Distributional Interpretation of Control Strategies for a Multiplicative Observation Noise System

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by

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Research Project

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Abstract

We consider the stabilization of a discrete-time linear system in the presence of continuous multiplicative observation noise. Previous work has explored time-varying periodic non-linear control approaches for this problem. To understand the information-gathering role of control in this problem, this report explicitly computes how the conditional density of the state of the system evolves given the observations.

The calculations suggest a novel control strategy that chooses the control equal to the maximum a-posteriori estimate for the state. We show that as $n \to \infty$ this control strategy does indeed drive the system state to 0 almost surely.

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1 Introduction

1.1 Multiplicative noise control system

Throughout this work, we will consider a scalar system with state $X_n \in \mathbb{R}$ at time $n \in \mathbb{Z}_{\geq 0}$, control U_n, observations Y_n, and multiplicative noise C_n. The initial state X_0 and the noises C_n are random variables (r.v.s) with corresponding distributions $\mathcal{D}_X$ and $\mathcal{D}_C$. The system is defined as follows in eq. (1).

$$\begin{cases} X_{n+1} = X_n + U_n & X_0 \sim \mathcal{D}_X \\ Y_n = C_n X_n & C_n \sim \mathcal{D}_C \text{ i.i.d.} \end{cases} \tag{1}$$

The control at a time n, U_n, is chosen causally as a function of $Y_0^n := (Y_0, Y_1, \ldots, Y_n)$. $\mathcal{D}_X$ and $\mathcal{D}_C$ are known to the controller. However, the controller does not have access to the realizations of the initial state ($X_0 = x_0$) nor multiplicative noises ($C_0 = c_0, C_1 = c_1, \ldots, C_n = c_n$). We denote r.v. realizations in lowercase. The objective of studying this system is to understand strategies for stabilizing the system, i.e., achieving $\lim_{n\to\infty} \mathbb{E}\left[|X_n|^2\right] = 0$.

A secondary objective is to understand the maximal rate at which the state of the system in eq. (1) can decay to zero. As discussed in [1, 2] this decay rate can be directly related to the maximum growth factor, a, that can be tolerated by the system $X_{n+1} = aX_n + U_n$, under the same observation model as eq. (1).

1.2 Related work

This work builds on ideas regarding the multiplicative noise control system discussed in [1]. In [1], the significance of active estimation of X_n by leveraging controls, U_n, is established. We elaborate more on this point by examining how controls $U_0, U_1, \ldots, U_{N-1}$ alter the conditional density of the initial state given our observations, i.e. $f(X_0|Y_0^N)$.

It was noted in [1] that nonlinear control strategies outperform linear ones for stabilizing the system in eq. (1).

In [2, 3] the problem of finding further control strategies that achieve better convergence rates was addressed by training neural networks to stabilize eq. (1). Then functions were fit to the neural network functions to see if such strategies were interpretable.

In [2] learning was useful for finding nonlinear strategies that achieved improved decay rates over the first handcrafted proof of concept controller given in [1]. The controllers found in [2] were interpretable in the sense of being expressible in the form of common functions. However, they were not interpretable in precisely how they were stabilizing the system and gaining more knowledge of the state from observations Y_n. We explore this question of interpreting how the controllers in [2] are actively learning through exploring the evolution of the maximum a-posteriori estimator of the state given observations.

2 Density Changes Under Control

2.1 Introduction

Our main goal in this section is to set up preliminaries that will allow us to show that the control strategy based on MAP estimation of the state stabilizes the system in the next section. To do so, we examine the qualitative effects of controls, U_n, on the information we have regarding our initial state, X_0, or latest state at some time N, i.e. X_N, for the system in eq. (1). This information comes in the form of conditional densities of X_0 or X_N given observations, Y_0^N, from which we can derive guarantees that a MAP-based control strategy stabilizes the system. We restrict our discussion to when X_0 and C_n are drawn from a uniform distribution on $[-1, 1]$, i.e. $\mathcal{D}_X = \text{Unif}[-1, 1]$ and $\mathcal{D}_C = \text{Unif}[-1, 1]$. The boundedness of the densities of these distributions restricts the realizable values of X_0 and X_N upon conditioning. This makes the behavior of control strategies interpretable. As we will shortly see, information regarding X_0 as captured by its joint density with observations, Y_0^N, is equivalent to the information regarding X_N conditioned on Y_0^N.

2.2 Derivation of System State Joint Densities and Conditional Densities

In this section, we derive expressions for the densities of (X_0, Y_0^N), (X_N, Y_0^N), $Y_0^N|X_0$, $X_0|Y_0^N$, and $X_N|Y_0^N$ which will enable examining how control gives us further information through our observations. To do this, we must first establish a lemma and some definitions.

Lemma 2.1. *Let $X_1^N = (X_1, \dots, X_N)$ and $Y = g(X_1^N)$ be r.v.s where $g : \mathbb{R}^N \to \mathbb{R}$ is a deterministic function of X_1^N. Then the distribution function of $Y|X_1^N$ is:*

$$\mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) = \begin{cases} 1 & y \ge g(x_1^N) \\ 0 & y < g(x_1^N) \end{cases}. \tag{2}$$

Proof. Consider $Y|X_1^N$. Since $Y = g(X_1^N)$, upon conditioning on $X_1^N = x_1^N$, Y takes value $g(x_1^N)$ with probability 1, i.e., $\mathbb{P}(Y = g(x_1^N)|X_1^N = x_1^N) = 1$. Consider the distribution function for the case that $y \ge g(x_1^N)$.

$$\mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) = \mathbb{P}\big(Y \le g(x_1^N)|X_1^N = x_1^N\big) + \mathbb{P}\big(g(x_1^N) < Y \le y|X_1^N = x_1^N\big) \tag{3}$$
$$\mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) \ge \mathbb{P}\big(Y \le g(x_1^N)|X_1^N = x_1^N\big) \tag{4}$$
$$\mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) \ge \mathbb{P}\big(Y = g(x_1^N)|X_1^N = x_1^N\big) + \mathbb{P}\big(Y < g(x_1^N)|X_1^N = x_1^N\big) \tag{5}$$
$$\mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) \ge \mathbb{P}\big(Y = g(x_1^N)|X_1^N = x_1^N\big) \tag{6}$$
$$\mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) \ge 1 \tag{7}$$
$$\implies \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) = 1 \tag{8}$$

Consider the distribution function for the case that $y < g(x_1^N)$.

$$1 = \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) + \mathbb{P}\big(Y > y|X_1^N = x_1^N\big) \tag{9}$$
$$1 = \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) + \mathbb{P}\big(g(x_1^N) > Y > y|X_1^N = x_1^N\big) + \mathbb{P}\big(Y \le g(x_1^N)|X_1^N = x_1^N\big) \tag{10}$$
$$1 \ge \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) + \mathbb{P}\big(Y \le g(x_1^N)|X_1^N = x_1^N\big) \tag{11}$$
$$1 \ge \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) + \mathbb{P}\big(Y = g(x_1^N)|X_1^N = x_1^N\big) + \mathbb{P}\big(Y < g(x_1^N)|X_1^N = x_1^N\big) \tag{12}$$
$$1 \ge \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) + \mathbb{P}\big(Y = g(x_1^N)|X_1^N = x_1^N\big) \tag{13}$$
$$1 \ge \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) + 1 \tag{14}$$
$$0 \ge \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) \tag{15}$$
$$\implies \mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) = 0 \tag{16}$$

Thus $\mathbb{P}\big(Y \le y|X_1^N = x_1^N\big) = \begin{cases} 1 & y \ge g(x_1^N) \\ 0 & y < g(x_1^N) \end{cases}$. □

Definition 2.1 (Controller with all available memory)**.** For the system in eq. (1), we say that we are using a **controller with all available memory** iff every control at time n, i.e. U_n, is a deterministic function of all available observations, i.e.,

$$\forall n : U_n = g_n(Y_0^n), g_n : \mathbb{R}^{n+1} \to \mathbb{R}. \tag{17}$$

Lemma 2.2 (Conditional density of $Y_0^N|X_0$)**.** *For the system in eq. (1), assume that X_0 has density $f_X(x)$, the C_n have density $f_C(c)$, and we use a controller with all available memory, $u_n = g_n(y_0^n)$. Then, we have the following $f(Y_0^N|X_0)$:*

$$f(Y_0^N = y_0^N|X_0 = x) = \prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x+\sum_{i=0}^{n-1} u_i}\right)}{|x + \sum_{i=0}^{n-1} u_i|} = \prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x+\sum_{i=0}^{n-1} g_i(y_0^i)}\right)}{|x + \sum_{i=0}^{n-1} g_i(y_0^i)|}. \tag{18}$$

Proof. By Bayes rule, we have the following:

$$f(Y_0^N = y_0^N | X_0 = x) = \left(\prod_{n=1}^{N} f(Y_n = y_n | X_0 = x, Y_0^{n-1} = y_0^{n-1}) \right) f(Y_0 = y_0 | X_0 = x). \tag{19}$$

Consider some arbitrary observation $Y_n = C_n(X_0 + \sum_{i=0}^{n-1} U_i) = C_n(X_0 + \sum_{i=0}^{n-1} g_i(Y_0^i))$ at times $n > 0$, and this observation under conditioning on the initial state and past observations: $Y_n | \{X_0 = x, Y_0^{n-1} = y_0^{n-1}\}$. Since the U_i, $0 \le i < n$ are functions of Y_0^i we see that the only leftover randomness comes from C_n: $Y_n | \{X_0 = x, Y_0^{n-1} = y_0^{n-1}\} = \left(x + \sum_{i=0}^{n-1} g_i(y_0^i) \right) \cdot C_n$. We have a similar result for $Y_0 | \{X_0 = x\} = x \cdot C_0$. Thus we have the following conditional densities:

$$f(Y_n = y_n | X_0 = x, Y_0^{n-1} = y_0^{n-1}) = \frac{f_C\left(\frac{y_n}{x + \sum_{i=0}^{n-1} g_i(y_0^i)} \right)}{|x + \sum_{i=0}^{n-1} g_i(y_0^i)|} \tag{20}$$

$$f(Y_0 = y_0 | X_0 = x) = \frac{f_C(\frac{y_0}{x})}{|x|}. \tag{21}$$

The above eqs. (20) and (21) follow from the fact that for some r.v. X with density $f_X(x)$, the r.v. $Y = aX$ with $a \in \mathbb{R}$ has density $f_Y(y) = \frac{f_X(\frac{y}{a})}{|a|}$. Substituting the RHS expressions of eqs. (20) and (21) into eq. (19) yields the desired result. □

Lemma 2.3 (Joint density of (X_0, Y_0^N))**.** *For the system in eq. (1), assume that X_0 has density $f_X(x)$, the C_n have density $f_C(c)$, and we use a controller with all available memory, $u_n = g_n(y_0^n)$. Then, we have the following joint density for (X_0, Y_0^N):*

$$f(X_0 = x, Y_0^N = y_0^N) = \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x + \sum_{i=0}^{n-1} u_i} \right)}{|x + \sum_{i=0}^{n-1} u_i|} \right) f_X(x) = \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x + \sum_{i=0}^{n-1} g_i(y_0^i)} \right)}{|x + \sum_{i=0}^{n-1} g_i(y_0^i)|} \right) f_X(x) \tag{22}$$

(Note: we take as convention for a sum $S = \sum_{i=L}^{H} a_i$ that if $L > H$, $S = 0$.)

Proof. By Bayes rule, we have the following:

$$f(X_0 = x, Y_0^N = y_0^N) = f(Y_0^N = y_0^N | X_0 = x) f(X_0 = x) \tag{23}$$

$$= f(Y_0^N = y_0^N | X_0 = x) f_X(x) \tag{24}$$

Substituting in the result of lemma 2.2, $f(Y_0^N | X_0)$, into eq. (24) yields the joint density. □

Lemma 2.4 (Joint density of (X_N, Y_0^N))**.** *For the system in eq. (1), assume that X_0 has density $f_X(x)$, the C_n have density $f_C(c)$, and we use a controller with all available memory, $u_n = g_n(y_0^n)$. Then, we have the following joint density for X_N, Y_0^N:*

$$f(X_N = x, Y_0^N = y_0^N) = \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x - \sum_{i=n}^{N-1} u_i} \right)}{|x - \sum_{i=n}^{N-1} u_i|} \right) f_X\left(x - \sum_{i=0}^{N-1} u_i \right) \tag{25}$$

$$= \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x - \sum_{i=n}^{N-1} g_i(y_0^i)} \right)}{|x - \sum_{i=n}^{N-1} g_i(y_0^i)|} \right) f_X\left(x - \sum_{i=0}^{N-1} g_i(y_0^i) \right). \tag{26}$$

(Note: we take as convention for a sum $S = \sum_{i=L}^{H} a_i$ that if $L > H$, $S = 0$.)

