Gordian: Formal Reasoning Based Outlier Detection for Secure Localization

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Accurate localization from Cyber-Physical Systems (CPS) is a critical enabling technology for context aware applications and CPS control. As localization plays an increasingly safety-critical role, location systems must be able to identify and eliminate faulty measurements to prevent dangerously inaccurate localization. In this paper we consider the range-based localization problem and propose a method to detect coordinated adversarial corruption on anchor positions and distance measurements. Our algorithm, Gordian, rapidly finds attacks by identifying geometric inconsistencies at the graph level without requiring assumptions about hardware, ranging mechanisms or cryptographic protocols. We give necessary conditions for which attack detection is guaranteed to be successful in the noiseless case, and use that intuition to extend Gordian to the noisy case where fewer guarantees are possible. In simulations generated from real-world sensor noise, we empirically show Gordian's trilateration counterexample generation procedure enables rapid attack detection even for combinatorially difficult problems.

CCS Concepts: • **Computer systems organization** → *Reliability*; *Sensor networks*;

Additional Key Words and Phrases: Secure Localization, Noisy Sensor Network Localization, Approximate Graph Embedding, Structural Rigidity, Semidefinite Programming, Semidefinite Relaxation, Satisfiability Modulo Convex Optimization

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1 INTRODUCTION

Accurate real-world location data from Cyber-Physical Systems (CPS) are already crucial for CPS control and location-based applications, and stand to become even more important as systems with dynamic context-aware functionality become more commonplace. However, many applications that depend on location data are subject to dangerous failure-modes when the CPS providing location information fails. For example in the hospital tracking context, a localization system failure could be life threatening. Likewise, a platoon of military vehicles trying to localize themselves with respect to each other while some of the cars are being hijacked cannot afford location errors. Due to demonstrated vulnerabilities in Global Navigation Satellite System (GNSS) spoofing [Humphreys et al. 2008], an adversary may mislead or attempt to crash an autonomous vehicle or robot in a similar ploy.

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Fortunately, many localization systems with sensors embedded in the environment resemble distributed sensor networks (eg. [Lazik et al. 2015]) where a measurement failure at a particular sensor may be isolated from other sensors. This distributed sensing can provide a great deal of redundancy in position estimation to aid in detecting and eliminating location-attacks and coordinated sensor failures. We explore this scenario in this paper through the lens of range-based localization: distance measurements are known between sensors, only some of which have known locations, and we seek to localize the entire network in the presence of adversarially corrupted measurements.

We propose Gordian[1], an attack and coordinated sensor failure detection algorithm. Detecting and mitigating attacks on sensor data is, in general, a combinatorial problem [Pasqualetti et al. 2013], which has been typically addressed by either brute force search, suffering from scalability issues [Pasqualetti et al. 2013], or via convex relaxations using algorithms that can terminate in polynomial time [Fawzi et al. 2014] but are not necessarily sound. On the other side, recent advances in combinatorial search techniques and in particular those used in Satisfiability Modulo Theories (SMT) solvers showed combinatorial problems can be cast into smaller problems that can be solved efficiently. As we demonstrate in this paper, these new Satisfiability Modulo Convex (SMC) solving techniques [Shoukry et al. 2017] are a neat fit for range-based attack detection.

This paper presents the following contributions:

- Sufficient topological and combinatorial conditions for attack detection in a noiseless network.
- Gordian, the first provably sound and complete coordinated attack detection algorithm over well-formed noiseless networks.
- A novel trilateration counterexample generation procedure for Gordian's SMT solving architecture that makes combinatorially intractable attack detection problems practically solvable in reasonable time.
- A localization algorithm appropriate for noise on both distance measurements *and* anchor coordinates.
- A generalization of the standard graph-embeddability problems studied in noiseless localization to a notion of *approximate embeddability*, appropriate for noise.
- A convex decision procedure for testing approximate embeddability of noisy networks at a desired confidence level, enabling the extension of Gordian to the noisy case.

In Section 2 we introduce conventional and secure localization algorithms, define our formal model of localization problems and introduce rigidity theory. Next in Section 3 we define our threat model, the attack detection problem, and identify an attack tolerance property for which we prove sufficient topological and combinatorial conditions. We present Gordian's architecture in Section 4, prove the algorithm sound and complete, and elucidate the algorithms for embeddability testing and counterexample generation. Section 5 extends localization, embeddability testing, and Gordian to the noisy case, which we evaluate in Section 6. We conclude in Section 7.

2 BACKGROUND AND RELATED WORK

Researchers have proposed a wide variety of localization schemes [Liu et al. 2007] including techniques as diverse as RF signal strength and fingerprinting [Honkavirta et al. 2009; Savvides et al. 2001], propagation time of an ultrasonic pulse [Lazik et al. 2015; Priyantha et al. 2000], and range-free techniques[He et al. 2003]. Many of these techniques assume complete trust of the entire localization system. The field of secure localization goes a step further to explore methods that work in the presence of malicious attacks.

[1]The name Gordian is a reference to a legend of Alexander the Great in which he "untied" the impossible Gordian Knot by slicing it in half with his sword.

2.1 Localization Networks

We apply a simple and common model [Anderson et al. 2010; Biswas and Ye 2004; Eren et al. 2004; Liu et al. 2008; Moore et al. 2004; Savvides et al. 2001; So and Ye 2007; Yang et al. 2013a] of range-based localization problems: Incomplete pairwise distance measurements are known between devices. Some devices, referred to as anchors, have known positions and other unlocalized devices do not.

More formally, we assume a set of n nodes representing sensors $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ with $S = \{1, 2, ..., n\}$ being their index set. $S_a \subseteq S$ represents anchor nodes, defined as nodes with *a priori* measured locations $p_a : S_a \to \mathbb{R}^2$. The rest, $S_x = S \setminus S_a$, have unspecified locations. An undirected graph $G = (S, E)$ is a natural model for the topology of a sensor network where the sensor nodes are treated as graph nodes and edges E represent measured internode distances. Let $E_a \subseteq E$ represent the edges (i, j) such that both $i, j \in S_a$ and $E_x = E \setminus E_a$. The weighted extension of G is given as $G_d = (S, E, W)$ where $W : E \to \mathbb{R}^+$. Combining all of the above, we formally define:

Definition 2.1 (Localization network). *A localization network N is a tuple (S, E, W, p_a) such that S is a set of nodes, E is a set of edges, $W : E \to \mathbb{R}^+$, $p_a : S_a \to \mathbb{R}^2$, where $S_a \subseteq S$.*

Let the function $p : \mathcal{S} \to \mathbb{R}^2$, be a "placement", assigning coordinates $p(s_i) \in \mathbb{R}^2$ to each sensor node. Now assume p^* is a placement representing ground truth positions of the nodes. The euclidean distance from node s_i to s_j is given by $d_{ij} = \|p^*(s_i) - p^*(s_j)\|_2$. With $\restriction$ as the set restriction operator, we are now equipped to pose the *exact* (noiseless) *localization* problem as follows:

Problem 2.2 (Exact localization). *Given a localization network $N = (S, E, W, p_a)$ with an unknown ground truth placement p^* s.t. $W(e_{ij}) = d_{ij}$ and $p_a = p^* \restriction_{S_a}$, find p^*.*

Fig. 1a shows an example localization network and placement which will be a long-running example in this paper. It has six anchors (nodes $1 - 6$) represented by squares and one non-anchor (node 7) represented by a circle. Lines in the diagram represent measured internode distances, where the length of a measurement (i.e. $W(e_{xy})$) from s_x to s_y corresponds to the length of the line from node x to node y. The placement of the nodes is intended to correspond to their position on the page. In other words this is an embedded graph, not an abstract graph representation. Lines from anchors to anchors are feinter than the lines to node 7 because we wish to draw attention to the latter. Fig. 1 (a) is an example of an embedding (also called a realization) because all the measurements are consistent with the placement. Were Fig. 1 (a) to be an inconsistent placement, one or more of the lines in the diagram would fail to meet up with a node at its endpoints. This is the case in Fig. 3.

2.2 Rigidity

Perhaps the most important question to ask about a localization network in the context of attack detection is whether or not p^* is the unique consistent placement with respect to N. This property is known as *unique localizability*. If a localization network has two distinct embeddings in the absence of attack, an adversary does not have to take any action to create ambiguity in the localization result as is the case for the two embeddings shown in Fig. 1. The hollow circle in Fig. 1b for node $7'$ represents the alternative consistent placement for $p(7) \neq p^*(7)$. Even without attacks, there are two ambiguous localization solutions.

