- Affine Transformations- Definition and Properties
- Projective Geometry- Definition and Properties
- Introduction to Homogenous Coordinates

1 Affine Transformation

> "Like Moliere's character who is pleasantly surprised to find that he has been speaking prose all his life, you may be pleasantly surprised to learn find that you have been learning affine geometry most of your life."

Theorem: A transformation such that the images of every three collinear points are themselves collinear is an affine transformation.

PROPERTIES PRESERVED

Parallelism

Collinearity - property of being a straight line

Incidence - whether or not a point lies on a curve

Ratio of length along a line preserved
—-A——B—————-C–D——-
—-A'–B'————C'-D'———

Ratio AB/CD = A'B'/C'D' preserved:

Midpoints : If α is an affine transformation and M is the midpoint of points A and B, then $\alpha(M)$ is the midpoint of $\alpha(A)$ and $\alpha(B)$.

One may see this theorem elegantly applied in proving that the medians of any triangle intersect. (Trivial in the case of an equilateral triangle, and any triangle may be considered an affine transformation of an equilateral)

PROPERTIES NOT PRESERVED

1. Length
2. Angle
3. Area

General Affine Transformation

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix} \begin{vmatrix} x1 \\ y1 \end{vmatrix} + \begin{vmatrix} d1 \\ d2 \end{vmatrix}$$

Question: If we know the correspondence between points, how can we estimate the Affine transform?

- We can estimate the shift (d) by taking the mean of a set of points and comparing it with the mean of the second set.
- To determine A, and if we have a lot of points, we can find the best match through a least squares fit.

min $\sum_{i=1}^{n} \|x' - Ax\|^2$

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EUCLIDEAN is a subset of AFFINE

AFFINE is a subset of PROJECTIVE

Whereas, Euclidean geometry formed a special case of Affine Geometry, Affine likewise forms a special case of the Projective Geometry. Since in Projective geometry, incidence and parallelism are not preserved, parallel lines interesect at infinity at an "ideal" point. This notion, which extends beyond Euclidean concepts, requires the introduction of HOMOGENOUS COORDINATES

2.1 Homogeneous Coordinates

"In homogeneous coordinates, we add a third coordinate to a point. Instead of being represented by a pair of number (x,y), each point is represented by a triplet (x,y,W). At the same time, we say that two sets of homogenous coordinates (x,y,W) and (x', y', W') represent the same point iff one is a multiple of the other"

(Computer Graphics- Foley et al)

Points in P2 projective space are represented as:

$$\begin{vmatrix} x1 \\ x2 \\ x3 \end{vmatrix}$$

If we divide through by x3 we get the point in $\Re2$

Where do the following two lines intersect in P2?

$$\begin{vmatrix} a1 & a2 & a3 \\ b1 & b2 & b3 \end{vmatrix} * \begin{vmatrix} x1 \\ x2 \\ x3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

Solve the system of equations. If a and b are not scalar multiples of eachother, they intersect when x3 = 0; ie the "ideal point."

In general the projective transformation P2 $\rightarrow$ P2 looks like:

$$\begin{vmatrix} x'1 \\ x'2 \\ x'3 \end{vmatrix} = \begin{vmatrix} P11 & P12 & P13 \\ P21 & P22 & P23 \\ P31 & P32 & P23 \end{vmatrix} * \begin{vmatrix} x1 \\ x2 \\ x3 \end{vmatrix}$$

A transformation of P2 $\rightarrow$ P2 has 8 paramters (not 9) since one is a scalar multiple.

2.2 Nice things about homogeneous Coordinates

1. If points are expressed in homogeneous coordinates, all three transformations can me treated as matrix multiplications.
2. In projective space, complicated expressions become simpler to handle
3. With regard to the upper left hand 2x2 submatrix:

$$\begin{vmatrix} P11 & P12 \\ P21 & P22 \end{vmatrix}$$

If this submatrix is orthogonal, the transformation preserves angles and lengths (ie Euclidean).

If the matrix is of the form:

$$\begin{vmatrix} P11 & P12 & P13 \\ P21 & P22 & P23 \\ 0 & 0 & 1 \end{vmatrix}$$

This corresponds to an arbitrary sequence of rotation, translation, and scale matrices. In otherwords, this is an affine transformation.

We can therefore see that the Affine is a special case of the projective transform.

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