Proof. For a r.v. X, we use the calligraphic $\mathcal{X}$ to indicate the support of the density of X. Using Bayes rule and the law of total probability, the joint density is as follows:

$$f\big(X_N = x, Y_0^N = y_0^N\big) = \int_{\mathcal{X}_0} f\big(X_N = x, X_0 = x', Y_0^N = y_0^N\big) dx' \tag{27}$$

$$= \int_{\mathcal{X}_0} f\big(X_N = x | X_0 = x', Y_0^N = y_0^N\big) f\big(X_0 = x', Y_0^N = y_0^N\big) dx'. \tag{28}$$

Since $X_N = X_0 + \sum_{i=0}^{N-1} g_i(Y_0^i)$ and is thus a function of X_0 and Y_0^{N-1}, we can apply lemma 2.1 to state that $\mathbb{P}\big(X_N \leq x | X_0 = x', Y_0^N = y_0^N\big) = \begin{cases} 1 & x \geq x' + \sum_{i=0}^{N-1} g_i(y_0^i) \\ 0 & x < x' + \sum_{i=0}^{N-1} g_i(y_0^i) \end{cases}$. The derivative of this probability is $f\big(X_N = x | X_0 = x', Y_0^N = y_0^N\big)$ in eq. (28), which will end up being a Dirac delta as seen below:

$$f\big(X_N = x, Y_0^N = y_0^N\big) = \int_{\mathcal{X}_0} \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}\big(X_N \leq x | X_0 = x', Y_0^N = y_0^N\big)\right) f\big(X_0 = x', Y_0^N = y_0^N\big) dx' \tag{29}$$

$$= \int_{\mathcal{X}_0} \delta\left(x - \left(x' + \sum_{i=0}^{N-1} g_i(y_0^i)\right)\right) f\big(X_0 = x', Y_0^N = y_0^N\big) dx' \tag{30}$$

$$= \int_{\mathcal{X}_0} \delta\left(\left(x - \sum_{i=0}^{N-1} g_i(y_0^i)\right) - x'\right) f\big(X_0 = x', Y_0^N = y_0^N\big) dx' \tag{31}$$

$$= \int_{\mathcal{X}_0} \delta\left(x' - \left(x - \sum_{i=0}^{N-1} g_i(y_0^i)\right)\right) f\big(X_0 = x', Y_0^N = y_0^N\big) dx' \tag{32}$$

$$= f\left(X_0 = x - \sum_{i=0}^{N-1} g_i(y_0^i), Y_0^N = y_0^N\right). \tag{33}$$

Substituting in the result of lemma 2.3 we have the following joint density:

$$f\big(X_N = x, Y_0^N = y_0^N\big) = \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x - \sum_{i=0}^{N-1} g_i(y_0^i) + \sum_{i=0}^{n-1} g_i(y_0^i)}\right)}{|x - \sum_{i=0}^{N-1} g_i(y_0^i) + \sum_{i=0}^{n-1} g_i(y_0^i)|}\right) f_X\left(x - \sum_{i=0}^{N-1} g_i(y_0^i)\right) \tag{34}$$

$$= \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x - \sum_{i=n}^{N-1} g_i(y_0^i)}\right)}{|x - \sum_{i=n}^{N-1} g_i(y_0^i)|}\right) f_X\left(x - \sum_{i=0}^{N-1} g_i(y_0^i)\right). \tag{35}$$

□

Note that $f(X_0, Y_0^N)$ is equivalent $f(X_N, Y_0^N)$ up to a shift by $\sum_{i=0}^{N-1} u_i$. Thus estimating X_N from observations Y_0^N using $f(X_N, Y_0^N)$ is equivalent to estimating X_0 from the Y_0^N using $f(X_0, Y_0^N)$. However, no such equivalency can be determined for the conditional distributions of $Y_0^N | X_0$ and $Y_0^N | X_N$, as the density of $Y_0^N | X_N$ will contain a density of X_N term as the difference from that of $Y_0^N | X_0$. There are thus three strategies of interest to examine for driving the state to zero.

We can now express the conditional densities of $X_0 | Y_0^N$ and $X_N | Y_0^N$ by utilizing Bayes rule.

Lemma 2.5 (Conditional density of $X_0 | Y_0^N$)**.** *For the system in eq. (1), assume that X_0 has density $f_X(x)$, the C_n have density $f_C(c)$, and we use a controller with all available memory, $u_n = g_n(y_0^n)$. Then $f(X_0 | Y_0^N)$ is given as:*

$$f(X_0 = x | Y_0^N = y_0^N) = \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x + \sum_{i=0}^{n-1} u_i}\right)}{|x + \sum_{i=0}^{n-1} u_i|}\right) f_X(x) \tag{36}$$

$$= \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x + \sum_{i=0}^{n-1} g_i(y_0^i)}\right)}{|x + \sum_{i=0}^{n-1} g_i(y_0^i)|}\right) f_X(x). \tag{37}$$

Proof. Apply Bayes rule to $f(X_0, Y_0^N)$.

$$f(X_0 = x | Y_0^N = y_0^N) = \frac{f(X_0 = x, Y_0^N = y_0^N)}{f(Y_0^N = y_0^N)}. \tag{38}$$

It suffices to substitute in $f(X_0, Y_0^N)$ from lemma 2.3. □

Lemma 2.6 (Conditional density of $X_N|Y_0^N$)**.** *For the system in eq. (1), assume that X_0 has density $f_X(x)$, the C_n have density $f_C(c)$, and we use a controller with all available memory, $u_n = g_n(y_0^n)$. Then, $f(X_N|Y_0^N)$ is given as:*

$$f(X_N = x|Y_0^N = y_0^N) = \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x - \sum_{i=n}^{N-1} u_i}\right)}{|x - \sum_{i=n}^{N-1} u_i|} \right) f_X\left(x - \sum_{i=0}^{N-1} u_i\right) \tag{39}$$

$$= \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x - \sum_{i=n}^{N-1} g_i(y_0^i)}\right)}{|x - \sum_{i=n}^{N-1} g_i(y_0^i)|} \right) f_X\left(x - \sum_{i=0}^{N-1} g_i(y_0^i)\right). \tag{40}$$

Proof. Apply Bayes rule to $f(X_N, Y_0^N)$.

$$f(X_N = x|Y_0^N = y_0^N) = \frac{f(X_N = x, Y_0^N = y_0^N)}{f(Y_0^N = y_0^N)}. \tag{41}$$

It suffices to substitute in the $f(X_N, Y_0^N)$ from lemma 2.4. □

2.3 Interpretation of Conditional Densities under Control

2.3.1 X_0 Estimation Perspective

As is often done in control problems, we consider the problem of estimating the initial state X_0 from $f(X_0|Y_0^N)$ to then use these estimates to control the system. We highlight the connection between estimation and control for our system which helps us later understand how MAP estimation helps with control. First, we make $f(X_0|Y_0^N)$ explicit by defining the indicator function:

Definition 2.2 (Indicator Function)**.** Let $\mathbb{1}\{x : P(x)\}$ denote the indicator function, where $P(x)$ is a proposition regarding x. The domain and range are specified as $\mathbb{1}\{x : P(x)\} : \mathcal{X} \to \{0, 1\}$. It takes on values as follows:

$$\mathbb{1}\{x : P(x)\} := \begin{cases} 1 & P(x) \text{ is true} \\ 0 & P(x) \text{ is false} \end{cases}. \tag{42}$$

We can now write the densities of X_0 and C_n, i.e. f_X and f_C. They are the same function and can be written in terms of $\mathbb{1}\{\cdot\}$ as:

$$f_X(x) = \frac{1}{2}\mathbb{1}\{x : |x| \le 1\} \tag{43}$$

$$f_C(c) = \frac{1}{2}\mathbb{1}\{c : |c| \le 1\}. \tag{44}$$

Thus from lemma 2.5 $f(X_0|Y_0^N)$ is a product of indicator functions:

$$f(X_0 = x|Y_0^N = y_0^N) = \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{f_C\left(\frac{y_n}{x + \sum_{i=0}^{n-1} u_i}\right)}{|x + \sum_{i=0}^{n-1} u_i|} \right) f_X(x) \tag{45}$$

$$= \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{\frac{1}{2}\mathbb{1}\left\{x : \left|\frac{y_n}{x + \sum_{i=0}^{n-1} u_i}\right| \le 1\right\}}{|x + \sum_{i=0}^{n-1} u_i|} \right) \left(\frac{1}{2}\mathbb{1}\{x : |x| \le 1\}\right). \tag{46}$$

The effect of the indicators in eq. (46) is to restrict the support of $f(X_0|Y_0^N)$. We denote the support of $f(X_0|Y_0^N)$ as $\mathcal{X}_{0|N}$. To see how $\mathcal{X}_{0|N}$ is restricted, we define the notion of an excision which comes from the noise density terms of the form $\mathbb{1}\left\{x : \left|\frac{y}{x-u}\right| \le 1\right\}$.

Definition 2.3 (Excision of a Function, $\mathcal{E}$)**.** The excision $\mathcal{E}$ of a function, $g : \mathbb{R} \to \mathbb{R}$, where g has support $\mathcal{X}$, is the interval $\mathcal{E} = \mathbb{R} \setminus \mathcal{X}$. We also say that g excises $\mathcal{E}$.

Thus a function of the form $\mathbb{1}\left\{x : \left|\frac{y}{x-u}\right| \leq 1\right\} = \mathbb{1}\{x : |y| \leq |x-u|\}$ with support $\mathcal{X} = \mathbb{R} \setminus (u-|y|, u+|y|)$ excises $\mathcal{E} = (u-|y|, u+|y|)$. Note that in eq. (46), the excisions from each observation density factor are centered at sums of the controls, i.e. $-\sum_{i=0}^{n-1} u_i$, with excision interval widths of $2|y_n|$. We define these excision centers to interpret them as estimates of X_0.

Definition 2.4 (Excision Centers, $\widetilde{U}_n$)**.** For the system in eq. (1), assume that $X_0 \sim$ Unif$[-1,1]$ and $C_n \sim$ Unif$[-1,1]$ i.i.d. with control sequence $U_0, U_1, \ldots$. We define the excision centers $\widetilde{U}_n$ as follows:

$$\widetilde{U}_n := -\sum_{i=0}^{n-1} U_i \quad \text{and} \quad \widetilde{U}_0 = 0. \tag{47}$$

Consider the evolution of the system state X_n over many iterations as below, where we write every X_n in terms of X_0 and all the controls:

$$\begin{aligned}
X_0 &= X_0 &(48)\\
X_1 &= X_0 + U_0 &(49)\\
X_2 &= X_0 + U_0 + U_1 &(50)\\
X_3 &= X_0 + U_0 + U_1 + U_2 &(51)\\
\implies X_n &= X_0 + \sum_{i=0}^{n-1} U_i. &(52)
\end{aligned}$$

Every state X_n is a sum of X_0 and an accumulation of controls U_i for $0 \leq i \leq n-1$. Utilizing the definition of $\widetilde{U}_n$ above, we can write every X_n in the following way:

$$X_n = X_0 - \widetilde{U}_n. \tag{53}$$

From eq. (53), we see that driving $X_n \to 0$ is equivalent to finding some choice of a sequence of $\widetilde{U}_n$ that best estimates the value of X_0. In doing so, each estimate of X_0 removes intervals from the support of $f(X_0|Y_0^N)$ of width $2|Y_n| = 2|C_n X_n| = 2|C_n(X_0 - \widetilde{U}_n)| \leq 2|X_0 - \widetilde{U}_n|$. To make explicit the effect of $\widetilde{U}_n$ on $f(X_0|Y_0^N)$, we write the support of $f(X_0|Y_0^N)$ in terms of $\widetilde{U}_n$ in the following lemma.

Lemma 2.7 (Support of $f(X_0|Y_0^N)$)**.** *For the system in eq. (1), assume that $X_0 \sim$ Unif$[-1,1]$ and $C_n \sim$ Unif$[-1,1]$ i.i.d., and we use a controller with all available memory with realizations $u_n = g_n(y_0^n)$. Then the support of $f(X_0 = x|Y_0^N = y_0^n)$, i.e. $\mathcal{X}_{0|N}$ is:*

$$\mathcal{X}_{0|N} = [-1,1] \setminus \left(\bigcup_{n=0}^{N} (\widetilde{u}_n - |y_n|, \widetilde{u}_n + |y_n|) \right). \tag{54}$$

Proof. Consider first $f(X_0|Y_0^N)$ as stated in eq. (46) with the uniform densities of X_0 and C_n substituted in:

$$\begin{aligned}
f(X_0 = x|Y_0^N = y_0^N) &= \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{\frac{1}{2}\mathbb{1}\left\{x : \left|\frac{y_n}{x+\sum_{i=0}^{n-1} u_i}\right| \leq 1\right\}}{|x + \sum_{i=0}^{n-1} u_i|} \right) \frac{1}{2}\mathbb{1}\{x : |x| \leq 1\} &(55)\\
&= \frac{1}{f(Y_0^N = y_0^N)} \left(\prod_{n=0}^{N} \frac{\frac{1}{2}\mathbb{1}\left\{x : \left|\frac{y_n}{x-\widetilde{u}_n}\right| \leq 1\right\}}{|x - \widetilde{u}_n|} \right) \frac{1}{2}\mathbb{1}\{x : |x| \leq 1\}. &(56)
\end{aligned}$$

In eq. (56) we used the definition of $\widetilde{U}_n$ to substitute expressions in terms of u_i in terms of the realization of $\widetilde{U}_n$, i.e. $\widetilde{u}_n$. The support of $f(X_0|Y_0^N)$ is determined by the support of $\mathbb{1}\{x : |x| \leq 1\}$ and excisions of the $\mathbb{1}\left\{x : \left|\frac{y_n}{x-\widetilde{u}_n}\right| \leq 1\right\}$ terms. The interval $[-1,1]$ is the support of $\mathbb{1}\{x : |x| \leq 1\}$. The $\mathbb{1}\left\{x : \left|\frac{y_n}{x-\widetilde{u}_n}\right| \leq 1\right\}$ terms have excisions $\mathcal{E}_n = (\widetilde{u}_n - |y_n|, \widetilde{u}_n + |y_n|)$. $f(X_0|Y_0^N)$ will be zero on all of the $\mathcal{E}_n$, and nonzero on whatever part is leftover in $[-1,1]$ once we have removed the excisions. So the support $\mathcal{X}_{0|N}$ of $f(X_0|Y_0^N)$ is thus:

$$\mathcal{X}_{0|N} = [-1,1] \setminus \left(\bigcup_{n=0}^{N} (\widetilde{u}_n - |y_n|, \widetilde{u}_n + |y_n|) \right). \tag{57}$$

□

The lemma above summarizes that using estimates of X_0 to query the system for more observations excludes where X_0 can be in the support of $f(X_0|Y_0^N)$ exactly around our estimates. Thus different choices of controls as functions of observations correspond to different estimation strategies for excluding where X_0 can be. However, since $f(X_0|Y_0^N)$ is not just its support, we next visualize these densities in the following subsections.