Eren et al. [Eren et al. 2004] identified Global Generic Rigidity (GGR) as the link between sensor network localization, the mathematical concept of a framework, and the theory of structures from mechanical, civil engineering, and physics. Intuitively, a framework is a collection of solid rods connected at flexible joints. A rigid framework will not deform

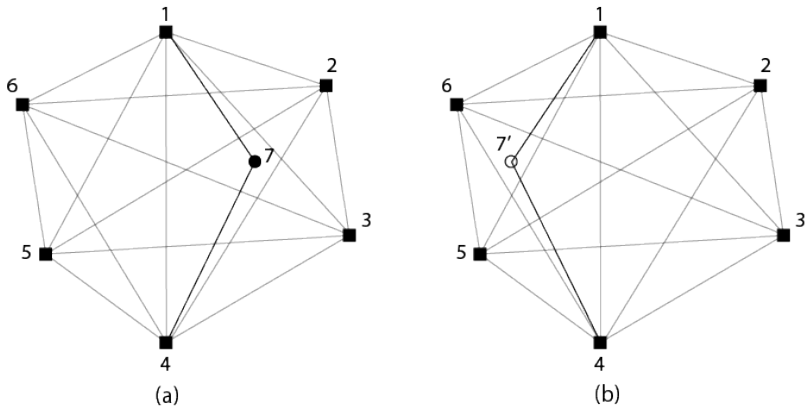

Fig. 1. Two different embeddings (a) and (b), for the same localization network that is rigid but not GGR.

when perturbed in a planar direction. A GGR framework is not only immune to planar deformation, it cannot be discontinuously manipulated (such as the 3-dimensional flip of node 7 along the line connecting nodes 1 and 4 in Fig. 1) into another configuration. Eren et al. also proved a network N is uniquely localizable if and only if its corresponding grounded graph (the framework obtained by treating a localization network's graph of measurements, G, as rods) is GGR. For simplicity we will write that N is GGR when its grounded graph is GGR. [Eren et al. 2004]

A globally rigid framework can be identified by topological conditions: G is *3-connected* (at least three edges must be removed to partition G) and G is *redundantly rigid* (any one edge may be removed from G and the resulting G' is still rigid) [Jackson and Jordan 2003]. We further assume that the frameworks under discussion are generic, i.e. the coordinates of p are algebraically independent over the rationals. Essentially, the generic requirement forces the framework's p to not be entirely co-linear or otherwise degenerate. The GGR property is efficiently testable by a randomized algorithm [Gortler et al. 2010]. We refer the interested reader to [Anderson et al. 2010; Eren et al. 2004] for a more thorough presentation on rigidity and localization.

2.3 Localization Algorithms

Localization algorithms attempt to solve the exact localization Problem 2.2. From a complexity perspective, just determining if a graph has an embedding (that preserves the d_{ij}) is known to be NP hard [Saxe 1979]. Researchers tackle intractability through two broad classes of methods: local methods that enable each node to determine its location from its neighbors without seeing the big picture and global methods that simultaneously localize all nodes from a perspective external to the network. Gordian makes use of both.

The most basic local algorithm, provided by Eren et al. is *iterative trilateration*, but it is only possible for a specific kind of *trilateration graph*. The definition of a trilateration graph is given in [Eren et al. 2004], but informally it can be thought of as a localization network that admits the following localization procedure: Begin with a set of three nodes $\{s_i, s_j, s_k\}$ with known locations (initially these can be anchor points). Find a node s_n that is currently without known location and is connected to three nodes s_i, s_j, and s_k in the known set. Draw three circles centered $p(i)$, $p(j)$, and $p(k)$ with radius d_{in}, d_{jn}, and d_{kn} respectively. The value of $p(n)$ is given by the unique intersection of the circles. Add s_n to the known set, and repeat the above procedure until all nodes are in the known set.

Eren et al. prove trilateration graphs are uniquely localizable (an important property for Gordian's counterexamples phase – see algorithm 2), and can be localized in polynomial time. However the procedure is incomplete in the sense that it fails to localize classes of uniquely localizable graphs such as bipartite and wheel graph networks [Moore et al. 2004].

Alternatives to the node-centric localization methods discussed above use some sort of optimization framework to localize all nodes at once [Anderson et al. 2010; Biswas et al. 2006; Biswas and Ye 2004; So and Ye 2007]. Although actual formulations vary, these approaches frame localization as an optimization problem and (very broadly speaking) aim to minimize the sum of some sort of squared errors resembling

$$\underset{p(i), i \in S_x}{\operatorname{argmin}} \sum_{(i,j) \in E_x} \frac{\left| \|(p(i) - p(j)\|_2^2 - W(e_{ij})^2 \right|}{W(e_{ij})^2}$$

$$s.t. \quad \forall i \in S_a \; p(i) = p_a(i) \tag{2.1}$$

where $p(i) \in \mathbb{R}^2$ is a decision variable representing the estimated location of sensor i, and $W(e_{ij})$ is the measured distance between i and j. Since distances are squared, the denominator $W(e_{ij})^2$ normalizes the influence of large edges in optimization.

These methods commonly rely on a relaxation of the general optimization problem in Equation 2.1 from a non-convex program in two dimensions to a Semidefinite Program (SDP) in a higher dimensional space (refer to (4.3) for the statement of the Biswas-Ye SDP relaxation (BY-SDP) used by Gordian). The relaxation was first proposed by Biswas and Ye [Biswas and Ye 2004] with good empirical performance, then proved to have important theoretical connections to rigidity and unique localizability [So and Ye 2007]. Most significantly, it is shown [So and Ye 2007, Theorem 4.2] that a two-dimension graph is uniquely localizable if and only if the max-rank BY-SDP solution is rank 2.

In this paper, we seek to address the *secure localization* problem with our Gordian algorithm. According to Zeng et al.'s survey [Zeng et al. 2013], secure localization methods in the literature fall into three broad categories: *prevention methods* which prevent the sensor network from collecting bad data in the first place, *detection methods* which identify and remove bad localization data, and *filtering methods* which are robust to bad localization as part of the localization procedure. Under this taxonomy, Gordian is a centralized detection method designed to identify and eliminate distance-measurement attacks (i.e. attacks that corrupt inter-node distance measurements) and anchor position change attacks before localization.

3 LOCALIZATION ATTACK DETECTION

As described in Sec. 2.2, even in the absence of attacks there are certain graph properties that must be present in the localization network for the localization problem to be well-posed. It should be no surprise then that stronger properties are required for the attack detection problem to be well-posed in the presence of attacks. Although conditions for the number of required non-malicious anchors are presented in [Zhong et al. 2008], and rigidity conditions for outlier detection are addressed in [Yang et al. 2013a,b] this paper is the first of our knowledge to give conditions and a systematic procedure for attack detection in the presence of adversarially *coordinated* sensor failures.

3.1 Threat Model

We consider two agents: a *localization system* that attempts to solve Problem 2.2 and an *adversary* that interferes by corrupting some of the measurements seen by the localization system with the goal of causing the system to incorrectly

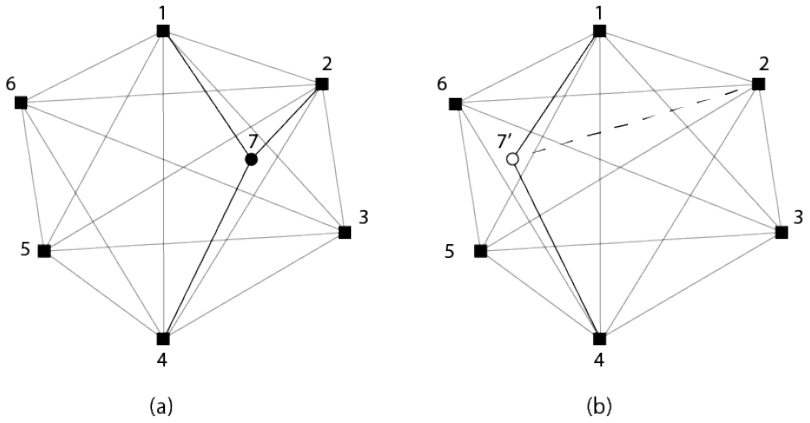

Fig. 2. An illustration of unique embeddings for a GGR localization network before attack (a) and after (b). The attack on edge (2,7), represented by the dashed line, still yields a consistent embedding in (b). This localization network is susceptible to undetectable attacks.

localize one or more sensors to the wrong locations. Thus, the adversary has potentially tampered with the localization system's ranging measurements[2] $m_{ij} = W(e_{ij}) * (1 + \delta_{ij})$ and anchor positions $a_i = p_a(i) + \alpha_i$. The $\delta_{ij} \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}^2$ values are controlled by the adversary. For now we assume a noiseless scenario: if an edge measurement m_{ij} (or an anchor position a_i) is attack-free or "clean", then $m_{ij} = d_{ij}$ ($a_i = p^*(i)$). Furthermore we assume from the system's perspective there is no otherwise distinguishing feature between clean and corrupted edges. We formalize an attack as an *attack profile* $\hat{\eta} = (b, c, m, a)$ where b is the set of attacked edges, c the set of attacked anchors, m the new weights on the attacked edges, and a the new locations of attacked anchors.

