \times 10^{-3} + j1.04 \times 10^{-2} \end{bmatrix} \quad (46)$$

The unilateral gain is calculated to be $U = 13.93$ dB and the maximum gain $G_{MAX} = 19.86$ dB. Fig. 2 shows the circuit for the embedding network.

The simulated gain and stability factor are shown in Fig. 3. As expected the maximum gain is 19.8 dB and the stability factor is unity at 60 GHz.

5 Conclusion

This report presented a theoretical derivation for the maximum achievable gain for a two port network which is equal to $4U$, where U is the unilateral gain of the network. The design procedure for embedding a lossless passive network was also shown and the procedure was verified using simulation results. It must be noted that the device is at the edge of being unconditionally stable and achieves this peak gain in a small operating frequency range. If the embedding network is lossy, then the unilateral gain is changed and the optimization is no longer valid. Since practical networks are lossy in nature, this procedure must be iterated to obtain the optimal solution.

References

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