2.3.2 Visualizing the density of $X_0|Y_0$ for $X_0 \sim$ Unif$[-1,1]$ and $C_n \sim$ Unif$[-1,1]$ i.i.d.

Using lemma 2.3 and indicators, we write $f(X_0|Y_0)$ explicitly:

$$f(X_0 = x|Y_0 = y) = \frac{1}{f(Y_0 = y)} \frac{1}{|x|} \cdot \frac{1}{2} \mathbb{1}\{x : |y| \leq |x|\} \cdot \frac{1}{2} \mathbb{1}\{x : |x| \leq 1\}. \tag{58}$$

There are four terms in our conditional density: an indicator function coming from the prior on X_0, an indicator function coming from our noise density which is even in the observation realization y, an envelope term $(\frac{1}{|x|})$ which originates as a normalization term for $f(Y_0|X_0)$, and lastly a Bayes rule density normalization term for the conditioning on Y_0. We plot the $X_0 = x$ dependent terms in fig. 1 below with a choice of $|y| = \frac{1}{2}$.

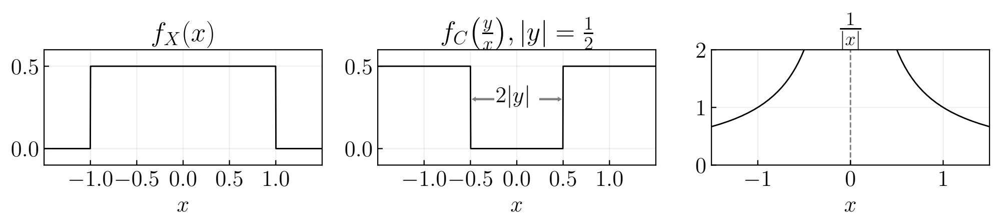

Figure 1: Graphs of the components of $f(X_0|Y_0 = \frac{1}{2})$ as functions of $X_0 = x$.

By looking at fig. 1 and fig. 2, where in fig. 2 we see the product of the graphs in fig. 1, we can see the significance of each of these terms. The $\mathbb{1}\{x : |x| \leq 1\}$ term, the indicator function originating from the

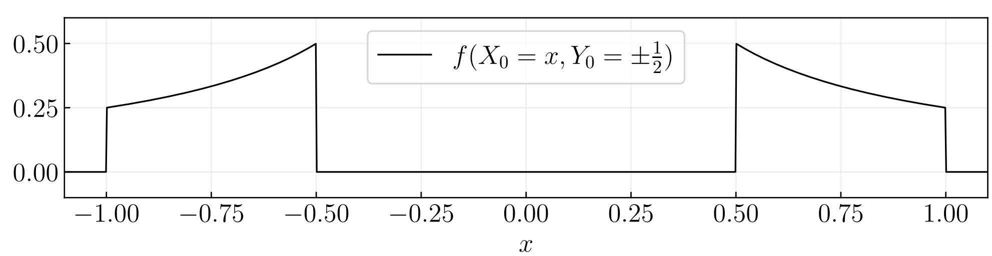

Figure 2: Graph of $f(X_0, Y_0)$ when $|Y_0| = \frac{1}{2}$, which is a product of the graphs in fig. 1.

density of X_0, is important because it constrains the support of the conditional density to realizable values of our initial state. The multiplicative observation has two contributions to the conditional density, the $\mathbb{1}\{x : |y| \leq |x|\}$ and $\frac{1}{|x|}$ terms. The indicator function coming from the noise density, $\mathbb{1}\{x : |y| \leq |x|\}$, when treated as a function of x has the support $(-\infty, -|y|] \cup [|y|, \infty)$. If we observe $Y_0 = y$, then this effectively excises $(-|y|, |y|)$ from the support of $f(X_0|Y_0 = y)$. This is true because $|C_0| \leq 1$: $Y_0 = C_0 X_0 \implies |Y_0| =$

$|C_0||X_0| \leq |X_0| \implies |Y_0| \leq |X_0|$. The last term, $\frac{1}{|x|}$, we refer to as the envelope due to its modulation by the indicator functions. In fig. 3 we see the components of $f(X_0|Y_0)$ superposed to show the modulating effect of X_0's density and the excision from the observation on the envelope.

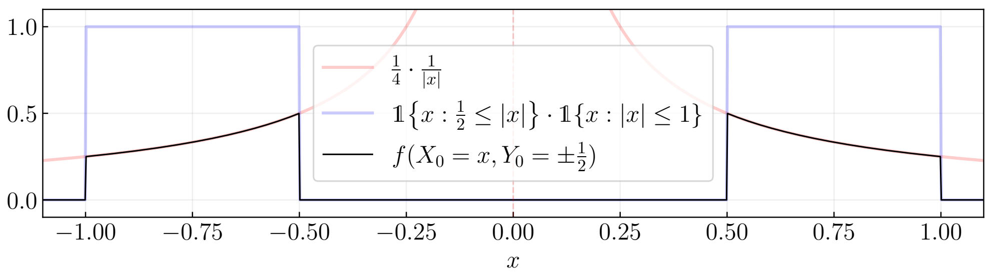

Figure 3: Same graph as in fig. 2 of $f(X_0, Y_0)$ when $|Y_0| = \frac{1}{2}$, but with envelope and modulating indicator functions superposed.

Since the normalization term $f(Y_0 = y)$ is not a function of x but is fixed once y is fixed, $f(X_0|Y_0)$ will have the same shape as $f(X_0, Y_0)$ as a function of x. Thus we continue by examining just $f(X_0, Y_0)$ as we vary the realization of Y_0. We can see the shape (up to scaling) of $f(X_0 = x|Y_0 = y)$ as y varies in fig. 4 below. We will now shortly see that the decomposition of our conditional density into an envelope modulated

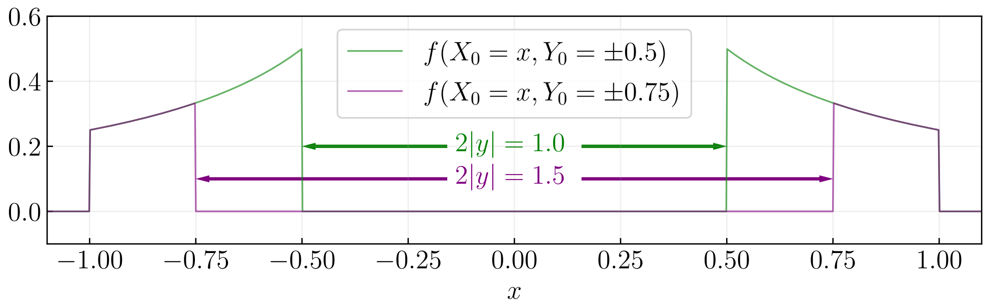

Figure 4: Slices of $f(X_0, Y_0)$ for different values of Y_0 to highlight that the difference in $f(X_0|Y_0 = y)$ for different values of y is different support excisions.

by excisions and the prior on X_0 continues to hold for when we have access to more observations as we apply control.

2.3.3 Visualizing the density of $X_0|Y_0, Y_1$ for $X_0 \sim$ Unif$[-1, 1]$ and $C_n \sim$ Unif$[-1, 1]$ i.i.d.

We now consider $f(X_0, Y_0, Y_1)$ when $X_0 \sim \text{Unif}[-1, 1]$ and $C_n \sim \text{Unif}[-1, 1]$ i.i.d.. Additionally, there is some control U_0 which determines $Y_1 = C_1 X_1 = C_1(X_0 + U_0)$. The joint density is as follows from lemma 2.3 and substituting in indicators:

$$f(X_0 = x, Y_0^1 = y_0^1) = \frac{\frac{1}{2}\mathbb{1}\left\{x : \left|\frac{y_1}{x+u_0}\right| \leq 1\right\}}{|x + u_0|} \frac{\frac{1}{2}\mathbb{1}\{x : \left|\frac{y_0}{x}\right| \leq 1\}}{|x|} f_X(x). \tag{59}$$

We see that the only difference of the $f(X_0, Y_0, Y_1)$ from $f(X_0, Y_0)$ is an extra factor of $f(Y_1 = y_1|X_0 = x, Y_0 = y_0) = \frac{\frac{1}{2}\mathbb{1}\left\{x : \left|\frac{y_1}{x+u_0}\right| \leq 1\right\}}{|x+u_0|}$. This new factor has two effects. The first is to change the envelope by adding a $\frac{1}{|x+u_0|}$ term, which has the same shape as $\frac{1}{|x|}$, but is centered at $x = -u_0$. This envelope change is important as it redistributes probability mass. The second effect is a new excision, $(-u_0 - |y_1|, -u_0 + |y_1|)$. We visualize the new envelope and new excision for choices of constant U_0. Below is the first such plot of

$f(X_0 = x, Y_0, Y_1)$ as a function of x in fig. 5, where values of $|y_0| = 0.25$, $U_0 = 0.75$, and $|y_1| = 0.125$, have been selected.

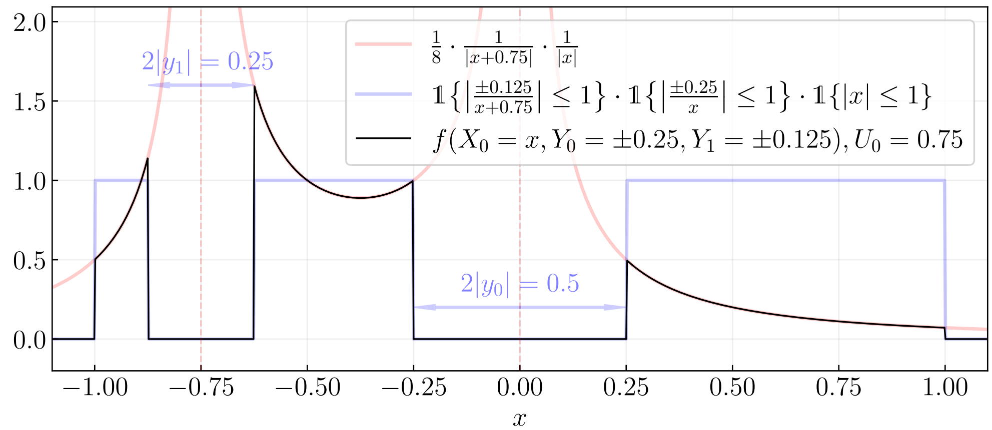

Figure 5: Plot of $f(X_0, Y_0, Y_1)$ with $|y_0| = 0.25$, $U_0 = 0.75$, and $|y_1| = 0.125$, with corresponding envelope and support functions superposed. Excisions are indicated by the observation labeled arrow breadths.

As seen in fig. 5 above, we see the noted changes from $f(X_0 = x, Y_0 = y_0)$ coming from the $f(Y_1 = y_1|X_0 = x, Y_0 = y_0)$ term. We see the new asymptote in the envelope located at $x = -U_0$. We also see the excisions located at $x = 0$ and $x = -U_0$, with the excision sizes being governed by the observation magnitudes. The combined effect of the envelope and excisions is that the density has local maxima or 'peaks' occurring at excision edges.