A localization network under attack profile $\hat{\eta}$ is modified with these observed values in the expected way: the graph G_d is weighted by m_{ij} instead of d_{ij} and anchor positions are given by $p_a(i) = a_i$ instead of $p_a(i) = p^*(i)$. The localization network N under attack $\hat{\eta}$ will be denoted $\hat{\eta}(N)$, where $\hat{\eta}(\cdot)$ is the *application* of an attack profile to a localization network. For example, in Fig. 2, the localization network N depicted in (a) has the correct distance between nodes 2 and 7: $\|p(2) - p(7)\|_2^2$. But (b) depicts an embedding of the attacked localization result $\hat{\eta}(N)$, where attack profile $\hat{\eta} = (\{(2, 7)\}, \emptyset, \{\|p(2) - p(7')\|_2^2\}, \emptyset)$.

3.2 Problem Formulation

In this section we introduce formal notation for discussing attacks in a localization network. If the system can identify the attack and remove all corrupted data from the localization network it observes, the system is free to solve Problem 2.2 using an ordinary, insecure, localization algorithm. We name this prior task the *attack detection* problem.

Given the graph (S, E) of a localization network, we define an *attack hypothesis* $\eta = (b, c)$ where $b \subseteq E, c \subseteq S_a$. Intuitively, this is a guess as to which anchors and edges are under attack. Clearly, for each true attack, an attack profile $\hat{\eta} = (b, c, m, a)$ induces an attack hypothesis $\eta = (b, c)$ by simply "remembering" the identities of attacked edges/anchors and "forgetting" the values of the attack. A natural partial order is defined on attack hypotheses, where $(b, c) \preceq (b', c')$ iff $b \subseteq b'$ and $c \subseteq c'$ (as set inclusion). We will say that $\hat{\eta}$ (or η) is an $(\bar{s}, \bar{t})$*-attack profile (hypothesis)* if $|b| \leq \bar{s}, |c| \leq t$.

[2]Ranging measurements are corrupted multiplicatively (not additively) to avoid cross terms between distance and error when squaring $W(e_{ij})^2$ in the noisy case (Section 5).

The space of all $(\bar{s}, \bar{t})$-attack profiles (hypotheses) for a localization network N will be denoted $\hat{H}_N(\bar{s}, \bar{t})$ ($H_N(\bar{s}, \bar{t})$), or, if N is clear from the context, just $\hat{H}(\bar{s}, \bar{t})$ ($H(\bar{s}, \bar{t})$). If $\hat{\eta}(N) = N$, we say that $\hat{\eta}$ is *trivial* with respect to N.

Given a localization network N and an attack hypothesis η, we denote $N \setminus \eta$ the localization network with the hypthesized attacked edges and anchors removed. Since an attack profile $\hat{\eta}$ induces an attack hypothesis η, we will sometimes abuse notation and write $N \setminus \hat{\eta}$ for an attack profile $\hat{\eta}$, which means we remove from N the attack hypothesis induced by $\hat{\eta}$. We pose the attack detection problem for secure localization as follows.

Problem 3.1 (Noiseless Attack Detection). *For a localization network N and an attack profile $\hat{\eta}$, given only $\hat{\eta}(N)$, find a hypothesis $\zeta \in H(\bar{s}, \bar{t})$ such that $\eta \preceq \zeta$ (where η is the hypothesis induced by $\hat{\eta}$).*

The crux of successful attack detection is to make the attacks stand out in some way from the clean measurements of the localization network. If corrupted edges and anchors make up only a small part of a localization network they can be identified as the minimal part of the network that is incompatible with the rest. This forms the general idea behind voting-based attack detection schemes, e.g., [Liu et al. 2008]. Drawing inspiration from Yang et al.'s rigidity-based outlier detection technique [Yang et al. 2013a], we use the *embeddability* of a localization network, to define an $(\bar{s}, \bar{t})$-attack tolerant (abbreviated as $(\bar{s}, \bar{t})$-AT) localization network that admits incompatibility-based attack identification for up to $\bar{s}$ distance measurement attacks and $\bar{t}$ anchor position attacks.

Definition 3.2 (Sub-localization Network). *A localization network $N' = (S, E', W', p'_a)$ is a sub-localization network of $N = (S, E, W, p_a)$ (denoted $N' \subseteq N$) when $E' \subseteq E$, $W' = W \restriction E'$, and $p'_a = p_a \restriction S'_a$. If in addition, $|E \setminus E'| \leq \bar{s}$ and $|S_a \setminus S'_a| \leq \bar{t}$, we write $N' \subseteq_{\bar{s},\bar{t}} N$.*

Given an attack profile (hypothesis) $\hat{\eta}$ on N, we define its restriction $\hat{\eta} \restriction_{N'}$ by simply removing any attacks on edges or anchors that are not in N'. By an abuse of notation, we will sometimes write $\hat{\eta}(N')$ instead of $\hat{\eta} \restriction_{N'} (N')$. A sub-localization network $N' \subseteq N$ also induces an attack hypothesis which we denote $\eta_{N'}$, such that $N' = N \setminus \eta_{N'}$. We can now define:

Definition 3.3 ($(\bar{s}, \bar{t})$-Attack Tolerance). *A localization network N is $(\bar{s}, \bar{t})$-attack tolerant if $\forall \hat{\eta} \in \hat{H}(\bar{s}, \bar{t})$ and $\forall \eta \in H(\bar{s}, \bar{t})$, for $N' = N \setminus \eta$ (in which case $N' \subseteq_{(\bar{s},\bar{t})} N$), if $\hat{\eta}(N')$ is consistent then $\hat{\eta}$ is trivial with respect to N', i.e. $\hat{\eta}(N') = N'$.*

Intuitively, the definition says that if we guess an incorrect $(\bar{s}, \bar{t})$ attack hypothesis and did not remove *all* of the attacked edges and anchors, the application of the real attack $\hat{\eta}$ to the resulting sub-localization network will necessarily leave it inconsistent. Therefore, attacks in an $(\bar{s}, \bar{t})$-AT localization network can be identified by consistency checking. Clearly, GGR is an insufficient criterion, as shown in Fig. 2b where the attacked localization network is still embeddable; attack tolerance requires something more.

3.3 Conditions for Attack Tolerance

We begin by first considering what it takes for a localization network to be $\bar{s}$-AT in the absence of anchors (and anchor attacks). In our analysis a $\bar{s}$-redundantly generically globally rigid (abbreviated as $\bar{s}$-GGR) network, is a localization network that remains GGR after the removal of up to any $\bar{s}$ edges from its graph. We start with the following lemma about $\bar{s}$-GGR localization networks, illustrated by Fig. 3:

Lemma 3.4. *Let N be a consistent $\bar{s}$-GGR localization network, $\hat{\eta} \in \hat{H}(\bar{s}, 0)$. $\hat{\eta}(N)$ is conistent if and only if $\hat{\eta}$ is trivial.*

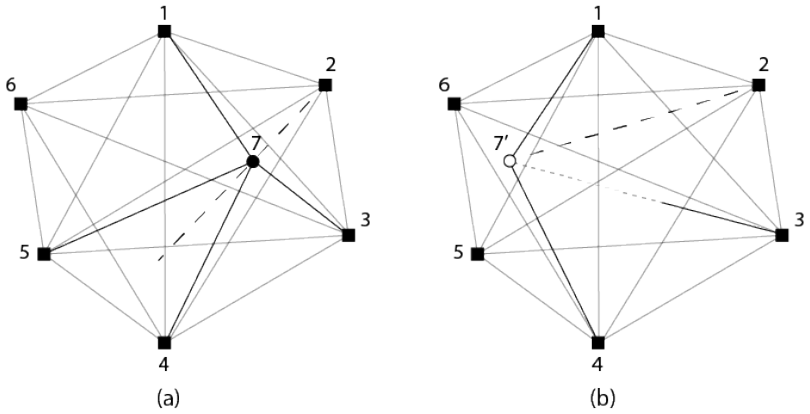

Fig. 3. Adding a redundant edge (3,7) to the example in Fig. 2 (b) satisfies Lemma 3.4. Neither the placement in (a) nor the placement in (b) is an embedding: (a) is incompatible with edge (2,7) and (b) is incompatible with (3,7)

Proof. Clearly, if $\hat{\eta}$ is trivial, $\hat{\eta}(N) = N$ is consistent. On the other hand, suppose $\hat{\eta}(N)$ is consistent. $\hat{\eta}(N) \setminus \hat{\eta} = N \setminus \hat{\eta}$ is also consistent (removing edges cannot make it any less consistent), and is GGR, since we removed no more than $\bar{s}$ edges from N. It therefore has exactly one realization (up to isometry), which induces the same edge weights on both N and $\hat{\eta}(N)$ □

Theorem 3.5. *If N is $2\bar{s}$-GGR, then N is $(\bar{s}, 0)$-AT.*

Proof. Let $N' \subseteq_{(\bar{s},0)} N$ and $\hat{\eta}$ an $(\bar{s}, 0)$-attack profile on N such that $\hat{\eta}(N')$ is consistent. N' is a consistent $\bar{s}$-GGR localization network, and therefore $\hat{\eta}(N') = N'$ by Lemma 3.4. □

The above results do not consider anchor attacks, but more importantly, they do not rely on the placement (or existence) of anchors at all, and are true up to isometry. We can therefore state the following.

