CS 280 Computer Vision

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1. Show that the vanishing points of lines on a plane lie on the vanishing line of the plane.

2. What is the silhouette of a sphere of radius r with its center at coordinates (x_0,0,z_0) under perspective projection on a planar surface? The perpendicular distance from the pinhole to the image plane is f.

3. In this exercise, the objective is to gain a richer understanding of the linearized projection map F_*^{-1}, both for spherical and planar perspective. For planar perspective you may consider typical choices of f, say 20 mm to 80 mm. Here are some situations:

   - Patches on the ground plane. Assume eye height of 2 m, and consider patches at varying distances and directions.
   - A patch directly 1 m ahead of the observer, as it is rotated about the Y axis. A rotation of about 3.7 degrees (why?) corresponds to the rotation needed to align the optical axes of the left and right eyes.

   A good way to visualize the efect of the mapping is to consider its effect on a gauge figure, such as a circular disk.

4. Verify Roderigues formula by considering the powers of the skew-symmetric matrix associated with the cross product with a vector.

5. Write Matlab functions for composing and inverting displacements of R^n. Apply these to a set of points and verify that distances are preserved. Explore the graphics and visualization tools in Matlab to show the points before and after the displacement has been applied.

6. Write a Matlab function for computing the orthogonal matrix A corresponding to rotation ϕ about the axis vector $\mathbf{s}$. Find the eigenvalues and eigenvectors of the orthogonal matrices and study any relationship to the axis vector. Verify the formula $\cos\phi = \frac{1}{2}\{\mathrm{trace}(A) - 1\}$

7. Write a Matlab function for the converse of that in the previous problem i.e. given an orthogonal matrix A, compute the skew-symmetric matrix B (and associated $\mathbf{s}$ and ϕ). Do these skew–symmetric matrices have any non-zero real eigenvalues? Why not?

8. For small ϕ, we can approximate an orthogonal matrix $A \approx I + \phi S$. Investigate the range ϕ over which this is a reasonable approximation. In structure-from-motion, we make such an assumption in going from a discrete to a differential approach.

9. Implement the equations which relate the point wise optical flow to the six parameters of rigid body translation and rotation, as well as the depth. Experiment with the *quiver* function in Matlab for displaying these flow fields. Construct displays for some interesting cases.

   A variant on this theme is to use natural perspective (projection on to a spherical surface). You can then construct pretty pictures of the magnitude of the flow as a function of θ, ϕ by using pseudocolor to map it onto the unit sphere.

10. As a test for the code that you have written in the previous exercise, suppose that I am driving my car along a straight stretch of freeway at a speed of 25 m/s. My eye height above the surface of the road is 1.25 m. What is the flow vector (in degrees/s)

    (a) At a point on the ground 25 m straight ahead.

    (b) At a point on the ground to my left at a distance of 25 m.

    (c) At points on the rear end of a 2 m wide car at a height of 1.25 m above the ground. This car has a headway of 25 m in front of me and is travelling at a speed of 20 m/s.

11. Implement the equations which relate the horizontal and vertical disparity for a binocular fixating system to δ, x, y. Experiment with the *quiver* function in Matlab for displaying these disparity fields. Construct displays for a plane slanted by 45 degrees around a vertical axis relative to the cyclopean visual axis. Assume interocular distance of 6 cm.

    (a) Vary fixation distance keeping gaze zero.

(b) Vary gaze angle keeping constant fixation distance, say 50 com.

12. Write functions for estimating the affine transformation between two sets of planar points given

    (a) Knowledge of the point correspondences.

    (b) Second moments before and after the affine transformation. In this case, you can estimate the transforms only upto an arbitrary orthogonal transform.

Relation Between Spherical and Planar Perspective Projection

Figure **??** depicts the situation. Let Σ be the viewsphere of unit radius centered at the origin of $\mathbf{R}^3$. Longitude and latitude on the earth suggest a convenient parametrization. We have the Cartesian coordinate system oriented as follows–the x-axis points due east, the y-axis points due north and the z-axis is oriented to make a right-handed coordinate system. The point $\mathbf{p}(\theta,\phi)$ of Σ with longitude θ $(-\pi/2 \leq \theta \leq \pi/2)$ and latitude ϕ $(-\pi/2 \leq \phi \leq \pi/2)$ has Euclidean coordinates

$$\mathbf{p}(\theta,\phi) = (\cos\phi\sin\theta, \sin\phi, \cos\phi\cos\theta)$$

We can measure angles and distances on the viewing sphere using the (orthonormal) spherical frame field

$$\begin{aligned}
\mathbf{E}_1 &= \cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{k} \\
\mathbf{E}_2 &= -\sin\phi\sin\theta\,\mathbf{i} + \cos\phi\,\mathbf{j} - \sin\phi\cos\theta\,\mathbf{k} \\
\mathbf{E}_3 &= \cos\phi\sin\theta\,\mathbf{i} + \sin\phi\,\mathbf{j} + \cos\phi\cos\theta\,\mathbf{k}
\end{aligned}$$

where $\mathbf{i},\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the x, y, and z axes respectively. $\mathbf{E}_1$ points due east along a circle of latitude, $\mathbf{E}_2$ points due north along a circle of longitude. At a point (θ,ϕ) on the viewsphere, the vectors $\mathbf{E}_1$, $\mathbf{E}_2$ provide an orthonormal basis for the tangent space at the point. We define θ_t, the tilt angle, as the counterclockwise rotation needed to align the $\mathbf{E}_1$, $\mathbf{E}_2$ vectors with the $\mathbf{t},\mathbf{b}$ vectors.