We visualize now what happens as we maintain the input but vary the observation magnitudes: $|y_0| = 0.5$, $U_0 = 0.75$, and $|y_1| = 0.25$ in fig. 6 below. While we maintain the same envelope from fig. 5, the difference of

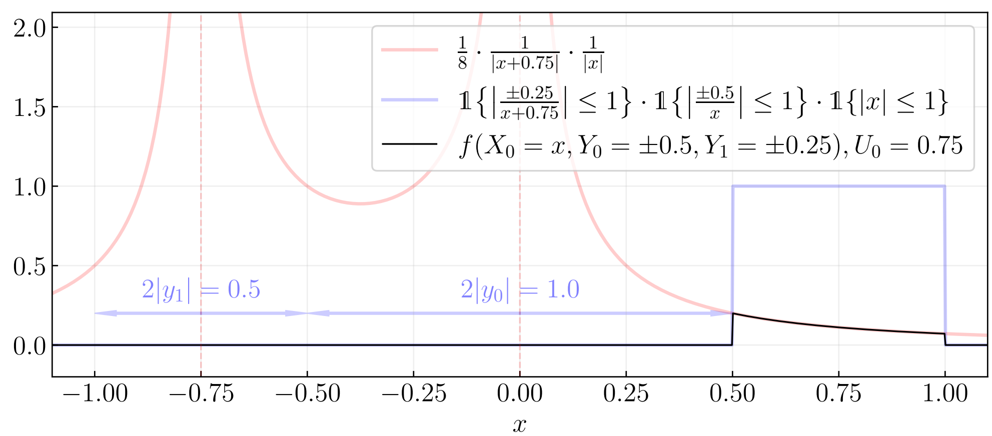

Figure 6: Plot of $f(X_0, Y_0, Y_1)$ with $|y_0| = 0.5$, $U_0 = 0.75$, and $|y_1| = 0.25$ with corresponding envelope and support functions superposed. Excisions are indicated by the observation labeled arrow breadths.

fig. 6 from fig. 5 is what part of $f(X_0, Y_0, Y_1)$ is excised. We see that with the larger observations, we only have the support of the density on the positive real line. Thus it is possible for our first choice of control and observations up until time $n = 1$ to indicate the sign of X_0. We now consider varying both U_0 as well as the observations Y_0 and Y_1 in fig. 7.

In fig. 7, we observe $|y_0| = 0.5$, then the control $U_0 = -0.5$ is applied, then we observe $|y_1| = 0.25$. Accordingly, the envelope differs from that of fig. 5 and fig. 6 with an asymptote located at $x = -U_0 = 0.5$, as well as an excision there of size $2|y_1| = 0.5$. Excisions may have overlap, and in particular, this is explained

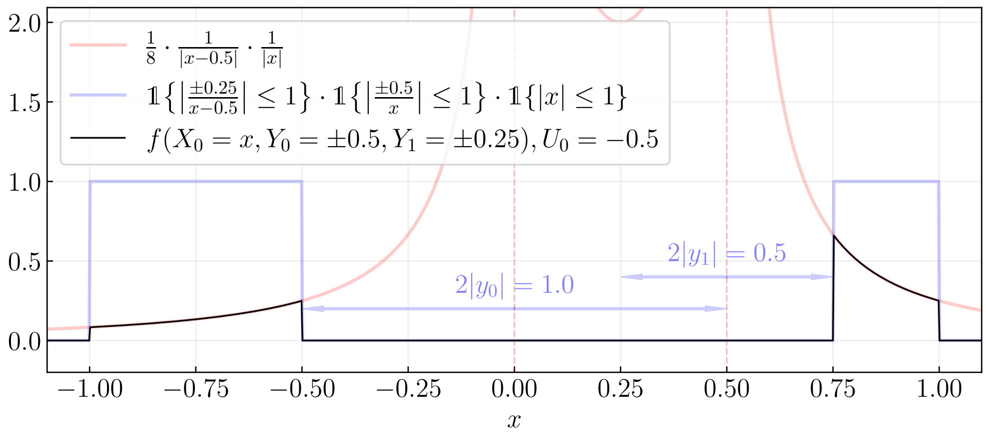

Figure 7: Plot of $f(X_0, Y_0, Y_1)$ with $|y_0| = 0.5$, $U_0 = -0.5$, and $|y_1| = 0.25$. Excisions are indicated by the observation labeled arrow breadths.

for fig. 7 by the fact that $U_0 = -|y_0|$ for this case. If we choose $U_0 = \pm|Y_0|$ we can center the second excision on an edge of the support of the density of $X_0|Y_0$, and therefore ensure that the support that remains when y_1 is realized is still exactly at most two disjoint intervals. This is because the control determines where the new excision in the density occurs, and thus choosing our control as a function of our first observation Y_0 means placing the location of the new excision relative to that of the first excision.

We saw in this subsection that our first control, U_0, will have two effects: the modification of the envelope and a yield of a new observation Y_1 which determines another excision's size. This pattern continues as we will see in the next sections.

2.3.4 Visualizing the density of $X_0|Y_0^N$ for $X_0 \sim$ Unif$[-1, 1]$ and $C_n \sim$ Unif$[-1, 1]$ i.i.d.

In this subsection, we visualize the shapes of $f(X_0|Y_0^N)$ for $N = 2, 3$ (i.e., having more than one control applied) with uniform initial state and noises to see that the pattern established in the previous subsections continues.

We start by first examining for when the controls are $U_0 = -0.5$ and $U_1 = -0.25$, and vary the observation realizations. With these controls, the excision centers are $\widetilde{U}_0 = 0$, $\widetilde{U}_1 = -U_0 = 0.5$, and $\widetilde{U}_2 = -(U_0 + U_1) = 0.75$, and thus we have three corresponding asymptotes in the envelope function. In fig. 8 we have observations $|y_0| = 0.5$, $|y_1| = 0.25$, and $|y_2| = 0.2$. In fig. 9 we have observations $|y_0| = 0.2$, $|y_1| = 0.15$, and $|y_2| = 0.2$.

We can see the guaranteed excision centered at $x = \widetilde{U}_0 = 0$ from the Y_0 observation. Additionally, these examples highlight that our choice of control functions $U_n = g_n(Y_0^n)$ decides whether we are guaranteed overlap - choosing constant controls will not allow the placement of excisions relative to previous excisions. If we wish to always have overlap and maintain a growing central excision, we should choose controls that result in the excision centers following a pattern of excision edges. Inspired by the appearance of fig. 8, choosing $\widetilde{U}_1 = |Y_0|$ and $\widetilde{U}_2 = |Y_0| + |Y_1|$ guarantees this behavior.

We now change the controls to $U_0 = -0.5$ and $U_1 = 1$ but maintain the same observation realizations as in fig. 9, $|y_0| = 0.2$, $|y_1| = 0.15$, and $|y_2| = 0.2$ and plot the density shape in fig. 10. The excision centers are now $\widetilde{U}_0 = 0$, $\widetilde{U}_1 = 0.5$, and $\widetilde{U}_2 = -0.5$.

It is visible from figs. 8 to 10 that the number of disjoint intervals of support possible depends on the overlap of excisions. When there is no overlap, it is possible to have $N + 2$ intervals, as each excision splits an interval into two, and there are $N + 1$ excisions corresponding to each of the $\widetilde{U}_n$ and $|Y_n|$.

Seeing these further examples of densities, there are a variety of strategies we can take to narrow down where X_0 lies. Possible strategies include excising inwards from the outside edges of $f(X_0|Y_0^N)$ $(x = \pm 1)$, outwards from the very first excision edges coming from the first observation Y_0 $(x = \pm|y_0|)$, and some specifications thereof as to which new location (left or right) we choose each time to excise from. The suggested two strategies happen to maintain a minimum number of intervals, but other more complex

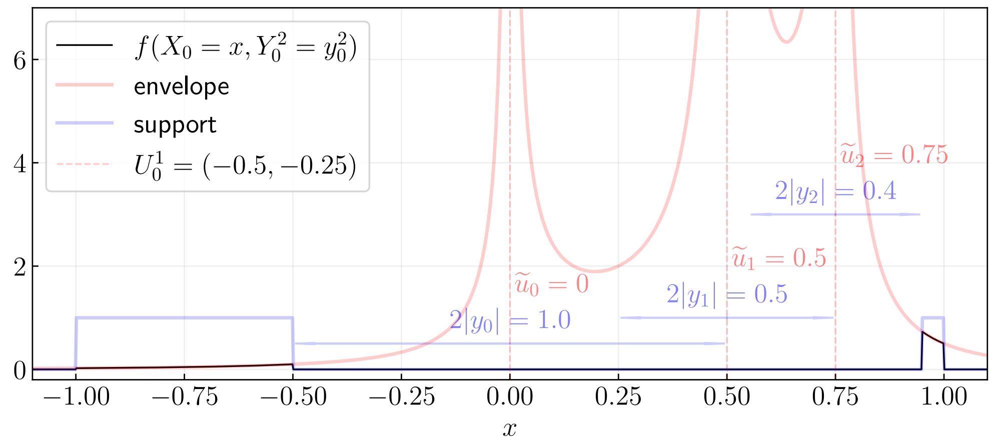

Figure 8: Plot of $f(X_0, Y_0^2)$ with $|y_0| = 0.5$, $|y_1| = 0.25$, and $|y_2| = 0.2$. Excisions are indicated by the observation labeled arrow breadths.

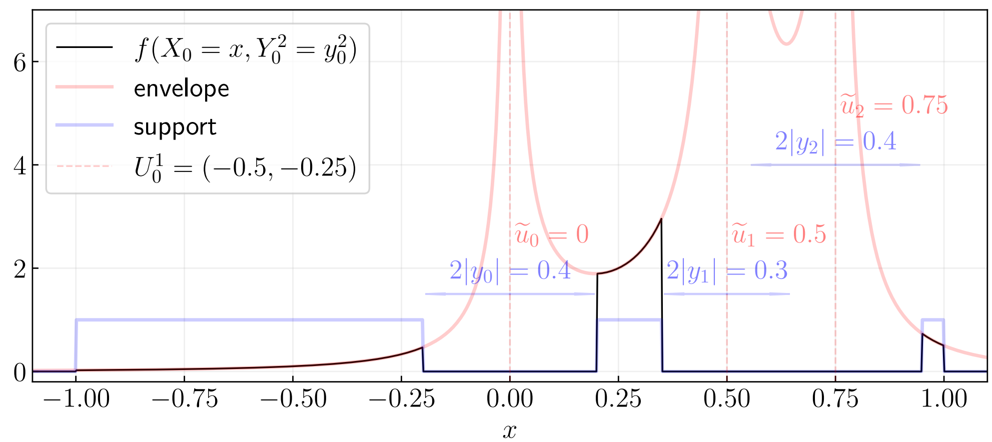

Figure 9: Plot of $f(X_0, Y_0^2)$ with $|y_0| = 0.2$, $|y_1| = 0.15$, and $|y_2| = 0.2$. Excisions are indicated by the observation labeled arrow breadths.

strategies may hop around creating more intervals of support. Since the sign of X_0 is unknown, another possible strategy is to completely remove half of the support to determine the sign of X_0, as this information may be useful. It is also of interest to leverage the envelope information we have access to as well, where new excisions can be placed at the highest peaks (i.e., excision edges of the density with maximum density value). We consider such a strategy in the next section.

The qualitative observations about the plots of the conditional densities $X_0|Y_0^N$ have some bearing on the interpretation of alternating strategies, i.e., strategies that are similar to $U_n = (-1)^n|Y_n|$. To see this, consider excision centers for up to time $N = 2$ when $U_n = (-1)^n|Y_n|$. We have excision centers $\widetilde{U}_0 = 0$, $\widetilde{U}_1 = -|Y_0|$, and $\widetilde{U}_2 = -|Y_0| + |Y_1|$. We see that the excision centered at $\widetilde{U}_1$ will be $(-|y_0| - |y_1|, -|y_0| + |y_1|)$. However, we also have the excision $(-|y_0|, |y_0|)$ at $\widetilde{U}_0$ to consider. If the realizations of X_0 and $|Y_1|$ are such to allow that $-|y_0| + |y_1| > |y_0|$, the union of the excisions at $\widetilde{U}_0$ and $\widetilde{U}_1$ result in an edge at $-|y_0| + |y_1|$, which happens to be chosen as the next excision center, $\widetilde{U}_2$, in the alternating strategy. Thus alternating strategies are similar in behavior to strategies that remove probability mass or support from the center of $f(X_0|Y_0^N)$'s support by alternating new excisions on the left and right central excision edges, but they may not do so 'greedily' by choosing the exact excision edge as we see that it depends on the specific realization of X_0 and the observation magnitudes.

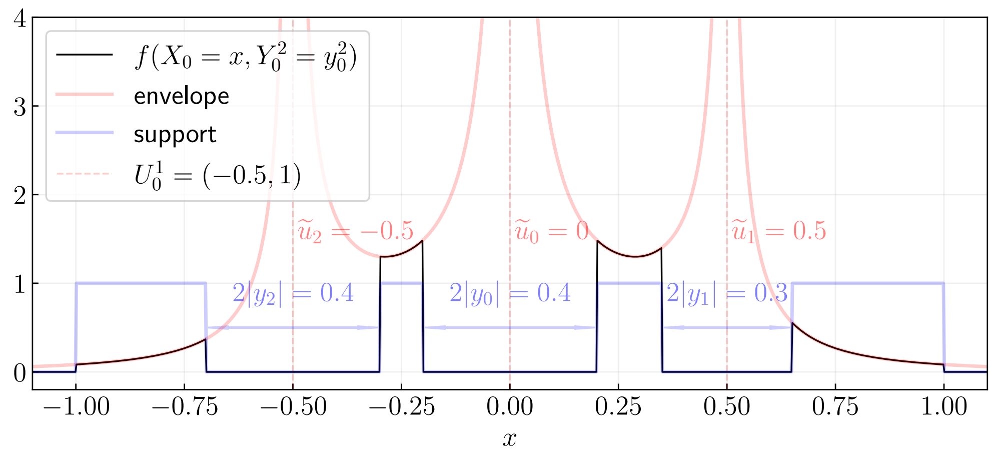

Figure 10: Plot of $f(X_0, Y_0^2)$ with $|y_0| = 0.2$, $|y_1| = 0.15$, and $|y_2| = 0.2$ with different controls from fig. 9. Excisions are indicated by the observation labeled arrow breadths.