Theorem 3.6. *If N is $2\bar{s}$-GGR and $|S_a| \geq 2\bar{t} + 3$, then N is $(\bar{s}, \bar{t})$-AT.*

Proof. We consider an attack $\hat{\eta} \in H(\bar{s}, \bar{t})$ and $N' \subseteq_{(s,t)} N$ such that $\hat{\eta}(N')$ is consistent. We must show the attack is trivial with respect to N'.

Clearly, since N' is $\bar{s}$-GGR the attack cannot contain any edges in N', or by Lemma 3.4 $\hat{\eta}(N')$ would be inconsistent. But without edge attacks, $\hat{\eta}(N')$ has exactly one realization up to isometry (it is $\bar{s}$-GGR, and in particular, GGR), and at least 3 unattacked anchors. These anchors uniquely localize the network, and so any anchor attack would make $\hat{\eta}(N')$ inconsistent. The attack is therefore trivial. □

Theorem 3.6 gives sufficient conditions for attack tolerance. A full characterization of the necessary and sufficient conditions for $(\bar{s}, \bar{t})$-AT attack detection are beyond the scope of this paper. The challenge of this characterization lies in evaluating the attack tolerance of a localization network that is not sufficiently redundantly GGR, but makes up the difference with anchor information. For an extreme example consider a sparsely connected localization network with no anchor attacks where every node is an anchor. Edge attacks are detectable but not by rigidity.

Algorithm 1 Attack Detection

procedure AttackDetection(localization network $\hat{\eta}(N)$, $\overline{s}$, $\overline{t}$)
  for $e_{ij} \in E$ and $a_k \in S_a$ **do**
    Declare pseudoboolean indicator variables b_{ij} and c_k ▷ 1 represents corrupted, 0 represents clean
  $C \leftarrow (\sum_{(i,j)\in E} b_{ij} \leq \overline{s}) \wedge (\sum_{k\in S_a} c_k \leq \overline{t})$ ▷ C is a set of pseudoboolean SAT clauses
  while Satisfiable(C) **do**
    AttackHypothesis $\zeta \leftarrow$ GetSatisfyingAssignment(C)
    (TestResult, EdgeResidues) $\leftarrow$ EmbeddabilityTest($\hat{\eta}(N) \setminus \zeta$)
    if TestResult = IsEmbeddable **then**
      return ζ ▷ $\eta \preceq \zeta$
    else
      NewC $\leftarrow$ GenCounterexamples ($\hat{\eta}(N) \setminus \zeta$, EdgeResidues)
      $C \leftarrow C \cup$ NewC $\cup (\bigvee_{(i,j)\in\zeta} b_{ij} \vee \bigvee_{k\in\zeta} c_k)$ ▷ C on next iteration includes counterexamples and ζ
  return Failure

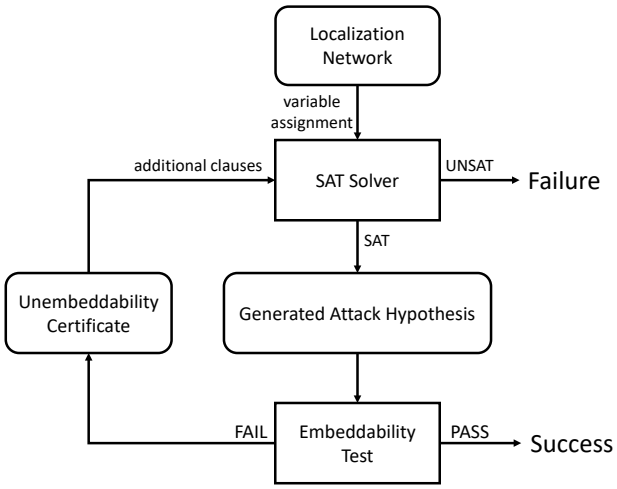

Fig. 4. An illustration of the steps and overall flow of Algorithm 1

4 THE GORDIAN ALGORITHM

Gordian is an algorithm designed to efficiently perform attack detection on finite localization networks, guaranteed to succeed on $(\overline{s}, \overline{t})$-AT localization networks. Gordian's design is inspired by Imhotep [Shoukry et al. 2015], Shoukry et al.'s lazy SMT solver for secure state estimation in the presence of attacks. Like Imhotep, Gordian identifies a combinatorial attack identification sub-problem that can be isolated from an otherwise convex optimization problem.

4.1 High Level Design

We present Gordian's high level process for solving the attack detection Problem 3.1 in Algorithm 1. Gordian's main steps are also graphically depicted in Fig. 4, and summarized here.

Given localization network N, GORDIAN first assigns a boolean variable to represent the corrupt/clean status for each of N's distance measurements and anchor positions. Next, GORDIAN assembles clauses made up of these variables to form a boolean satisfiability (SAT) problem[3], in such a way that a satisfying assignment to the variables represents a plausible attack hypothesis. GORDIAN solves the SAT problem and tests the attack hypothesis ζ obtained as the SAT solution. This is accomplished by testing the embeddability of $N' = N \setminus \zeta$, a network with the hypothesized attacked edges removed. If N' is embeddable, by Lemma 3.4 all remaining attacks are trivial and ζ may be returned as the Problem 3.1 solution. If N' is not embeddable, GORDIAN attempts to find small $N'' \subseteq N'$ that are also not embeddable. Algorithm 2 attempts to find these N'' and uses them to construct new SAT counterexample clauses which direct future iterations of GORDIAN to test attack hypotheses containing edges and anchors from N''. By Theorem .1 in the Appendix, assuming a well-posed input, GORDIAN will always terminate in the "Success" zone of Fig. 4 with a Problem 3.1 solution.

We walk through an iteration of GORDIAN's main loop on the long-running example[4] in Fig. 3. GORDIAN first assigns boolean variables to the anchors a_1, a_2, a_3, a_4, a_5, a_6 and the edges $e_{(1,2)}$, $e_{(1,3)}$, ... $e_{(4,7)}$. In our convention, the literal x refers to the positive occurance of variable x and x' as its negation. The clauses, C are initialized with $e_{(1,2)} + e_{(1,3)} + \ldots$ $e_{(4,7)} \leq \bar{s} = 1$ and $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq \bar{t} = 0$ to encode the assumption there are no more than $\bar{s}$ corrupted edges and no more than $\bar{t}$ corrupted anchors. The SAT solver immediately finds a satisfying assignment by setting every variable to false, signifying an empty attack hypothesis ζ. No edges or anchors are removed from $\hat{\eta}(N)$ on this iteration. An embeddability test is performed, and due to the inconsistency on edges connected to node 7, it fails. GORDIAN goes to the Trilateration Counterexamples (GENCOUNTEREXAMPLES) step starting with a randomly selected high residue edge, which happens to be (1,2). GORDIAN randomly selects edges (1,6) and (2,6) to complete the initial triangle. The induced sublocalization network for nodes 1, 2, and 6 is embeddable, so GORDIAN randomly selects node 7 to extend the counterexample. The induced sublocalization network for nodes 1, 2, 6, and 7 is still embeddable so GORDIAN tries again by randomly selecting node 3. Finally, the induced sublocalization network for nodes 1, 2, 3, 6 and 7 is *not* embeddable, so GORDIAN learns the clause $(a_1 \vee a_2 \vee a_3 \vee a_6 \vee e_{(1,2)} \vee e_{(1,3)} \vee e_{(1,6)} \vee e_{(1,7)} \vee e_{(2,3)} \vee e_{(2,6)} \vee e_{(2,7)} \vee e_{(3,6)} \vee e_{(3,7)})$, which contains every edge measurement and anchor position in the counterexample. On the next iteration of SAT solving, the new clause will lead to an attack hypothesis that sets at least one of those variables to true, shrinking the search space for the true attack.

4.2 Attack Hypothesis Generation

Assume we begin with a finite $(\bar{s}, \bar{t})$-AT localization network N that is corrupted by $\hat{\eta} \in \hat{H}(\bar{s}, \bar{t})$ (denoted $\hat{\eta}(N)$) as input. To model an attack hypothesis ζ on $\hat{\eta}(N)$, GORDIAN assigns pseudo-boolean indicator variables b_{ij} and c_k to the edges and anchors of its input. The attack hypothesis is a tuple of the edges and anchors for which the pseudo-boolean variables were set to 1. More formally $\zeta = (\{e_{ij} : b_{ij} = 1\}, \{a_k : c_k = 1\})$. This model allows attack hypotheses to be generated by a pseudo-boolean satisfiability (SAT) solver; initially only given the constraint there are fewer than $\bar{s}$ and $\bar{t}$ attacks, but over time accumulating counterexample clauses learned from EMBEDDABILITYTEST and GENCOUNTEREXAMPLES.

4.3 Embeddability Test

We will show in this section how to frame the embeddability (and localization) problem as a convex program, relying on a key result in the literature [So and Ye 2007]: Uniquely Localizable (UL) localization networks with no attacks

[3]Technically the constraints on line 4 of Algorithm 1 make this a pseudo-Boolean satisfiability problem, that is translatable into a boolean SAT problem.
[4]The example in Fig. 3 is just 1-GGR and not technically a well-formed input for $\bar{s} = 1$. GORDIAN can still run on such a localization network, just without the guarantee of correctness from Theorem 3.6.

can always be localized in the plane using the (BY-SDP) technique [Biswas et al. 2006] presented below. We take the contrapositive of this result to identify inconsistent localization networks.