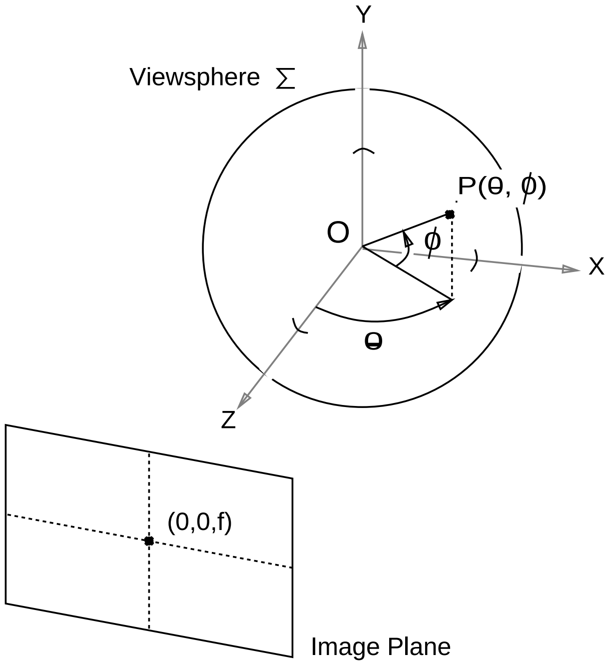

Figure 1: **Transforming image plane coordinates into longitude/lattitude spherical coordinates.** The optical center is at the origin, O, of the coordinate system. The Z-axis intersects the image plane $Z = f$ at the point $(0, 0, f)$. Point $P = P(\theta, \phi)$ lies on the surface of the unit viewsphere Σ. Point P corresponds to a point $Q = (X, Y, f)$ (not shown) in the image plane, i.e. point P is the intersection of ray $\overline{OQ}$ with the surface of the unit viewsphere. The projection of P onto the Z-X plane makes a counterclockwise angle of θ with respect to the Z-axis.

We next introduce the dual 1-forms of the $(\mathbf{E}_1, \mathbf{E}_2)$ basis, $D_1 = \cos\phi\, d\theta$ and $D_2 = d\phi$. If at a point (θ_0, ϕ_0), one takes a small step $(\Delta\theta, \Delta\phi)$, then the dual 1-forms tell us that the component of this vector in the $\mathbf{E}_1$ direction, Δx, is $\cos\phi_0\, \Delta\theta$ and the component in the $\mathbf{E}_2$ direction, Δy, is $\Delta\phi$.

We can identify the point $\mathbf{p}(\theta, \phi)$ of the viewsphere with the direction (X, Y, f) where X, Y are the coordinates of the projection on an image plane at distance f from the optical center. We then get the following simple relationship between θ, ϕ, and the image plane coordinates:

$$\theta = \tan^{-1}\frac{X}{f} \tag{1}$$

$$\phi = \sin^{-1}\frac{Y}{R} \tag{2}$$

where $R = \sqrt{X^2 + Y^2 + f^2}$. This defines a mapping from (X, Y) to (θ, ϕ) . We compute its Jacobian which can be used to approximate the step $(\Delta\theta, \Delta\phi)$, associated with the step in the image plane, $(\Delta X, \Delta Y)$.

$$\begin{bmatrix} \Delta\theta \\ \Delta\phi \end{bmatrix} \approx \begin{bmatrix} \frac{f}{X^2+f^2} & 0 \\ -\frac{XY}{R^2\sqrt{X^2+f^2}} & \frac{\sqrt{X^2+f^2}}{R^2} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix}$$

We next need to find the step $(\Delta x, \Delta y)$ on the viewsphere measured in the orthonormal basis $(\mathbf{E}_1, \mathbf{E}_2)$, associated with the step $(\Delta\theta, \Delta\phi)$. For this we use the 1-forms above. Thus the complete transformation between $(\Delta X, \Delta Y)$ on the image plane and $(\Delta x, \Delta y)$ on the viewing sphere is:

$$\begin{aligned} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} &\approx \begin{bmatrix} \cos\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{f}{X^2+f^2} & 0 \\ -\frac{XY}{R^2\sqrt{X^2+f^2}} & \frac{\sqrt{X^2+f^2}}{R^2} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} \\ &= \begin{bmatrix} \frac{f\cos\phi}{X^2+f^2} & 0 \\ -\frac{XY}{R^2\sqrt{X^2+f^2}} & \frac{\sqrt{X^2+f^2}}{R^2} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} \\ &= \begin{bmatrix} \frac{f}{R\sqrt{X^2+f^2}} & 0 \\ -\frac{XY}{R^2\sqrt{X^2+f^2}} & \frac{\sqrt{X^2+f^2}}{R^2} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} \\ &= J \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} \end{aligned} \tag{3}$$

Note that for the point $(0,0)$ where the optical axis intersects the image plane, J is just $1/f$ times the identity matrix and this transformation reduces to simply scaling $(\Delta X, \Delta Y)$ by a factor of $1/f$. For other points on the image plane, we get a distortion which becomes significant for large fields of view. Leonardo da Vinci was the first to point out this effect of position on the image plane. He drew a distinction between natural perspective (onto a sphere) and artificial perspective (onto a plane) as used in constructing a painting.

Converting from affine transforms $\overline{A}$ on the image plane to affine transforms $\overline{\overline{A}}$ measured with respect to the bases $(\mathbf{E}_{i1}, \mathbf{E}_{i2})$, $i = 1, 2$ on the image sphere is simple:

$$\overline{\overline{A}} = J_2 \bar{A} J_1^{-1} = G(\bar{A})$$

where J_i is as defined in equation **??**.