We can also better understand the need for nonlinearity in any control strategy and also that neural network learned control functions in [2] appear to be similar to functions of $|Y_n|$. Given that we lose sign information of X_n from multiplicative noise with zero mean, using $|Y_n|$ or functions of $|Y_n|$ allows us to pin new excisions relative to a choice of a specific sign of established excision edges.

In summary, we have seen more evidence for the general pattern of excisions centered at $\widetilde{U}_n$ with excision widths dependent on $|Y_n|$. It is now of interest to see if there are controls to guarantee efficient estimation for X_0 and possibly even convergence to X_0. There is a tradeoff to consider in that the more informative controls in the sense of those that excise the most support are those which choose locations where X_0 are not, in that the excisions are allowed to be bigger: $|Y_n| \leq |X_0 - \widetilde{U}_n|$.

3 MAP-based control

In this section, we investigate the generation of controls using the conditional densities of $X_0|Y_0^n$ and $X_n|Y_0^n$, and in particular using the maximizer of such densities. We first define the Maximum a-Posteriori (MAP) estimate of the state in the following way.

Definition 3.1 (State MAP Estimate, $\widehat{X}_{m|n}^{\text{MAP}}$)**.** For the system in eq. (1), assume we have observation realizations y_0^n and control realizations u_0^{n-1}. The MAP estimate of the state X_m as seen from time $n \geq m$, denoted by $\widehat{X}_{m|n}^{\text{MAP}}$, is the optimizer of $f(X_m|Y_0^n)$ in the following way:

$$\widehat{X}_{m|n}^{\text{MAP}}(y_0^n) := \arg\max_x f(X_m = x|Y_0^n = y_0^n). \tag{60}$$

We now define a control law using the MAP estimate of the state for all time:

Definition 3.2 (MAP Control Law)**.** We call the controller with all available memory that is the sequence of controls $\{U_n = -\widehat{X}_{n|n}^{\text{MAP}}(Y_0^n)\}_{n\geq 0}$ the MAP control law.

Due to our earlier observation about the joint densities of (X_0, Y_0^N) and (X_N, Y_0^N) being identical up to a shift of the sum of all controls in the state argument, it turns out that we can state a corresponding relationship between the MAP estimate of the initial and current states as seen from time N, $\widehat{X}_{0|N}^{\text{MAP}}$ and $\widehat{X}_{N|N}^{\text{MAP}}$.

Lemma 3.1 (Recentering Lemma)**.** *For the system in eq. (1), assume that X_0 has density $f_X(x)$, the C_n have density $f_C(c)$, with observation realizations y_0^N and control realizations $u_n = g_n(y_0^n), 0 \leq n \leq N-1$.*

Then at any time N,

$$\widehat{X}^{MAP}_{N|N}(y_0^N) = \widehat{X}^{MAP}_{0|N}(y_0^N) + \sum_{n=0}^{N-1} u_n. \tag{61}$$

Proof. We begin from the definition of $\widehat{X}^{\text{MAP}}_{N|N}$.

$$\widehat{X}^{\text{MAP}}_{N|N}\left(y_0^N\right) = \arg\max_x f\left(X_N = x | Y_0^N = y_0^N\right) \tag{62}$$

$$= \arg\max_x \frac{f\left(X_N = x, Y_0^N = y_0^N\right)}{f\left(Y_0^N = y_0^N\right)} \tag{63}$$

$$= \arg\max_x \frac{f\left(X_0 = x - \sum_{n=0}^{N-1} u_n, Y_0^N = y_0^N\right)}{f\left(Y_0^N = y_0^N\right)}. \tag{64}$$

We apply the relationship between the joint densities of (X_0, Y_0^N) and (X_N, Y_0^N) in eq. (64), as was originally shown in eq. (33). Since we have a shift by the constant $\sum_{n=0}^{N-1} u_n$ in the argument we are maximizing over, we can pull this out of the maximization in the following way:

$$\widehat{X}^{\text{MAP}}_{N|N}\left(y_0^N\right) = \left(\arg\max_{x'} \frac{f\left(X_0 = x', Y_0^N = y_0^N\right)}{f\left(Y_0^N = y_0^N\right)}\right) + \sum_{n=0}^{N-1} u_n \tag{65}$$

$$= \left(\arg\max_{x'} f\left(X_0 = x' | Y_0^N = y_0^N\right)\right) + \sum_{n=0}^{N-1} u_n \tag{66}$$

$$\widehat{X}^{\text{MAP}}_{N|N}\left(y_0^N\right) = \widehat{X}^{\text{MAP}}_{0|N}\left(y_0^N\right) + \sum_{n=0}^{N-1} u_n. \tag{67}$$

The last step follows from the definition of $\widehat{X}^{\text{MAP}}_{0|N}\left(y_0^N\right)$. □

Note that we call this the recentering lemma because the qualitative effect of controls is to recenter $f(X_0|Y_0^N)$ to yield $f(X_N|Y_0^N)$. Thus the locations of the maxima of these conditional densities also coincide up to the same shift. We are now interested in seeing how the MAP control law does in terms of subsequent estimates of the initial state with new information.

Theorem 3.2 (MAP Control is Successive Estimation of X_0)**.** *For the system in eq. (1), assume that X_0 has density $f_X(x)$, the C_n have density $f_C(c)$, and we use the MAP control law, with observation realizations y_0^n and control realizations $u_n = -\widehat{X}^{MAP}_{n|n}(y_0^n)$. Then for any time n with $\widehat{X}^{MAP}_{0|-1} := 0$, the following two relationships hold:*

$$X_{n+1} = X_0 - \widehat{X}^{MAP}_{0|n}(y_0^n) \tag{68}$$

$$u_n = \widehat{X}^{MAP}_{0|n-1}(y_0^{n-1}) - \widehat{X}^{MAP}_{0|n}(y_0^n). \tag{69}$$

Proof. We examine X_{n+1} under the MAP control law.

$$X_{n+1} = X_n + u_n \tag{70}$$

$$= X_n - \widehat{X}^{\text{MAP}}_{n|n}(y_0^n). \tag{71}$$

Apply the recentering lemma (lemma 3.1) and the fact that $X_n = X_0 + \sum_{i=0}^{n-1} u_i$.

$$X_{n+1} = \left(X_0 + \sum_{i=0}^{n-1} u_i\right) - \left(\widehat{X}^{\text{MAP}}_{0|n}(y_0^n) + \sum_{i=0}^{n-1} u_i\right) \tag{72}$$

$$X_{n+1} = X_0 - \widehat{X}^{\text{MAP}}_{0|n}(y_0^n). \tag{73}$$

This finishes the proof of eq. (68). The proof of eq. (69) follows from eq. (68) in the following way:

$$
\begin{aligned}
u_n &= X_{n+1} - X_n && (74)\\
&= \left(X_0 - \widehat{X}^{\mathrm{MAP}}_{0|n}(y_0^n)\right) - \left(X_0 - \widehat{X}^{\mathrm{MAP}}_{0|n-1}(y_0^{n-1})\right) && (75)\\
&= \widehat{X}^{\mathrm{MAP}}_{0|n-1}(y_0^{n-1}) - \widehat{X}^{\mathrm{MAP}}_{0|n}(y_0^n). && (76)
\end{aligned}
$$

Let us verify the above concretely for $n = 0$.

$$
u_0 \stackrel{?}{=} \widehat{X}^{\mathrm{MAP}}_{0|-1} - \widehat{X}^{\mathrm{MAP}}_{0|0}(y_0) = 0 - \widehat{X}^{\mathrm{MAP}}_{0|0}(y_0) = -\widehat{X}^{\mathrm{MAP}}_{0|0}(y_0). \tag{77}
$$

Since we assumed we were using the MAP control law, the above result is consistent with our assumptions. □

We now interpret the behavior of the sequence of such controls upon the conditional distribution over time in the following subsections, which helps give insight into both the structure of the estimates and later controls, and intuition for how sample paths of the state evolve.

3.1 MAP Control at time $n = 0$ when $X_0 \sim$ Unif$[-1, 1]$ and $C_n \sim$ Unif$[-1, 1]$ i.i.d.

From section 2.3.2, we saw that the first observation y_0 leads to $f(X_0|Y_0)$ with symmetry in $X_0 = x$, a support of $\mathcal{X}_0 = [-1, -|y_0|] \cup [|y_0|, 1]$, and peaks at $\pm|y_0|$.Thus we have that the first MAP estimate is $\widehat{X}^{\mathrm{MAP}}_{0|0}(y_0) = \pm|y_0|$. Then the first MAP control must be $u_0 = -\widehat{X}^{\mathrm{MAP}}_{0|0}(y_0) = \mp|y_0|$.

From the discussion in section 2.3.3 and the following sections, since the control is occurring at the edges of the first excision boundary from observation Y_0, the effect of this specific control is to expand the already excised support of the conditional density of X_0 in such a way that we do not see more than 2 disjoint intervals. This can be shown by examining $f(X_0|Y_0^1 = y_0^1)$. We choose $U_0 = -|Y_0| = -\widehat{X}^{\mathrm{MAP}}_{0|0}$.

Lemma 3.3. *For the system in eq. (1), assume that $X_0 \sim$ Unif$[-1, 1]$ and $C_n \sim$ Unif$[-1, 1]$ i.i.d. with $U_0 = -|Y_0|$. Then the support of $f(X_0|Y_0^1 = y_0^1)$, $\mathcal{X}_0$, is:*

$$
\mathcal{X}_0 = [-1, \min(-|y_0|, |y_0| - |y_1|)] \cup [|y_0| + |y_1|, 1]. \tag{78}
$$

We take as convention that for $[a, b]$ if $a > b$ then $[a, b] = \{\}$.

Proof. We can write $f(X_0|Y_0^1)$ with $U_0 = -|Y_0|$ from lemma 2.5:

$$
f(X_0 = x|Y_0^1 = y_0^1) = \frac{1}{f(Y_0^1 = y_0^1)} \frac{f_C\left(\frac{y_1}{x - |y_0|}\right)}{|x - |y_0||} \frac{f_C\left(\frac{y_0}{x}\right)}{|x|} f_X(x). \tag{79}
$$

Since we have uniform distributions, the conditional density is:

$$
f(X_0 = x|Y_0^1 = y_0^1) = \frac{1}{f(Y_0^1 = y_0^1)} \cdot \frac{\frac{1}{2}\mathbb{1}\{x : |y_1| \leq |x - |y_0||\}}{|x - |y_0||} \cdot \frac{\frac{1}{2}\mathbb{1}\{x : |y_0| \leq |x|\}}{|x|} \cdot \frac{1}{2}\mathbb{1}\{x : |x| \leq 1\} \tag{80}
$$

The support of the last factor on the RHS of eq. (80) is $\mathcal{X} = [-1, 1]$. We then have the excision of $\mathcal{E}_0 = (-|y_0|, |y_0|)$ from the third factor, and finally, the excision of $\mathcal{E}_1 = (|y_0| - |y_1|, |y_0| + |y_1|)$ from the second factor. It is always the case that $|y_0| + |y_1| \geq |y_0|$, so that when considering $\mathcal{X}_0 = \mathcal{X} - (\mathcal{E}_0 \cup \mathcal{E}_1)$, we have that $\mathcal{E}_0 \cup \mathcal{E}_1 = (\min(-|y_0|, |y_0| - |y_1|), |y_0| + |y_1|)$. This will also be a singular interval because $\mathcal{E}_1$ is centered on $|y_0|$, which is one of the edges of $\mathcal{E}_0$. Thus,

$$
\begin{aligned}
\mathcal{X}_0 &= [-1, 1] - (\min(-|y_0|, |y_0| - |y_1|), |y_0| + |y_1|) && (81)\\
&= [-1, \min(-|y_0|, |y_0| - |y_1|)] \cup [|y_0| + |y_1|, 1] && (82)
\end{aligned}
$$

□

We can now glean some qualitative characteristics of our new $f(X_0|Y_0^1)$ under our first MAP control. If we look at eq. (80), we see that the signs of the y_0^1 do not matter as all observation realizations have absolute values. In particular, this symmetry also applies to the prior on the observations, $f(Y_0^1 = y_0^1)$, which is simply the joint expression to the right with absolute values with X_0 integrated out. This means that our next estimate $\widehat{X}_{0|1}^{\text{MAP}}(y_0^1)$, which is a result of optimizing over the conditional density, is, therefore, symmetric in y_0 and y_1, i.e., it suffices to look at $\widehat{X}_{0|1}^{\text{MAP}}(y_0^1)$ over the first quadrant in y_0^1. Additionally, if $|y_0|+|y_1| \geq 1$, we have excision of the entirety of the positive support (up to measurability). This suggests that we have some non-zero probability, $\mathbb{P}(|Y_0|+|Y_1| \geq 1 | U_0 = -|Y_0|)$, that the sign is made known for certain with the first two observations given our first MAP control. We compute this probability below.