Let the unknown $p(i)$ for $i \in S_x$ be decision variables and define $X = [p(1), p(2), ..., p(|S_x|)] \in \mathbb{R}^{2\times|S_x|}$ as the matrix of decision variables obtained by stacking the first and second coordinates of the $p(i)$. Also, let $v_i \in \{0,1\}^n$ be a unit column vector whose ith component is 1 and all other components 0, and $a_j \in \mathbb{R}^2$ be the position of anchor node j.

We define $g_{ij} = \begin{bmatrix} v_i - v_j & 0 \end{bmatrix}^\top$ if both s_i and s_j are sensors, and $g_{ij} = \begin{bmatrix} v_i & -a_j \end{bmatrix}^\top$ if either of s_i and s_j is an anchor. Now, the squares of sensor-sensor distances and sensor-anchor distances can be uniformly represented as

$$\|p(i) - p(j)\|^2 = \left| g_{ij}^\top \begin{bmatrix} \mathbf{X}^\top\mathbf{X} & \mathbf{X}^\top \\ \mathbf{X} & \mathbf{I}_2 \end{bmatrix} g_{ij} \right| \tag{4.1}$$

where $\mathbf{I}_2$ is a 2×2 identity matrix. With this representation for the aggregate $\|p(i) - p(j)\|^2$, we can set up an optimization problem of the form in (2.1). It is worthy to note that, the sensor set and the anchor set vary with the attack hypothesis. A hypothetically attacked anchor will be treated as a sensor with unknown location in the optimization problem.

$$\begin{aligned} \underset{\mathbf{X},\mathbf{Y}}{\text{minimize}} \quad & \sum_{(i,j)\in E} \underbrace{\frac{\left| g_{ij}^\top \begin{bmatrix} \mathbf{Y} & \mathbf{X}^\top \\ \mathbf{X} & \mathbf{I}_2 \end{bmatrix} g_{ij} - W(e_{ij})^2 \right|}{W(e_{ij})^2}}_{residue_{(i,j)}} \\ \text{subject to} \quad & \mathbf{Y} = \mathbf{X}^\top\mathbf{X}. \end{aligned} \tag{4.2}$$

We observe that the objective in Problem 4.2 is a summation over the residues on each edge (i, j) which will attain a minimum of zero, i.e. all residues are zeros, if there exists an embedding. However, Problem 4.2 is not convex, because the constraint $\mathbf{Y} = \mathbf{X}^\top\mathbf{X}$ restricts the rank of $\mathbf{Y}$ to be 2.

Biswas and Ye's solution [Biswas et al. 2006] to this dilemma is to lift the feasible set to a higher dimension by relaxing the constraint to $\mathbf{Y} \succeq \mathbf{X}^\top\mathbf{X}$ [Biswas and Ye 2004]. This yields the following SDP[5] solvable in polynomial time by interior point methods [Boyd and Vandenberghe 2004]. We refer to this relaxation as the BY-SDP.

$$\begin{aligned} \underset{\mathbf{X},\mathbf{Y}}{\text{minimize}} \quad & \sum_{(i,j)\in E_x} \frac{|g_{ij}^\top \mathbf{Z} g_{ij} - W(e_{ij})^2|}{W(e_{ij})^2} \\ \text{subject to} \quad & \mathbf{Z} \doteq \begin{bmatrix} \mathbf{Y} & \mathbf{X}^\top \\ \mathbf{X} & \mathbf{I}_2 \end{bmatrix} \succeq 0. \end{aligned} \tag{4.3}$$

Conceptually, this relaxation allows the solver to localize each sensor in a higher-dimension space $\mathbb{R}^{|S_x|}$ instead of in $\mathbb{R}^2$ [Biswas et al. 2006]. If the residues are all zero and $\mathbf{Y}$ has rank 2, up to some numerical errors, we can certify that the localization network is embeddable in $\mathbb{R}^2$. If localization is desired, the component of $\mathbf{Z}$ corresponding to $\mathbf{X}$ can be read off as the projection of the high dimensional solution back down to the plane of the anchors.

4.4 Trilateration Counterexamples

Without an efficient way to reduce the search space of attack hypotheses, Algorithm 1 reduces to a brute force search over all $\binom{|E|}{|\bar{s}|} \cdot \binom{|S_a|}{|\bar{t}|}$ candidates for ζ. But if Algorithm 1 finds $\hat{\eta}(N)\setminus\zeta$ unembeddable, often only a small $\hat{\eta}(N') \subset \hat{\eta}(N)\setminus\zeta$

[5]The normalizing factor $W(e_{ij})^2$ in the denominator is not explicitly given by Biswas and Ye, but it is implicitly equivalent to the arbitrary multiplicative weights in [Biswas et al. 2006].

causes unembeddability. If Algorithm 2 can find an $\hat{\eta}(N')$, Algorithm 1 can reduce its search space dramatically by learning a counterexample clause constructed by taking the "or" of the boolean variables for nodes and edges in $\hat{\eta}(N')$. With the new clause, all attack hypotheses on future iterations must offer an explanation of why N' was unembeddable, focusing the search.

Our heuristic approach, presented in Algorithm 2, for finding small $\hat{\eta}(N')$ is motivated by the observation that high residue values from localization tend to occur in the *vicinity* of the attacks in the graph. As clean trilateration localization networks are GGR, Eren et al.'s iterative trilateration method [Eren et al. 2004] is the basis for Algorithm 2. Starting with three nodes, Algorithm 2 extends a candidate $\hat{\eta}(N')$ one node at a time until $\hat{\eta}(N')$ is determined to be unembeddable[6] or a maximum subgraph size (8 nodes in our implementation) is reached. We memorize calls to the EmbeddabilityTest procedure to save runtime.

ImportanceSamplingByResidue produces a starting point for a trilateration subgraph by weighting every edge in the graph by its residue, normalizing weights into a probability distribution summing to 1, and then sampling an edge from that distribution. ImportanceSamplingByConnectingResidues is a similar algorithm that samples nodes with at least three connections to the current trilateration subgraph. The weight assigned to a node is equal to the sum of edge residues on edges connecting it to the trilateration subgraph.

Algorithm 2 Trilateration Counterexamples

procedure GenCounterexamples(localization network N, `EdgeResidues`)
  $C \leftarrow \emptyset$ ▷ C is the set of counterexamples
  for 1 : `NumberOfIterations` **do** ▷ `NumberOfIterations` is a parameter
    $(i, j) \leftarrow$ ImportanceSamplingByResidue(`EdgeResidues`)
    `ThirdNodeCandidates` $\leftarrow \{k : k \in S \,\&\, (i, k), (j, k) \in E\}$
    $k \leftarrow$ ImportanceSamplingByConnectingResidues(`ThirdNodeCandidates, EdgeResidues`)
    $N' \leftarrow$ NewSubLocalizationNetworkInducedBy$_N(\{i, j, k\})$
    while $|S'| <$ `MaxSubgraphSize` **do** ▷ `MaxSubgraphSize` is a parameter
      if EmbeddabilityTest(N') = `NotEmbeddable` **then**
        $C \leftarrow C \cup (\bigvee_{(i,j)\in E'} b_{ij} \vee \bigvee_{k\in S'_a} c_k)$ ▷ At least one edge or anchor in N' is corrupted
        break
      `NextNodeCands` $\leftarrow \{n : n \in S, \exists\, i, j, k \in S' \text{ s.t. } (i, n), (j, n), (k, n) \in E\}$
      `n` $\leftarrow$ ImportanceSamplingByConnectingResidues(`NextNodeCands, EdgeResidues`)
      $N' \leftarrow$ NewSubLocalizationNetworkInducedBy$_N(S' \cup n)$
  return C

5 NOISY GORDIAN

Real world sensors are imperfect, introducing small errors into their measurements which shouldn't qualify as attacks. We now consider the Noisy Localization and Noisy Attack Detection problems where small errors are expected on uncorrupted values. With an extension of Gordian's embeddability test (section 4.3) to this noisy case, we show how Gordian can be used to identify violations of sensor noise assumptions.

5.1 Noisy Localization Networks

To review our previous terminology, for localization network $N = (S, E, W, p_a)$, the Exact Localization Problem 2.2 presupposes the weights $W(e_{ij}) = d_{ij}$ and anchor positions $p_a = p^* \restriction_{S_a}$. An Attack Profile $\hat{\eta} = (b, c, m, a)$ substitutes $m_{ij} = d_{ij} * (1 + \delta_{ij})$ in place of weight $W(e_{ij})$ for $e_{ij} \in b$ and anchor positions $a_i = p^*(i) + \alpha_i$ for $p_a \in c$.

[6] Non-zero BY-SDP residues are sufficient for showing unembeddability.

We now introduce Noise Profiles $\hat{\nu} = (b, c, m, a)$ as a specialized form of Attack Profile, where δ_{ij} and α_i are not arbitrarily determined by an adversary, but instead drawn from zero-mean probability distributions as $\delta_{ij} \sim \Theta_{ij}$ and $\alpha_i \sim \Phi_i$ with the symbol $\sim$ standing for the "is distributed according to" relation. A noise profile may be applied to a noiseless localization network N as $\hat{\nu}(N)$ and may be composed with an (adversarial) attack profile $\hat{\eta}$ to represent a both corrupted and noisy localization network as $\hat{\eta}(\hat{\nu}(N))$.

Since our noise assumptions give anchor positions the possibility of error (like in GNSS sensors), we give the "softened" anchor constraints version of Equation 2.1. The new constant λ weights the relative significance of distance measurements against anchor positions in the objective. In this paper we, somewhat arbitrarily, use $\lambda = 5$.

$$\underset{p(i),\, i \in S}{\text{minimize}} \sum_{(i,j) \in E} \frac{\left| \|(p(i) - p(j)\|_2^2 - W(e_{ij})^2 \right|}{W(e_{ij})^2} + \lambda \sum_{k \in S_a} \|(p(k) - p_a(k)\|_2^2. \tag{5.1}$$

Observe the change from S_x and E_x in Equation 2.1 to S and E in the first summation of Equation 5.1. S changes because anchor positions are now decision variables. E changes because inter-anchor measurements are useful when anchors can "float". When anchor positions were hard constraints, the inter-anchor distance measurements taken by sensors were irrelevant and hence excluded via E_x.

5.2 Approximate Embeddings

We cannot simply extend Problem 2.2 to the noisy case by treating $\hat{\nu}(N)$ as input in place of N, because even attack-free noisy localization networks are almost never consistent. As Anderson et al. observe, solving Problem 2.2 for a noisy GGR localization network is equivalent to finding the solution to an over-determined system of polynomial equations (the distance constraints), and such solutions do not in general exist [Anderson et al. 2010]. Therefore we define noisy localization as the solution to the optimization problem in Equation 2.1.

Problem 5.1 (Noisy Localization). *Given $\hat{\eta}(N)$ for localization network N and noise profile $\hat{\eta}$, find the solution to (2.1).*

Framing noisy localization directly as an optimization problem is common in the literature (e.g. [Biswas and Ye 2004; Savvides et al. 2001; Wang et al. 2008]). In this paper as we consider anchor noise, we instead use Equation 5.1 and solve the similar Problem 5.2.

Problem 5.2 (Noisy Localization with Soft Anchors). *Given $\hat{\eta}(N)$ for localization network N and noise profile $\hat{\eta}$, find the solution to (5.1).*

Problems 5.2 and 5.1 are not given in terms of finding p^*; however Anderson et al. show (a statement similar to) Problem 5.1 has useful properties such as a unique minimum for a uniquely localizable N when noise is sufficiently small [Anderson et al. 2010]. Therefore a solution to Problems 5.2 and 5.1 can be thought of as an *approximate* solution to Problem 2.2.

We seek to generalize the notion of a noiseless embedding to an *approximate* embedding in an analogous way by examining residues. As a noiseless embedding has zero residues, an approximate embedding should have low residues. But how low?

Consider $N = (S, E, W, p_a)$ and a ground truth placement p^* s.t. $W(e_{ij}) = d_{ij}$ and $p_a = p^* \restriction_{S_a}$. In the remainder of this section we explain how to set residue thresholds for approximate embeddability in such a way that p^* is an

approximate embedding of $\hat{v}(N)$ at high confidence in the absence of an attack. If some N' were not approximately embeddable in this way, with high confidence $N' \neq \hat{v}(N)$, implying N' contains an attack.

We have already defined an edge $residue_{(i,j)}$ from Equation 4.2. Now with noise on anchors we may also define an anchor $residue_i$ as $\|(p(k) - p_a(k)\|_2^2$. Note for both edges and anchors, residues are defined with respect to a particular placement p.

Definition 5.3 (γ-β-Embedding). *A placement p is a γ-β-Embedding with respect to localization network N if*

(1) $\sum_{(i,j)\in E} residue_{(i,j)} \leq \gamma_e$.
(2) $\sum_{k\in S_a} residue_k \leq \gamma_a$.
(3) $\forall (i,j) \in E\ residue_{(i,j)} \leq \beta_{ij}$.
(4) $\forall k \in S_a\ residue_k \leq \beta_k$.

Conditions (1) and (2) appear in the the objective function for noiseless embeddability testing with soft anchor constraints in Equation 5.1. The β parameter used for conditions (3) and (4) captures the idea that bounded noise distributions Θ and Φ should yield bounded residues. The choice of particular conditions for Definition 5.3 was motivated by the following observations about the residues at p^*:

Since $d_{ij}^2 = \|(p^*(i) - p^*(j)\|_2^2$, and d_{ij}^2 appears in every term of noisy $W(e_{ij})$, it can be factored out, which yields

$$\begin{aligned} residue_{ij} = \frac{\left|d_{ij}^2 - m_{ij}^2\right|}{m_{ij}^2} &= \frac{\left|d_{ij}^2 - (d_{ij}^2(1+\delta_{ij})^2)\right|}{d_{ij}^2(1+\delta_{ij})^2} \\ &= \frac{\left|1 - (1+\delta_{ij})^2)\right|}{(1+\delta_{ij})^2} \end{aligned} \tag{5.2}$$

The same is true for $p^*(i)$ in

$$residue_k = \|p^*(k) - (p^*(k) + \alpha_k)\|_2^2 = \|\alpha_k\|_2^2. \tag{5.3}$$

These factored residue expressions contain no instances of d_{ij} nor p^*, yet represent the value of residues when $p = p^*$. Therefore, *it is not necessary to know p^* to calculate the residues at p^**. Noisy p^* residues may be determined directly from Θ and Φ.

These observations enable the selection of γ and β values G and B such that p^* is G-B-embeddable at high confidence. Monte Carlo simulation is a simple and effective method for settling on G and B values appropriate for Θ and Φ. The algorithm is straightforward: for a sufficiently large number of trials, sample δ and α values and compute $\sum_{(i,j)\in E} residue_{(i,j)}$ and $\sum_{k\in S_a} residue_k$. Looking at the histogram of residue sums across the trials, pick a high percentile (e.g. the 99.9th percentile) and use the corresponding values for G_e and G_a. The same algorithm works for B_e and B_a, and as we show in Section 6 is applicable to arbitrary Θ and Φ such as data sets from real-world sensors.

5.3 Approximate Embeddability Test

We next show how an SDP formulation for noisy localization may be adapted to G-B-embeddability testing via a constraint satisfaction problem (CSP) with a semidefinite relaxation.

The BY-SDP procedure (4.3) was originally intended as a solution to Problem 5.1 for noisy localization networks and is empirically effective with noisy measurements [Biswas and Ye 2004]. We give the soft-anchor variant of the method for Problem 5.2 in Equation 5.4. This alternative formulation can be intuitively understood as treating anchor nodes as unlocalized nodes with an anchor residue term in the objective penalizing the anchor's distance from its nominal

position. Formally, a modification must be made to matrix $\mathbf{X}$ by including anchor nodes as decision variables. Here $\mathbf{X} = [p(1), p(2), ..., p(|S|)] \in \mathbb{R}^{2\times|S|}$. Let $g_{ij} = \begin{bmatrix} v_i - v_j & 0 \end{bmatrix}^\top$ *for all* s_i and s_j. Separate the other case of g from before to $f_i = \begin{bmatrix} v_i & -a_j \end{bmatrix}^\top$ for $s_k \in S_a$. The definitions of $\mathbf{Y}$ and $\mathbf{Z}$ are as before, but with respect to the new $\mathbf{X}$.

$$\underset{\mathbf{Z}\succeq 0}{\text{minimize}} \sum_{(i,j)\in E} \frac{|g_{ij}^\top \mathbf{Z} g_{ij} - W(e_{ij})^2|}{W(e_{ij})^2} + \lambda \sum_{k\in S_a} f_k^\top \mathbf{Z} f_k \tag{5.4}$$

The first summation expresses inter-node residues changed to sum over E in place of E_x because inter-anchor measurements provide information when anchor positions are not hard constraints. The second summation expresses anchor residues ($\|(p(i) - p_a(i)\|_2^2$). As before, the semidefinite relaxation of Z changes this from an nonconvex rank-constrained problem to an SDP problem. To obtain the location estimate, read $\mathbf{X}^*$ off the minimizer $\mathbf{Z}^*$.