Lemma 3.4. *Consider the system in eq. (1) with $U_0 = -\widehat{X}_{0|0}^{MAP} = -|Y_0|$. Then $sgn(X_0) = -1$ is known for certain with probability $\mathbb{P}(|Y_0|+|Y_1| \geq 1 | U_0 = -|Y_0|) \approx 0.1038$.*

Proof. We wish to compute $\mathbb{P}(|Y_0|+|Y_1| \geq 1 | U_0 = -|Y_0|)$. It is from $f(X_0, Y_0, Y_1)$ as follows:

$$\mathbb{P}(|Y_0|+|Y_1| \geq 1 | U_0 = -|Y_0|) = 4\mathbb{P}(Y_0 + Y_1 \geq 1 | U_0 = -|Y_0|) \tag{83}$$

$$= 4\int_{-1}^{1}\int_{0}^{1}\int_{1-y_0}^{1+y_0} \frac{f_C\left(\frac{y_1}{x-|y_0|}\right)}{|x-|y_0||}\frac{f_C\left(\frac{y_0}{x}\right)}{|x|} f_X(x)\,dy_1 dy_0 dx \tag{84}$$

$$= 4\int_{-1}^{1}\int_{0}^{1} \frac{f_C\left(\frac{y_0}{x}\right)}{|x||x-|y_0||} f_X(x)\int_{1-y_0}^{1+y_0} f_C\left(\frac{y_1}{x-|y_0|}\right) dy_1 dy_0 dx. \tag{85}$$

We simplify the probability to $4\mathbb{P}(Y_0 + Y_1 \geq 1 | U_0 = -|Y_0|)$ in the first step due to the aforementioned symmetry with the absolute values on y_0^1 in the $f(X_0, Y_0, Y_1)$ and due to the evenness of $f_C(c)$. The upper bound on y_1 also comes from the fact that $|y_1| \leq 1 + |y_0|$, which reduces to $y_1 \leq 1 + y_0$ when $y_0, y_1 \geq 0$. As a function of y_1, $f_C\left(\frac{y_1}{x-|y_0|}\right)$ takes value $\frac{1}{2}$ on support $[-|x-|y_0||, |x-|y_0||]$. We have the following possibilities:

$$\int_{1-y_0}^{1+y_0} f_C\left(\frac{y_1}{x-|y_0|}\right) dy_1 = \begin{cases} 0 & 1-y_0 \geq |x-|y_0|| \\ \frac{1}{2}(|x-|y_0|| - (1-y_0)) & 1-y_0 < |x-|y_0||, 1+y_0 \geq |x-|y_0|| \\ y_0 & 1+y_0 < |x-|y_0|| \end{cases}. \tag{86}$$

Due to the value of 0 of the above sub-integral, we can modify the bounds on x and y_0 to exclude the non-contributing region, as well as split the integral based on the latter two cases. We plot the regions below in fig. 11.

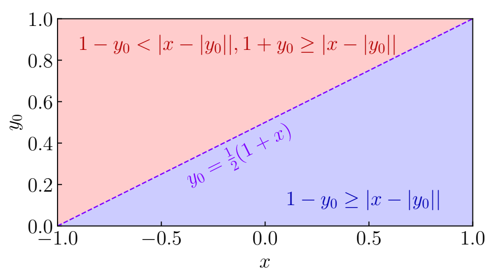

Figure 11: Plots of the regions of eq. (86) within $\{(x, y_0) \in [-1,1] \times [0,1]\}$. The relevant support of the result is in red.

Plotting the regions specified by the cases indicate that the region specified by the last case, $\{(x, y_0)|1 + y_0 < |x-|y_0||\}$, has no intersection with $\{(x, y_0)|(x, y_0) \in [-1,1] \times [0,1]\}$, and the region specified by the second condition, $\{(x, y_0)|1-y_0 < |x-|y_0||, 1+y_0 \geq |x-|y_0||\}$, has an intersection with $\{(x, y_0)|(x, y_0) \in [-1,1] \times [0,1]\}$ that is writable as $\{(x, y_0)|\frac{1}{2}(1+x) < y_0 \leq 1, -1 \leq x \leq 1\}$. Thus the probability we wish to compute becomes:

$$\mathbb{P}(|Y_0|+|Y_1|\geq 1) = 4\int_{-1}^{1}\int_{\frac{1}{2}(1+x)}^{1} \frac{f_C\left(\frac{y_0}{x}\right)}{|x||x-|y_0||} f_X(x)\cdot\frac{1}{2}(|x-|y_0||-(1-y_0))dy_0dx \tag{87}$$

$$= 2\int_{-1}^{1}\int_{\frac{1}{2}(1+x)}^{1} \frac{f_C\left(\frac{y_0}{x}\right)}{|x|} f_X(x)\left(1-\frac{1-y_0}{|x-|y_0||}\right)dy_0dx \tag{88}$$

$$= 2\int_{-1}^{1}\int_{\frac{1}{2}(1+x)}^{1} \frac{f_C\left(\frac{y_0}{x}\right)}{|x|}\frac{1}{2}\left(1-\frac{1-y_0}{|x-|y_0||}\right)dy_0dx \tag{89}$$

$$= \int_{-1}^{1}\int_{\frac{1}{2}(1+x)}^{1} \frac{f_C\left(\frac{y_0}{x}\right)}{|x|}\left(1-\frac{1-y_0}{|x-|y_0||}\right)dy_0dx. \tag{90}$$

We modify the bounds again by using that the support of $f_C\left(\frac{y_0}{x}\right)$ as a function of y_0 is $[-|x|,|x|]$. Since $|x|\leq 1$, we must modify the upper bound of y_0. We have the support of the integrand where the lower bound of integration on y_0 remains below the upper bound of the support of $f_C\left(\frac{y_0}{x}\right)$, i.e. $\frac{1}{2}(1+x)<|x|$. We have that either $\frac{1}{2}(1+x)<x$ or $\frac{1}{2}(1+x)<-x$, which imply $x>1$ or $x<-\frac{1}{3}$. The integrand support region of eq. (90) appears as in fig. 12. Thus the probability is now:

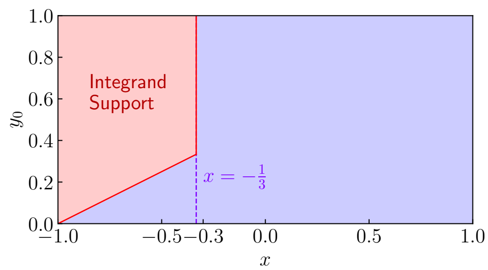

Figure 12: Support of the integral in eq. (90) with restrictions coming from $f_C\left(\frac{y_0}{x}\right)$ within $\{(x,y_0)\in[-1,1]\times[0,1]\}$.

$$\mathbb{P}(|Y_0|+|Y_1|\geq 1) = \frac{1}{2}\int_{-1}^{-\frac{1}{3}}\int_{\frac{1}{2}(1+x)}^{|x|}\frac{1}{|x|}\left(1-\frac{1-y_0}{|x-|y_0||}\right)dy_0dx \tag{91}$$

$$= \frac{1}{2}\int_{-1}^{-\frac{1}{3}}\int_{\frac{1}{2}(1+x)}^{-x}\frac{1}{-x}\left(1-\frac{1-y_0}{y_0-x}\right)dy_0dx \tag{92}$$

$$= \frac{1}{2}\int_{-1}^{-\frac{1}{3}}\int_{\frac{1}{2}(1+x)}^{-x}\frac{1}{x}\left(\frac{1-x+x-y_0}{y_0-x}-1\right)dy_0dx \tag{93}$$

$$= \frac{1}{2}\int_{-1}^{-\frac{1}{3}}\int_{\frac{1}{2}(1+x)}^{-x}\frac{1}{x}\left(\frac{1-x}{y_0-x}-2\right)dy_0dx \tag{94}$$

$$= \frac{1}{2}\int_{-1}^{-\frac{1}{3}}\frac{1-x}{x}\int_{\frac{1}{2}(1+x)}^{-x}\frac{1}{y_0-x}dy_0dx - \int_{-1}^{-\frac{1}{3}}\frac{1}{x}\int_{\frac{1}{2}(1+x)}^{-x}dy_0dx \tag{95}$$

$$= \frac{1}{2}\int_{-1}^{-\frac{1}{3}}\frac{1-x}{x}\int_{\frac{1}{2}(1+x)}^{-x}\frac{1}{y_0-x}dy_0dx + \int_{-1}^{-\frac{1}{3}}\frac{1+3x}{2x}dx \tag{96}$$

$$= \frac{1}{2}\int_{-1}^{-\frac{1}{3}}\frac{1-x}{x}\log\left|\frac{-2x}{\frac{1}{2}(1-x)}\right|dx + \frac{1}{2}\log\left(\frac{1}{3}\right)+1 \tag{97}$$

$$= 1-\frac{1}{3}\log(3)+\frac{1}{2}\int_{-1}^{-\frac{1}{3}}\frac{1-x}{x}\log\left(\frac{4x}{x-1}\right)dx \tag{98}$$

$$\approx 0.1038 \tag{99}$$

We do not have an exact expression as the last summand above is not analytically integrable. □

Next, we consider the behavior of the next MAP control, $U_1 = -\widehat{X}^{\mathrm{MAP}}_{1|1}(Y_0^1) = -\widehat{X}^{\mathrm{MAP}}_{0|1}(Y_0^1) + \widehat{X}^{\mathrm{MAP}}_{0|0}(Y_0)$.

3.2 MAP Control at time $n = 1$ when $X_0 \sim$ Unif$[-1,1]$ and $C_n \sim$ Unif$[-1,1]$ i.i.d.

We start by considering the realizable values of Y_0^1 that come from the first MAP Control, $U_0 = -\widehat{X}^{\mathrm{MAP}}_{0|0}(Y_0) = -|Y_0|$. Since $Y_1 = C_1(X_0 - |Y_0|)$, we know that $|Y_1| \le 1 + |Y_0|$. This upper bound on $|Y_1|$ is the maximum magnitude that $X_1 = X_0 - |Y_0|$ can take. Thus the next MAP Control, $U_1 = -\widehat{X}^{\mathrm{MAP}}_{1|1}(Y_0^1)$ varies over the region $\{(y_1, y_0) : 0 \le |y_0| \le 1, 0 \le |y_1| \le 1 + |y_0|\}$ for which we plot just the first quadrant below in fig. 13 due to aforementioned symmetry.

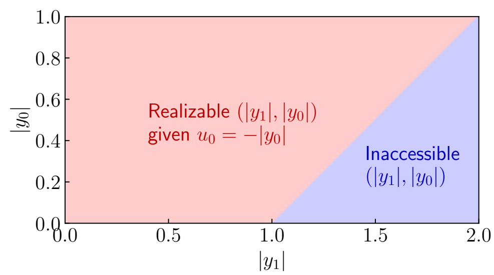

Figure 13: Region of definition for the MAP control $u_1 = -\widehat{X}^{\mathrm{MAP}}_{1|1}(y_0^1)$.

Lemma 3.5. *Consider the function* $f(x) = \frac{1}{|x||x-a|}$ *where* $a \ge 0$ *with support* $(-\infty, 0) \cup (a, \infty)$. *Then for* $x_1 \in (-\infty, 0)$ *and* $x_2 \in (a, \infty)$ *we have that:*

$$\arg \min_{x \in \{x_1, x_2\}} \left| \frac{a}{2} - x \right| = \arg \max_{x \in \{x_1, x_2\}} f(x). \tag{100}$$

The interpretation of the lemma above is that the minimizer of the distance from an axis of symmetry coincides with the maximizer of function as the function has decaying tails. This lemma is useful for computing the time $n = 1$ MAP control because our $f(X_0|Y_0^1)$ has an envelope that is proportional to the function $f(x)$ above.

Proof. First, we show that $f(x)$ is symmetric about $x = \frac{a}{2}$.

$$g(x) = f\left(x + \frac{a}{2}\right) = \frac{1}{|x + \frac{a}{2}||x - \frac{a}{2}|} = g(-x) \tag{101}$$

Then, we show that $f(x)$ is decreasing on (a, ∞).

$$f(x) = \frac{1}{x(x-a)}, x > a \tag{102}$$

$$= \frac{1}{x^2 - ax} \tag{103}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = -\frac{1}{(x^2 - ax)^2} \cdot (2x - a) \tag{104}$$

Since the above derivative is strictly negative for $x > a$, we have that our function is decreasing on (a, ∞).

WLOG consider the following inequality:

$$f(x_2) > f(x_1) \tag{105}$$
$$f(x_2) > f\Big(\frac{a}{2} + \frac{a}{2} - x_1\Big) \tag{106}$$
$$x_2 < a - x_1 \tag{107}$$
$$x_2 - \frac{a}{2} < \frac{a}{2} - x_1 \tag{108}$$
$$\Big|x_2 - \frac{a}{2}\Big| < \Big|\frac{a}{2} - x_1\Big| \tag{109}$$
$$\Big|\frac{a}{2} - x_2\Big| < \Big|\frac{a}{2} - x_1\Big|. \tag{110}$$

eq. (106) follows from symmetry, and eq. (107) follows from that $f(x)$ is decreasing on (a, ∞) with $x_1 \in (-\infty, 0) \iff a - x_1 \in (a, \infty)$. ☐

We now compute $U_1 = -\widehat{X}^{\text{MAP}}_{1|1} = \widehat{X}^{\text{MAP}}_{0|0} - \widehat{X}^{\text{MAP}}_{0|1}$, by deriving $\widehat{X}^{\text{MAP}}_{0|1}$ using lemmas 3.5 and 3.3.