The important thing about Equation 5.4 from an approximate embeddability testing perspective is that it contains expressions for each edge and anchor residue. This is everything we need to test the G-B-embeddability of network N:

$$\sum_{(i,j)\in E} \frac{\left|g_{ij}^\top \mathbf{Z} g_{ij} - W(e_{ij})^2\right|}{W(e_{ij})^2} \leq G_e \tag{5.5a}$$

$$\sum_{k\in S_a} f_k^\top \mathbf{Z} f_k \leq G_a \tag{5.5b}$$

$$\frac{\left|g_{ij}^\top \mathbf{Z} g_{ij} - W(e_{ij})^2\right|}{W(e_{ij})^2} \leq B_{ij}, \forall (i,j) \in E \tag{5.5c}$$

$$f_k^\top \mathbf{Z} f_k \leq B_k, \forall k \in S_a, \tag{5.5d}$$

If the above constraints are not feasible with respect to any $\mathbf{Z} \succeq 0$, we can then conclude that the given N is not G-B-embeddable; however, if the constraints are feasible, we cannot be sure whether N is truly G-B-embeddable in 2 dimensions or simply appears as such due to the relaxation of the rank constraint on Z.

5.4 Noisy Attack Detection

Noisy attack detection generalizes from the noiseless case.

Problem 5.4 (Noisy Attack Detection). *For a localization network N, noise profile $\hat{\nu}$, and attack profile $\hat{\eta}$, given only $\hat{\eta}(\hat{\nu}(N))$, find a hypothesis $\zeta \in H(\bar{s}, \bar{t})$ such that $\hat{\eta}(\hat{\nu}(N)) \setminus \zeta$ is G-B-embeddable.*

We implemented Noisy Gordian by replacing the embeddability test in section 4.3 with Monte Carlo calculation of G and B and the CSP[7] in Equation 5.5. Noisy Gordian solves Problem 5.4 up to the SDP relaxation of its embeddability test. By the selection of parameters G and B, such a hypothesis exists (when $\zeta = \hat{\eta}$) with high confidence for $\hat{\eta}(\hat{\nu}(N)) \setminus \zeta$, and lines 7 and 8 of Algorithm 1 ensure Noisy Gordian will not terminate until it has found one. We classify non-trivial attacks (larger in magnitude than plausible noise from $\hat{\nu}$) into two categories: *G-B-effective* and *G-B-compatible*.

- *G-B-***Effective Attacks**: $\hat{\eta}$, yielding G-B-unembeddable $\hat{\eta}(\hat{\nu}(N))$.
- *G-B-***Compatible Attacks**: non-trivial $\hat{\eta}$, yielding G-B-embeddable $\hat{\eta}(\hat{\nu}(N))$.

[7] (5.5) is a constraint satisfaction problem, and does not produce meaningful residues as expected by Line 7 of Algorithm 1 in equation 5.5. As a workaround we generate residues by first solving a noisy localization problem (5.4) whenever residues are required.

BM	#Nodes	#Edges	$\bar{s}$	$\bar{t}$	BY-SDP (s)	ESDP (s)	$+\delta'_{ij}$	$-\delta'_{ij}$	$\|\alpha_i\|$	Noisy (s)	Correct	Nearly Correct
1	20	176	1	2	65.87	81.67	2.1	2.8	1.6	142.40	100%	0%
2	20	170	2	2	73.54	89.65	2.2	1.9	1.4	199.23	100%	0%
3	20	170	2	3	66.22	64.74	1.9	2.0	1.5	197.77	100%	0%
4	20	160	1	3	61.01	50.87	4.5	*	1.5	152.96	80%	20%
5	20	171	4	2	74.96	91.53	2.8	1.7	1.7	354.38	60%	0%
6	20	167	4	3	74.06	93.39	1.9	2.9	1.7	*	0%	40%
7	30	369	4	3	163.39	208.32	2.4	2.3	1.6	480.68	100%	0%
8	30	397	7	0	2140.79	211.23	2.1	1.8	*	218.74	80%	0%
9	30	411	6	0	104.62	174.55	2.8	2.2	*	446.83	80%	0%
10	30	381	6	2	190.13	259.04	1.9	2.1	1.8	811.40	80%	0%
11	30	368	0	5	46.00	76.81	*	*	1.6	142.25	100%	0%
12	30	372	2	7	160.67	201.96	2.3	2.0	1.4	645.57	80%	0%
13	40	716	5	7	1186.59	454.86	2.2	2.0	1.5	2671.00	100%	0%
14	40	723	5	3	267.82	382.80	1.9	2.0	1.6	1786.09	100%	0%
15	40	501	2	2	145.65	142.42	2.0	1.9	1.6	422.64	100%	0%
16	60	908	2	2	404.21	303.91	2.0	3.6	1.8	1398.83	60%	40%
17	70	1070	2	2	722.18	754.97	2.0	1.3	1.7	1383.09	80%	0%
18	80	1258	2	2	366.71	319.39	2.5	*	1.6	969.74	80%	0%

Table 1. Gordian runtime on BenchMarks (BM) using either BY-SDP or ESDP embeddability tests in the noiseless case, maximum undetectable attacks in the noisy case, and noisy runtime with accuracy. Noiseless runtimes are averages over 6 trials. For comparison, a brute-force (no trilateration counterexamples) BY-SDP trial took 3193.20 seconds on benchmark 1 and over 72 hours on benchmark 2 without terminating. δ' and α values represent maximum undetectable attacks. A * in the table is a placeholder for an impossible to collect value, either because there were no attacked edges/anchors in the benchmark or all attacked edges in the benchmark were too short to determine an effective negative attack. Noisy runtimes are averages over the fully correct runtimes found over 5 trials. Nearly correct trials had at most 1 misidentified attack. Benchmark 6 had no fully correct (all attacks identified) runtimes and hence no data point. Our explanation for this is below.

When Noisy Gordian solves Problem 5.4 it identifies G-B-effective attacks. Any unidentified G-B-compatible attacks are limited in size by the uncorrupted part of the localization network: If p is a G-B-embedding of $\hat{\eta}(\hat{v}(N))$, p must also be a G-B-embedding of $\hat{\eta}(\hat{v}(N)) \setminus \hat{\eta}$. This is obvious with respect to definition 5.3 where all conditions true for the residues of $\hat{\eta}(\hat{v}(N))$ must also hold for the subset of those residues in $\hat{\eta}(\hat{v}(N)) \setminus \hat{\eta}$. Therefore "$G$-$B$-compatible" attacks can only be as bad as the worst G-B-embeddable p of $\hat{\eta}(\hat{v}(N)) \setminus \hat{\eta}$, which the adversary does not influence.

Put simply, for geometric configurations or noise conditions where you wouldn't expect a "good" noisy localization result, you shouldn't expect a "good" attack detection result (i.e with limited G-B compatible attacks). In the world of navigation systems, Geometric Dilution of Precision (GDP) [Langley 1999] describes how the geometry of noisy range measurements can affect localization precision for a single sensor. Even for GGR localization networks, poor geometry on measurements may result in poor localization precision and poor attack detection. The theoretical aspects of this subject are worthy of further research, but for now we estimate the role of geometry and network topology empirically in our experimental evaluation by searching for maximum size G-B-compatible attacks on our benchmarks.

6 RESULTS

As the correctness of Gordian and related algorithms are proved in the previous sections, our empirical evaluation is focused on evaluating performance. In this section we evaluate the runtime of noiseless and noisy Gordian.

6.1 Experimental Setup

We implemented Gordian in Matlab 2016b, using the Yalmip toolbox [Löfberg 2004] to model the Semidefinite Programming (SDP)s. Our implementation uses MOSEK [noa 2011] and SAT4J's pseudoboolean solver [Le Berre 2010] as the underlying SDP and Boolean Satisfiability Problem (SAT) solvers. Our testing platform is a Windows machine with a quad-core Intel Core i7-4700MQ CPU and 16GB memory.

We generated benchmark graphs by randomly dropping nodes onto a 15 by 15 grid, and connecting those points with inter-node distances within a fixed connection radius[8]. Attacks consist of $\bar{s}$ randomly selected edges and $\bar{t}$ randomly selected anchors. In each benchmark, $2\bar{t}+3$ nodes are randomly determined to be anchors.

We model Θ with data obtained from Lab 11, the authors of the SurePoint Localization System [Kempke et al. 2016]. SurePoint uses DecaWave DW1000 ultra-wideband transceivers to achieve highly accurate ranging data in a real world enviornment. SurePoint achieves a median *additive* error of 0.08m with -0.59m and 0.31m as 95th percentile errors after proccessing.

A noise profile $\hat{v}$ was determined in the following way. Anchors were moved in a uniformly random direction a distance sampled from a truncated normal distribution clipped at 1.0 with mean 0 and standard deviation 0.2. Bounded additive noise was sampled uniformly at random from the SurePoint data set after eliminating the 405 out of 73414 outlier[9] values of noise with absolute error value greater than 0.7. We used these thresholds (1.0 and 0.7) to calculate B_{ij} and B_i values and Monte Carlo simulation to determine the 99.9th percentile of residue sums to set G_e and G_a values for G-B-Embeddability and Noisy Gordian experiments.

6.2 Noiseless Gordian

Our goal in experimental evaluation of the Noiseless Gordian algorithm is to evaluate the efficiency of Gordian with noiseless embeddability tests on localization networks with random attacks. The result for each benchmark is listed in Table 1.

In addition to the BY-SDP noiseless embeddabilty test, we also evaluate a (potentially faster) noiseless embeddability test built from Wang et al.'s alternative *edge*-centric (ESDP) relaxation of equation 4.2. Like the BY-SDP's rank condition for embeddability, Theorem 4.2 from [Wang et al. 2008] asserts that in the absence of attack, when the diagonal elements of $\mathbf{Y}-\mathbf{X}^\top\mathbf{X}$ are zero, all nodes have been localized to their true locations. Since an unembeddable localization network cannot simultaneously localize nodes to their true locations and achieve zero residues, the combination of zero residues and zero $trace(\mathbf{Y}-\mathbf{X}^\top\mathbf{X})$ certifies an embeddable ESDP solution.

In our trials of Gordian, BY-SDP vs ESDP embeddability tests usually result in roughly comparable runtimes, in line with the expectation of the SDP trending faster on dense localization networks with fewer nodes and the ESDP trending faster on sparse localization networks with more nodes [Wang et al. 2008]. Benchmarks 8 and 13 show instances where the ESDP achieved a much faster runtime than the BY-SDP. This may suggest in these cases a difference in the utility of ESDP residues and BY-SDP residues for trilateration counterexample generation.

Numerical errors and solver inaccuracies were significant challenges for implementation of embeddability tests in our noiseless experiments where picking thresholds for "nearly zero" and "nearly rank two" required fine-tuning.

[8]Due to scaling by $W(e_{ij})$ in the denominator of edge residues, a short edge with large noise can overpower the other residues. To prevent this problem in our evaluation we enforce a minimum measurement length of 1m before noise and 0.3m after noise.
[9]We eliminate outliers from the noise distribution to ensure that the only outliers in our experiments are the attacks we deliberately apply to our benchmarks.

However these concerns are dwarfed by measurement error in the noisy case, which we consider for the remainder of our evaluation.

6.3 *G-B*-Embeddability

We identified two options for relating additive SurePoint noise ($m_{ij} = d_{ij} + \delta'_{ij}$) to the multiplicative noise model ($m_{ij} = d_{ij}(1+\delta_{ij})$) assumed for Equation 5.2: obtain an equivalent $\delta_{ij} = \delta'_{ij}/d_{ij}$ or directly estimate d_{ij} as $d_{ij} = m_{ij} - \delta'_{ij}$ to calculate residue estimates from $residue_{ij} = \left|d_{ij}^2 - m_{ij}^2\right|/m_{ij}^2$ and the m_{ij} values on hand. We use this last approach in our evaluation to determine G values from Monte Carlo simulation.

The forth section of Table 1 addresses the performance of the *G-B* embeddability test[10] from equation 5.5. As discussed in Section 5.4, noise levels and the particular geometry of a benchmark determine whether an attack is *G-B* compatible (not detectable) or *G-B* effective (detectable). As we can only evaluate the runtime of Gordian when it has effective attacks to detect, we first set out to estimate the minimum size of an effective attack on our benchmarks.

This was accomplished by starting with the attack free and noise free underlying N of a benchmark, applying SurePoint and anchor noise $\hat{\nu}(N)$, arbitrarily selecting a single attacked edge/attacked anchor and corrupting it with a positive edge attack ($m_{ij} = d_{ij} + \delta'_{ij}$), a negative edge attack ($m_{ij} = d_{ij} - \delta'_{ij}$), or an anchor attack ($a_i = p_a(i) + \alpha_i$) to produce $\hat{\eta}(\hat{\nu}(N))$. Starting at the noise bound we incrementally increased the magnitude of the attack ($|\delta'_{ij}|$ or $\|\alpha_i\|$) until the resulting $\hat{\eta}(\hat{\nu}(N))$ became not *G-B*-Embeddable. The results in the 4th section of table 1 are averages across 5 trials of this process. This methodology produced useful estimates for the minimum size of a *G-B*-effective attack in these benchmarks given our noise conditions.

6.4 Noisy Gordian

As discussed, noisy Gordian is only able to detect *G-B*-effective attacks. Therefore for each edge attack we set $\delta'_{ij} \in [5,6]$ and for each anchor attack $\|\alpha_i\|_2 \in [5,6]$ so all attacks would likely be over the threshold of *G-B*-effectiveness. Edge attacks were given a 50% chance of being positive or negative[11]. In this way $\hat{\eta}$ was fixed for each benchmark. We generated 5 noise profiles $\hat{\nu}$ and for each profile evaluated a trial of Noisy Gordian on each $\hat{\eta}(\hat{\nu}(N))$ of our benchmarks.

The final section of table 1 contains the results of this experiment. The average runtimes across trials yielding fully correct attack detection results are presented in the first column. This allows a nearly apples-to-apples comparison against runtimes in the noiseless case where correct attack detection results are always produced. Overall, Noisy Gordian is significantly slower than corresponding noiseless problems, but is still much faster than brute force.

In the majority of trials, Noisy Gordian successfully identified the full attack. When Noisy Gordian failed in our experiments, either it confused *no more than one* uncorrupted edge or anchor with a corrupted edge or anchor, or the *G-B*-embeddability test misclassified an embeddable sublocalization network as unembeddable. In the latter case, Noisy Gordian learns a counterexample clause that makes the SAT problem on line 5 of algorithm 1 unsatisfiable (which is always a risk considering the 99.9% confidence intervals on G). This was what happened with benchmark 6. By repeating the tests for benchmark 6 with a bigger (99.99%) confidence interval, the results improved to 0% correct and 80% nearly correct. In summary, *over all experiments, Noisy Gordian either produced a correct result, a nearly correct result with a single error, or failed while indicating no result could be found at all.*

[10] We implemented and evaluated an ESDP-style approximate embeddability test too, but determined the ESDP's weaker relaxation has too many false negatives, i.e. localization networks which only appear embeddable under the ESDP relaxation, to justify its use.
[11] If a negative attack is impossible because d_{ij} is small and edges cannot have negative length, it is changed to a positive attack

7 CONCLUSION

We have presented Gordian, an attack detection algorithm at the distance-graph abstraction level. In the noiseless case, we proved Gordian sound and complete for $(\bar{s}, \bar{t})$-AT input. By generalizing localization network embeddability testing to approximate embeddability testing in the noisy case, we likewise extend Gordian to Noisy Gordian. We evaluate our algorithms on noisy attack detection problems constructed from the error distribution of a real-world localization system, demonstrating the practical utility of our formal reasoning based outlier detection scheme.

Gordian and Noisy Gordian consistently finished combinatorially difficult attack detection benchmarks with a high success rate on the order of minutes instead of days. These algorithms are appropriate as a periodic check to identify bad actors in a long-running sensor network or general network of cyber-physical systems. Once adversarial data are removed, localization can be made more trustworthy and more accurate.

APPENDIX

Theorem .1. *Given a finite $(\bar{s}, \bar{t})$-AT localization network $N = (S, E, W, p_a)$ and a ground truth placement p^* s.t. $W(e_{ij}) = d_{ij}$ and $p_a = p^* \restriction_{S_a}$ and $\hat{\eta} \in \hat{H}(\bar{s}, \bar{t})$, Algorithm 1 is a sound and complete procedure for solving Problem 3.2 when given input $\hat{\eta}(N)$.*

Proof. (Soundness) The algorithm returns with an attack hypothesis $\eta_{N'}$ only if $\hat{\eta}(N')$ is consistent. But since N was $(\bar{s}, \bar{t})$-AT, that means by definition that $\hat{\eta}$ is trivial with respect to N', and so $\hat{\eta} \subseteq \eta_{N'}$.

(Completeness) Let η be the attack hypothesis induced by $\hat{\eta}$. If on line 4 we choose an $(\bar{s}, \bar{t})$-attack hypothesis $\zeta \succeq \eta$, $\hat{\eta}(N) \setminus \zeta$ is consistent and the algorithm ends. Denote by C_i the set of clauses at iteration i and let $SAT(C_i)$ be the set of satisfiable assignments to C on iteration i. Since C_0 simply bounds the number of attacks, clearly $\eta \in SAT(C)$. C_{i+1} is constructed $C_{i+1} \supseteq C_i$ (line 12) to forbid a reassignment on future iterations matching ζ (line 12 last term). Specifically $\zeta \in SAT(C_i)$, but $\zeta \notin SAT(C_{i+1})$, and $SAT(C_{i+1}) \subset SAT(C_i)$. $SAT(C_0)$ is finite, and so we just have to show that if $\eta \in SAT(C_i)$ and the algorithm doesn't end, $\eta \in SAT(C_{i+1})$.

On every iteration where the algorithm doesn't end we add clauses to C of the form $(\bigvee_{(i,j)\in \mathsf{E}} b_{ij} \vee \bigvee_{k\in \mathsf{Sa}} c_k)$ for some inconsistent sub-localization network. At least one of the b_{ij}, c_k must be in the true attack hypothesis η, or the sub-localization network would be consistent (the unattacked localization network was consistent, and removing edges or anchors can't make it less consistent). And so η satisfies the new added clause. □

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