Theorem 3.6. *For the system in eq. (1), assume that $X_0 \sim Unif[-1, 1]$ and $C_n \sim Unif[-1, 1]$ i.i.d. with $U_0 = -|Y_0|$ with observation realizations y_0^1. If $|y_0| + |y_1| > 1$, then $\widehat{X}^{MAP}_{0|1} = \min(-|y_0|, |y_0| - |y_1|)$, i.e. the sign changes: $\widehat{X}^{MAP}_{0|0} = |y_0| \geq 0 \implies \widehat{X}^{MAP}_{0|1} = \min(-|y_0|, |y_0| - |y_1|) \leq 0$.*

The insight for this case comes from the intuition that a complete excision of the positive support of $f(X_0|Y_0^1)$ only leaves a negative value to be considered.

Proof. The hypotheses of lemma 3.3 are satisfied. So the support of $f(X_0 = x|Y_0^1 = y_0^1)$ is as follows:

$$\mathcal{X}_0 = [-1, \min(-|y_0|, |y_0| - |y_1|)] \cup [|y_0| + |y_1|, 1] \tag{111}$$
$$= [-1, \min(-|y_0|, |y_0| - |y_1|)] \cup \{\} \tag{112}$$
$$= [-1, \min(-|y_0|, |y_0| - |y_1|)]. \tag{113}$$

Since we have $f(X_0 = x|Y_0 = y_0^1) \propto \frac{1}{|x||x-|y_0||}$, by lemma 3.5 the maximizer must be $\widehat{X}^{\text{MAP}}_{0|1} = \min(-|y_0|, |y_0| - |y_1|)$ as this is the closest value to the axis of symmetry $x = \frac{|y_0|}{2}$ on the remaining negative support. ☐

Theorem 3.7. *For the system in eq. (1), assume that $X_0 \sim Unif[-1, 1]$ and $C_n \sim Unif[-1, 1]$ i.i.d. with $U_0 = -|Y_0|$ with observation realizations y_0^1. If $|y_0| + |y_1| < 1$ and $|y_0| > |y_1|$, then $\widehat{X}^{MAP}_{0|1} = |y_0| + |y_1|$*

Proof. The hypotheses of lemma 3.3 are satisfied. So the support of $f(X_0 = x|Y_0^1 = y_0^1)$ is as follows:

$$\mathcal{X}_0 = [-1, \min(-|y_0|, |y_0| - |y_1|)] \cup [|y_0| + |y_1|, 1]. \tag{114}$$

In particular, because $|y_0| + |y_1| < 1$, we do not excise the positive support. By lemma 3.5 it suffices to check which of $\min(-|y_0|, |y_0| - |y_1|)$ and $|y_0| + |y_1|$ maximizes $f(X_0 = x|Y_0 = y_0^1) \propto \frac{1}{|x||x-|y_0||}$. We can resolve the minimum expression by using our assumption:

$$|y_0| > |y_1| \tag{115}$$
$$|y_0| > \frac{1}{2}|y_1| \tag{116}$$
$$2|y_0| > |y_1| \tag{117}$$
$$-2|y_0| < -|y_1| \tag{118}$$
$$-|y_0| < |y_0| - |y_1|. \tag{119}$$

We apply lemma 3.5 and compare the distances of now $-|y_0|$ and $|y_0|+|y_1|$ to $\frac{|y_0|}{2}$:

$$\left|\frac{|y_0|}{2} - (-|y_0|)\right| \overset{?}{<} \left|\frac{|y_0|}{2} - (|y_0|+|y_1|)\right| \tag{120}$$

$$\left||y_0| + \frac{|y_0|}{2}\right| \overset{?}{<} \left||y_1| + \frac{|y_0|}{2}\right| \tag{121}$$

$$|y_0| \overset{?}{<} |y_1|. \tag{122}$$

Since our assumption $|y_0| > |y_1|$ resolves the inequality above, we know that $|y_0|+|y_1|$ is closer to $\frac{|y_0|}{2}$ than $-|y_0|$, meaning that the maximizer of $f(X_0|Y_0^1)$ is $\widehat{X}_{0|1}^{\text{MAP}} = |y_0|+|y_1|$. □

It is of note that $U_1 = -\widehat{X}_{1|1}^{\text{MAP}}(Y_0^1)$ is discontinuous along $|y_1| = |y_0|, |y_0|+|y_1| \leq 1$. This is because the state estimate is not a function as a result of an optimization (multiple maximizers may exist), with the core insight being that the excision edge peaks of $f(X_0|Y_0^1)$ may indicate the same maximal density for an estimate of either sign. Thus the MAP control strategy cannot be learned with standard ReLU feedforward networks which are piece-wise linear continuous functions.

3.3 Interpreting MAP-based control with all available memory

In this section, we prove that the MAP control strategy of $U_n = -\widehat{X}_{n|n}^{\text{MAP}}$ leads to $\widehat{X}_{0|n}^{\text{MAP}}$ converging to X_0 almost surely and X_n converging to 0 almost surely. We use the intuition gained from the previous subsections of an outward growing excision from the center of $f(X_0|Y_0^N)$. We first define the candidates for the argument of $f(X_0|Y_0^N)$ that maximize $f(X_0|Y_0^N)$ which correspond to the edges of the central excision.

Definition 3.3 (Positive and Negative Maxima Candidates, $\widehat{X}_{0|n}^{+}$, $\widehat{X}_{0|n}^{-}$)**.** For the system in eq. (1), assume that $X_0 \sim \text{Unif}[-1,1]$ and $C_n \sim \text{Unif}[-1,1]$ i.i.d. with $U_n = -\widehat{X}_{n|n}^{\text{MAP}}$. We define the positive maximum candidate, $\widehat{X}_{0|n}^{+}$, and the negative maximum candidate, $\widehat{X}_{0|n}^{-}$, iteratively in the following way:

$$\widehat{X}_{0|n}^{+} := \max\left(\widehat{X}_{0|n-1}^{\text{MAP}} + |Y_n|, \widehat{X}_{0|n-1}^{+}\right) \tag{123}$$

$$\widehat{X}_{0|n}^{-} := \min\left(\widehat{X}_{0|n-1}^{\text{MAP}} - |Y_n|, \widehat{X}_{0|n-1}^{-}\right). \tag{124}$$

Theorem 3.8 ($\widehat{X}_{0|n}^{\text{MAP}} \to X_0$ almost surely)**.** *For the system in eq. (1), assume that $X_0 \sim$ Unif$[-1,1]$ and $C_n \sim$ Unif$[-1,1]$ i.i.d. with $U_n = -\widehat{X}_{n|n}^{MAP}$. Then $\widehat{X}_{0|n}^{MAP} \to X_0$ almost surely, and $X_n \to 0$ almost surely.*

Proof. We first prove that $\widehat{X}_{0|n}^{\text{MAP}} \to X_0$ almost surely. Without loss of generality, consider when $X_0 > 0$.

First, we note that $\widehat{X}_{0|n}^{+}$ is increasing by its definition:

$$\widehat{X}_{0|n}^{+} = \max\left(\widehat{X}_{0|n-1}^{\text{MAP}} + |Y_n|, \widehat{X}_{0|n-1}^{+}\right) \geq \widehat{X}_{0|n-1}^{+}. \tag{125}$$

Next, we note that $\widehat{X}_{0|n}^{-}$ is decreasing by its definition:

$$\widehat{X}_{0|n}^{-} = \min\left(\widehat{X}_{0|n-1}^{\text{MAP}} - |Y_n|, \widehat{X}_{0|n-1}^{-}\right) \leq \widehat{X}_{0|n-1}^{-}. \tag{126}$$

Let us show by induction that $X_0 > 0$ is an upper bound to both the above sequences, i.e., $\forall n$, $X_0 \geq \widehat{X}_{0|n}^{+}$ and $X_0 \geq \widehat{X}_{0|n}^{-}$. We have that the base case holds:

$$\widehat{X}_{0|0}^{+} = |Y_0| = |C_0 X_0| = |C_0| X_0 \leq X_0 \tag{127}$$

$$\widehat{X}_{0|0}^{-} = -|Y_0| = -|C_0 X_0| \leq 0 \leq X_0 \tag{128}$$

Now, assume the inductive hypothesis that $X_0 \geq \widehat{X}^+_{0|n-1}$ and $X_0 \geq \widehat{X}^-_{0|n-1}$. Since $\widehat{X}^-_{0|n-1}$ is decreasing, the next term $\widehat{X}^-_{0|n}$ is upper bounded by X_0, i.e. $\widehat{X}^-_{0|n} \leq \widehat{X}^-_{0|n-1} \leq X_0$. Now, we have to verify that $\widehat{X}^+_{0|n} \leq X_0$. When $\widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n| > \widehat{X}^+_{0|n-1}$:

$$\begin{aligned}
\widehat{X}^+_{0|n} &= \max\left(\widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n|, \widehat{X}^+_{0|n-1}\right) && (129)\\
&= \widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n| && (130)\\
&= \widehat{X}^{\text{MAP}}_{0|n-1} + \left|C_n\left(X_0 - \widehat{X}^{\text{MAP}}_{0|n-1}\right)\right| && (131)\\
&= \widehat{X}^{\text{MAP}}_{0|n-1} + |C_n|\left(X_0 - \widehat{X}^{\text{MAP}}_{0|n-1}\right) && (132)\\
&= (1 - |C_n|)\widehat{X}^{\text{MAP}}_{0|n-1} + |C_n|X_0 && (133)\\
&\leq (1 - |C_n|)X_0 + |C_n|X_0 && (134)\\
&= X_0 && (135)\\
&\implies \widehat{X}^+_{0|n} \leq X_0. && (136)
\end{aligned}$$

Note that we can apply the inequality $\widehat{X}^+_{0|n-1} \leq X_0$ or $\widehat{X}^-_{0|n-1} \leq X_0$ to get eq. (134) as $\widehat{X}^{\text{MAP}}_{0|n-1} = \widehat{X}^+_{0|n-1}$ or $\widehat{X}^{\text{MAP}}_{0|n-1} = \widehat{X}^-_{0|n-1}$ and we have a convex combination of X_0 and $\widehat{X}^{\text{MAP}}_{0|n-1}$ as $0 \leq |C_n| \leq 1$.

We consider the other case on $\widehat{X}^+_{0|n}$, when $\widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n| \leq \widehat{X}^+_{0|n-1}$:

$$\begin{aligned}
\widehat{X}^+_{0|n} &= \max\left(\widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n|, \widehat{X}^+_{0|n-1}\right) && (137)\\
&= \widehat{X}^+_{0|n-1} && (138)\\
&\leq X_0 && (139)\\
&\implies \widehat{X}^+_{0|n} \leq X_0. && (140)
\end{aligned}$$

Thus we have shown that $\forall n$, $X_0 \geq \widehat{X}^+_{0|n}$ and $X_0 \geq \widehat{X}^-_{0|n}$.

The strategy for the rest of the proof is as follows: we wish to establish that $\widehat{X}^{\text{MAP}}_{0|n}$ converges to $\widehat{X}^+_{0|n}$ almost surely, and that $\widehat{X}^+_{0|n}$ converges to X_0 almost surely. We first show the former, then show the latter by the monotone convergence theorem by proving that X_0 is the least upper bound of $\widehat{X}^+_{0|n}$ with probability 1.

We will now show that $\widehat{X}^{\text{MAP}}_{0|n}$ converges to $\widehat{X}^+_{0|n}$ almost surely. To do this, it suffices to show that $\widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^-_{0|n}$ only finitely many times.

Thus, we wish to show the following equality holds:

$$\mathbb{P}\left(\lim_{n\to\infty} \widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^+_{0|n} | X_0 > 0\right) = \mathbb{P}\left(\widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^-_{0|n} \text{ finitely many times} | X_0 > 0\right) = 1. \tag{141}$$

We proceed by using a proof by contradiction. If possible, let $\widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^-_{0|n}$ infinitely often with probability 1. Consider such a sample path where $\widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^-_{0|n}$ infinitely often. This sample path will have a sequence of strictly increasing time indices, $k_1, k_2, \ldots$, at which $\widehat{X}^{\text{MAP}}_{0|k_i} = \widehat{X}^-_{0|k_i}$. For the transition from time k_i to

$k_i + 1$:

$$\begin{aligned}
\widehat{X}^-_{0|k_i+1} &= \min\left(\widehat{X}^{\text{MAP}}_{0|k_i} - |Y_{k_i+1}|, \widehat{X}^-_{0|k_i}\right) && (142)\\
&= \min\left(\widehat{X}^-_{0|k_i} - |Y_{k_i+1}|, \widehat{X}^-_{0|k_i}\right) && (143)\\
&= \widehat{X}^-_{0|k_i} - |Y_{k_i+1}| && (144)\\
&= \widehat{X}^-_{0|k_i} - \left|C_{k_i+1}\left(X_0 - \widehat{X}^{\text{MAP}}_{0|k_i}\right)\right| && (145)\\
&= \widehat{X}^-_{0|k_i} - \left|C_{k_i+1}\left(X_0 - \widehat{X}^-_{0|k_i}\right)\right| && (146)\\
&= \widehat{X}^-_{0|k_i} - |C_{k_i+1}|\left(X_0 - \widehat{X}^-_{0|k_i}\right) && (147)\\
&= (1 + |C_{k_i+1}|)\widehat{X}^-_{0|k_i} - |C_{k_i+1}|X_0. && (148)
\end{aligned}$$

Since the k_i are strictly increasing ($k_{\{i+1\}} \geq k_i + 1$) and $\widehat{X}^-_{0|n}$ is decreasing, we have from eq. (148) that:

$$\widehat{X}^-_{0|k_{\{i+1\}}} \leq \widehat{X}^-_{0|k_i+1} = (1 + |C_{k_i+1}|)\widehat{X}^-_{0|k_i} - |C_{k_i+1}|X_0. \tag{149}$$

Thus, we can chain these lower bounds starting from k_1 to obtain the following inequality:

$$\forall i \geq 1 : \widehat{X}^-_{0|k_i} \leq \left(\prod_{j=1}^{i-1} \left(1 + |C_{k_j+1}|\right)\right)\widehat{X}^-_{0|k_1} - X_0 \sum_{j=1}^{i-1}\left(|C_{k_j+1}| \prod_{l=j+1}^{i-1} (1 + |C_{k_l+1}|)\right). \tag{150}$$

The chaining involves multiplying eq. (149) by $1 + |C_{k_i+1}| \geq 1$, and then adding $-|C_{k_i+1}|X_0$ to both sides to apply each $\widehat{X}^-_{0|k_i}$ lower bound. Since we assumed that $X_0 > 0$, we have that $\widehat{X}^-_{0|k_1} < 0$ with probability 1 due to the following chain of inequalities and the fact that $\mathbb{P}(C_0 = 0) = 0$:

$$\widehat{X}^-_{0|k_1} \leq \widehat{X}^-_{0|0} = -|Y_0| = -|C_0|X_0 \leq 0. \tag{151}$$

Additionally, $\forall k_j : 1 + |C_{k_j+1}| > 1$ with probability 1 as $\mathbb{P}\big(C_{k_j+1} = 0\big) = 0$. Thus $\widehat{X}^-_{0|k_i}$ should grow without bound with probability 1. There exists some first $k^\star$ such that $\widehat{X}^-_{0|k^\star} < -1$. Then, the support of $f(X_0|Y_0^{k^\star})$ has had the interval $[-1, 0]$ excised so that thereafter $\forall n \geq k^\star : \widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^+_{0|n}$, which is a contradiction. Thus we have $\widehat{X}^{\text{MAP}}_{0|n} \to \widehat{X}^+_{0|n}$ almost surely.

We now want to show that X_0 is the least upper bound on $\widehat{X}^+_{0|n}$ with probability 1 to apply the monotone convergence theorem. X_0 is not the least upper bound iff $\widehat{X}^+_{0|n}$ stops strictly increasing prematurely, i.e. the case $\widehat{X}^+_{0|n} = \widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n| > \widehat{X}^+_{0|n-1}$ occurs finitely many times and $\widehat{X}^+_{0|n} < X_0$ with probability 1.

We proceed by using a proof by contradiction. If possible, let X_0 not be the least upper bound on $\widehat{X}^+_{0|n}$ with probability 1. It follows that $\widehat{X}^+_{0|n} = \widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n| > \widehat{X}^+_{0|n-1}$ only finitely many times with probability 1. Then, there is a time N_{stop} such that $\forall n \geq N_{\text{stop}} : \widehat{X}^+_{0|n} < X_0$ and $\forall n \geq N_{\text{stop}} : \widehat{X}^{\text{MAP}}_{0|n-1} + |Y_n| \leq \widehat{X}^+_{0|n-1}$.

By our proof that $\widehat{X}^{\text{MAP}}_{0|n} \to \widehat{X}^+_{0|n}$ almost surely, we know there is a time N_+ such that $\forall n \geq N_+$: $\widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^+_{0|n}$.

Consider times $n \geq \max(N_{\text{stop}}, N_+)$. $\forall n \geq \max(N_{\text{stop}}, N_+)$, we have that:

$$\widehat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}| \leq \widehat{X}_{0|n}^{+} \tag{152}$$

$$\widehat{X}_{0|n}^{+} + |Y_{n+1}| \leq \widehat{X}_{0|n}^{+} \tag{153}$$

$$|Y_{n+1}| \leq 0 \tag{154}$$

$$|C_{n+1}X_{n+1}| \leq 0 \tag{155}$$

$$|C_{n+1}(X_0 - \widehat{X}_{0|n+1}^{\text{MAP}})| \leq 0 \tag{156}$$

$$|C_{n+1}(X_0 - \widehat{X}_{0|n+1}^{+})| \leq 0 \tag{157}$$

$$|C_{n+1}|(X_0 - \widehat{X}_{0|n+1}^{+}) \leq 0 \tag{158}$$

$$|C_{n+1}| \leq 0 \tag{159}$$

$$|C_{n+1}| = 0 \tag{160}$$

Note that eqs. (153) and (157) apply because we have that $n \geq N_+$, and that eq. (158) applies because $n \geq N_{\text{stop}}$ so that $X_0 > \widehat{X}_{0|n+1}^{+}$. However, $|C_{n+1}| = 0$ occurring $\forall n \geq \max(N_{\text{stop}}, N_+)$ is a zero probability event. This contradicts our starting assumption that we had a probability 1 event (of X_0 not being a least upper bound on $\widehat{X}_{0|n}^{\text{MAP}}$).

Thus X_0 is the least upper bound on $\widehat{X}_{0|n}^{+}$ with probability 1, and so by the monotone convergence theorem ($\widehat{X}_{0|n}^{+}$ is increasing, $\widehat{X}_{0|n}^{+}$ has least upper bound X_0), we can conclude that $\widehat{X}_{0|n}^{+}$ converges to X_0 almost surely when $X_0 > 0$. Finally, since $\widehat{X}_{0|n}^{\text{MAP}}$ converges to $\widehat{X}_{0|n}^{+}$ almost surely, then $\widehat{X}_{0|n}^{\text{MAP}}$ must also converge to X_0 almost surely when $X_0 > 0$, i.e. $\mathbb{P}\Big(\lim_{n\to\infty} \widehat{X}_{0|n}^{\text{MAP}} = X_0 | X_0 > 0\Big) = 1$.

We now revisit our assumption that $X_0 > 0$. We can for $X_0 < 0$ work through a nearly identical argument, except with $\widehat{X}_{0|n}^{+}$ and $\widehat{X}_{0|n}^{-}$ swapped in considerations ($\widehat{X}_{0|n}^{\text{MAP}} \to \widehat{X}_{0|n}^{-} \to X_0$ almost surely). So we should see that $\mathbb{P}\Big(\lim\limits_{n\to\infty} \widehat{X}_{0|n}^{\text{MAP}} = X_0 | X_0 < 0\Big) = 1$ as well. By the law of total probability:

$$\mathbb{P}\Big(\lim_{n\to\infty} \widehat{X}_{0|n}^{\text{MAP}} = X_0\Big) = \frac{1}{2}\mathbb{P}\Big(\lim_{n\to\infty} \widehat{X}_{0|n}^{\text{MAP}} = X_0 | X_0 > 0\Big) + \frac{1}{2}\mathbb{P}\Big(\lim_{n\to\infty} \widehat{X}_{0|n}^{\text{MAP}} = X_0 | X_0 < 0\Big) = 1. \tag{161}$$

Note we omit conditioning on $X_0 = 0$ as this is a zero probability event. Since $X_n = X_0 - \widehat{X}_{0|n-1}^{\text{MAP}}$ by theorem 3.2, $X_n \to 0$ almost surely as well. ☐

Theorem 3.9 (Second Moment Convergence Rate of MAP Control)**.** *For the system in eq. (1), assume that $X_0 \sim Unif[-1, 1]$ and $C_n \sim Unif[-1, 1]$ i.i.d. with $U_n = -\widehat{X}_{n|n}^{MAP}$. Then there exists a r.v. time N_{MAP} after which we stabilize X_n with second moment rate $\frac{1}{3}$, i.e.:*

$$\frac{\mathbb{E}\big[X_{n+2}^2 | n \geq N_{MAP}\big]}{\mathbb{E}\big[X_{n+1}^2 | n \geq N_{MAP}\big]} = \frac{1}{3}. \tag{162}$$

Proof. Without loss of generality, consider the case that $X_0 > 0$ and thus $\widehat{X}_{0|n}^{+} \to X_0$, surely from below by the previous proof of theorem 3.8. Under this condition, we know there exists almost surely some time N_+ such that $\forall n \geq N_+$, we have $\widehat{X}_{0|n}^{\text{MAP}} = \widehat{X}_{0|n}^{+}$. We wish to consider such a trajectory of $\widehat{X}_{0|n}^{\text{MAP}}$ for $n \geq N_+$.

We consider the iteration step in terms of $\widehat{X}_{0|n}^{+}$ first for some $n \geq N_+$:

$$\widehat{X}_{0|n+1}^{+} = \max\left(\widehat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}|, \widehat{X}_{0|n}^{+}\right) \tag{163}$$

$$\widehat{X}_{0|n+1}^{\text{MAP}} = \max\left(\widehat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}|, \widehat{X}_{0|n}^{\text{MAP}}\right) \tag{164}$$

We can change the $\widehat{X}^{+}_{0|n+1}$ and $\widehat{X}^{+}_{0|n}$ terms to $\widehat{X}^{\text{MAP}}_{0|n+1}$ and $\widehat{X}^{\text{MAP}}_{0|n}$ in eq. (164) because we are at or have exceeded the time N_+, and $\forall n \geq N_+ : \widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^{+}_{0|n}$.

$$\widehat{X}^{+}_{0|n+1} = \widehat{X}^{\text{MAP}}_{0|n} + |Y_{n+1}| \tag{165}$$

$$= \widehat{X}^{\text{MAP}}_{0|n} + |C_{n+1}X_{n+1}| \tag{166}$$

$$-\widehat{X}^{\text{MAP}}_{0|n+1} = -\widehat{X}^{\text{MAP}}_{0|n} - |C_{n+1}|X_{n+1} \tag{167}$$

$$X_0 - \widehat{X}^{\text{MAP}}_{0|n+1} = X_0 - \widehat{X}^{\text{MAP}}_{0|n} - |C_{n+1}|X_{n+1} \tag{168}$$

$$X_{n+2} = X_{n+1} - |C_{n+1}|X_{n+1} \tag{169}$$

$$= (1 - |C_{n+1}|)X_{n+1}. \tag{170}$$

Note that $X_{n+1} = X_0 - \widehat{X}^{\text{MAP}}_{0|n} = X_0 - \widehat{X}^{+}_{0|n} \geq 0$ in eq. (166). Thus we have shown that for $n \geq N_+$, that $X_{n+2} = (1 - |C_{n+1}|)X_{n+1}$. We may now compute a second moment rate of the state X_n for $n \geq N_+$:

$$\frac{\mathbb{E}\big[X^2_{n+2}|n \geq N_+\big]}{\mathbb{E}\big[X^2_{n+1}|n \geq N_+\big]} = \frac{\mathbb{E}\Big[(1 - |C_{n+1}|)^2 X^2_{n+1}|n \geq N_+\Big]}{\mathbb{E}\big[X^2_{n+1}|n \geq N_+\big]} \tag{171}$$

$$= \frac{\mathbb{E}\Big[(1 - |C_{n+1}|)^2\Big]\mathbb{E}\big[X^2_{n+1}|n \geq N_+\big]}{\mathbb{E}\big[X^2_{n+1}|n \geq N_+\big]} \tag{172}$$

$$= \mathbb{E}\Big[(1 - |C_{n+1}|)^2\Big] = \frac{1}{3}. \tag{173}$$

Note that C_{n+1} is independent of X_{n+1}, and also does not depend on N_+. The last equality comes from $1 - |C_{n+1}| \sim \text{Unif}[0, 1]$ which has second moment $\frac{1^2+1\cdot 0+0^2}{3} = \frac{1}{3}$. For the case $X_0 < 0$, we can consider a time N_- such that for $n \geq N_-$, we have $\widehat{X}^{\text{MAP}}_{0|n} = \widehat{X}^{-}_{0|n}$. Finally, define N_{MAP}:

$$N_{\text{MAP}} := \begin{cases} N_+ & \text{if } X_0 > 0 \\ N_- & \text{if } X_0 < 0 \\ 0 & \text{if } X_0 = 0 \end{cases}. \tag{174}$$

□

The above theorem informs us that we have a decent second moment convergence rate of $\widehat{X}^{\text{MAP}}_{n|0}$ to X_0 once our controller is confident regarding the sign of X_0. However, this says nothing about how quickly the MAP control resolves the sign of X_0 which must be characterized by the statistics of N_{MAP}. Simulation of the MAP strategy indicates that this rate is achieved fairly quickly. We can thus conjecture that $\mathbb{E}[N_{\text{MAP}}]$ is small ($e.g. that \mathbb{E}[N_{\text{MAP}}] < 10$) and that its tail decays quickly.

4 Conclusion

In this report we have given a proof of a method that utilizes $f(X_0|Y_0^N)$ to generate a MAP estimate of the initial state, $\widehat{X}^{\text{MAP}}_{0|N}$, to control the system in eq. (1). On the way, we have gained insights that allow us to interpret the results of prior related work. As future steps, we wish to also leverage the insights gained from density changes of $f(X_0|Y_0^N)$ to make controllers based on using the other densities $f(Y_0^N|X_0)$ and $f(Y_0^N|X_N)$, as well as seeing if the MAP strategy is similarly interpretable and provably stabilizing for Gaussian distributions of X_0 and C_n.

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